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Integral Equations and Operator Theory

Mapping properties of the Boltzmann collision operator

R. Duduchava and S. Rjasanow

Abstract. In the present paper we study the mapping properties of the non-linear Boltzmann collision operator on a scale of weighted Bessel potential spaces.

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Introduction

We consider the classical Boltzmann equation for a simple, dilute gas of particles [4]

$$f_t + (v, \mathrm{grad}_x f) = Q(f,f) \tag{0.1}$$

which describes the time evolution of the particle density $f(t,x,v)$

$$f \;:\; \mathbb{R}_+ \times \Omega \times \mathbb{R}^3 \to \mathbb{R}_+ .$$

Here $\mathbb{R}_+$ denotes the set of non-negative real numbers and $\Omega \subset \mathbb{R}^3$ is a domain in physical space. The right-hand side of equation (0.1), known as the collision integral or the collision term, is of the form

$$Q(f,f)(v) = \int\limits_{\mathbb{R}^3}\int\limits_{S^2} B(v,w,e)\Big(f(v')f(w') - f(v)f(w)\Big)\, de\, dw\,. \tag{0.2}$$

Note that $Q(f,f)$ depends on t and x only as parameters, so we have omitted this dependence in (0.2) for conciseness. The following notations have been used in (0.2): $v,w \in \mathbb{R}^3$ are the pre-collision velocities, $e \in S^2 \subset \mathbb{R}^3$ is a unit vector, $v',w' \in \mathbb{R}^3$ are the post-collision velocities and $B(v,w,e)$ is the collision kernel. The operator $Q(f,f)$ represents the change of the distribution function $f(t,x,v)$

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due to the binary collisions between particles. A single collision results in a change of the velocities of the colliding partners $v, w \rightarrow v', w'$ with

$$v' = \frac{1}{2}\Big(v + w + |u|\,e\Big),\;\; w' = \frac{1}{2}\Big(v + w - |u|\,e\Big)\,,$$

where $u = v - w$ denotes the relative velocity. The Boltzmann equation (0.1) is subjected to an initial condition

$$f(0, x, v) = f_0(x, v),\;\; x \in \Omega,\;\; v \in \mathbb{R}^3$$

and to the boundary conditions on $\Gamma = \partial\Omega$. The kernel $B(v, w, e)$ can be written as

$$B(v, w, e) = B(|u|, \mu) = |u|\sigma(|u|, \mu),\;\; \mu = \cos(\theta) = \frac{(u, e)}{|u|}\,.$$

The function $\sigma : \mathbb{R}_+ \times [-1, 1] \rightarrow \mathbb{R}_+$ is the differential cross-section and θ is the scattering angle. Some special models for the kernel are as follows:

1. The **hard spheres model** is described by the kernel

$$B(|u|, \mu) = \frac{d^2}{4}|u|\,,$$

   where d denotes the diameter of the particles.
2. The kernel

$$B(|u|, \mu) = |u|^{1 - 4/m} g_m(\mu)\,,\;\; m > 1 \tag{0.3}$$

   corresponds to the **inverse power cut-off potential** (see [6]) of the interaction. m denotes the order of the potential and $g_m \in \mathbb{L}_1([-1, 1])$ is a given function of the scattering angle only.
3. The special case of $m = 4$ in (0.3) corresponds to the **Maxwell pseudo-molecules** with

$$B(|u|, \mu) = g_4(\mu)\,.$$

   The collision kernel $B(|u|, \mu)$ here does not depend on the relative speed $|u|$.
4. The **Variable Hard Spheres model** (VHS) (see [1]) has an isotropic kernel

$$B(|u|, \mu) = C_\lambda |u|^\lambda\,,\;\; -3 < \lambda \le 1. \tag{0.4}$$

   The model includes as particular cases the hard spheres model for $\lambda = 1$ and the Maxwell pseudo–molecules for $\lambda = 0$.

The collision integral (0.2) decomposes into the natural gain and loss parts

$$Q(f, f)(v) = Q_+(f, f)(v) - Q_-(f, f)(v)\,,$$

where the bilinear operators $Q_+(\cdot, \cdot)\,,\; Q_-(\cdot, \cdot)$ are

$$Q_+(f, g)(v) = \int\limits_{\mathbb{R}^3}\int\limits_{S^2} B(|u|, \mu) f(v') g(w')\, de\, dw \tag{0.5}$$

and

$$Q_-(f,g)(v) = \int\limits_{\mathbb{R}^3}\int\limits_{S^2} B(|u|,\mu) f(v) g(w)\, de\, dw\,. \tag{0.6}$$

We will also consider the linear operators $Q_+(f)[\cdot]$ and $Q_-(f)[\cdot]$ acting on g for a fixed function f. Before we begin the study of the mapping properties of the operators $Q_+(f)$ and $Q_-(f)$, we discuss the results known from the literature.

A. Bobylev and V. Vedenjapin [2] proved the following pointwise estimate for the gain part of the collision operator

$$Q_+(f,f)(v) \leq C\, \|f \,|\, \mathbb{L}_\infty\| \int\limits_{\mathbb{R}^3} B_{tot}(|v-w|) f(w)\, dw \tag{0.7}$$

with

$$B_{tot}(|v-w|) = \int\limits_{\mathbb{S}^2} B(|u|,\mu)\, de\,.$$

For Maxwell pseudo-molecules (0.7) leads to the boundedness

$$\|Q_+(f,f) \,|\, \mathbb{L}_\infty\| \leq C\, B_{tot}\, \|f \,|\, \mathbb{L}_\infty\|\, \|f \,|\, \mathbb{L}_1\|\ .$$

T. Gustafsson [8] considered the weighted spaces

$$\mathbb{L}_p^{\langle\nu\rangle} = \mathbb{L}_p^{\langle\nu\rangle}(\mathbb{R}^3) = \left\{g:\ \mathbb{R}^3 \to \mathbb{C}, \langle\cdot\rangle^\nu g \in \mathbb{L}_p(\mathbb{R}^3)\right\}\ ,\ \langle v\rangle^\nu := (1+|v|^2)^{\nu/2}$$

and the following kernels

$$B(|u|,\mu) = |u|^\lambda g(\mu)\,,\ 0 < \lambda \leq 1\,,\ g \in \mathbb{L}_1([-1,1])\,. \tag{0.8}$$

He proved that the operator

$$Q_+\ :\ \left(\mathbb{L}_1^{\langle\nu+\lambda\rangle} \bigcap \mathbb{L}_p^{\langle\nu+\lambda\rangle}\right) \times \left(\mathbb{L}_1^{\langle\nu+\lambda\rangle} \bigcap \mathbb{L}_p^{\langle\nu+\lambda\rangle}\right) \to \mathbb{L}_p^{\langle\nu\rangle} \tag{0.9}$$

is bounded for the weighted $\mathbb{L}_p$ spaces with $1 \leq p < \infty$ and $0 \leq \nu < \infty$. As we see, T. Gustafsson proved that Q_+ is an operator of the order 0.

P.L. Lions [10] proved the estimates

$$\begin{aligned} \left\|Q_+(f,g) \,|\, \mathbb{W}^1\right\| &\leq C\, \|f \,|\, \mathbb{L}_1\|\, \|g \,|\, \mathbb{L}_2\|\ , \\ \left\|Q_+(f,g) \,|\, \mathbb{W}^1\right\| &\leq C\, \|f \,|\, \mathbb{L}_2\|\, \|g \,|\, \mathbb{L}_1\|\ , \end{aligned} \tag{0.10}$$

provided the collision kernel $B(|u|,\mu)$ satisfies

$$B(|u|,\mu) \in \mathbb{C}_0^\infty(\mathbb{R}_+ \times [-1,1])\,; \tag{0.11}$$

i.e. kernels are infinitely smooth with respect to both variables $|u|$ and μ and have compact supports with respect to the variable u. It is easy to ascertain that the conditions in (0.11) are too restrictive to cover the models of interaction described above. The estimates (0.10) can be written in an equivalent and compact form as the continuity of the mapping

$$Q_+\ :\ \mathbb{L}_1 \times \mathbb{L}_2 \to \mathbb{W}^1\ , \tag{0.12}$$

where $\mathbb{W}^1 := \mathbb{W}^1(\mathbb{R}^3)$ is the Sobolev space (see Section 1). If $f \in \mathbb{L}_1$ is fixed, the boundedness (0.10) shows that $Q_+(f)$ is an operator of the order -1.

J. Struckmeier [13] proved the boundedness property of the gain term of the Boltzmann collision operator as being similar to (0.9) in the case $B(|u|, \mu) = const$ which corresponds to the Maxwell molecules

$$Q_+ \ : \ \left(\mathbb{L}_\infty \bigcap \mathbb{L}_1\right) \times \left(\mathbb{L}_\infty \bigcap \mathbb{L}_1\right) \to \mathbb{L}_\infty \, .$$

B. Wennberg [17] proved the boundedness property of the operator $Q_+(f)$ as being similar to (0.12) for the collision kernel (0.8)

$$Q_+ \ : \ \left(\mathbb{L}_1^{\langle\nu+1\rangle} \bigcap \mathbb{L}_p^{\langle\nu+1\rangle}\right) \times \left(\mathbb{L}_1^{\langle\nu+1\rangle} \bigcap \mathbb{L}_p^{\langle\nu+1\rangle}\right) \to \mathbb{W}^{1,\langle\nu\rangle} \tag{0.13}$$

if there are the restrictions

$$\frac{1}{2} < \lambda \leq 1 \, , \ p > \frac{6}{2\lambda - 1} \, .$$

F. Bouchut and L. Desvillettes [3] proved the following smoothing property of the operator Q_+

$$Q_+ \ : \ \mathbb{L}_2^{\langle\nu+1\rangle} \times \mathbb{L}_2^{\langle\nu+1\rangle} \to \mathbb{W}_0^1 \, , \qquad \nu \geq 0$$

for the collision kernel (0.8) with $\lambda \geq -3/2$ and $g \in \mathbb{L}_2([-1,1])$. We use the notation $\mathbb{W}_0^1$ for the homogeneous Sobolev space considered in [3]. In contrast the usual Sobolev space $\mathbb{W}^1$ the homogeneous space is defined with a seminorm, containing $\mathbb{L}_p$ norms of only the derivatives of functions. This makes an essential difference between the spaces and for the corresponding boundedness properties.

