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Jörg Eschmeier

Let $H^2(D)$ be the Hardy space on a bounded strictly pseudoconvex domain $D \subset \mathbb{C}^n$ with smooth boundary. Using Gelfand theory and a spectral mapping theorem of Andersson and Sandberg [2] for Toeplitz tuples with H^∞-symbol, we show that a Toeplitz tuple $T_f = (T_{f_1}, \ldots, T_{f_m}) \in L(H^2(\sigma))^m$ with symbols $f_i \in H^\infty + C$ is Fredholm if and only if the Poisson-Szegö extension of f is bounded away from zero near the boundary of D. Corresponding results are obtained for the case of Bergman spaces. Thus we extend results of McDonald [9] and Jewell [7] to systems of Toeplitz operators.

1 Introduction

Let $D \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with smooth boundary. Extending results of McDonald [9] for the unit ball, Jewell proved in [7] that a Toeplitz operator T_f with symbol in $H^\infty + C$ on the Bergman space or Hardy space over D is Fredholm if and only if f, or its Poisson-Szegö extension in the case of the Hardy space, is bounded away from zero near the boundary of D. A basic ingredient of the proof was the observation that, for every multiplicative linear functional φ of $H^\infty(D)$ belonging to the fibre of the maximal ideal space of $H^\infty(D)$ over a boundary point $\lambda \in \partial D$ and any function $f \in H^\infty(D)$, the value $\varphi(f)$ belongs to the cluster set of f at λ.

In the present note we replace single Fredholm operators T_f by tuples $T_f = (T_{f_1}, \ldots, T_{f_m})$ of Toeplitz operators with symbol $f \in (H^\infty + C)^m$. If the above cluster value property of $H^\infty(D)$ were known to be true for tuples $f \in H^\infty(D)^m$ instead of single functions, then the methods from [7] could be extended in a straightforward way to calculate the essential spectrum of the essentially commuting multioperator T_f. However, the cluster value property for finite tuples in $H^\infty(D)$ is equivalent to the validity of the Corona Theorem for $H^\infty(D)$. This equivalence is well known and follows, for instance, as a direct application of Theorem 1 from [5].

In the following we show that, in spite of this difficulty, properties of the Poisson transform and suitable results from Gelfand theory can be used to prove the spectral mapping formula

$$\sigma_e(T_f) = \bigcap(\overline{f(U \cap D)};\ U \supset \partial D \text{ open})$$

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for Toeplitz tuples T_f with symbol in $f \in (H^\infty + C)^m$ on Hardy and Bergman spaces over strictly pseudoconvex domains. Here again, in the Hardy-space case, the symbol f has to be interpreted as the Poisson-Szegö extension of f.

2 Preliminaries

Let $D \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with smooth boundary and let $H^2(D)$ be the Hardy space on D. Since the point evaluation at every point of D is continuous on $H^2(D)$, the space $H^2(D)$ is an analytic functional Hilbert space and hence prossesses a reproducing kernel $K : D \times D \to \mathbb{C}$. Let σ be the normalized surface measure on ∂D. We shall identify $H^2(D)$ with its image $H^2(\sigma)$ under the isometry

$$H^2(D) \to L^2(\sigma), \ f \mapsto f^*$$

associating with each function $f \in H^2(D)$ its non-tangential boundary value f^*. For $z \in D$, consider the function

$$P(z, \cdot) = \frac{|K(\cdot, z)^*|^2}{K(z, z)} \in L^2(\sigma).$$

As usual, we call P the Poisson-Szegö kernel and define the Poisson-Szegö integral of a function $f \in L^2(\sigma)$ by

$$\mathscr{P}[f] : D \to \mathbb{C}, \ z \mapsto \int_{\partial D} f P(z, \cdot) d\sigma.$$

The Poisson-Szegö integral reproduces functions in $H^2(D)$. For $f \in C(\partial D)$ the Poisson-Szegö integral entends to a function $F \in C(\overline{D})$ with $F|\partial D = f$ (see[10] or [8] for both properties).

