Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint Nr. 323

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Saarbrücken 2013

Fachrichtung 6.1 – Mathematik
Universität des Saarlandes

Preprint No. 323
submitted: April 29, 2013

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Characterization of trace spaces of $\boldsymbol{H}(\mathbf{curl}, \Omega)$ on curvilinear Lipschitz polyhedral domains Ω

Trace spaces of the energy space $\mathbf{H}(\mathbf{curl}, \Omega)$ appear naturally in variational formulations that result from the Maxwell equations. Their characterization is a topic that has been intensively studied in the past ten years. The first contributions treat the case of smooth manifolds [3, 4, 6] whereas Lipschitz domains Ω are considered in recent papers [2]. Although focus is made on the smooth case, Nédélec's monograph is a pathleading reference as most basic definitions of tangent vector spaces and surface differential operators are presented. And thus, the generalizations presented in [2] build up nicely on Nédélec's exposition. The characterization of trace spaces of $\mathbf{H}(\mathbf{curl}, \Omega)$ for general Lipschitz domains as presented in [2] is rather abstract and there was the need to find yet another access leading an intuitive understanding of the regularity in these function spaces. In the pioneering paper [1] explicit characterizations of the trace spaces of $\mathbf{H}(\mathbf{curl}, \Omega)$ are given under the assumption that Γ be the boundary of a Lipschitz polyhedral domain Ω. Explicit means that the regularity of the functionals is captured by integral expressions which allow for numerical evaluation. The current developments regarding high order methods require generalizations of the explicit characterizations for curvilinear Lipschitz polyhedral domains Ω.
Let Ω be a curvilinear Lipschitz polyhedral domain whose boundary Γ allows for an exact description by a regular mesh of the following kind.

Definition 1 Let Γ be the boundary of an open curvilinear Lipschitz polyhedral domain. We assume that the following representation of Γ is given by

$$\Gamma = \bigcup_{i=1}^{N} \overline{\Gamma_i} = \bigcup_{i=1}^{N} \Gamma_i \cup \bigcup_{i=1}^{N_e} e_i \cup \bigcup_{i=1}^{N_v} v_i \,. \tag{1}$$

Here $\{\Gamma_i\}_{i=1}^N$ denotes the set of all elements (faces) with smooth parametrisations of the following kind

$$\hat{\boldsymbol{F}}_i \colon \hat{T} \to \Gamma_i \,, \quad \hat{\boldsymbol{F}}_i \in \mathcal{C}^\infty(\overline{\hat{T}})^3 \,, \tag{2}$$

the set $\{e_i\}_{i=1}^{N_e}$ contains all edges with smooth parametrisations of the following kind

$$\hat{\boldsymbol{E}}_i \colon (0,1) \to e_i \,, \quad \hat{\boldsymbol{E}}_i \in \mathcal{C}^\infty\big([0,1]\big)^3 \,. \tag{3}$$

The vertices in the mesh are collected in the set $\{v_i\}_{i=1}^{N_v}$, respectively.

The parametrisations (2) can naturally be extended to parametrisations of the closed elements $\overline{\Gamma_i}$. In a regular mesh, the resulting edge parametrisations are compatible with (3) meaning that they coincide up to orientation. Thus,

Γ is globally continuous and it is given piecewise by triangular elements. The elements are in general curved as their parametrisations are non-linear functions.
This setting often allows to specify the abstract definitions of function spaces because relevant regularity properties can be made explicit exploiting the simple geometrical situation. As an example, consider a function $\varphi \in \mathrm{H}^{1/2}(\Gamma)$. The restriction of φ to an element is meaningful and we write $\varphi_i = \varphi\big|_{\Gamma_i}$. Now, consider the open domain $\Gamma_{i_1} \cup e_i \cup \Gamma_{i_2}$, where Γ_{i_1} and Γ_{i_2} are neighbouring elements and e_i denotes their common edge. Then, the weak continuity of φ across the edge e_i means that the following condition is necessarily fulfilled [1, 5]

$$\mathcal{N}_i(\varphi) = \int\limits_{\Gamma_{i_1}} \int\limits_{\Gamma_{i_2}} \frac{\left|\varphi_{i_1}(x) - \varphi_{i_2}(y)\right|^2}{|x-y|^3} \,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(y) < \infty\,. \tag{4}$$

