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Abstract

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ and a function $u_0 \in W^1_1(\Omega;\mathbb{R}^N)$ we consider the minimization problem

$$(\mathcal{P}) \qquad \int_\Omega f(\nabla u)\, dx \to \min \text{ in } u_0 + \overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)$$

where $f\colon \mathbb{R}^{nN} \to [0,\infty)$ is a strictly convex integrand. Let $\mathcal{M}$ denote the set of all L^1-cluster points of minimizing sequences of problem $(\mathcal{P})$. We show that the geometric relaxation of problem $(\mathcal{P})$ coincides with the relaxation based on the notion of the extended Lagrangian, moreover, we prove that the elements u of $\mathcal{M}$ are in one-to-one correspondence with the solutions of the relaxed problems.

AMS Subject Classification: 49
Key words: variational problems, linear growth, generalized minimizers, relaxation, functions of bounded variation

1 Introduction

In this note we are concerned with variational problems of linear growth defined on spaces of vector-valued functions which are usually handled by introducing a suitable relaxed version of the problem or by passing to some dual variational formulation. There exist two — at least formally — different approaches to a reasonable concept of relaxation, the first one being preferred in connection with problems of minimal surface type, the second one occuring in the theory of perfect plasticity. For experts in the theory of relaxation it might be obvious that both points of view lead to the same result but we did not find a rigorous proof in the literature and so we decided to sketch the arguments in the present paper.

To be precise, let us first fix our notation. In what follows Ω and $\widehat{\Omega}$ denote bounded Lipschitz domains in $\mathbb{R}^n$, $n \geq 2$, such that $\Omega \Subset \widehat{\Omega}$. Given boundary values u_0 of Sobolev class $W^1_1(\Omega;\mathbb{R}^N)$ we may extend u_0 to the domain $\widehat{\Omega}$ such that $u_0 \in \overset{\circ}{W}{}^1_1(\widehat{\Omega};\mathbb{R}^N)$ and let

$$(1.1) \qquad BV_{u_0}(\Omega;\mathbb{R}^N) := \Big\{ u \in BV(\widehat{\Omega};\mathbb{R}^N) : u = u_0 \text{ on } \widehat{\Omega} - \Omega \Big\}$$

where $BV(\widehat{\Omega};\mathbb{R}^N)$ is the space of functions of bounded variation (see e.g. [Giu] or [AFP]). Suppose that we are given a strictly convex (in the sense of definition) function $f\colon \mathbb{R}^{nN} \to [0,\infty)$ of linear growth, i.e. f satisfies

$$(1.2) \qquad a|Z| - b \;\leq\; f(Z) \;\leq\; A|Z| + B \quad \text{for all } Z \in \mathbb{R}^{nN}$$

with some positive constants a, A, b, B and we further assume that $f(0) = 0$. As a matter of fact the variational problem

$$(\mathcal{P}) \qquad J[u] := \int_\Omega f(\nabla u)\, dx \to \min \quad \text{in } u_0 + \overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^N)$$

in general fails to have a solution but from (1.2) it follows that the set

$$\mathcal{M} := \Big\{ u \in BV(\Omega;\mathbb{R}^N) \quad : \quad u \text{ is a } L^1\text{-cluster point of some minimizing sequence to problem } (\mathcal{P}) \Big\}$$

of generalized minimizers is non-empty. Next we recall the concept of the "geometric" relaxation of problem $(\mathcal{P})$ involving the Dirichlet boundary data u_0 via the space $BV_{u_0}(\Omega;\mathbb{R}^N)$ (see (1.1)) which is discussed in the paper [GMS] of Giaquinta, Modica and Souček. For functions $w \in BV(\widehat{\Omega};\mathbb{R}^N)$ let

$$\widehat{J}\,[w,\widehat{\Omega}] = \int_{\widehat{\Omega}} f(\nabla^a w)\, dx \; + \; \int_{\widehat{\Omega}} f_\infty\left(\frac{\nabla^s w}{|\nabla^s w|}\right) d|\nabla^s w|$$

where $\nabla^a w$ denotes the absolutely continuous part of ∇w with respect to Lebesgue's measure, $\nabla^s w$ is the singular part of ∇w, and f_∞ denotes the recession function of f. Given these definitions, the lower semicontinuity theorem of Reschetnyak (see [Re]) immediately implies the existence result [GMS], Theorem 1.3:

THEOREM 1.1 *There exists a minimum point for the problem*

$$\widehat{J}\,[\cdot,\widehat{\Omega}] \to \min \;\; \textit{in } BV_{u_0}(\Omega;\mathbb{R}^N).$$