For $f \in L^\infty(\sigma)$, we define the Toeplitz operator $T_f \in L(H^2(\sigma))$ and the Hankel operator $H_f \in L(H^2(\sigma), L^2(\sigma))$ with symbol f by

$$T_f = PM_f|H^2(\sigma) \ \text{ and } \ H_f = (1 - P)M_f|H^2(\sigma).$$

Here $P : L^2(\sigma) \to H^2(\sigma)$ denotes the orthogonal projection and $M_f : L^2(\sigma) \to L^2(\sigma)$, $g \mapsto fg$, is the operator of multiplication with f. For $z \in D$, let $k_z = K(\cdot, z)^*/\|K(\cdot, z)\|_{H^2(\sigma)}$ be the normalized reproducing kernel vector at the point z. The Berezin transform of an operator $T \in L(H^2(\sigma))$ is the function

$$\Gamma(T) : D \to \mathbb{C}, \ z \mapsto \langle Tk_z, k_z\rangle.$$

The Berezin transform of the Toeplitz operator T_f with symbol $f \in L^\infty(\sigma)$ coincides with its Poisson-Szegö integral

$$\Gamma(T_f)(z) = \langle T_f k_z, k_z\rangle_{H^2(\sigma)} = \int_{\partial D} f P(z, \cdot) d\sigma.$$

It is well known (see for instance [5]) that the Berezin transform of a compact operator $K \in L(H^2(\sigma))$ vanishes on the boundary of D in the sense that

$$\lim_{z \to \partial D} \Gamma(K)\,(z) = 0.$$

For a given subset $S \subset L^\infty(\sigma)$, the Toeplitz algebra with symbol class S is the closed subalgebra of $L(H^2(\sigma))$ defined by

$$\mathcal{T}(S) = \overline{\text{alg}}\,\{T_f;\ f \in S\}.$$

Important choices for S are the set of all bounded analytic functions (or better their boundary values), which will be denoted by $H^\infty = H^\infty(\sigma)$ in the sequel, and the class $C = C(\partial D)$ consisting of all complex-valued continuous functions on ∂D. A result of Aytuna and Chollet [1], generalizing a corresponding observation of Rudin for the unit ball, shows that $H^\infty + C = H^\infty(\sigma) + C(\partial D) \subset L^\infty(\sigma)$ is a closed subalgebra. It is known (see for instance [4]) that the Toeplitz algebra $\mathcal{T}(H^\infty + C)$ contains the set $\mathcal{K}(H^2(\sigma))$ of all compact operators and that the map

$$\tau : H^\infty + C \to \mathcal{T}(H^\infty + C)/\mathcal{K}(H^2(\sigma)),\ f \mapsto T_f + \mathcal{K}(H^2(\sigma))$$

is an isometric isomorphism of Banach algebras. In particular, Toeplitz tuples $T_f = (T_{f_1}, \ldots, T_{f_m})$ with symbols $f_i \in H^\infty + C$ essentially commute in the sense that the commutators

$$[T_{f_i}, T_{f_j}] = T_{f_i}T_{f_j} - T_{f_j}T_{f_i} \quad (1 \leq i, j \leq m)$$

are compact.
The Koszul complex (cf. [6])

$$K^\bullet(T, H) : 0 \to \Lambda^0(H) \xrightarrow{\delta_T^0} \Lambda^1(H) \xrightarrow{\delta_T^1} \cdots \xrightarrow{\delta_T^{n-1}} \Lambda^n(H) \to 0$$

of an essentially commuting tuple $T \in L(H)^m$ of bounded operators on a Hilbert space H is an essential complex of Hilbert spaces in the sense that $\delta_T^{i+1} \circ \delta_T^i$ is compact for every i. The tuple T is called Fredholm if the Koszul complex $K^\bullet(T, H)$ possesses an essential homotopy, that is, there are bounded operators $\epsilon^i : \Lambda^i(H) \to \Lambda^{i-1}(H)$ with

$$\epsilon^{i+1}\delta_T^i - \delta_T^{i-1}\epsilon^i - 1_{\Lambda^i(H)} \in \mathcal{K}(\Lambda^i(H))$$

for all i. One can show (Lemma 2.6.10 and Theorem 10.2.5 in [6]) that the tuple T is Fredholm if and only if the Koszul complex $K^\bullet(L_T, \mathcal{C}(H))$ of the commuting tuple $L_T = (L_{T_1}, \ldots, L_{T_m})$ consisting of the left multiplication operators

$$L_{T_i} : \mathcal{C}(H) \to \mathcal{C}(H),\ [A] \mapsto [T_i A]$$

with T_i on the Calkin algebra $\mathcal{C}(H) = L(H)/\mathcal{K}(H)$ is exact. The essential spectrum of an essentially commuting tuple $T \in L(H)^m$ is defined as

$$\sigma_e(T) = \{z \in \mathbb{C}^m;\ K^\bullet(z - T, H) \text{ is not Fredholm }\} = \sigma(L_T, \mathcal{C}(H)),$$

where $\sigma(L_T, \mathcal{C}(H))$ denotes the Taylor spectrum [11] of the commuting tuple $L_T \in L(\mathcal{C}(H))^m$.