Using a different method, X. Lu [11] proved the smoothness result as being similar to (0.13) for the gain term of the Boltzmann equation.

In the present paper we prove the following boundedness property of the operator $Q_+(f)$ in the scale of weighted Bessel potential spaces: the operators

$$Q_+ \ : \ \mathbb{H}^{\sigma,\langle\nu\rangle} \times \mathbb{H}^{s,\langle\nu\rangle} \ \to \mathbb{H}^{s+1} \, , \quad -\frac{1}{2} < \lambda \leq 1 \, , \quad \nu > \frac{3}{2} + \lambda \, , \tag{0.14}$$

$$Q_+ \ : \ \mathbb{H}_\infty^{\sigma,\langle\nu\rangle} \times \mathbb{H}^{s,\langle\nu\rangle} \ \to \mathbb{H}^{s+1} \, , \quad -2 < \lambda \leq 1 \, , \quad \nu > 3 + \lambda \, , \tag{0.15}$$

$$\text{for all} \quad \sigma \geq 0 \, , \quad -\sigma \leq s \leq \sigma$$

are bounded. In particular, are bounded

$$Q_+ \ : \ \mathbb{L}_2^{\langle\nu\rangle} \times \mathbb{L}_2^{\langle\nu\rangle} \ \to \mathbb{H}^1 \, , \qquad -\frac{1}{2} < \lambda \leq 1 \, , \quad \nu > \frac{3}{2} + \lambda \, , \tag{0.16}$$

$$Q_+ \ : \ \mathbb{L}_\infty^{\langle\nu\rangle} \times \mathbb{L}_2^{\langle\nu\rangle} \ \to \mathbb{H}^1 \, , \qquad -2 < \lambda \leq 1 \, , \quad \nu > 3 + \lambda \, . \tag{0.17}$$

Note that constraints imposed on the density f in (0.14) are less restrictive on the behavior of the density at infinity than the classical condition of finite kinetic energy in the phase space $\Omega \times \mathbb{R}^3$:

$$\int_{\Omega}\int_{\mathbb{R}^3} |v|^2 f(t,x,v)\, dv\, dx < \infty\,. \tag{0.18}$$

In fact, if $f(t,x,v) = \mathcal{O}(\langle v\rangle^{-\mu})$, then conditions in (0.16) imply $2(\nu-\mu) < -3$ or $\mu > \frac{3}{2} + \nu > 3 + \lambda$, while condition (0.18) implies $2 - \mu < -3$ or $\mu > 5$ even if $\Omega \subset \mathbb{R}^3$ is a compact set.

The paper is organised as follows. In Section 1 we introduce the function spaces, the three-dimensional Fourier transform and the pseudodifferential operators. Furthermore we formulate the mapping properties of the pseudodifferential operators using the asymptotic behaviour of their symbols. Then, in Section 2, we deal with the gain part of the collision operator, construct its adjoint and prove the main boundedness result formulated in (0.14). In Section 3 we consider the loss part of the collision operator.

1. Preliminaries

Let $g: \mathbb{R}^3 \to \mathbb{C}$ be a complex-valued function, $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$ a multi-index of nonnegative integers and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. We use $\partial^\alpha g$ to denote a mixed partial derivative of g of the order $|\alpha|$

$$\partial^\alpha g = \frac{\partial^{|\alpha|} g}{\partial^{\alpha_1} v_1 \partial^{\alpha_2} v_2 \partial^{\alpha_3} v_3}\,.$$

We will use the inequality $\alpha \preceq \beta$ for two multi-indices in the following sense

$$\alpha \preceq \beta \;\Leftrightarrow\; \alpha_j \le \beta_j\,, \quad j = 1,2,3\,.$$

Later we will need the Leibnitz formula for the multidimensional derivative of the product of two functions f and g

$$\partial^\alpha (f\cdot g) = \sum_{\beta \preceq \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} (\partial^\beta f)\cdot(\partial^{\alpha-\beta} g) \tag{1.1}$$

with the binomial coefficients

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\alpha_1!\,\alpha_2!\,\alpha_3!}{\beta_1!\,\beta_2!\,\beta_3!\,(\alpha_1-\beta_1)!\,(\alpha_2-\beta_2)!\,(\alpha_3-\beta_3)!}\,.$$

1.1. Function spaces

Let $1 \le p < \infty$. The **classical** $\mathbb{L}_p = \mathbb{L}_p(\mathbb{R}^3)$ **space** consists of functions g having the property that the following Lebesgue integral is finite

$$\mathbb{L}_p = \mathbb{L}_p(\mathbb{R}^3) = \Big\{ g:\ \mathbb{R}^3 \to \mathbb{C}\,,\ \int_{\mathbb{R}^3} |g(v)|^p\, dv < \infty \Big\}.$$

The $\mathbb{L}_p$ norm of the function g is defined by

$$\|g\,|\,\mathbb{L}_p\| = \left(\int\limits_{\mathbb{R}^3} |g(v)|^p\,dv\right)^{1/p}. \tag{1.2}$$

For $p=\infty$, we use the canonical generalisation of (1.2)

$$\|g\,|\,\mathbb{L}_\infty\| = \operatorname{ess}\sup_{v\in\mathbb{R}^3} |g(v)|\,.$$

The following abbreviation will often be used

$$\langle v\rangle^\nu = (1+|v|^2)^{\nu/2}\,,\quad v\in\mathbb{R}^3\,,\quad \nu\in\mathbb{R}\,.$$

The symbol $|v| = (v_1^2+v_2^2+v_3^2)^{1/2}$ denotes here the length of the three-dimensional vector v. The **weighted** $\mathbb{L}_p^{\langle\nu\rangle}$ **spaces** are defined for $1\le p\le\infty$ as follows

$$\mathbb{L}_p^{\langle\nu\rangle} = \mathbb{L}_p^{\langle\nu\rangle}(\mathbb{R}^3) = \Big\{g:\ \mathbb{R}^3\to\mathbb{C},\ \langle\cdot\rangle^\nu g\in\mathbb{L}_p\Big\}\,.$$

The corresponding norm is

$$\Big\|g\,|\,\mathbb{L}_p^{\langle\nu\rangle}\Big\| = \|\langle\cdot\rangle^\nu g\,|\,\mathbb{L}_p\|\,.$$

The **classical Sobolev space** $\mathbb{W}_p^m = \mathbb{W}_p^m(\mathbb{R}^3)$ consists of functions with the following property

$$\mathbb{W}_p^m = \mathbb{W}_p^m(\mathbb{R}^3) = \Big\{g:\ \mathbb{R}^3\to\mathbb{C},\ \partial^\alpha g\in\mathbb{L}_p\,,\ \forall\alpha:\ |\alpha|\le m\Big\}. \tag{1.3}$$

The norm in the Sobolev space $\mathbb{W}_p^m$ is defined as follows

$$\big\|g\,|\,\mathbb{W}_p^m\big\| = \left(\sum_{|\alpha|\le m} \|\partial^\alpha g\,|\,\mathbb{L}_p\|^p\right)^{1/p}. \tag{1.4}$$

The corresponding **weighted Sobolev space** $\mathbb{W}_p^{m,\langle\nu\rangle} = \mathbb{W}_p^{m,\langle\nu\rangle}(\mathbb{R}^3)$ is defined via

$$\mathbb{W}_p^{m,\langle\nu\rangle} = \mathbb{W}_p^{m,\langle\nu\rangle}(\mathbb{R}^3) = \Big\{g:\ \mathbb{R}^3\to\mathbb{C},\ \partial^\alpha(\langle\cdot\rangle^\nu g)\in\mathbb{L}_p\,,\ \forall\alpha:\ |\alpha|\le m\Big\} \tag{1.5}$$

and has the norm

$$\Big\|g\,|\,\mathbb{W}_p^{m,\langle\nu\rangle}\Big\| = \left(\sum_{|\alpha|\le m} \|\partial^\alpha(\langle\cdot\rangle^\nu g)\,|\,\mathbb{L}_p\|^p\right)^{1/p}. \tag{1.6}$$

For the Sobolev spaces $\mathbb{W}_2^m$ and $\mathbb{W}_2^{m,\langle\nu\rangle}$ the notations $\mathbb{W}^m$ and $\mathbb{W}^{m,\langle\nu\rangle}$ are used respectively.