Besides the classical Sobolev space $\mathrm{H}^{1/2}(\Gamma)$ [5], we work with vector-valued spaces as $\mathbf{H}^{1/2}(\Gamma) = \mathrm{H}^{1/2}(\Gamma)^3$ and spaces of square integrable tangent vector fields, namely,

$$\begin{aligned} \mathbf{L}^2_{\mathrm{t}}(\Gamma) &= \left\{ \boldsymbol{\varphi} \in \mathrm{L}^2(\Gamma)^3 : \ \boldsymbol{\varphi} \cdot \boldsymbol{n}|_\Gamma = 0 \text{ almost everywhere} \right\}, \\ \mathbf{H}^{1/2}_{-}(\Gamma) &= \left\{ \boldsymbol{\varphi} \in \mathbf{L}^2_{\mathrm{t}}(\Gamma) : \ \boldsymbol{\varphi}_i \in \mathrm{H}^{1/2}(\Gamma_i)^3,\ 1 \le i \le N \right\} . \end{aligned}$$

The trace spaces of $\mathbf{H}(\mathbf{curl}, \Omega)$ are the functional spaces that render the trace operators γ_R and γ_D surjective [1, 2], i.e.,

$$\begin{aligned} \gamma_R &\colon \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)\,, \\ \gamma_D &\colon \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)\,. \end{aligned}$$

It has been shown in [2] that it holds

$$\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) = \mathbf{grad}_\Gamma\, \mathrm{H}^{1/2}(\Gamma) \,+\, \mathbf{H}^{1/2}_{\|}(\Gamma) \tag{5}$$

$$\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = \mathbf{curl}_\Gamma\, \mathrm{H}^{1/2}(\Gamma) \,+\, \mathbf{H}^{1/2}_{\perp}(\Gamma)\,. \tag{6}$$

Thus, either trace space of $\mathbf{H}(\mathbf{curl}, \Omega)$ can be characterized as algebraic sum of two spaces. The topology of the first results from the classical Sobolev space $\mathrm{H}^{1/2}(\Gamma)$ together with the properties of the surface differential operators $\mathbf{curl}_\Gamma$ and $\mathbf{grad}_\Gamma$ [2]. Explicit characterizations of the second spaces for the case of Lipschitz polyhedral domains Ω have been developed in [1].
The objective of this paper is to generalize the characterizations of $\mathbf{H}^{1/2}_{\|}(\Gamma)$ and $\mathbf{H}^{1/2}_{\perp}(\Gamma)$, respectively, such that curvilinear Lipschitz polyhedral domains Ω are also covered. Let us, therefore, consider two neighbouring elements Γ_{i_1}

and Γ_{i_2} with common edge e_i. Without restriction of any kind we assume that

$$e_i = \Big\{ x \in \Gamma,\ x = \hat{\boldsymbol{E}}_i(\xi_1) = \hat{\boldsymbol{F}}_{i_1}(\xi) = \hat{\boldsymbol{F}}_{i_2}(\xi),\ \xi_1 \in (0,1),\ \xi_2 = 0 \Big\}\,. \tag{7}$$

The tangent planes at every point on Γ_{i_1} and Γ_{i_2}, respectively, are spanned by the tangent vector fields as introduced in the following definition.

Definition 2 The natural tangent vector fields are given in terms of the local parametrisations, i.e.,

$$\{\boldsymbol{\epsilon}_{1,i_1}, \boldsymbol{\epsilon}_{2,i_1}\} \ \text{ on } \ \Gamma_{i_1} \quad \text{with} \quad \boldsymbol{\epsilon}_{l,i_1}(\xi) \ = \ \frac{\partial \hat{\boldsymbol{F}}_{i_1}(\xi)}{\partial \xi_l}\,, \quad l = 1, 2\,, \tag{8}$$

$$\{\boldsymbol{\epsilon}_{1,i_2}, \boldsymbol{\epsilon}_{2,i_2}\} \ \text{ on } \ \Gamma_{i_2} \quad \text{with} \quad \boldsymbol{\epsilon}_{l,i_2}(\xi) \ = \ \frac{\partial \hat{\boldsymbol{F}}_{i_2}(\xi)}{\partial \xi_l}\,, \quad l = 1, 2\,. \tag{9}$$