3 Main Result

To prove the spectral mapping theorem for the essential spectrum of Toeplitz tuples with symbol in $H^\infty + C$, we need a result on the asymptotic multiplicativity of the Poisson-Szegö transform.

1 Lemma. *For $f, g \in H^\infty + C$, the Poisson-Szegö transform satisfies*

$$\lim_{z \to \partial D} |\mathscr{P}[fg](z) - \mathscr{P}[f](z)\mathscr{P}[g](z)| = 0.$$

Proof. We need some results on the Berezin transform that are implicitly contained in [3]. Since every point $z \in \partial D$ is a peak point for the Banach algebra $A(D) = \{f \in C(\overline{D});\ f|D \,\text{holomorphic}\}$, it follows that $A(D)$ is a pointed function algebra in the sense of [3] (see Definition 2.1 and Theorem 2.3 in [3]). It is elementary to check that the Hardy space $H^2(D)$ is a quasi-free Hilbert module over $A(D)$ as defined in [3].

For $z \in D$, consider the isometry

$$V_z : \mathbb{C} \to H^2(\sigma),\ t \mapsto tk_z.$$

The mapping $P_z = V_z V_z^*$ is the orthogonal projection onto the one-dimensional subspace of $H^2(\sigma)$ spanned by k_z. For given operators $S, T \in L(H^2(\sigma))$, the estimate

$$\begin{aligned} |\Gamma(ST)(z) - \Gamma(S)(z)\Gamma(T)(z)| &= |(V_z^* STV_z - V_z^* SP_z TV_z)(1)| \\ &= |V^* S[T, P_z]V_z(1)| \le \|S\|\ \|[T, P_z]\| \end{aligned}$$

holds for every point $z \in D$. For $\alpha \in \partial D$, the set of all operators $T \in L(H^2(\sigma))$ with the property that

$$\lim_{z \to \alpha} \|[T, P_z]| = 0$$

is a C^*-algebra containing the Toeplitz algebra $\mathcal{T}(A(D)|\partial D) = \mathcal{T}(C)$ (see the proof of Theorem 3.2 in [3]). An elementary compactness argument shows that $\lim_{z \to \partial D} \|[T, P_z]\| = 0$ for every operator $T \in \mathcal{T}(C)$. Therefore the relation

$$\lim_{z \to \partial D} |\Gamma(ST)(z) - \Gamma(S)(z)\Gamma(T)(z)| = 0$$

holds for any pair of operators $S \in L(H^2(\sigma)), T \in \mathcal{T}(C)$. Since for $g \in C$ and $f \in L^\infty(\sigma)$, the semi-commutator $T_f T_g - T_{fg} = PM_f H_g$ is compact (Theorem 4.2.17 in [12]), it follows that

$$\begin{aligned} |\mathscr{P}[fg](z) - \mathscr{P}[f](z)\mathscr{P}[g](z)| &\le \Gamma(T_{fg} - T_f T_g)(z)| \\ &+ |\Gamma(T_f T_g)(z) - \Gamma(T_f)(z)\Gamma(T_g)(z)| \end{aligned}$$

tends to zero as $z \to \partial D$. Using in addition the fact that $\mathscr{P}[fg] = \mathscr{P}[f]\mathscr{P}[g]$ for $f, g \in H^\infty$, one easily deduces the assertion. □

We begin by proving one half of our spectral mapping theorem in a particular situation. For simplicity, we use the notation $F = \mathscr{P}[f] = (\mathscr{P}[f_1], \ldots, \mathscr{P}[f_m])$ for the Poisson-Szegö transform of a tuple $f = (f_1, \ldots, f_m) \in L^\infty(\sigma)^m$.

2 Lemma. *For given $g \in (H^\infty)^r$, $h \in C^s$ and $f = (g, h)$, the spectral inclusion*

$$\bigcap(\overline{F(U \cap D)};\ U \supset \partial D \,\text{open}) \subset \sigma_e(T_f)$$

holds.