Remark 1. *The following norm in* $\mathbb{W}_p^{m,\langle\nu\rangle}$ *is equivalent to norm* (1.6)*:*

$$\Big\|g\,|\,\mathbb{W}_p^{m,\langle\nu\rangle}\Big\| = \left(\sum_{|\alpha|\le m} \|\langle\cdot\rangle^\nu\partial^\alpha g\,|\,\mathbb{L}_p\|^p\right)^{1/p}.$$

The **Schwartz space** $\mathbb{S} = \mathbb{S}(\mathbb{R}^3)$ of rapidly decreasing smooth test functions is defined as follows:

$$\mathbb{S} = \mathbb{S}(\mathbb{R}^3) = \left\{ g \in \mathbb{C}^\infty(\mathbb{R}^3) : \ |\langle v\rangle^m \partial^\beta g(v)| \leq C_{m,\beta} \right\},$$

with arbitrary $m \in \mathbb{N}_0$, $\beta \in \mathbb{N}_0^3$, $v \in \mathbb{R}^3$ and with some positive constants $C_{m,\beta}$. A sequence $\{g_n\}$, $n \in \mathbb{N}$ of functions from $\mathbb{S}$ is said to converge to zero ($g_n \to 0$) in $\mathbb{S}$ if for each compact set $\Omega \subset \mathbb{R}^3$, and for all $m \in \mathbb{N}_0$, $\beta \in \mathbb{N}_0^3$ the sequences $\{\langle v\rangle^m \partial^\beta g_n\}$, $n \in \mathbb{N}$ converge to zero uniformly in Ω. The adjoint space $\mathbb{S}' = \mathbb{S}'(\mathbb{R}^3)$ is called the **space of tempered distributions**. If, for example, $\varphi \in \mathbb{C}(\mathbb{R}^3)$ is a continuous function with the property

$$\varphi(v) = \mathcal{O}(\langle v\rangle^a), \quad |v| \to \infty$$

for some $a \in \mathbb{R}$, then φ defines a regular distribution over $\mathbb{S}$ as follows:

$$\langle g, \varphi\rangle_{\mathbb{L}_2} = \int\limits_{\mathbb{R}^3} g(v)\,\overline{\varphi(v)}\,dv, \quad \forall g \in \mathbb{S}.$$

We will use the same notation even for a non–regular distribution $\varphi \in \mathbb{S}'$, bearing in mind the duality between the space of test functions and the space of distributions under the integral. The **space** $\mathbb{C}_0^\infty = \mathbb{C}_0^\infty(\mathbb{R}^3)$ of smooth test functions with compact supports is a proper subset of $\mathbb{S}$ and its dual space of distributions $\mathbb{D} = \mathbb{D}(\mathbb{R}^3)$ contains the space of tempered distributions as a proper subset

$$\mathbb{C}_0^\infty(\mathbb{R}^3) \subset \mathbb{S}(\mathbb{R}^3) \subset \mathbb{S}'(\mathbb{R}^3) \subset \mathbb{D}'(\mathbb{R}^3).$$

Let $\mu = m + \nu$, where $m = 0, 1, \ldots$ is an integer and $0 \leq \nu < 1$. The space of Hölder functions $\mathbb{C}^\mu$ is defined as follows

$$\mathbb{C}^\mu = \mathbb{C}^\mu(\mathbb{R}^3) = \left\{ g \in \mathbb{C}(\mathbb{R}^3) \ : \ \|g \,|\, \mathbb{C}^\mu\| < \infty \right\}$$

and is endowed with the norm

$$\|g \,|\, \mathbb{C}^\mu\| := \sum_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^3} |\partial^\alpha g(x)| + \sum_{|\alpha| = m} \|\partial^\alpha g \,|\, \mathbb{C}^\nu\|,$$

$$\|\varphi \,|\, \mathbb{C}^\nu\| := \sup_{x \in \mathbb{R}^3} |\varphi(x)| + \sup_{\substack{x,h \in \mathbb{R}^3 \\ h \neq 0}} \frac{|\varphi(x+h) - \varphi(x)|}{|h|^\nu}.$$

1.2. The Fourier transform and further spaces

The three-dimensional Fourier transform of the function g is defined as

$$\hat{g}(\xi) = \mathcal{F}_{v \to \xi}[g(v)](\xi) = \int\limits_{\mathbb{R}^3} g(v) e^{\imath (v, \xi)}\,dv,$$

where (v, ξ) denotes the three-dimensional scalar product. The corresponding inverse Fourier transform is then

$$g(v) = \mathcal{F}^{-1}_{\xi \to v}[\hat{g}(\xi)](v) = \frac{1}{(2\pi)^3} \int\limits_{\mathbb{R}^3} \hat{g}(\xi) e^{-\imath (\xi, v)}\,d\xi. \tag{1.7}$$

The Fourier transform $\hat{g}$ exists, at least, for $g \in \mathbb{L}_1$. It is well known that the Schwartz space $\mathbb{S}$ is invariant under the Fourier transform $\mathcal{F}$ and under its inverse $\mathcal{F}^{-1}$

$$\mathcal{F}^{\pm 1} : \ \mathbb{S} \to \mathbb{S} \, . \tag{1.8}$$

The further properties of the Fourier transform are

$$\begin{aligned} \mathcal{F}_{v\to\xi}[\partial^\alpha g(v)](\xi) &= (-\imath\, \xi)^\alpha \mathcal{F}_{v\to\xi}[g](\xi) \, , \\ \partial^\alpha \mathcal{F}_{v\to\xi}[g(v)](\xi) &= (\imath)^{|\alpha|} \mathcal{F}_{v\to\xi}[v^\alpha g(v)](\xi) \, , \end{aligned} \tag{1.9}$$

which hold for the arbitrary test function $g \in \mathbb{S}$. From the well known Plancherel equality

$$\langle f, g \rangle_{\mathbb{L}_2} = \int\limits_{\mathbb{R}^3} f(v) \, \overline{g(v)} \, dv = (2\pi)^{-3} \langle \hat{f}, \hat{g} \rangle_{\mathbb{L}_2} \tag{1.10}$$

which holds for every $f, g \in \mathbb{L}_2$ we obtain the well known Parseval identity

$$\| f \,|\, \mathbb{L}_2 \| = (2\pi)^{-3/2} \left\| \hat{f} \,|\, \mathbb{L}_2 \right\| \, . \tag{1.11}$$

Thus, the mappings $(2\pi)^{-3/2}\mathcal{F}$ and $(2\pi)^{3/2}\mathcal{F}^{-1}$ are isometrical isomorphisms in $\mathbb{L}_2$. The Fourier transform of a tempered distribution $\varphi \in \mathbb{S}'$ is given by the following definition

$$\langle g, \hat{\varphi} \rangle_{\mathbb{L}_2} = \langle \hat{g}, \varphi \rangle_{\mathbb{L}_2} \, , \ \forall g \in \mathbb{S}$$

and has the property

$$\mathcal{F}^{\pm 1} : \ \mathbb{S}' \to \mathbb{S}' \, .$$

With the help of the Fourier transform we define the **Bessel potential** space $\mathbb{H}^s_p = \mathbb{H}^s_p(\mathbb{R}^3)$, $s \in \mathbb{R}$, $1 < p < \infty$ of the tempered distributions by

$$\mathbb{H}^s_p = \mathbb{H}^s_p(\mathbb{R}^3) = \left\{ \ \varphi \in \mathbb{S}' \ : \ \|\varphi \,|\, \mathbb{H}^s_p\| < \infty \right\} ,$$

where the norm in $\mathbb{H}^s_p$ is defined as follows:

$$\|\varphi \,|\, \mathbb{H}^s_p\| := \left(\int\limits_{\mathbb{R}^3} \left| \mathcal{F}^{-1}_{\xi\to y} \left[\langle \xi \rangle^s \hat{\varphi}(\xi) \right](y) \right|^p \, dy \right)^{\frac{1}{p}} \, .$$

In a particular case $p = 2$, due to (1.11), the norm in the space $\mathbb{H}^s = \mathbb{H}^s_2(\mathbb{R}^3)$ acquires a simpler form (modulo the factor $(2\pi)^{-3/2}$)

$$\|\varphi \,|\, \mathbb{H}^s\| = \left(\int\limits_{\mathbb{R}^3} \langle \xi \rangle^{2s} |\hat{\varphi}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} \, . \tag{1.12}$$

Finally, for all $s,\nu \in \mathbb{R}$ we define the **weighted Bessel potential** space $\mathbb{H}_p^{s,\langle\nu\rangle} = \mathbb{H}_p^{s,\langle\nu\rangle}(\mathbb{R}^3)$ (or $\mathbb{H}^{s,\langle\nu\rangle} = \mathbb{H}_2^{s,\langle\nu\rangle}(\mathbb{R}^3)$ when $p=2$) via

$$\mathbb{H}_p^{s,\langle\nu\rangle} = \mathbb{H}_p^{s,\langle\nu\rangle}(\mathbb{R}^3) := \left\{\varphi \in \mathbb{S}' \;:\; \left\|\varphi \,|\, \mathbb{H}_p^{s,\langle\nu\rangle}\right\| := \left\|\langle\cdot\rangle^\nu \varphi \,|\, \mathbb{H}_p^s\right\| < \infty\right\}. \tag{1.13}$$

For an integer $s = m \in \mathbb{N}$ the Bessel potential spaces $\mathbb{H}_p^m$ and $\mathbb{H}_p^{s,\langle\nu\rangle}$ become the classical Sobolev spaces $\mathbb{W}_p^m$ and $\mathbb{W}_p^{m,\langle\nu\rangle}$ (see (1.3) and (1.5) respectively) with the equivalent norms (1.4) and (1.6) (see [16, § 2.5.6]). The following embedding property of the weighted Bessel potential spaces is almost trivial:

$$\mathbb{H}_p^{s,\langle\nu\rangle} \subseteq \mathbb{H}_p^s \subseteq \mathbb{H}_p^{s,\langle\mu\rangle}, \quad \forall s \in \mathbb{R}, \quad \forall \nu \geq 0, \quad \forall \mu \leq 0, \quad \forall p \in [1,\infty),$$

while the next one is less trivial and is known as the Sobolev lemma (see [15, § 2.7.1]):

$$\mathbb{H}_p^{s+\frac{3}{p}}(\mathbb{R}^3) \subset \mathbb{C}^s(\mathbb{R}^3), \quad \forall s > 0, \quad \forall p \in [1,\infty). \tag{1.14}$$

Note that if $\mathbb{X}^*$ denotes the dual (adjoint) space to a Banach space $\mathbb{X}$, then

$$\left(\mathbb{H}_p^{s,\langle\nu\rangle}\right)^* = \mathbb{H}_{p'}^{-s,\langle-\nu\rangle}, \quad p' := \frac{p}{p-1}$$

and, in particular, $\left(\mathbb{H}^{s,\langle\nu\rangle}\right)^* = \mathbb{H}^{-s,\langle-\nu\rangle}$. Here we expose a minimal information about interpolation of operators which suffices to our purposes. For details we refer the reader to [16].