Let us fix some function $w \in BV_{u_0}(\Omega;\mathbb{R}^N)$. Observing

$$\nabla^s w \llcorner \partial\Omega = (u_0 - w) \otimes \nu \; \mathcal{H}^{n-1} \llcorner \partial\Omega,$$

ν denoting the outward unit normal vector to $\partial\Omega$, we may write

$$\begin{aligned} \widehat{J}\,[w,\widehat{\Omega}] \; &= \; \int_\Omega f(\nabla^a w)\, dx \; + \int_\Omega f_\infty\left(\frac{\nabla^s w}{|\nabla^s w|}\right) d|\nabla^s w| \\ &\quad + \int_{\partial\Omega} f_\infty\Big((u_0 - w)\otimes\nu\Big)\, d\,\mathcal{H}^{n-1} + \int_{\widehat{\Omega}-\Omega} f(\nabla u_0)\, dx, \end{aligned} \tag{1.3}$$

and since the last integral on the right-hand side of (1.3) is constant we drop this term and introduce the energy K: $BV(\Omega;\mathbb{R}^N) \to \mathbb{R}$,

$$\begin{aligned} K[u] \; &:= \; \int_\Omega f(\nabla^a u)\, dx + \int_\Omega f_\infty\left(\frac{\nabla^s u}{|\nabla^s u|}\right)\, d|\nabla^s u| \\ &\quad + \int_{\partial\Omega} f_\infty\Big((u_0 - u)\otimes\nu\Big)\, d\,\mathcal{H}^{n-1}, \end{aligned} \tag{1.4}$$

hence

$$\widehat{J}\,[w,\widehat{\Omega}] = K[w_{|\Omega}] + const$$

for any $w \in BV_{u_0}(\Omega;\mathbb{R}^N)$. Conversely, given $u \in BV(\Omega;\mathbb{R}^N)$, we let $\widehat{u}$ denote the extension via u_0 to the domain $\widehat{\Omega}$ and get

$$\widehat{J}\,[\widehat{u},\widehat{\Omega}] = K[u] + const.$$

The properties of the functional K defined in (1.4) are summarized in

THEOREM 1.2

i) The minimization problem

$$(\widehat{\mathcal{P}}) \qquad K[w] \to \min \ \textit{ in } BV(\Omega;\mathbb{R}^N)$$

admits at least one solution.

ii) $\displaystyle\inf_{w\in u_0+\overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^N)} J[w] = \inf_{w\in BV(\Omega;\mathbb{R}^N)} K[w]$

iii) $u \in \mathcal{M} \iff u$ *is K-minimizing.*

We note that the proof of Theorem 1.2 is based on Theorem 1.1 together with some minor adjustments of the arguments given in [Giu] for the minimal surface case. A short outline will be presented in the next section.

In order to introduce the notion of relaxation via some suitable Lagrangian function (following the lines of Seregin [Se], see [FS] for a complete list of references, and of Strang and Temam [ST]) we observe

$$J[u] = \sup_{\tau\in L^\infty(\Omega;\mathbb{R}^{nN})} l(u,\tau) \ \text{ for all } \ u \in W_1^1(\Omega;\mathbb{R}^N)$$

where

$$l(u,\tau) = \int_\Omega \tau : \nabla u\ dx - \int_\Omega f^*(\tau)\ dx,\ u \in W_1^1(\Omega;\mathbb{R}^N),\ \tau \in L^\infty(\Omega;\mathbb{R}^{nN}),$$

is the Lagrangian, and f^* denotes the conjugate function of f. Quoting [ET] we remark that the dual problem to $(\mathcal{P})$, i.e. the problem

$$(\mathcal{P}^*) \qquad \begin{aligned} R \quad &\to \ \max \ \text{ on } L^\infty(\Omega;\mathbb{R}^{nN}),\\ R[\tau] \ &:= \inf_{u\in u_0+\overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^N)} l(u,\tau), \end{aligned}$$

admits a solution, moreover we have

$$\inf_{w\in u_0+\overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^N)} J[w] = \max_{\tau\in L^\infty(\Omega;\mathbb{R}^{nN})} R[\tau].$$

For functions $u \in u_0 + \overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^N)$ and tensors τ from the space

$$\mathcal{U} := \Big\{ \sigma \in L^\infty(\Omega;\mathbb{R}^{nN}) : \operatorname{div}\sigma \in L^n(\Omega;\mathbb{R}^N) \Big\}$$

it is easy to check that

$$l(u,\tau) = \int_\Omega \operatorname{div}\tau \cdot (u_0 - u)\, dx - \int_\Omega f^*(\tau)\, dx + \int_\Omega \tau : \nabla u_0 \, dx =: \widetilde{l}(u,\tau),$$

and the extended Lagrangian $\widetilde{l}(u,\tau)$ makes sense in case $u \in BV(\Omega;\mathbb{R}^N)$. Finally, we use the extended Lagrangian to define

$$\widetilde{J}\,[w] := \sup_{\tau \in U} \widetilde{l}(w,\tau), \;\; w \in BV(\Omega;\mathbb{R}^N). \tag{1.5}$$

The next result is essentially due to Seregin (see, e.g. [Se], we just added the fact that $\tilde{J}$-minimizers lie in the set $\mathcal{M}$), but with the help of Theorem 1.4 below it can be reduced to Theorem 1.2.