The tangent vector fields actually span tangent planes at points x on the physical boundary Γ meaning that

$$\tilde{\boldsymbol{\epsilon}}_{l,i_j}(x) = \boldsymbol{\epsilon}_{l,i_j}\Big(\hat{\boldsymbol{F}}_{i_j}^{-1}(x) \Big), \quad l = 1, 2,\ j = 1, 2\,.$$

As the physical point x can uniquely identified with its parameter coordinate ξ, we will skip the tilde in the following and denote the tangent vector fields at a point $x \in \Gamma_{i_j}$ simply by $\boldsymbol{\epsilon}_{l,i_j}(x)$.
Note that the tangent vector fields are non-constant vector fields. We call them natural or intrinsic as they are neither normalized nor point-wise orthogonal with respect to the Euclidean metric. However, to obtain a notion of an Euclidean orthogonality we introduce the so-called cotangent vector fields on the curvilinear elements.

Definition 3 The cotangent vector fields are the set of tangent vector fields which are point-wise orthogonal to the tangent vector fields, i.e.,

$$\{\boldsymbol{\epsilon}^{1,i_1}, \boldsymbol{\epsilon}^{2,i_1}\} \ \text{ on } \ \Gamma_{i_1} \quad \text{such that} \quad \boldsymbol{\epsilon}^{l,i_1} \cdot \boldsymbol{\epsilon}_{k,i_1} = \delta_{lk}\,, \quad k, l = 1, 2\,, \tag{10}$$

$$\{\boldsymbol{\epsilon}^{1,i_2}, \boldsymbol{\epsilon}^{2,i_2}\} \ \text{ on } \ \Gamma_{i_2} \quad \text{such that} \quad \boldsymbol{\epsilon}^{l,i_2} \cdot \boldsymbol{\epsilon}_{k,i_2} = \delta_{lk}\,, \quad k, l = 1, 2\,. \tag{11}$$

The cotangent vector fields are obviously extrinsically defined because the Euclidean metric is used in (10) and (11).
Now, we have the tools at hand to characterize the spaces $\mathbf{H}_{\|}^{1/2}(\Gamma)$, $\mathbf{H}_{\perp}^{1/2}(\Gamma)$ in case Γ be the boundary of a curvilinear Lipschitz polyhedral domain.

Definition 4 Let Ω be the boundary of a curvilinear Lipschitz polyhedral domain. The space $\mathbf{H}_{\|}^{1/2}(\Gamma)$ contains functionals $\boldsymbol{\varphi} \in \mathbf{H}_{-}^{1/2}(\Gamma)$ which exhibit a weak $\|$-continuity, i.e., for all edges e_i in Γ it holds

$$\mathcal{N}_i^{\|}(\boldsymbol{\varphi}) = \int_{\Gamma_{i_1}} \int_{\Gamma_{i_2}} \frac{\left|\boldsymbol{\varphi}_{i_1}(x) \cdot \boldsymbol{\epsilon}_{1,i_1}(x) - \boldsymbol{\varphi}_{i_2}(y) \cdot \boldsymbol{\epsilon}_{1,i_2}(y)\right|^2}{|x-y|^3} \, \mathrm{d}\sigma(x) \, \mathrm{d}\sigma(y) < \infty \,. \tag{12}$$

The space $\mathbf{H}_{\perp}^{1/2}(\Gamma)$ contains functionals $\boldsymbol{\varphi} \in \mathbf{H}_{-}^{1/2}(\Gamma)$ which exhibit a weak $\perp$-continuity, i.e., for all edges e_i in Γ it holds

$$\mathcal{N}_i^{\perp}(\boldsymbol{\varphi}) = \int_{\Gamma_{i_1}} \int_{\Gamma_{i_2}} \frac{\left|J_{\Gamma_{i_1}} \boldsymbol{\epsilon}^{2,i_1}(x) \cdot \boldsymbol{\varphi}_{i_1}(x) - J_{\Gamma_{i_2}} \boldsymbol{\epsilon}^{2,i_2}(y) \cdot \boldsymbol{\varphi}_{i_2}(y)\right|^2}{|x-y|^3} \, \mathrm{d}\sigma(x) \, \mathrm{d}\sigma(y) < \infty \,. \tag{13}$$