Let $\mathbb{G}$ be a category of Banach spaces $\mathbb{X}$ embedded in a common linear space $\mathbb{X} \subset \mathbb{V}$. By an interpolation functor is meant a mapping which associates to a pair $\{\mathbb{X}_0, \mathbb{X}_1\}$ in $\mathbb{G}$ (called an interpolation pair) a function space $\mathbb{F}(\{\mathbb{X}_0, \mathbb{X}_1\})$ such that

1. $\mathbb{X}_0 \cap \mathbb{X}_1 \subset \mathbb{F}(\{\mathbb{X}_0, \mathbb{X}_1\}) \subset \mathbb{X}_0 + \mathbb{X}_1$;
2. For any two interpolation pairs $\{\mathbb{X}_0, \mathbb{X}_1\}$, $\{\mathbb{Y}_0, \mathbb{Y}_1\}$ in $\mathbb{G}$ any operator

$$\mathcal{A} \in \mathcal{L}(\mathbb{X}_0, \mathbb{Y}_0) \cap \mathcal{L}(\mathbb{X}_1, \mathbb{Y}_1)$$

restricted to $\mathbb{F}(\{\mathbb{X}_0, \mathbb{X}_1\})$ belongs to $\mathcal{L}(\mathbb{F}(\{\mathbb{X}_0, \mathbb{X}_1\}), \mathbb{F}(\{\mathbb{Y}_0, \mathbb{Y}_1\}))$.

Furthermore, if there is a constant $C > 0$ such that for any interpolation pairs $\{\mathbb{X}_0, \mathbb{X}_1\}$ and $\{\mathbb{Y}_0, \mathbb{Y}_1\}$ in $\mathbb{G}$ and for any operator $\mathcal{A} \in \mathcal{L}(\mathbb{X}_0, \mathbb{Y}_0) \cap \mathcal{L}(\mathbb{X}_1, \mathbb{Y}_1)$ the inequality

$$\|\mathcal{A}\big|\mathcal{L}(\mathbb{F}(\{\mathbb{X}_0, \mathbb{X}_1\}), \mathbb{F}(\{\mathbb{Y}_0, \mathbb{Y}_1\}))\| \leq C\|\mathcal{A}\big|\mathcal{L}(\mathbb{X}_0, \mathbb{Y}_0)\|^{1-\theta} \, \|\mathcal{A}\big|\mathcal{L}(\mathbb{X}_1, \mathbb{Y}_1)\|^\theta$$

holds, then $\mathbb{F}$ is said to be an interpolation functor of type θ for $0 \leq \theta \leq 1$. There are known many interpolation functors in the literature, but we will apply only a single one (see [16, § 2.4.7] for this result): for arbitrary $s_0, s_1 \in \mathbb{R}$, $1 \leq p \leq \infty$ the complex interpolation functor gives

$$\mathbb{F}(\{\mathbb{H}_p^{s_0}, \mathbb{H}_p^{s_1}\}) := \left[\mathbb{H}_p^{s_0}, \mathbb{H}_p^{s_1}\right]_\theta = \mathbb{H}_p^s, \tag{1.15}$$

where

$$s = (1-\theta)s_0 + \theta s_1, \qquad 0 \leq \theta \leq 1. \tag{1.16}$$

1.3. Pseudodifferential operators

In this subsection we give some basic definitions and properties of pseudodifferential operators (PsDOs for short) for our subsequent applications.

The Hörmander class of symbols $\mathbb{S}^r_{0,0}(\mathbb{R}^3) = \mathbb{S}^r(\mathbb{R}^3)$, $r \in \mathbb{R}$, which is encountered most frequently in the classical theory of PsDOs, consists of functions $a(v, \xi)$ with the following estimates:

$$|\partial^\alpha_\xi \partial^\beta_v a(v, \xi)| < C_{\alpha,\beta} \langle\xi\rangle^r\,, \quad \forall\alpha, \beta \in \mathbb{N}^3\,, \quad \forall v, \xi \in \mathbb{R}^3\,.$$

For a symbol $a \in \mathbb{S}^r(\mathbb{R}^3)$ the pseudodifferential operator

$$\mathcal{A} = \mathcal{A}(v, D) : \ \mathbb{S} \to \mathbb{S}'$$

is defined as follows

$$\mathcal{A}(v, D)[g](v) = \frac{1}{(2\,\pi)^3} \int\limits_{\mathbb{R}^3} a(v, \xi) \mathcal{F}_{z\to\xi}[g(z)](\xi) e^{-\imath(\xi, v)}\, d\xi\,,\ \forall g \in \mathbb{S}\,.$$

If a symbol $a(v, \xi) = a(\xi)$ is independent of the variable v, we deal with a pure convolution operator.

According the Calderon-Vaillancourt theorem (see, e.g., [14, Ch. XII]) the inclusion $a \in \mathbb{S}^r(\mathbb{R}^3)$ ensures that the pseudodifferential operator

$$\mathcal{A}(v, D) : \ \mathbb{H}^s \to \mathbb{H}^{s-r} \tag{1.17}$$

is bounded for all $s \in \mathbb{R}$.

The operators which are connected with the Boltzmann equations are different from the classical PsDOs because they have non-smooth symbols. Most general results on the boundedness of such PsDOs are available in [12]. Again, these results are useless for our purposes because they require some restricted smoothness of symbols in the variable v.

The operators which we encounter in our investigation admit representation as a Bochner integral from parameter-dependent convolution operators (see [7]). To define such a parameter-dependent operator we consider a symbol $a \in \mathbb{C}(\mathbb{R}^3, \mathbb{S}')$, which is a continuous function $a\ :\ \mathbb{R}^3 \to \mathbb{S}'(\mathbb{R}^3)$, and write

$$\mathcal{A}(u, D)[g](v) = \mathcal{F}^{-1}_{\xi\to v}[a(u, \xi)\widehat{g}(\xi)](u, v)\,,\ \forall g \in \mathbb{S}\,. \tag{1.18}$$

It is easy to ascertain that the definition (1.18) is correct, producing the operator $\mathcal{A}(u, D)[g](v) : \ \mathbb{S}(\mathbb{R}^3) \to \mathbb{S}'(\mathbb{R}^3 \times \mathbb{R}^3)$.

The next lemma is a simple consequence of the Parseval identity (1.11) and plays an essential role in the subsequent section.

Lemma 2. *Let $r \in \mathbb{R}$ and assume the symbol $a(u, \xi)$ is uniformly bounded*

$$\operatorname*{ess\ sup}_{u\in\mathbb{R}^3} |a(u, \xi)| \le C_a \langle\xi\rangle^r\,,\ \forall\xi \in \mathbb{R}^3\,. \tag{1.19}$$

Then the pseudodifferential operator $\mathcal{A}$ defined as

$$\mathcal{A}(u, D)[g](v) = \mathcal{F}^{-1}_{\xi\to v}\left[a(u, \xi)\mathcal{F}_{z\to\xi}\left[g(z)\right](\xi)\right](u, v) \tag{1.20}$$

is bounded

$$\mathcal{A}(u,D) \ : \ \mathbb{H}^r \to \mathbb{L}_2$$

in the following sense

$$\operatorname{ess}\sup_{u\in\mathbb{R}^3} \|\mathcal{A}(u,D)[g] \,|\, \mathbb{L}_2\| \leq C_a \, \|g \,|\, \mathbb{H}^r\| \, . \tag{1.21}$$

Proof. Using (1.20) and the Parseval identity (1.11) we obtain

$$\begin{aligned}\|\mathcal{A}(u,D)[g] \,|\, \mathbb{L}_2\| &= \left\|\mathcal{F}^{-1}_{\xi\to v}\left[a(u,\xi)\hat{g}(\xi)\right] \,|\, \mathbb{L}_2\right\| \\ &= (2\pi)^{-3/2}\, \|a(u,\cdot)\hat{g}(\cdot) \,|\, \mathbb{L}_2\| \\ &= (2\pi)^{-3/2}\, \left\|a(u,\cdot)\langle\cdot\rangle^{-r}\langle\cdot\rangle^{r}\hat{g}(\cdot) \,|\, \mathbb{L}_2\right\| \, .\end{aligned}$$

Taking the supremum with respect to u, using (1.19) and the definition (1.12) of the weighted Sobolev norm in $\mathbb{H}^r_2 = \mathbb{H}^r$ leads to the final estimate

$$\begin{aligned}\operatorname{ess}\sup_{u\in\mathbb{R}^3} \|\mathcal{A}(u,D)[g] \,|\, \mathbb{L}_2\| &\leq (2\pi)^{-3/2} C_a \, \|\langle\cdot\rangle^{r}\hat{g}(\cdot) \,|\, \mathbb{L}_2\| \\ &= C_a \left\|\mathcal{F}^{-1}_{\xi\to v}\left[\langle\xi\rangle^{r}\hat{g}(\xi)\right] \,|\, \mathbb{L}_2\right\| = C_a \, \|g \,|\, \mathbb{H}^r\| \, . \quad \blacksquare\end{aligned}$$

2. The gain part of the collision integral

The next lemma is, perhaps, well known for experts in Boltzmann equation. Since we were not able to find a relevant reference in literature and in our proofs we quote Lemma 3 several times, below we expose the result with a proof.