THEOREM 1.3

i) The minimization problem

$$(\widetilde{\mathcal{P}}) \qquad \widetilde{J}[w] \to \min \;\; \text{in } BV(\Omega;\mathbb{R}^N)$$

admits a solution.

ii) $\displaystyle \inf_{w \in u_0 + \overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^N)} J[w] = \inf_{w \in BV(\Omega;\mathbb{R}^N)} \widetilde{J}[w].$

iii) $u \in \mathcal{M} \iff u$ *is* $\widetilde{J}$*-minimizing.*

So, according to Theorem 1.2 and 1.3 problem $(\widehat{\mathcal{P}})$ as well as problem $(\widetilde{\mathcal{P}})$ is a suitable relaxed version of the original problem $(\mathcal{P})$ in the sense that we have the properties stated in *ii*) and *iii*). Now, in order to get a complete picture of the situation, we formulate our main result.

THEOREM 1.4 *On the space $BV(\Omega;\mathbb{R}^N)$ the functionals K and $\widetilde{J}$ defined in (1.4) and (1.5) coincide.*

The rest of the paper is organized as follows: in Section 2 we give a short proof of Theorem 1.2, in Section 3 we prove the identity $\widetilde{J} = K$, and in a final section we present a brief application of our results.

2 Proof of Theorem 1.2

Part i) of Theorem 1.2 is an immediate consequence of Theorem 1.1 and the definition of the functional K. For the minimal surface case, i.e. $N = 1$ and $J[u] = \int_\Omega \sqrt{1+|\nabla u|^2}\, dx$, ii) is established in [Giu], Proposition 14.3, p.161. In the general case we first observe that we have $K[w] = J[w]$ for $w \in u_0 + \overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^N)$, thus

$$\inf_{u_0+\overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^N)} J \;\geq\; \inf_{BV(\Omega;\mathbb{R}^N)} K.$$

For the opposite inequality we use a lemma which in slightly different forms occurs in many textbooks, see e.g. [Giu] or [AFP]. Unfortunately we could not find an explicit reference for the statement given below.

LEMMA 2.1 *Let $w \in BV(\Omega;\mathbb{R}^N)$ and consider its extension*

$$\widehat{w} = \begin{cases} w & on \quad \Omega, \\ u_0 & on \quad \widehat{\Omega} - \Omega. \end{cases}$$

Then there exists a sequence $\{w_m\}$ in $u_0 + C_0^\infty(\Omega;\mathbb{R}^N)$ such that if $m \to \infty$ (and if we extend w_m by u_0 to $\widehat{\Omega}$)

a) $w_m \to \widehat{w}$ *in* $L^1(\widehat{\Omega};\mathbb{R}^N)$,

b) $\displaystyle\int_{\widehat{\Omega}} \sqrt{1+|\nabla w_m|^2}\, dx \to \int_{\widehat{\Omega}} \sqrt{1+|\nabla \widehat{w}|^2}$.

Assume for the moment that Lemma 2.1 is true, fix $w \in BV(\Omega,\mathbb{R}^N)$ and define $\{w_m\}$ as above. Then, from Reschetnyak's continuity theorem (see [Re], compare also [AG], Theorem 2.1 and Proposition 2.2) we deduce

$$\widehat{J}[\widehat{w},\widehat{\Omega}] \;=\; \lim_{m\to\infty} \widehat{J}[w_m,\widehat{\Omega}],$$

hence $K[w] = \lim_{m\to\infty} K[w_{m|\Omega}]$, and clearly

$$K[w_{m|\Omega}] \;=\; J[w_{m|\Omega}] \;\geq\; \inf_{u_0+\overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^N)} J$$

which proves ii) of the theorem. Suppose that $u \in BV(\Omega;\mathbb{R}^N)$ is K-minimizing. We apply Lemma 2.1 with w replaced by u and get for the corresponding approximating sequence $\{w_m\} \in BV(\widehat{\Omega};\mathbb{R}^N)$

$$J[w_{m|\Omega}] \;=\; K[w_{m|\Omega}] \underset{m\to\infty}{\longrightarrow} K[u],$$