Proof. Suppose that T_f is Fredholm. It suffices to show that $F = \mathscr{P}[f]$ is bounded away from zero close to the boundary of D. Since T_f is Fredholm, the row multiplication

$$H^2(\sigma)^m \xrightarrow{T_f} H^2(\sigma)$$

with $m = r + s$ has finite-codimensional range. The orthogonal projection $Q \in L(H^2(\sigma))$ to the kernel of the operator $T_f T_f^*$ has finite rank and $T_f T_f^* + Q$ is bounded below. Hence there is a constant $c > 0$ with

$$T_f T_f^* + Q \geq c 1_{H^2(\sigma)}.$$

Since the Berezin transform $\Gamma(Q)(z)$ tends to zero as z approaches the boundary of D, there is an open neighbourhood U of ∂D such that

$$\sum_{i=1}^{m} \Gamma(T_{f_i} T_{f_i}^*)(z) = \Gamma(T_f T_f^*) \geq c/2$$

for all $z \in U \cap D$. An elementary calculation (Lemma 7 in [5]) yields that

$$\Gamma(T_{g_i} T_{g_i}^*) = |G_i|^2 \quad (i = 1, \ldots, r)$$

on D. Since $T_{h_i} T_{h_i}^* - T_{|h_i|^2}$ is compact and since by Lemma 1

$$\mathscr{P}[|h_i|^2](z) - |\mathscr{P}[h_i](z)|^2 \xrightarrow{z \to \partial D} 0,$$

it follows that

$$\Gamma(T_{h_i} T_{h_i}^*)(z) - |H_i(z)|^2 \xrightarrow{z \to \partial D} 0 \quad (i = 1, \ldots, s).$$

Summarizing we obtain that

$$\sum_{i=1}^{m} |F_i(z)|^2 - \Gamma(T_f T_f^*)(z) = \sum_{i=1}^{m} (|H_i(z)|^2 - \Gamma(T_{h_i} T_{h_i}^*)(z)) \to 0$$

as z approaches the boundary of D. Thus the assertion follows. $\square$

To prepare the proof of the opposite inclusion, we recall some results from Gelfand theory. Consider a unital algebra homomorphism $\Phi : \mathcal{M} \to L(X)$ from a unital commutative Banach algebra $\mathcal{M}$ into the algebra of all bounded operators on a Banach space X. A spectral system on $B = \overline{\Phi(\mathcal{M})}$ is a rule σ that assigns to each finite tuple $a \in B^r$ a compact subset $\sigma(a) \subset \mathbb{C}^r$ which is contained in the joint spectrum $\sigma_B(a) = \{z \in \mathbb{C}^r;\ 1 \notin \sum_{i=1}^r (z_i - a_i)B\}$ of a in B and which is compatible with projections in the sense that

$$p(\sigma(a, b)) = \sigma(a) \text{ and } q(\sigma(a, b)) = \sigma(b),$$

where p and q are the projections of C^{r+s} onto its first r and last s coordinates.

For a given set M, let us denote by $c(M)$ the set of all finite tuples of elements in M. Standard results going back to J.L. Taylor (see, e.g., Proposition 2.6.1 in [6]) show that, for a spectral system σ as above, the set

$$\Delta_{\Phi,\sigma} = \{\lambda \in \Delta_{\mathcal{M}};\ \hat{f}(\lambda) \in \sigma(\Phi(f)) \text{ for all } f \in c(\mathcal{M})\}$$

is the unique closed subset of the maximal ideal space $\Delta_{\mathcal{M}}$ of $\mathcal{M}$ with $\hat{f}(\Delta_{\Phi,\sigma}) = \sigma(\Phi(f))$ for all $f \in c(\mathcal{M})$. Here $\Phi(f) = (\Phi(f_1), \ldots, \Phi(f_r))$ and the Gelfand transform $\hat{f} = (\hat{f}_1, \ldots, \hat{f}_r)$ are formed componentwise for $f \in \mathcal{M}^r$.

Let $\Phi_0 : \mathcal{M} \to L(X)$ be the restriction of Φ to a unital closed subalgebra $\mathcal{M}_0 \subset \mathcal{M}$, and let σ_0 denote the spectral system on $B_0 = \overline{\Phi(\mathcal{M}_0)}$ obtained by restricting σ. An elementary exercise, using the uniqueness property of Δ_{Φ_0,σ_0}, shows that the restriction map

$$r : \Delta_{\Phi,\sigma} \to \Delta_{\Phi_0,\sigma_0},\ \lambda \mapsto \lambda|\mathcal{M}_0$$

is well defined, surjective and continuous (relative to the Gelfand topologies).