Lemma 3. *An arbitrary partial derivative of the functions $Q_\pm(f,g)(v)$ can be represented as*

$$\partial^\alpha Q_\pm(f,g)(v) = \sum_{\beta\preceq\alpha} \begin{pmatrix}\alpha\\ \beta\end{pmatrix} Q_\pm(\partial^\beta f, \partial^{\alpha-\beta} g)(v) \, . \tag{2.1}$$

Proof. Using the identity

$$\partial^\alpha Q(f,g)(v) = \partial^\alpha_z Q(f,g)(v+z)\Big|_{z=0} \, ,$$

the invariance of the collision integral with respect to the Galileo transformation

$$Q(f,g)(v+z) = \int_{\mathbb{R}^3}\int_{S^2} B(|u|,\mu)\Big(f(v'+z)g(w'+z) - f(v+z)g(w+z)\Big)\, de\, dw$$

and the Leibnitz formula (1.1) we immediately obtain the required property (2.1). $\blacksquare$

Corollary 4. *The following important boundedness properties hold:*

$$Q_+ \, , \ Q_- \, , \ Q \ : \ \mathbb{S}\times\mathbb{S} \to \mathbb{S} \, . \tag{2.2}$$

Proof. Since the proofs for the operators Q_+, Q_- and Q are exactly the same, we will consider only the operator Q_+. Due to definition (1.6) and to property (2.1) we get

$$\begin{aligned}
\left\| Q_+(f)[g] \,|\, \mathbb{W}_2^{m+1,\langle\nu\rangle} \right\|^2 &= \sum_{|\alpha|\le m+1} \left\| \partial^\alpha Q_+(f)[g] \,|\, \mathbb{L}_2^{\langle\nu\rangle} \right\|^2 \\
&= \sum_{|\alpha|\le m+1} \left\| \sum_{\beta\preceq\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} Q_+(\partial^\beta f)[\partial^{\alpha-\beta} g] \,\Big|\, \mathbb{L}_2^{\langle\nu\rangle} \right\|^2 \\
&\le \sum_{|\alpha|\le m+1} \left(\sum_{\beta\preceq\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \left\| Q_+(\partial^\beta f)[\partial^{\alpha-\beta} g] \,|\, \mathbb{L}_2^{\langle\nu\rangle} \right\| \right)^2 .
\end{aligned}$$

Now using the boundedness (0.9) we obtain the estimate

$$\begin{aligned}
\left\| Q_+(f)[g] \,|\, \mathbb{W}_2^{m+1,\langle\nu\rangle} \right\|^2 &\le C \sum_{|\alpha|,|\beta|\le m} \left\| \partial^\beta f \,|\, \mathbb{L}_1^{\langle\nu+\lambda\rangle} \right\|^2 \left\| \partial^\alpha g \,|\, \mathbb{L}_2^{\langle\nu+\lambda\rangle} \right\|^2 \\
&= C \left\| f \,|\, \mathbb{W}_1^{m,\langle\nu+\lambda\rangle} \right\|^2 \left\| g \,|\, \mathbb{W}_2^{m,\langle\nu+\lambda\rangle} \right\|^2 \qquad (2.3)
\end{aligned}$$

for some positive constant C. (2.3) can be interpreted as the following boundedness property

$$Q_+ \;:\; \mathbb{W}_1^{m,\langle\nu+\lambda\rangle} \times \mathbb{W}_2^{m,\langle\nu+\lambda\rangle} \longrightarrow \mathbb{W}_2^{m+1,\langle\nu\rangle} \,, \quad \forall m \in \mathbb{N}_0 \,, \,, \forall \nu \in \mathbb{R}_+ \,.$$

Due to the Sobolev embedding lemma (see (1.14))

$$\begin{aligned}
\mathbb{W}_2^{m,\langle\nu\rangle} &\subset \mathbb{C}^{\ell,\langle\nu\rangle} \,, \; \ell = m - \frac{3}{2} \,, \qquad (2.4) \\
\mathbb{W}_1^{m,\langle\nu\rangle} &\subset \mathbb{C}^{\ell,\langle\nu\rangle} \,, \; \ell = m - 3 \,,
\end{aligned}$$

i.e.

$$\varphi \in \mathbb{W}_2^{m,\langle\nu\rangle} \Longrightarrow \langle v\rangle^\nu \varphi \in \mathbb{C}^{m-\ell,\langle\nu\rangle} \Longleftrightarrow \varphi \in \mathbb{C}^{m-\ell} \,.$$

Therefore,

$$\bigcap_{m\in\mathbb{N}_0,\, \nu\in\mathbb{R}_+} \mathbb{W}_2^{m,\langle\nu\rangle} = \bigcap_{\ell\in\mathbb{N}_0,\, \nu\in\mathbb{R}_+} \mathbb{C}^{\ell,\langle\nu\rangle} = \mathbb{S}$$

and from (2.4) we obtain the required boundedness (2.2). ∎

In order to study the mapping properties of the linear operator $Q_+(f)$ we find the explicit form of the adjoint operator $Q_+^*(f)$.

Lemma 5. *The adjoint operator $Q_+^*(f)$ to $Q_+(f)$ defined in* (0.5) *can be written as*

$$Q_+^*(f)[h](v) = \int_{\mathbb{R}^3} \overline{f(v-u)} \mathcal{A}(u, D)[h](v) \, du \qquad (2.5)$$

where the parameter-dependent pseudodifferential operator $\mathcal{A}$ (cf. (1.20)*) has the following symbol*

$$a(u,\xi) = e^{\imath \frac{1}{2}(u,\xi)} \int\limits_{S^2} B(|u|,\mu)\, e^{\imath \frac{1}{2}|u|(e,\xi)} de \tag{2.6}$$

and B is the collision kernel.

Proof. Using the well known identity

$$\langle Q_+(f,g)\,,\, h\rangle = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\int\limits_{S^2} B(|u|,\mu) f(w)\, g(v)\, \overline{h(w')}\, de\, dv\, dw$$

we obtain

$$\langle Q_+(f)[g], h\rangle_{\mathbb{L}_2} = \langle g, Q_+^*(f)[h]\rangle_{\mathbb{L}_2}$$

with

$$Q_+^*(f)[h](v) = \int\limits_{\mathbb{R}^3}\int\limits_{S^2} B(|u|,\mu)\overline{f(w)}\, h(w')\, de\, dw\,.$$

The inverse Fourier transform (1.7) for the function $h(w')$

$$h(w') = \frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3} \hat{h}(\xi) e^{-\imath(w',\xi)}\, d\xi$$

leads after the substitution $w = v - u\,, dw = du$ to

$$Q_+^*(f)[h](v) = \int\limits_{\mathbb{R}^3} \overline{f(v-u)} \left(\frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3} a(u,\xi)\hat{h}(\xi)\, e^{-\imath(v,\xi)}\, d\xi \right) du$$

with $a(u,\xi)$ defined in (2.6). Thus the proof is accomplished. ∎

Remark 6. *The property* (2.1) *is also valid for the operator $Q_+^*(f)$. This can be checked directly using the representation* (2.5)

$$\begin{aligned} \partial^\alpha Q_+^*(f)[g](v) &= \sum_{\beta\preceq\alpha} \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \int\limits_{\mathbb{R}^3} \overline{\partial^\beta f(v-u)}\mathcal{A}(u,D)[\partial^{\alpha-\beta} g](v)\, du \\ &= \sum_{\beta\preceq\alpha} \begin{pmatrix}\alpha\\ \beta\end{pmatrix} Q_+^*(\partial^\beta f)[\partial^{\alpha-\beta} g](v)\,. \end{aligned} \tag{2.7}$$

We have used the obvious commutativity of the differentiation and of the convolution-type integral operator $\mathcal{A}$.

The main result we can derive from representation (2.5) is the following.

Theorem 7. *Assume the symbol $a(u,\xi)$ in* (2.5) *can be estimated as*

$$|a(u,\xi)| \le C_a \frac{|u|^\lambda}{|u|\,|\xi| + 1}\,. \tag{2.8}$$

1. *If*

$$f \in \mathbb{L}_2^{\langle\nu\rangle} \quad \textit{and} \quad -\frac{1}{2} < \lambda \leq 1\,, \quad \nu > \frac{3}{2} + \lambda\,, \tag{2.9}$$

then the operators

$$Q_+^*(f) \quad : \quad \mathbb{H}^{-1} \to \mathbb{L}_2^{\langle-\nu\rangle}\,, \tag{2.10}$$

$$Q_+(f) \quad : \quad \mathbb{L}_2^{\langle\nu\rangle} \to \mathbb{H}^1 \tag{2.11}$$

are bounded.

2. *If*

$$f \in \mathbb{L}_\infty^{\langle\nu\rangle} \quad \textit{and} \quad -2 < \lambda \leq 1\,, \quad \nu > 3 + \lambda\,, \tag{2.12}$$

then the operators

$$Q_+^*(f) \quad : \quad \mathbb{H}^{-1} \to \mathbb{L}_2^{\langle-\nu\rangle}\,, \tag{2.13}$$

$$Q_+(f) \quad : \quad \mathbb{L}_2^{\langle\nu\rangle} \to \mathbb{H}^1 \tag{2.14}$$

are bounded.

Proof. Let us rewrite (2.5) in the following equivalent form

$$Q_+^*(f)[g](v) = \int\limits_{\mathbb{R}^3} \overline{f(v-u)} \frac{\langle u\rangle}{|u|^{1-\lambda}} \mathcal{A}_1(u,D)[g](v)\,du\,,$$

where the symbol of the operator $\mathcal{A}_1$ can be estimated as follows

$$|a_1(u,\xi)| \leq C_a \frac{|u|}{\langle u\rangle} \frac{1}{|u|\,|\xi|+1}\,. \tag{2.15}$$

This symbol is uniformly bounded with respect to the parameter u

$$\operatorname*{ess\,sup}_{u\in\mathbb{R}^3} |a_1(u,\xi)| \leq 2^{\frac{1}{2}} C_a \langle\xi\rangle^{-1}\,. \tag{2.16}$$

and therefore fulfills the conditions of Lemma 2. To prove (2.16) we consider first the case $|\xi| \leq 1$ and apply the obvious inequalities

$$|u| \leq \langle u\rangle\,, \qquad \frac{1}{|u|\,|\xi|+1} \leq 1 \leq 2^{\frac{1}{2}} \langle\xi\rangle^{-1}$$

to (2.15). Now let $|\xi| \geq 1$. Then we rewrite (2.15) in the following form

$$|a_1(u,\xi)| \leq C_a \frac{1}{\langle u\rangle} \frac{|u|}{|u|\,|\xi|+1}$$

and obtain the estimate (2.16) since

$$\frac{1}{\langle u\rangle} \leq 1\,, \qquad \frac{|u|}{|u|\,|\xi|+1} = \frac{1}{|\xi|+1/|u|} \leq |\xi|^{-1} \leq 2^{\frac{1}{2}} \langle\xi\rangle^{-1}\,.$$