thus

$$J[w_{m|\Omega}] \to \inf_{u_0+\overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^N)} J$$

on account of $ii)$. Hence $\{w_{m|\Omega}\}$ is a J-minimizing sequence which is of class $u_0 + \overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^N)$ such that $w_{m|\Omega} \to u$ in $L^1(\Omega;\mathbb{R}^N)$. This implies $u \in \mathcal{M}$. Conversely, consider $u \in \mathcal{M}$ being the L^1-limit of some J-minimizing sequence $\{u_m\} \in u_0 + \overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^N)$. Since K is lower semicontinuous with respect to this convergence, we find

$$K[u] \;\le\; \liminf_{m\to\infty} K[u_m] \;=\; \liminf_{m\to\infty}\; J[u_m] \;=\; \inf_{u_0+\overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^N)} J$$

and by $ii)$ u is seen to be K-minimizing. $\square$

Let us now come to the **Proof of Lemma 2.1.** First we recall the definition of the measure

$$B \longmapsto \int_B \sqrt{1+|\nabla u|^2},$$

where B is a Borel subset of $\widehat{\Omega}$ and u denotes a function in $BV(\widehat{\Omega};\mathbb{R}^N)$. We let

$$\int_B \sqrt{1+|\nabla u|^2} \;:=\; \int_B \sqrt{1+|\nabla^a u|^2}\,dx + |\nabla^s u|(B) \tag{2.1}$$

which is in accordance with the general concept of applying a convex function to a measure (see [DT]). It is easy to see that assumption $b)$ of Lemma 2.1 implies the weaker condition

$$\int_{\widehat{\Omega}} |\nabla w_m|\,dx \longrightarrow \int_{\widehat{\Omega}} |\nabla \widehat{w}|,$$

for example we may quote [AG], Proposition 2.2, with the choice $F(P) := |P|$. It should also be noted that a version of our approximation lemma involving condition $b)$ occurs in [AG], Proposition 2.3. During the proof of Lemma 2.1 we replace (2.1) by the following equivalent representation which for example can be deduced by applying [DT], Proposition 1.2, to the function $f_0(P) = \sqrt{1+|P|^2} - 1$. For u and B as in (2.1) we have

$$\begin{aligned} \int_B \sqrt{1+|\nabla u|^2} \;&=\; \mathcal{L}^n(B) - \mathcal{L}^n(\widehat{\Omega}) \\ &\quad+ \sup_{\tau\in C_0^\infty(\widehat{\Omega};\mathbb{R}^{nN}),|\tau|\le 1} \left\{ \int_B \tau : \nabla u + \int_{\widehat{\Omega}} \sqrt{1-|\tau|^2}\,dx \right\}. \end{aligned} \tag{2.2}$$

For notational simplicity we restrict ourselves to the case when $\Omega = B_1 = B_1(0)$ and $\widehat{\Omega} = B_2 = B_2(0)$, the general situation is reduced to this special setting via a covering argument, for details we refer to [B2]. Let us fix $w \in BV(B_1;\mathbb{R}^N)$ and let

$$\widehat{w}(z) = \begin{cases} w(z), & \text{if } |z| < 1,\\ u_0(z), & \text{if } |z| \geq 1, \end{cases}$$

where we agree to define $\widehat{w}(z) = 0$ for $|z| \geq 2$. For $\delta \in (0,1)$ close to 1 let

$$v_\delta(z) := (\widehat{w} - u_0)\left(\frac{z}{\delta}\right) + u_0(z). \tag{2.3}$$

Note that $v_\delta(z) = u_0(z)$ for $|z| \geq \delta$. For $\tau \in C_0^\infty(B_2;\mathbb{R}^{nN})$, $|\tau| \leq 1$, we deduce $\big(\widetilde{\tau}(z) := \tau(\delta z)\big)$

$$\begin{aligned}
&\int_{B_2} \nabla v_\delta : \tau + \int_{B_2} \sqrt{1-|\tau|^2}\, dx\\
&\quad = \delta^{n-1}\left[\int_{B_{\frac{2}{\delta}}} \widetilde{\tau} : \nabla \widehat{w} \;+\; \int_{B_{\frac{2}{\delta}}} \sqrt{1-|\widetilde{\tau}|^2}\, dx\right]\\
&\qquad +\delta^{n-1}\left[\delta^{1-n}\int_{B_2} \sqrt{1-|\tau|^2}\, dx \;-\; \int_{B_{\frac{2}{\delta}}} \sqrt{1-|\widetilde{\tau}|^2}\, dx\right]\\
&\qquad - \int_{B_2} \left[\delta^{-1}\nabla u_0\left(\frac{y}{\delta}\right) - \nabla u_0(y)\right] : \tau(y)\, dy \;=:\; (I) \;+\; (II) \;-\; (III),
\end{aligned}$$