Definition 5 Let us assume that Ω is an open curvilinear Lipschitz polyhedral domain (1). The mesh-dependent norms $||\cdot||^2_{\|,1/2,\Gamma}$, $||\cdot||^2_{\perp,1/2,\Gamma}$ are defined as follows

$$||\boldsymbol{\varphi}||^2_{\|,1/2,\Gamma} := \sum_{i=1}^{N} ||\boldsymbol{\varphi}||^2_{1/2,\Gamma_i} + \sum_{i=1}^{N_e} \mathcal{N}_i^{\|}(\boldsymbol{\varphi}) \,, \tag{14}$$

$$||\boldsymbol{\varphi}||^2_{\perp,1/2,\Gamma} := \sum_{i=1}^{N} ||\boldsymbol{\varphi}||^2_{1/2,\Gamma_i} + \sum_{i=1}^{N_e} \mathcal{N}_i^{\perp}(\boldsymbol{\varphi}) \,. \tag{15}$$

The spaces $\mathbf{H}_{\|}^{1/2}(\Gamma)$ and $\mathbf{H}_{\perp}^{1/2}(\Gamma)$ are Hilbert spaces when endowed with the norms (14) and (15), respectively. The proof of the Hilbert space property is presented in [1]. The trace operator γ_R is furthermore surjective on $\mathbf{H}_{\|}^{1/2}(\Gamma)$.

Proposition 0.1 *The mapping $\gamma_R : \mathbf{H}^1(\Omega) \to \mathbf{H}_{\|}^{1/2}(\Gamma)$ is linear continuous and surjective.*

Proof The proof given here follows [1]. The main difference is that, according to (12), non-constant tangent vector fields must be considered.
The continuity of the mapping follows from the continuity of the standard trace operator from $\mathbf{H}^1(\Omega)$ to $\mathbf{H}^{1/2}(\Gamma)$ and the continuous embedding $\mathbf{H}_{\|}^{1/2}(\Gamma) \hookrightarrow \mathbf{H}_{-}^{1/2}(\Gamma)$.
The surjectivity is proved by the construction of a compatible normal component at every element Γ_i such that the resulting global vector-valued function lies in $\mathbf{H}^{1/2}(\Gamma)$. The latter can then be extended by standard arguments to a

function in $\mathbf{H}^1(\Omega)$. Let us be given a partition of unity over Γ by three sets of Lipschitz functions $\{\chi_{v_i}\}_{i=1}^{N_v}$, $\{\chi_{e_i}\}_{i=1}^{N_e}$ and $\{\chi_{\Gamma_i}\}_{i=1}^{N}$ such that

$$\forall\, x \in \Gamma : \quad \sum_{i=1}^{N_v} \chi_{v_i}(x) + \sum_{i=1}^{N_e} \chi_{e_i}(x) + \sum_{i=1}^{N} \chi_{\Gamma_i}(x) = 1\,.$$

By help of this partitioning, the construction of the normal component boils down to three different cases, namely, extension in the neighbourhood of an open element Γ_i, in the neighbourhood of an edge e_i and in the neighbourhood of a vertex v_i.

In the neighbourhood of an open element Γ_i, the extension in normal direction is chosen zero.

The situation is different in the neighbourhood of an edge as the extension in normal direction is in general non-trivial. Let us fix an edge e_i with neighbouring elements Γ_{i_1} and Γ_{i_2}. Different from [1] and according to (12), we assume that a tangent vector field $\boldsymbol{\varphi} \in \mathbf{H}_{\|}^{1/2}(\Gamma)$ is given in terms of natural cotangent vector fields, i.e.,

$$\boldsymbol{\varphi} = \begin{cases} \boldsymbol{\varphi}_{i_1} = \varphi_{1,i_1}\boldsymbol{\epsilon}^{1,i_1} + \varphi_{2,i_1}\boldsymbol{\epsilon}^{2,i_1} & \text{on } \Gamma_{i_1}\,, \\ \boldsymbol{\varphi}_{i_2} = \varphi_{1,i_2}\boldsymbol{\epsilon}^{1,i_2} + \varphi_{2,i_2}\boldsymbol{\epsilon}^{2,i_2} & \text{on } \Gamma_{i_2}\,. \end{cases} \tag{16}$$