As before, let $H^2(\sigma)$ be the Hardy space on a bounded strictly pseudoconvex domain $D \subset \mathbb{C}^n$ with smooth boundary. We apply the above remarks to the Banach algebras $\mathcal{M}_0 = H^\infty$, $\mathcal{M} = H^\infty + C$ and the algebra homomorphism $\Phi : \mathcal{M} \to L(\mathcal{C}(H^2(\sigma)))$, $f \mapsto L_{T_f}$, mapping $f \in \mathcal{M}$ to the operator L_{T_f} of left multiplication with T_f on the Calkin algebra $\mathcal{C}(H^2(\sigma))$. Let σ be the spectral system on $B = \overline{\Phi(\mathcal{M})}$ associating with each tuple $a \in B^r$ its Taylor spectrum as a commuting tuple of bounded operators on $\mathcal{C}(H^2(\sigma))$. We write σ_0 for the restriction of σ to $B_0 = \overline{\Phi(\mathcal{M}_0)}$.

Recall that, for a tuple $f \in c(L^\infty(\sigma))$, we write $F = \mathscr{P}[f]$ for its Poisson-Szegö transform. As usual we shall identify functions $f \in H^\infty(\sigma)$ with their Poisson-Szegö transform $F = \mathscr{P}[f] \in H^\infty(D)$. It was shown by Andersson and Sandberg [2] (Theorem 1.2) that the spectral mapping fomula

$$\sigma(\Phi(f)) = \sigma_e(T_f) = \bigcap \left(\overline{f(U \cap D)};\ U \supset \partial D \text{ open}\right)$$

holds for every tuple $f \in c(H^\infty)$. Let $\pi = (\pi_1, \ldots, \pi_n)$ be the tuple of coordinate functions. Using Theorem 1 in [5] we obtain that

$$\hat{f}(\lambda) \in \bigcap \left(\overline{f(U \cap D)};\ U \text{ open neighbourhood of } \hat{\pi}(\lambda)\right)$$

for $f \in c(H^\infty)$ and every functional $\lambda \in \Delta_{\Phi_0,\sigma_0}$.

3 Proposition. *For $g \in (H^\infty)^r$, $h \in C^s$ and $f = (g, h)$, the spectral inclusion formula*

$$\sigma_e(T_f) \subset \bigcap \left(\overline{F(U \cap D)};\ U \supset \partial D \text{ open}\right)$$

holds.

Proof. Suppose that $0 \in \sigma_e(T_f)$. It suffices to show that 0 is contained in the intersection on the right. By the remarks preceding the proposition there is a functional $\lambda \in \Delta_{\Phi,\sigma}$ with $0 = \hat{f}(\lambda) = (\hat{g}(\lambda), \hat{h}(\lambda))$. Since $\lambda|C \in \Delta_C$, there is a point $z_0 \in \partial D$ with

$$\lambda(\varphi) = \varphi(z_0) \quad (\varphi \in C).$$

In particular, it follows that $\lim_{z \to z_0} H(z) = h(z_0) = 0$. The above cited results from [2] and [5] imply that

$$0 = \hat{g}(\lambda) \in \bigcap \left(\overline{g(U \cap D)};\ U \text{open neighbourhood of } z_0 \right).$$

Hence there is a sequence $(z_k)_{k \geq 1}$ in D with $\lim_{k \to \infty} z_k = z_0$ and

$$\lim_{k \to \infty} (g(z_k),\ H(z_k)) = 0.$$

This observation completes the proof. □

Our next aim is to show that Lemma 2 and Proposition 3 remain true for arbitrary symbols $f \in (H^\infty + C)^m$.

4 Theorem. *For $f \in (H^\infty + C)^m$, the formula*

$$\sigma_e(T_f) = \bigcap \left(\overline{F(U \cap D)};\ U \supset \partial D \text{ open} \right)$$

holds.