where

$$\begin{aligned}
(I) &\leq \delta^{n-1}\int_{B_{\frac{2}{\delta}}} \sqrt{1+|\nabla \widehat{w}|^2},\\
\big|(III)\big| &\leq \varepsilon, \text{ if } \delta \text{ is sufficiently close to 1},\\
\big|(II)\big| &= \left|\int_{B_2} \sqrt{1-|\tau|^2}\, dx - \frac{1}{\delta}\int_{B_2} \sqrt{1-|\tau|^2}\, dx\right|\\
&\leq \left(\tfrac{1}{\delta} - 1\right) \mathcal{L}^n(B_2).
\end{aligned}$$

From the explicit formula (2.1) for the measure $\int \sqrt{1+|\nabla \widehat{w}|^2}$ (which according to our convention we regard as a measure on $\mathbb{R}^n$) and the properties of $\widehat{w}$ (there is no mass of $\nabla \widehat{w}$ on ∂B_2) we see

$$\lim_{\delta \uparrow 1} \delta^{n-1}\int_{B_{\frac{2}{\delta}}} \sqrt{1+|\nabla \widehat{w}|^2} \;=\; \int_{B_2} \sqrt{1+|\nabla \widehat{w}|^2},$$

thus

$$\begin{aligned}\int_{B_2}\sqrt{1+|\nabla v_\delta|^2} &= \sup\Bigg\{\int_{B_2}\nabla v_\delta : \tau \\ &\quad +\int_{B_2}\sqrt{1-|\tau|^2}\,dx : \tau\in C_0^\infty(B_2;\mathbb{R}^{nN}), |\tau|\le 1\Bigg\} \\ &\le \varepsilon+\left(\tfrac{1}{\delta}-1\right)\,\mathcal{L}^n(B_2) \;+\; \delta^{n-1}\int_{B_{\frac{2}{\delta}}}\sqrt{1+|\nabla\widehat{w}|^2}\end{aligned}$$

and therefore (since $\varepsilon > 0$ was arbitrary)

$$\limsup_{\delta\uparrow 1}\int_{B_2}\sqrt{1+|\nabla v_\delta|^2} \;\le\; \int_{B_2}\sqrt{1+|\nabla\widehat{w}|^2}.$$

From $v_\delta \longrightarrow \widehat{w}$ in $L^1(B_2)$ together with the lower semicontinuity of $\int_{B_2}\sqrt{1+|\cdot|^2}$ we infer the opposite inequality.

So, for $\varepsilon > 0$ given, we can find δ close to 1 such that the function $v = v_\delta$ from (2.3) satisfies

$$\text{(2.4)}\qquad \begin{cases} v(z)=u_0(z) \text{ for } |z|\ge\delta, \quad \displaystyle\int_{B_2}|v-\widehat{w}|\,dz\le\varepsilon, \\ \left|\displaystyle\int_{B_2}\sqrt{1+|\nabla v|^2}-\int_{B_2}\sqrt{1+|\nabla\widehat{w}|^2}\right| \;\le\; \varepsilon. \end{cases}$$

Let $(\ldots)^\rho$ denote the mollification of a function with respect to a smoothing kernel ω_ρ. With v from (2.4) we let

$$w_\rho \;:=\; u_0+(v-u_0)^\rho,$$

i.e. w_ρ equals u_0 + the mollification of the scaled function $z\mapsto(\widehat{w}-u_0)(\frac{z}{\delta})$. We have $(v-u_0)^\rho(z)=0$ if $|z|\ge\delta+\rho$, and if we assume $\rho+\delta<1$, the function $(v-u_0)^\rho$ is of class $C_0^\infty(B_1;\mathbb{R}^N)$. Moreover, we deduce from (2.4)

$$\text{(2.5)}\qquad \int_{B_2}|w_\rho-\widehat{w}|\,dx \;\le\; 2\varepsilon \text{ if } \rho\le\rho(\varepsilon,\delta).$$

Using the inequality

$$\int_{B_2}\sqrt{1+|\nabla w_\rho|^2}\,dx \;\le\; \int_{B_2}\sqrt{1+|\nabla v^\rho|^2}\,dx \;+\; \int_{B_2}|\nabla u_0-\nabla u_0^\rho|\,dx$$

it suffices to discuss

$$\int_{B_2} \sqrt{1+|\nabla v^\rho|^2}\, dx \;=\; \sup\Big\{ \int_{B_2} \tau : \nabla v^\rho \, dx + \int_{B_2} \sqrt{1-|\tau|^2}\, dx : \\ \tau \in C_0^\infty(B_2;\mathbb{R}^{nN}), |\tau| \le 1 \Big\}.$$