Our objective is to construct a function $\boldsymbol{u} \in \mathbf{H}^{1/2}(\Gamma)$,

$$\boldsymbol{u} = \begin{cases} \boldsymbol{u}_{i_1} = \varphi_{1,i_1}\boldsymbol{\epsilon}^{1,i_1} + \varphi_{2,i_1}\boldsymbol{\epsilon}^{2,i_1} + \varphi_{3,i_1}\boldsymbol{n}_{i_1} & \text{on } \Gamma_{i_1}\,, \\ \boldsymbol{u}_{i_2} = \varphi_{1,i_2}\boldsymbol{\epsilon}^{1,i_2} + \varphi_{2,i_2}\boldsymbol{\epsilon}^{2,i_2} + \varphi_{3,i_2}\boldsymbol{n}_{i_2} & \text{on } \Gamma_{i_2}\,, \end{cases} \tag{17}$$

such that $\gamma_R\,\boldsymbol{u} = \boldsymbol{\varphi}$. The continuity conditions for $\boldsymbol{u}$ can be formulated with respect to any basis of $\mathbb{R}^3$. Without restriction of any kind we choose a moving frame spanned by the vectors $\{\boldsymbol{\epsilon}_{1,i_1}, \boldsymbol{\epsilon}_{2,i_1}, \boldsymbol{n}_{i_1}\}$. Thus, $\boldsymbol{u} \in \mathbf{H}^{1/2}(\Gamma)$ in the neighbourhood of e_i if (4) holds for all linear independent components of $\boldsymbol{u}$ across e_i, i.e.,

$$\int\limits_{\Gamma_{i_1}}\int\limits_{\Gamma_{i_2}} \frac{\left|\boldsymbol{u}_{i_1}(x)\cdot\boldsymbol{\epsilon}_{1,i_1}(x) - \boldsymbol{u}_{i_2}(y)\cdot\boldsymbol{\epsilon}_{1,i_1}(x)\right|^2}{|x-y|^3}\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(y) < \infty\,, \tag{18}$$

$$\int\limits_{\Gamma_{i_1}}\int\limits_{\Gamma_{i_2}} \frac{\left|\boldsymbol{u}_{i_1}(x)\cdot\boldsymbol{\epsilon}_{2,i_1}(x) - \boldsymbol{u}_{i_2}(y)\cdot\boldsymbol{\epsilon}_{2,i_1}(x)\right|^2}{|x-y|^3}\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(y) < \infty\,, \tag{19}$$

$$\int\limits_{\Gamma_{i_1}}\int\limits_{\Gamma_{i_2}} \frac{\left|\boldsymbol{u}_{i_1}(x)\cdot\boldsymbol{n}_{i_1}(x) - \boldsymbol{u}_{i_2}(y)\cdot\boldsymbol{n}_{i_1}(x)\right|^2}{|x-y|^3}\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(y) < \infty\,. \tag{20}$$

Recall that the tangent vector field $\boldsymbol{\epsilon}_{1,i_1}$ is continuous across e_i as $\boldsymbol{\epsilon}_{1,i_1}\big|_{e_i} = \boldsymbol{\epsilon}_{1,i_2}\big|_{e_i}$ due to (7). Thus, there exists a constant c such that for all $x \in e_i$ it holds

$$\frac{\left|\boldsymbol{u}_{i_1}(x)\cdot\boldsymbol{\epsilon}_{1,i_1}(x) - \boldsymbol{u}_{i_2}(y)\cdot\boldsymbol{\epsilon}_{1,i_1}(x)\right|^2}{|x-y|^3} \leq \underbrace{\frac{\left|\varphi_{1,i_1}(x) - \varphi_{1,i_2}(y)\right|^2}{|x-y|^3}}_{<\infty \text{ by } (12)} + \frac{c}{|x-y|}$$