To prove (2.10) we proceed as follows. Let f_ν denote $\langle\cdot\rangle^\nu f(\cdot)$. Then we obtain

$$\begin{aligned}
&\left\| Q_+^*(f)[g] \,|\, \mathbb{L}_2^{\langle-\nu\rangle} \right\| \\
&\quad\le \left\{ \int\limits_{\mathbb{R}^3} \left[\int\limits_{\mathbb{R}^3} \langle v\rangle^{-\nu} \frac{\langle u\rangle}{|u|^{1-\lambda}} \langle v-u\rangle^{-\nu} |f_\nu(v-u)| |\mathcal{A}_1(u,D)[g](v)| \, du \right]^2 dv \right\}^{\frac12} \\
&\quad\le 2^{\frac{\nu}{2}} \left\{ \int\limits_{\mathbb{R}^3} \left[\int\limits_{\mathbb{R}^3} \frac{\langle u\rangle^{1-\nu}}{|u|^{1-\lambda}} |f_\nu(v-u)| |\mathcal{A}_1(u,D)[g](v)| \, du \right]^2 dv \right\}^{\frac12} .
\end{aligned}$$

Above we have applied the inequality

$$\langle u-v\rangle^{-\nu} \le 2^{\frac{\nu}{2}} \langle u\rangle^{-\nu} \langle v\rangle^{\nu}, \quad \forall u,v \in \mathbb{R}^3, \tag{2.17}$$

which is an alternative form of the Peetre inequality

$$\langle u+v\rangle^{\mu} \le 2^{\frac{|\mu|}{2}} \langle u\rangle^{|\mu|} \langle v\rangle^{\mu}, \ \forall u,v \in \mathbb{R}^3, \ \mu \in \mathbb{R}. \tag{2.18}$$

Further we apply the Schwarz inequality and accomplish the estimates

$$\begin{aligned}
&\left\| Q_+^*(f)[g] \,|\, \mathbb{L}_2^{\langle-\nu\rangle} \right\| \\
&\quad\le 2^{\frac{\nu}{2}} \left[\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} |f_\nu(v-u)|^2 \, du \int\limits_{\mathbb{R}^3} \frac{\langle u\rangle^{2(1-\nu)}}{|u|^{2(1-\lambda)}} \left|\mathcal{A}_1(u,D)[g](v)\right|^2 \, du \, dv \right]^{\frac12} \\
&\quad\le 2^{\frac{\nu}{2}} \| f_\nu \,|\, \mathbb{L}_2 \| \left[\int\limits_{\mathbb{R}^3} \frac{\langle u\rangle^{2(1-\nu)}}{|u|^{2(1-\lambda)}} \left(\int\limits_{\mathbb{R}^3} \left|\mathcal{A}_1(u,D)[g](v)\right|^2 \, dv \right) du \right]^{\frac12} \\
&\quad\le 2^{\frac{\nu}{2}} \| f \,|\, \mathbb{L}_2^{\langle\nu\rangle} \| \left[\int\limits_{\mathbb{R}^3} \frac{\langle u\rangle^{2(1-\nu)}}{|u|^{2(1-\lambda)}} \, du \right]^{\frac12} \operatorname*{ess\,sup}_{u\in\mathbb{R}^3} \| \mathcal{A}(u,D)[g] \,|\, \mathbb{L}_2 \| \\
&\quad\le 2^{\frac{\nu+1}{2}} C_a C_{\lambda,\nu,1} \| f \,|\, \mathbb{L}_2^{\langle\nu\rangle} \| \| g \,|\, \mathbb{H}^{-1} \| ,
\end{aligned} \tag{2.19}$$

due to Lemma 2 (see (1.21) and (2.16)). Moreover, the conditions in (2.9) ensure the convergence of the following integral

$$C_{\lambda,\nu,1}^2 := \int\limits_{\mathbb{R}^3} \frac{\langle u\rangle^{2(1-\nu)}}{|u|^{2(1-\lambda)}} \, du = 2\pi \frac{\Gamma\left(\lambda+\frac12\right)\Gamma\left(\nu-\lambda-\frac32\right)}{\Gamma\left(\nu-1\right)} . \tag{2.20}$$

The inequality (2.19) accomplishes the proof of (2.10). To prove (2.13) we start similarly to the foregoing case. Since $f \in \mathbb{L}_\infty^{\langle\nu\rangle}$, using

$$|f(v)| \le \| f \,|\, \mathbb{L}_\infty^{\langle\nu\rangle} \| \langle v\rangle^{-\nu}$$

we get

$$\begin{aligned}
&\left\| Q_+^*(f)[g] \,|\, \mathbb{L}_2^{\langle -\nu\rangle} \right\| \\
&\le \|f \,|\, \mathbb{L}_\infty^{\langle \nu\rangle}\| \left\{ \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}^3} \langle v\rangle^{-\nu} \frac{\langle u\rangle}{|u|^{1-\lambda}} \langle v-u\rangle^{-\nu} \mathcal{A}_1(u,D)[g](v)| \, du \right]^2 dv \right\}^{\frac{1}{2}} \\
&\le 2^{\frac{\nu}{2}} \|f \,|\, \mathbb{L}_\infty^{\langle \nu\rangle}\| \left\{ \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}^3} \frac{\langle u\rangle^{1-\nu}}{|u|^{1-\lambda}} |\mathcal{A}_1(u,D)[g](v)| \, du \right]^2 dv \right\}^{\frac{1}{2}} .
\end{aligned}$$

Above we have applied the inequality (2.17). Next we apply the inequality

$$\left\{ \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}^3} w(u,v)\, du \right]^2 dv \right\}^{\frac{1}{2}} \le \int_{\mathbb{R}^3} \left\{ \int_{\mathbb{R}^3} w^2(u,v)\, dv \right\}^{\frac{1}{2}} du\,, \tag{2.21}$$

proved in [9, Theorem 202] for a non-negative function $w(u,v) \ge 0$, and accomplish the estimates

$$\begin{aligned}
\left\| Q_+^*(f)[g] \,|\, \mathbb{L}_2^{\langle -\nu\rangle} \right\| &\le 2^{\frac{\nu}{2}} \|f \,|\, \mathbb{L}_\infty^{\langle \nu\rangle}\| \int_{\mathbb{R}^3} \frac{\langle u\rangle^{1-\nu}}{|u|^{1-\lambda}} \left[\int_{\mathbb{R}^3} |\mathcal{A}_1(u,D)[g](v)|^2 \, dv \right]^{\frac{1}{2}} du \\
&\le 2^{\frac{\nu}{2}} \|f \,|\, \mathbb{L}_\infty^{\langle \nu\rangle}\| \left[\int_{\mathbb{R}^3} \frac{\langle u\rangle^{1-\nu}}{|u|^{1-\lambda}} \, du \right] \operatorname{ess}\sup_{u\in\mathbb{R}^3} \|\mathcal{A}(u,D)[g] \,|\, \mathbb{L}_2\| \\
&\le 2^{\frac{\nu+1}{2}} C_a C_{\lambda,\nu,2} \|f \,|\, \mathbb{L}_\infty^{\langle \nu\rangle}\| \|g \,|\, \mathbb{H}^{-1}\|
\end{aligned} \tag{2.22}$$

due to Lemma 2 (see (1.21) and (2.16)). Moreover, the conditions in (2.12) ensure the convergence of the integral (cf. (2.20))

$$C_{\lambda,\nu,2} := \int_{\mathbb{R}^3} \frac{\langle u\rangle^{1-\nu}}{|u|^{1-\lambda}} \, du = 2\pi \frac{\Gamma\left(\frac{\lambda+2}{2}\right) \Gamma\left(\frac{\nu-\lambda-3}{2}\right)}{\Gamma\left(\frac{\nu-1}{2}\right)} .$$

The inequality (2.22) accomplishes the proof of (2.13). (2.11) and (2.14) follows from (2.10) and (2.13) by duality because $\mathbb{H}^1$ and $\mathbb{L}_2^{\langle \nu\rangle}$ are the dual spaces to $\mathbb{H}^{-1}$ and $\mathbb{L}_2^{\langle -\nu\rangle}$, respectively. ∎

Lemma 8. *For the VHS model* (0.4) *the symbol* $a(u,\xi)$ *defined in* (2.6) *can be written explicitly:*

$$a(u,\xi) = 4\pi\, C_\lambda e^{i\frac{1}{2}(u,\xi)} |u|^\lambda \operatorname{sinc}\left(\frac{1}{2}|u|\,|\xi| \right) , \tag{2.23}$$

where the notation

$$\operatorname{sinc}(y) = \frac{\sin(y)}{y}$$

has been used. The symbol $a(u,\xi)$ *in* (2.23) *fulfils the estimate* (2.8).

Proof. Using the obvious inequality

$$|\operatorname{sinc}(y)| \leq \frac{3}{2y+1}\,,\ y \geq 0$$

we deduce from (2.23) the estimate (2.8) with the constant $C_a = 12\pi C_\lambda$. ∎

Lemma 9. *If the collision kernels of the inverse power potential type (cf.* (0.3)*)*

$$B(|u|,\mu) = |u|^\lambda g_m(\mu)\,,\ \lambda = 1 - 4/m\,,\ m > 1$$

has the additional property

$$g_m \in \mathbb{H}_1^a([-1,1])\,,\ a > 1\,, \tag{2.24}$$

for its symbol $a(u,\xi)$ *there holds estimate* (2.8).