Let us fix a smooth tensor τ with compact support in B_2. Recalling $v = u_0$ for $|z| \ge \delta$ and $|\nabla v|(\mathbb{R}^n - B_2) = 0$ we get

$$\int_{B_2} \tau : \nabla v^\rho \, dx \;=\; \int_{B_2} \tau^\rho : \nabla v \;=\; \int_{B_{2+\rho}} \tau^\rho : \nabla v,$$

hence (observe $\tau^\rho \in C_0^\infty(B_{2+\rho};\mathbb{R}^{nN})$, $|\tau^\rho| \le 1$)

$$\begin{aligned} &\int_{B_2} \tau : \nabla v^\rho \;+\; \int_{B_2} \sqrt{1-|\tau|^2}\, dx \\ &\quad= \int_{B_{2+\rho}} \tau^\rho : \nabla v \;+\; \int_{B_{2+\rho}} \sqrt{1-|\tau^\rho|^2}\, dx \;+ \int_{B_2} \sqrt{1-|\tau|^2}\, dx \\ &\qquad - \int_{B_{2+\rho}} \sqrt{1-|\tau^\rho|^2}\, dx \\ &\quad\le \int_{B_{2+\rho}} \sqrt{1+|\nabla v|^2} \;+\; \int_{B_2} \sqrt{1-|\tau|^2}\, dx \;-\; \int_{B_{2+\rho}} \sqrt{1-|\tau^\rho|^2}\, dx. \end{aligned}$$

As remarked above it is immediate (see (2.1)) that

$$\lim_{\rho \downarrow 0} \int_{B_{2+\rho}} \sqrt{1+|\nabla v|^2} \;=\; \int_{B_2} \sqrt{1+|\nabla v|^2}.$$

Since $P \mapsto \sqrt{1-|P|^2}$, $|P| \le 1$, is concave, Jensen's inequality implies

$$\begin{aligned} &\int_{B_2} \sqrt{1-|\tau|^2}\, dx \;-\; \int_{B_{2+\rho}} \sqrt{1-|\tau^\rho|^2}\, dx \\ &\qquad\le \int_{B_2} \sqrt{1-|\tau|^2}\, dx \;-\; \int_{B_{2+\rho}} \Big(\sqrt{1-|\tau|^2}\Big)^\rho \, dx. \end{aligned}$$

Finally we have

$$
\begin{aligned}
\int_{B_2} \sqrt{1-|\tau|^2}\,dx &= \int_{B_2}\left(\int_{B_\rho(x)} \omega_\rho(x-y)\,dy\right)\sqrt{1-|\tau(x)|^2}\,dx\\
&= \int_{B_2}\left(\int_{B_{2+\rho}} \omega_\rho(x-y)\,dy\right)\sqrt{1-|\tau(x)|^2}\,dx\\
&= \int_{B_{2+\rho}}\left(\int_{B_2} \omega_\rho(x-y)\,\sqrt{1-|\tau(x)|^2}\,dx\right)dy\\
&\le \int_{B_{2+\rho}}\left(\sqrt{1-|\tau|^2}\right)^\rho (y)\,dy,
\end{aligned}
$$

and we get

$$
\int_{B_2}\sqrt{1+|\nabla v^\rho|^2}\,dx \;\le\; \int_{B_{2+\rho}}\sqrt{1+|\nabla v|^2},
$$

hence for $\rho \le \rho(\varepsilon,\delta)$

$$
\int_{B_2}\sqrt{1+|\nabla w_\rho|^2}\,dx \;\le\; 2\varepsilon + \int_{B_2}\sqrt{1+|\nabla v|^2} \tag{2.6}
$$

Now let $\varepsilon = \frac{1}{m} \in \mathbb{N}$, and calculate $\delta_m \uparrow 1$ according to (2.4). The sequence $\rho_m = \rho_m(\delta_m)$ is defined such that (2.5) and (2.6) are true. Let $w_m := w_{\rho_m}$. Our construction implies $w_m - u_0 \in C_0^\infty(B_1;\mathbb{R}^N)$, (2.5) gives $w_m \to w$ in $L^1(B_2;\mathbb{R}^N)$, and from (2.4) and (2.6) we deduce

$$
\limsup_{m\to\infty}\int_{B_2}\sqrt{1+|\nabla w_m|^2}\,dx \;\le\; \int_{B_2}\sqrt{1+|\nabla\widehat{w}|^2},
$$

the opposite inequality being a consequence of lower semicontinuity. This completes the proof of Lemma 2.1. □

3 Proof of Theorem 1.4

Under the assumptions concerning our integrand f we have

LEMMA 3.1 *The conjugate function f^* is essentially smooth, i.e. f^* is a proper convex function and for $D := \operatorname{int}(\operatorname{dom} f^*)$ we have*

a) D is non-empty.