This means that (18) is necessarily fulfilled by any $\boldsymbol{\varphi} \in \mathbf{H}_{\|}^{1/2}(\Gamma)$. Different from (18), the remaining conditions (19) and (20) serve to determine contributions to the unknown scalar-valued functionals $\varphi_{3,i_1}, \varphi_{3,i_2} \in H^{1/2}(\Gamma)$ in the neighbourhood of e_i. Due to (10), the integrand in (19) contains only φ_{3,i_2} and, thus, (19) is sufficient to determine an appropriate contribution φ_{3,i_2}. Afterwards, φ_{3,i_1} can be defined through condition (20).
Similar arguments apply for the extension in the neighbourhood of a vertex. We consider a vertex v_i. The set of elements that share v_i is denoted $\{\Gamma_{i_l}\}_{l=1}^{n}$ where $n \geq 3$. As in the previous case, on each edge, the two conditions (19) and (20) are necessary and sufficient to determine φ_{3,i_1} and φ_{3,i_2} in the neighbourhood of this particular edge. Different from before is that for each Γ_{i_l} there are two edges and, thus, two extensions each of which belongs to one of the edges. In order to combine them to one functional that is weakly continuous across both edges, a simple blending function is used [1].

Proposition 0.2 *The mapping* $\gamma_D : \mathbf{H}^1(\Omega) \to \mathbf{H}_{\perp}^{1/2}(\Gamma)$ *is linear and surjective.*

Proof Let $\boldsymbol{\varphi}_{\perp} \in \mathbf{H}_{\perp}^{1/2}(\Gamma)$. Consider $\boldsymbol{\varphi}_{\|} = \boldsymbol{\varphi}_{\perp} \times \boldsymbol{n} \in \mathbf{H}_{\|}^{1/2}(\Gamma)$. According to the preceding proposition there exists an extension $\boldsymbol{U} \in \mathbf{H}^1(\Omega)$ with $\gamma_R \boldsymbol{U} = \boldsymbol{\varphi}_{\|}$ and, therefore, $\gamma_D \boldsymbol{U} = \boldsymbol{\varphi}_{\perp}$.

Definition 4 is a generalization of the setting treated in [1]. Assume that Γ is the boundary of a Lipschitz polyhedral domain and consider two neighbouring elements Γ_{i_1} and Γ_{i_2} with common edge e_i. The elements are plane triangles and, thus, it is possible to fix on the both elements piecewise constant tangent vectors which are orthonormal with respect to the Euclidean metric, namely,

$$\begin{aligned} \{\boldsymbol{\tau}_i, \boldsymbol{\tau}_{i_1}\} \text{ on } \Gamma_{i_1} \quad \text{with } \boldsymbol{\tau}_i &= \frac{1}{J_{e_i}}\boldsymbol{\epsilon}_{1,i_1}, \quad \boldsymbol{\tau}_{i_1} = \frac{J_{\Gamma_{i_1}}}{J_{e_i}}\boldsymbol{\epsilon}^{2,i_1}, \\ \{\boldsymbol{\tau}_i, \boldsymbol{\tau}_{i_2}\} \text{ on } \Gamma_{i_2} \quad \text{with } \boldsymbol{\tau}_i &= \frac{1}{J_{e_i}}\boldsymbol{\epsilon}_{1,i_2}, \quad \boldsymbol{\tau}_{i_2} = \frac{J_{\Gamma_{i_2}}}{J_{e_i}}\boldsymbol{\epsilon}^{2,i_2}. \end{aligned}$$

Here J_{e_i} denotes the Jacobian of the edge parametrisation (3) and $J_{\Gamma_{i_l}}$, $l = 1, 2$, the Jacobians of the element parametrisations (2). Accordingly, the functionals $\mathcal{N}_i^{\|}$ and $\mathcal{N}_i^{\perp}$ simplify to the following expressions

$$\mathcal{N}_i^{\|}(\boldsymbol{\varphi}) = J_{e_i} \int\limits_{\Gamma_{i_1}} \int\limits_{\Gamma_{i_2}} \frac{\left|\boldsymbol{\varphi}_{i_1}(x) \cdot \boldsymbol{\tau}_i - \boldsymbol{\varphi}_{i_2}(y) \cdot \boldsymbol{\tau}_i\right|^2}{|x-y|^3} \, \mathrm{d}\sigma(x) \, \mathrm{d}\sigma(y) \,, \tag{21}$$

$$\mathcal{N}_i^{\perp}(\boldsymbol{\varphi}) = J_{e_i} \int\limits_{\Gamma_{i_1}} \int\limits_{\Gamma_{i_2}} \frac{\left|\boldsymbol{\tau}_{i_1} \cdot \boldsymbol{\varphi}_{i_1}(x) - \boldsymbol{\tau}_{i_2} \cdot \boldsymbol{\varphi}_{i_2}(y)\right|^2}{|x-y|^3} \, \mathrm{d}\sigma(x) \, \mathrm{d}\sigma(y) \,. \tag{22}$$