Proof. Let $f = g + h \in (H^\infty + C)^m$ be given with $g \in (H^\infty)^m$ and $h \in C^m$. Using a particular case of the analytic spectral mapping theorem for the Taylor spectrum, we obtain that

$$\sigma_e(T_f) = \sigma_e(T_g + T_h) = \sigma(L_{T_g} + L_{T_h})$$

$$= \{z + w;\ (z, w) \in \sigma(L_{T_g}, L_{T_h})\} = \{z + w;\ (z, w) \in \sigma_e(T_g, T_h)\}.$$

If $(z, w) \in \sigma_e(T_g, T_h)$, then by Proposition 3 there is a sequence (u_k) in D converging to some point $u \in \partial D$ such that

$$(z, w) = \lim_{k \to \infty} (G, H)\ (u_k).$$

But then

$$z + w = \lim_{k \to \infty} (G + H)\ (u_k) = \lim_{k \to \infty} F(u_k).$$

Hence $\sigma_e(T_f)$ is contained in the intersection on the right-hand side. Conversely, if ξ is a point in the intersection on the right-hand side, then there is a sequence (u_k) in D converging to a point $u \in \partial D$ such that

$$\xi = \lim_{k \to \infty} F(u_k) = \lim_{k \to \infty} (G(u_k) + H(u_k)).$$

But then $w = \lim_{k \to \infty} H(u_k) = h(u)$ exists and hence also $z = \lim_{k \to \infty} G(u_k)$ exists. By Lemma 2 we know that $(z, w) \in \sigma_e(T_g, T_h)$. Hence $\xi = z + w \in \sigma_e(T_f)$ as was to be shown. □

For a tuple $T = (T_1, \ldots, T_n) \in L(H)^n$ of operators on a Hilbert space H, the right essential spectrum $\sigma_{re}(T)$ is usually defined as the set of all points $z \in \mathbb{C}^n$ for which the range of the row multiplication

$$H^n \xrightarrow{(z_1 - T_1, \ldots, z_n - T_n)} H$$

is not finite codimensional or, equivalently, the row multiplication

$$\mathcal{C}(H)^n \xrightarrow{(z_1 - L_{T_1}, \ldots, z_n - L_{T_n})} \mathcal{C}(H)$$

is not onto (see e.g. Lemma 2.6.10 in [6] for the equivalence). Hence the right essential spectrum $\sigma_{re}(T)$ of T coincides with the right spectrum $\sigma_r(L_T, \mathcal{C}(H))$ of the multiplication tuple L_T on the Calkin algebra. Since Lemma 2 remains true with $\sigma_e(T_f)$ replaced by $\sigma_{re}(T_f)$ (see the proof of Lemma 2) and since the analytic spectral mapping formula used in the proof of Theorem 4 also holds for the right Taylor spectrum (Corollary 2.6.8 in [6]), we obtain the following consequence.

5 Corollary. *For $f \in (H^\infty + C)^m$, the formula*

$$\sigma_e(T_f) = \sigma_{re}(T_f) = \bigcap \left(\overline{F(U \cap D)};\ U \supset \partial D \text{ open} \right)$$

holds. $\square$

Our main result (Theorem 4) can also be proved for Toeplitz tuples $T_f \in L(L^2_a(D))^m$ with symbol $f \in (H^\infty(D) + C(\overline{D})^m$ on the Bergman space $L^2_a(D) = \{f \in \mathscr{O}(D);\ \|f\|^2 = \int_D |f|^2 d\lambda < \infty\}$ formed with respect to the volume measure λ on a strictly pseudoconvex domain $D \subset \mathbb{C}^n$ with smooth boundary. It suffices to replace the spectral mapping formula for Toeplitz tuples with H^∞-symbol of Andersson and Sandberg from [2] by the corresponding spectral mapping formula for the Bergman space (Theorem 8.2.6 in [6]) and to replace the Poisson-Szegö transform by the Poisson-Bergman transform. All properties needed for the Poisson-Bergman integral can be found in [8]. We only state the corresponding result in the Bergman case.

6 Theorem. *Let $D \subset \mathbb{C}$ be a bounded strictly pseudoconvex domain with smooth boundary. Then for $f \in (H^\infty(D) + C(\overline{D}))^m$, the essential spectrum of the Toeplitz tuple $T_f \in L(L^2_a(D))^m$ on the Bergman space $L^2_a(D)$ is given by*

$$\sigma_e(T_f) = \sigma_{re}(T_f) = \bigcap \left(\overline{f(U \cap D)};\ U \supset \partial D \text{ open} \right).$$

$\square$

The reader should observe that, since the Poisson-Bergman transform of a continuous function $h \in C(\overline{D})^m$ extends to a continuous function $H \in C(\overline{D})^m$ with $H|\partial D = h|\partial D$, the intersection on the right-hand side does not change when f is replaced by its Poisson-Bergman transform F.

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