Moreover

b) f^* is differentiable throughout D and

c) $\lim_{i\to\infty}|\nabla f^*(Q_i)| = +\infty$, whenever $\{Q_i\}$ is a sequence in D converging to a boundary point Q of D.

Proof of Lemma 3.1. As a convex and finite function on $\mathbb{R}^{nN}$, f is continuous (see [Ro], Corollary 10.1.1, p.83). For a proper convex function, closedness is the same as lower semicontinuity ([Ro], p.52), hence in particular f is closed. As a strictly convex function f clearly is essentially strictly convex, thus we may apply Theorem 26.3, p.253, of [Ro] to see that f^* is essentially smooth. $\square$

REMARK 3.1 *From the linear growth condition (1.2) it follows that* $\operatorname{dom} f^*$ *is a bounded set, compare e.g. [DT], Section 1.2, moreover* $f^* \geq 0$ *and* $f^*(0) = 0$.

Let $L(c) := \{Q \in \mathbb{R}^{nN} : f^*(Q) \leq c\}$, $c \in \mathbb{R}$. We claim that the definition (1.5) of $\widetilde{J}$ can be rewritten as

$$\begin{aligned} \widetilde{J}[w] \;&=\; \sup\big\{\widetilde{l}(w,\tau) : \tau \in C^\infty(\overline{\Omega};\mathbb{R}^{nN}), \\ &\tau(x) \in L(c) \text{ on } \overline{\Omega} \text{ for some } c = c(\tau) \in \mathbb{R}\big\},\ w \in BV(\Omega;\mathbb{R}^N). \end{aligned} \tag{3.1}$$

Let us fix some point $Q_0 \in D$ and a function $w \in BV(\Omega;\mathbb{R}^N)$. For $\varepsilon > 0$ we choose $\tau = \tau_\varepsilon \in \mathcal{U}$ such that

$$\widetilde{J}[w] - \widetilde{l}(w,\tau) < \varepsilon,$$

in particular we may assume that $\tau(x) \in \operatorname{dom} f^*$ a.e. For a sequence $\lambda_k \in (0,1)$, $\lambda_k \to 1$ as $k \to \infty$, we then let

$$\tau_k \;:=\; (1-\lambda_k)Q_0 + \lambda_k\tau.$$

Since Q_0 belongs to the interior of $\operatorname{dom} f^*$, we can find an open ball B around Q_0 compactly contained in D. Let C denote the union of all segments $\overline{P\tau(x)}$ with $P \in B$. Clearly C is contained in the convex set $\operatorname{dom} f^*$, and any point $Q \in \overline{Q_0\tau(x)}$ — different from $\tau(x)$ in case $\tau(x) \in \partial\operatorname{dom} f^*$ — belongs to the interior of $\operatorname{dom} f^*$, thus $\tau_k(x) \in D$ for almost all x and any k. Next we prove the existence of numbers $\gamma_k > 0$ such that

$$\operatorname{dist}(\tau_k, \partial\operatorname{dom} f^*) \geq \gamma_k \ \text{ for all } k \in \mathbb{N} \tag{3.2}$$

almost everywhere. Let

$$\varepsilon_k \;:=\; \frac{1}{2}\,\min\{\operatorname{dist}(B, \partial\operatorname{dom} f^*), (1-\lambda_k)\operatorname{rad}(B)\}.$$

Case 1.

$$\operatorname{dist}(\tau(x), \partial\operatorname{dom} f^*) \leq \varepsilon_k.$$

Then

$$\begin{aligned}\operatorname{dist}(\tau_k(x), \partial\mathrm{dom}\, f^*) &\geq |\tau_k(x) - \tau(x)| - \operatorname{dist}(\tau(x), \partial\mathrm{dom}\, f^*) \\ &\geq (1-\lambda_k)|\tau(x) - Q_0| - \varepsilon_k,\end{aligned}$$

and

$$|\tau(x) - Q_0| \geq rad\ (B),$$

since $\tau(x) \in B$ would imply

$$\operatorname{dist}(\tau(x), \partial\mathrm{dom}\, f^*) \geq \operatorname{dist}(B, \partial\mathrm{dom}\, f^*) > \varepsilon_k$$

contradicting our assumption in Case 1. Thus

$$\operatorname{dist}\big(\tau_k(x), \partial\mathrm{dom}\, f^*\big) \geq (1-\lambda_k)\ \mathrm{rad}\,(B) - \varepsilon_k \geq \frac{1}{2}\,(1-\lambda_k)\ \mathrm{rad}\,(B).$$