Proof. The symbol $a(u,\xi)$ now has the following form

$$a(u,\xi) = e^{\imath\frac{1}{2}(u,\xi)}|u|^\lambda \int\limits_{S^2} g_m(\mu) e^{\imath\frac{1}{2}|u|(e,\xi)}\,de\,.$$

Thus it is sufficient to estimate the integral over the unit sphere

$$\int\limits_{S^2} g_m(\mu)\,e^{\imath\frac{1}{2}|u|(e,\xi)}\,de \leq C_a \frac{1}{|u|\,|\xi|+1}\,. \tag{2.25}$$

Assuming $|u|\,|\xi| \leq 1$ we immediately obtain this estimate with the constant

$$C_1 = 4\pi\,\|g_m\,|\,\mathbb{L}_1([-1,1])\|\,.$$

The estimate for $|u|\,|\xi| \geq 1$ is a little more delicate. We use the following parametrisation of the unit sphere in (2.25)

$$e = Q\begin{pmatrix} \cos\varphi\,\sin\theta \\ \sin\varphi\,\sin\theta \\ \cos\theta \end{pmatrix}\,,\ 0 \leq \varphi < 2\pi\,,\ 0 \leq \theta \leq \pi\,,\ de = \sin\theta\,d\varphi\,d\theta\,,$$

where the orthogonal 3×3 matrix Q is compiled from the following columns

$$\frac{(\xi\times u)\times\xi}{|\xi\times u|\,|\xi|}\,,\ \frac{\xi\times u}{|\xi\times u|}\,,\ \frac{\xi}{|\xi|}\,.$$

Thus we get

$$Q^T\xi = |\xi|\Big(0,0,1\Big)^T\,,\ Q^T u = \frac{1}{|\xi|}\Big(|\xi\times u|, 0, (\xi,u)\Big)^T$$

and

$$\mu = \mu(\varphi,\theta) = \frac{(u,e)}{|u|} = \frac{1}{|\xi|\,|u|}\Big(|\xi\times u|\cos\varphi\,\sin\theta + (\xi,u)\cos\theta\Big)\,.$$

The integral (2.25) transforms into

$$\int_0^{2\pi}\int_0^{\pi} g_m\Big(\mu(\varphi,\theta)\Big) e^{\imath\frac{1}{2}|u|\,|\xi|\cos\theta}\sin\theta\, d\varphi\, d\theta\,.$$

Integration by parts with respect to θ leads to

$$\int_0^{\pi} g_m\Big(\mu(\varphi,\theta)\Big) e^{\imath\frac{1}{2}|u|\,|\xi|\cos\theta}\sin\theta\, d\theta = -\frac{2\imath}{|u|\,|\xi|}\Big[g_m(\mu)e^{\imath\frac{1}{2}|u|\,|\xi|\cos\theta}\Big]_0^{\pi}$$

$$+\frac{2\imath}{|u|\,|\xi|}\int_0^{\pi} g_m'\Big(\mu(\varphi,\theta)\Big)\frac{\partial\mu(\varphi,\theta)}{\partial\theta}e^{\imath\frac{1}{2}|u|\,|\xi|\cos\theta}\, d\theta\,.$$

The Sobolev lemma in the one-dimensional case reads as follows: (cf. (1.14))

$$\mathbb{H}^s_p(\mathbb{R})\subset\mathbb{C}(\mathbb{R})\,,\ \ s>1/p\,.$$

Thus condition (2.24) means that the function g_m is continuous on $[-1,1]$ and therefore $|g_m(\mu)|\le C_0$. Using the inequalities

$$\Big|g_m\Big(\mu(\varphi,\theta)\Big)\Big|\le C_0\,,\ \int_0^{\pi}\Big|g_m'\Big(\mu(\varphi,\theta)\Big)\Big|\,d\theta\le C_3\,,\ \left|\frac{\partial\mu(\varphi,\theta)}{\partial\theta}\right|\le 2$$

we get the estimate

$$\left|\int_{S^2} g_m(\mu)e^{\imath\frac{1}{2}|u|\big(e,\xi\big)}de\right|\le\frac{C_4}{|u|\,|\xi|}\le\frac{2C_4}{|u|\,|\xi|+1}\,,$$

with $C_4=8\pi(C_0+C_3)$. Thus (2.8) holds with the constant $C_a=\max(C_1,C_4)\,.$ ∎

Now we can prove the main result of the paper.

Theorem 10. *Let the collision kernel $B(|u|,\mu)$ be such that estimate* (2.8) *holds and*

$$\sigma\ge 0\,,\quad -\sigma\le s\le\sigma\,.$$

1. *If*

$$-\frac{1}{2}<\lambda\le 1\,,\quad \nu>\frac{3}{2}+\lambda \tag{2.26}$$

then the operators

$$Q_+\quad:\quad \mathbb{H}^{\sigma,\langle\nu\rangle}\times\mathbb{H}^{s,\langle\nu\rangle}\to\mathbb{H}^{s+1}\,, \tag{2.27}$$

$$Q_+^*\quad:\quad \mathbb{H}^{\sigma,\langle\nu\rangle}\times\mathbb{H}^{s-1}\to\mathbb{H}^{s,\langle-\nu\rangle} \tag{2.28}$$

are bounded.

2. *If*

$$-2 < \lambda \leq 1\,, \quad \nu > 3 + \lambda \tag{2.29}$$

then the operators

$$Q_+ \quad : \quad \mathbb{H}_\infty^{\sigma,\langle\nu\rangle} \times \mathbb{H}^{s,\langle\nu\rangle} \to \mathbb{H}^{s+1}\,, \tag{2.30}$$

$$Q_+^* \quad : \quad \mathbb{H}_\infty^{\sigma,\langle\nu\rangle} \times \mathbb{H}^{s-1} \to \mathbb{H}^{s,\langle-\nu\rangle} \tag{2.31}$$

are bounded.

Proof. If $s = \sigma = 0$ the proposed boundedness properties have already been proved in Theorem 7. For non-negative integers $s = \sigma = 0, 1, \ldots$ the proof follows with the help of (2.1) and (2.7).

By interpolation (see (1.15)-(1.16)) we derive the proposed boundedness properties for arbitrary $s, \sigma > 0$ fixing g and f one after another, we make $Q_+(f, g)$ into a linear operator $Q_+(f)[g]$ or $Q_+(g)[f]$, applied to g and f respectively. To apply the interpolation result (1.15)-(1.16) we still need to remove the weight. For this we note that the operator

$$Q_+(f) \; : \; \mathbb{H}_p^{\omega,\langle\gamma\rangle} \to \mathbb{H}_p^{\delta,\langle\mu\rangle} \tag{2.32}$$

is bounded if and only if the operator

$$Q_+^{-\gamma,\mu}(f) \; : \; \mathbb{H}_p^{\omega} \to \mathbb{H}_p^{\delta} \tag{2.33}$$

is bounded, where

$$Q_+^{-\gamma,\mu}(f)[g](v) := \langle v\rangle^\mu Q_+(f)[\langle\cdot\rangle^{-\gamma} g](v)\,,$$

$$p = 2, \infty\,, \quad \omega, \delta \in \mathbb{R} \quad \text{and either} \quad \gamma = \nu,\ \mu = 0 \quad \text{or} \quad \gamma = 0,\ \mu = -\nu\,.$$

Now applying the interpolation (1.15)-(1.16) we accomplish the proof for $0 \leq s \leq \sigma$ For Q_+.

For Q_+^* and positive parameters $0 \leq s \leq \sigma$ the proof is similar.

Since the spaces $\mathbb{H}^{s,\langle\nu\rangle}$ and $\mathbb{H}^{-s,\langle-\nu\rangle}$ are dual for arbitrary $s, \nu \in \mathbb{R}$, from (2.27) and (2.28) we get by duality that the operators

$$Q_+^* \quad : \quad \mathbb{H}^{\sigma,\langle\nu\rangle} \times \mathbb{H}^{-s-1} \to \mathbb{H}^{-s,\langle-\nu\rangle}\,,$$

$$Q_+ \quad : \quad \mathbb{H}^{\sigma,\langle\nu\rangle} \times \mathbb{H}^{-s,\langle\nu\rangle} \to \mathbb{H}^{-s+1}$$

are bounded. These are the boundedness results (2.28) and (2.27) for negative $-\sigma \leq s < 0$, respectively.

The boundedness results (2.30) and (2.31) for negative $-\sigma \leq s < 0$ follow from (2.31) and (2.30) for positive $0 < s \leq \sigma$ by duality. ∎

3. The loss part of the collision integral

The bilinear operator Q_-, which corresponds to the loss part of the collision integral defined in (0.6), can be written in the following form

$$Q_-(f,g)(v) = \int_{\mathbb{R}^3} B_{tot}(|v-w|)f(v)g(w)\,dw = f(v)\,\mathcal{B}[g](v)\,,$$

where the linear integral operator $\mathcal{B}$

$$\mathcal{B}[g](v) = \int_{\mathbb{R}^3} B_{tot}(|v-w|)g(w)\,dw \tag{3.1}$$

is of the convolution type. In order to study the mapping properties of the operator (3.1) we need to investigate the kernel. For the inverse power potential model (cf. (0.3)) the kernel B_{tot} is

$$\begin{aligned} B_{tot} &= |u|^{1-4/m} \int_{S^2} g_m\left(\frac{(u,e)}{|u|}\right) de = g_{m,tot}|u|^{1-4/m}\,, \qquad (3.2)\\ g_{m,tot} &= 2\pi \int_{-1}^{1} g_m(\mu)\,d\mu \end{aligned}$$

and with $\lambda = 1-4/m$ the operator $\mathcal{B}$ takes the following form

$$\mathcal{B}[g](v) = g_{m,tot} \int_{\mathbb{R}^3} |v-w|^\lambda g(w)\,dw\,, \quad -3 < \lambda \le 1\,.$$

In the special case of the Maxwell pseudo-molecules the integral operator (3.1) degenerates into the functional

$$\mathcal{B}[g](v) = g_{4,tot} \int_{\mathbb{R}^3} g(w)\,dw = \varrho\, g_{4,tot}\,,$$

where ϱ denotes the "density" which corresponds to the function g.

The mapping properties of the operator $\mathcal{B}$ can now be formulated as follows.