A scaling by the constant edge-based Jacobian J_{e_i} leads to the functionals which were originally presented in [1], i.e.,

$$\int\limits_{\Gamma_{i_1}} \int\limits_{\Gamma_{i_2}} \frac{\left|\boldsymbol{\varphi}_{i_1}(x) \cdot \boldsymbol{\tau}_i - \boldsymbol{\varphi}_{i_2}(y) \cdot \boldsymbol{\tau}_i\right|^2}{|x-y|^3} \, \mathrm{d}\sigma(x) \, \mathrm{d}\sigma(y) \,, \tag{23}$$

$$\int\limits_{\Gamma_{i_1}} \int\limits_{\Gamma_{i_2}} \frac{\left|\boldsymbol{\tau}_{i_1} \cdot \boldsymbol{\varphi}_{i_1}(x) - \boldsymbol{\tau}_{i_2} \cdot \boldsymbol{\varphi}_{i_2}(y)\right|^2}{|x-y|^3} \, \mathrm{d}\sigma(x) \, \mathrm{d}\sigma(y) \,. \tag{24}$$

Note that in (23) and (24) geometrical information about the edge e_i is used, namely, the orthonormal system relies on the normalized edge tangent vector $\boldsymbol{\tau}_i$. This is not the case in the generalized functionals (21) and (22) that result from (12) and (13), respectively. For curvilinear elements, the tangent and cotangent vector fields are non-constant and the edge tangent vector is, therefore, not meaningful on the element interiors. Ultimately, (23) and (24) are restricted to the boundaries of polyhedral Lipschitz domains.
There are more reasons that motivate to establish the characterizations (12) and (13). Recall the decompositions (5) and (6), i.e., the trace spaces $\mathbf{H}_{\|}^{1/2}(\Gamma)$ and $\mathbf{H}_{\perp}^{1/2}(\Gamma)$ are actually auxiliary spaces for the definition of the trace spaces that belong to $\mathbf{H}(\mathbf{curl}, \Omega)$. From classical discretization theory and differential geometry it is known that the transformation behaviour of the vector-valued surface differential operators $\mathbf{grad}_\Gamma$ and $\mathbf{curl}_\Gamma$ differ from each other [2]. Let $\varphi \in H^{1/2}(\Gamma)$ be given piecewise in terms of parameter coordinates, i.e.,

$$\varphi\big|_{\Gamma_i}(x) = \varphi\big(\hat{\boldsymbol{F}}_i^{-1}(x)\big) = \hat{\varphi}(\xi), \quad \xi = (\xi_1, \xi_2)^\top \in \hat{T} \,. \tag{25}$$

Then, the surface gradient of φ is locally given in terms of cotangent vector fields, i.e.,

$$\mathbf{grad}_\Gamma \varphi\big|_{\Gamma_i} = \frac{\partial \hat{\varphi}}{\partial \xi_1} \boldsymbol{\epsilon}^{1,i} + \frac{\partial \hat{\varphi}}{\partial \xi_2} \boldsymbol{\epsilon}^{2,i} \,, \tag{26}$$

whereas the vector-valued surface curl operator reads

$$\mathbf{curl}_\Gamma \varphi\big|_{\Gamma_i} = \frac{1}{J_{\Gamma_i}} \left(\frac{\partial \hat{\varphi}}{\partial \xi_2} \boldsymbol{\epsilon}_{1,i} - \frac{\partial \hat{\varphi}}{\partial \xi_1} \boldsymbol{\epsilon}_{2,i} \right) . \tag{27}$$

By means of (26) and (27) together with (5) and (6) the characterizations (12) and (13) are natural. From a differential point of view the point-wise scalar multiplication of a tangent vector with a cotangent vectors corresponds to the evaluation of differential form on a vector field. In this context, the surface gradient leads to a differential one form and the vector-valued surface curl operator leads to a multivector one field. The projections that appear in (12) and (13) are evaluation of differential forms leading to a characterization of the regularity in the trace space $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ in terms of its dual space $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ and vice versa.

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