Lemma 11. *Assume*

$$\mu > 3 - \frac{3}{q} + |\lambda| = \frac{3}{q'} + |\lambda| \ge \frac{3}{q'} + \lambda > 0 \tag{3.3}$$

with $q' = \dfrac{q}{q-1}$ *and* $1 \le q \le \infty$. *Then*

$$\mathcal{B} \,:\, \mathbb{L}_q^{\langle\mu\rangle} \to \mathbb{L}_\infty^{\langle-\lambda\rangle}$$

is continuous and the inequality

$$\left\| \mathcal{B}[g] \,|\, \mathbb{L}_\infty^{\langle-\lambda\rangle} \right\| \le C_{1,\lambda,\mu,q} \left\| g \,|\, \mathbb{L}_q^{\langle\mu\rangle} \right\|$$

holds for all $g(v) \in \mathbb{L}_q^{\langle\mu\rangle}$.

Proof. We suppose $1 \leq q < \infty$. For $q = \infty$ the proof is essentially the same with obvious modifications concerning the supremum norm

$$\left\| g \,|\, \mathbb{L}_\infty^{\langle \mu \rangle} \right\| = \operatorname*{ess\,sup}_{v \in \mathbb{R}^3} |\langle v \rangle^\mu g(v)| \,.$$

We proceed with the Hölder inequality as follows:

$$\begin{aligned} \left|\langle v \rangle^{-\lambda} \mathcal{B}[g](v)\right| &= g_{m,tot} \int_{\mathbb{R}^3} \langle v \rangle^{-\lambda} |v-w|^\lambda \langle w \rangle^{-\mu} \langle w \rangle^\mu g(w)\, dw \\ &\leq g_{m,tot} \left(\int_{\mathbb{R}^3} \left(\frac{|v-w|^\lambda}{\langle v \rangle^\lambda \langle w \rangle^\mu} \right)^{q'} dw \right)^{\frac{1}{q'}} \left\| g \,|\, \mathbb{L}_q^{\langle \mu \rangle} \right\| \,. \end{aligned}$$

For $\lambda > 0$ we use the substitution $\tilde{w} = v - w$, $d\tilde{w} = dw$ in the last integral. Removing the tilde sign it turns out with (cf. (2.17))

$$|v-w|^\lambda \leq \langle v-w \rangle^\lambda \leq 2^{\lambda/2} \langle v \rangle^\lambda \langle w \rangle^\lambda$$

that the integral

$$\int_{\mathbb{R}^3} \left(\frac{|v-w|^\lambda}{\langle v \rangle^\lambda \langle w \rangle^\mu} \right)^{q'} dw \leq 2^{\frac{q'\lambda}{2}} \int_{\mathbb{R}^3} \langle w \rangle^{(\lambda-\mu)q'} dw < \infty$$

is finite because of the assumption of the lemma $(\mu - \lambda) q' > 3$.

For $\lambda < 0$ we similarly find with

$$|v|^{-\lambda} \leq 2^{-\lambda/2} \langle v-w \rangle^{-\lambda} \langle v \rangle^{-\lambda}$$

and using the substitution $\tilde{w} = v - w$, $d\tilde{w} = dw$ again

$$\int_{\mathbb{R}^3} \left(\frac{|v-w|^\lambda}{\langle v \rangle^\lambda \langle w \rangle^\mu} \right)^{q'} dw \leq \int_{\mathbb{R}^3} \frac{\langle v-w \rangle^{-q'\lambda}}{|v-w|^{-q'\lambda} \langle w \rangle^{q'(\lambda+\mu)}} dw < \infty \,.$$

The last integral converges for $w \to v$ because of the assumption $-q'\lambda < 3$ and for $w \to \infty$ because $q'(\lambda + \mu) > 3$ (see (3.3)).

The remark that for $\lambda = 0$ the function $\mathcal{B}[g](v)$ is constant (see (3.3)) completes the proof with the final estimate

$$\left|\langle v \rangle^{-\lambda} \mathcal{B}[g](v)\right| \leq C_{1,\lambda,\mu,q} \left\| g \,|\, \mathbb{L}_q^{\langle \mu \rangle} \right\| \,, \; -3 < \lambda \leq 1 \,.$$

∎

Remark 12. *The condition* (3.3) *is not restrictive for the solution of the Boltzmann equation $f(t,x,v) \geq 0$ which represents the distribution of particles in the phase space $\Omega \times \mathbb{R}^3$ and, therefore, $f(t,x,v)$ maintains a finite kinetic energy* (0.18).

Corollary 13. *If the condition* (3.3) *holds, the operator*

$$\mathcal{B} \;:\; \mathbb{H}_q^{s,\langle \nu \rangle} \to \mathbb{H}_\infty^{s,\langle -\lambda \rangle}$$

is bounded for all $s \geq 0$.

Proof. For an integer $s = m \in \mathbb{N}_0$ the proof is a direct consequence of the foregoing lemma because

$$\partial^\alpha \mathcal{B}[g](v) = \mathcal{B}[\partial^\alpha g](v)\,,\ \alpha \in \mathbb{N}_0^3\,,\ |\alpha| \leq m$$

(see (2.1)). For arbitrary $s \geq 0$ the proof then follows by interpolation (see (1.15) and (1.16)).

■

Corollary 14. *Let* (3.3) *hold and* $0 \leq s \leq \sigma\,,\ 1 \leq p \leq \infty\,,\ \nu \in \mathbb{R}$. *Then the bilinear operator*

$$Q_- \ :\ \mathbb{H}_p^{\sigma,\langle\mu\rangle} \times \mathbb{H}_q^{s,\langle\nu\rangle} \to \mathbb{H}_p^{s,\langle\nu-\lambda\rangle} \tag{3.4}$$

is bounded.

In particular, the loss term (0.6) *of the Boltzmann collision integral* (0.2) *has the following boundedness property*

$$Q_- \ :\ \mathbb{H}_p^{\sigma,\langle\nu\rangle} \times \mathbb{H}_p^{s,\langle\nu\rangle} \to \mathbb{H}_p^{s,\langle\nu-\lambda\rangle}\ ,$$

f or all $0 \leq s \leq \sigma$ *provided the conditions* (3.3) *hold with* $q = p$ *and* $\mu = \nu$.

Proof. First let us prove the following assertion:

$$a \in \mathbb{H}_\infty^{\sigma,\langle\gamma\rangle},\ \varphi \in \mathbb{H}_p^{s,\langle\nu\rangle} \quad \text{yield} \quad a\varphi \in \mathbb{H}_p^{s,\langle\nu+\gamma\rangle}\ . \tag{3.5}$$

The assertion can easily be verified for integers $s, \sigma = 0, 1, \ldots,\ s \leq \sigma$. Now we fix $a \in \mathbb{W}_\infty^{m,\langle\mu\rangle}$, $m \in \mathbb{N}_0$, interpret (3.5) as a boundedness of the multiplication operator aI, and extend the boundedness property to an arbitrary $0 \leq \sigma \leq m$ by interpolation (1.15) and (1.16). After this we fix $\varphi \in \mathbb{H}_\infty^{s,\langle\nu\rangle}$ and extend similarly the boundedness property for arbitrary $0 \leq s \leq \sigma$. This accomplishes the proof of (3.5).

For integers $s = n,\ \sigma = m \in \mathbb{N}_0$ the proof of the asserted boundedness (3.4) is a direct consequence of property (2.1) and property (3.5).

For arbitrary $0 \leq s \leq \sigma$ the proof then follows by interpolation, applied twice as in the proof of assertion (3.5). ■

Remark 15. *It can be proved that the operator*

$$\mathcal{B}\ :\ \mathbb{H}_{q,com}^s \to \mathbb{H}_{q,loc}^{s+3+\lambda}$$

is bounded for arbitrary $s \in \mathbb{R}$. *In fact, the symbol* $a(\xi)$ *of the operator of the convolution type* $\mathcal{B}$

$$\mathcal{B}[g](v) = \int\limits_{\mathbb{R}^3} B_{tot}(|v-w|) f(w)\, dw$$

can be computed as the Fourier transform of its kernel

$$a(\xi) = \mathcal{F}_{u\to\xi}[B_{tot}(|u|)](\xi)$$

(cf. (3.1),(3.2)*). Thus the symbol of the operator $\mathcal{B}_0$ can be written as*

$$a(\xi) = g_{m,tot} \int\limits_{\mathbb{R}^3} |u|^\lambda e^{\imath(u,\xi)}\, du\,.$$

The result is (see e.g. [5]*)*

$$a(\xi) = \begin{cases} -(2\pi)^2 g_{4,tot} \dfrac{\delta'(|\xi|)}{|\xi|} & , \quad \textit{for } \lambda = 0\,, \\ \\ -4\pi\,(\lambda+1)\Gamma(\lambda+1)\sin\left(\frac{\lambda\pi}{2}\right) g_{m,tot} \dfrac{1}{|\xi|^{\lambda+3}} & , \quad \textit{for } \lambda \neq 0\,. \end{cases}$$

In the case of the hard spheres model ($\lambda = 1$) we get

$$a(\xi) = -\frac{8\pi^2 d^2}{|\xi|^4}\,.$$

Thus the symbol $a(\xi)$ always has singularity at $\xi = 0$. By cutting out the neighbourhood of 0*, with the help of a cut-off function with a compact support we decompose the operator $\mathcal{B}$ in a sum*

$$\mathcal{B} = \mathcal{B}^{(1)} + \mathcal{B}^{(2)}\,,$$

where $\mathcal{B}^{(1)}$ has no more singularity at 0 *and, having the order $-3-\lambda$, maps*

$$\mathcal{B}^{(1)}\,:\;\mathbb{H}^s_{q,com} \to \mathbb{H}^{s+3+\lambda}_{q,loc}\,.$$

The operator $\mathcal{B}_0^{(2)}$ is smoothing

$$\mathcal{B}^{(2)}\,:\;\mathbb{H}^s_{q,com} \to \mathbb{C}^\infty \subset \mathbb{H}^{s+3+\lambda}_{q,loc}$$

because the symbol has a compact support, but functions $\mathcal{B}^{(2)}[f](v)$ might have problems with integration at infinity.

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