Case 2.

$$\operatorname{dist}\big(\tau(x), \partial\mathrm{dom}\, f^*\big) \geq \varepsilon_k.$$

By the choice of ε_k the open ball B' of radius ε_k around Q_0 belongs to D, the same is true for the ball B'' of radius ε_k centered at $\tau(x)$. Let Z denote the union of all segments $\overline{PQ}$ with $P \in B'$, $Q \in B''$. Clearly $Z \subset \operatorname{int}(\mathrm{dom}\, f^*)$ and it is immediate that the distance of the segment $\overline{Q_0\tau(x)}$ to $\partial\mathrm{dom}\, f^*$ is bounded below by ε_k, hence $\operatorname{dist}\big(\tau_k(x), \partial\mathrm{dom}\, f^*\big) \geq \varepsilon_k$. This implies (3.2) with $\gamma_k = \varepsilon_k$.

Remarking that $K_k := \{Q \in \mathrm{dom}\, f^* : \operatorname{dist}(Q, \partial\mathrm{dom}\, f^*) \geq \gamma_k\}$ is a compact set contained in $\operatorname{int}(\mathrm{dom}\, f^*)$, continuity of f^* on $\operatorname{int}(\mathrm{dom}\, f^*)$ implies the existence of a real number c_k such that $f^* \leq c_k$ on K_k, and from (3.2) we deduce

$$\tau_k \in L(c_k) \ \text{ almost everywhere.} \tag{3.3}$$

From

$$f^*(\tau_k) \leq (1-\lambda_k) f^*(Q_0) + \lambda_k f^*(\tau)$$

and the choice of τ we get

$$\begin{aligned}\widetilde{l}(w, \tau_k) &\geq \lambda_k\, \widetilde{l}(w, \tau) + (1-\lambda_k)\ \Big(- |\Omega| f^*(Q_0) + \int_\Omega Q_0 : \nabla u_0\ dx\Big) \\ &\geq \lambda_k\big(\widetilde{J}[w] - \varepsilon\big) + (1-\lambda_k)\ \Big(- |\Omega| f^*(Q_0) + \int_\Omega Q_0 : \nabla u_0\ dx\Big),\end{aligned}$$

therefore

$$\widetilde{J}\,[w] \leq 2\varepsilon + \widetilde{l}(w, \tau_k) \tag{3.4}$$

for all $k \gg 1$. Let us fix such an integer k. In order to verify our claim (3.1), we apply a modification of the approximation Lemma A.1.1 of [FS] to the tensor $\sigma := \tau_k$. This is necessary since it is not clear that the construction provided in [FS] preserves condition (3.3).

LEMMA 3.2 *Suppose that $\sigma \in \mathcal{U}$ satisfies $\sigma(x) \in L(c)$ for some $c \in \mathbb{R}$. Then a sequence $\sigma_m \in C^\infty(\overline{\Omega}; \mathbb{R}^N)$ exists such that*

$$
\begin{array}{llcll}
i) & \sigma_m & \to & \sigma & \textit{a.e. and in } L^t(\Omega; \mathbb{R}^{nN}) \textit{ for all } t < \infty; \\[2ex]
ii) & \operatorname{div} \sigma_m & \to & \operatorname{div} \sigma & \textit{in } L^n(\Omega; \mathbb{R}^N); \\[2ex]
iii) & \sigma_m & \overset{*}{\rightharpoonup} & \sigma & \textit{in } L^\infty(\Omega; \mathbb{R}^{nN}); \\[2ex]
iv) & \sigma_m(x) & \in & L(c) & \textit{for all } x \in \overline{\Omega},\ m \in \mathbb{N}.
\end{array}
$$

Proof of Lemma 3.2. We use a construction due to [A], p.170. Since $\partial\Omega$ is Lipschitz, we can cover $\partial\Omega$ by open sets $V_1, \ldots, V_r$ such that after rotation V_j takes the form

$$V_j = \{x \in \mathbb{R}^n : |(x_1, \ldots, x_{n-1})| < r_j, |x_n - g_j(x_1, \ldots, x_{n-1})| < h_j\}$$

where g_j is a Lipschitz function. Moreover, we have

$$
\begin{array}{ccccccc}
x_n & = & g_j(x_1, \ldots, x_{n-1}) & & & \Longrightarrow & x \in \partial\Omega, \\[2ex]
0 & < & x_n - g_j(x_1, \ldots, x_{n-1}) & < & h_j & \Longrightarrow & x \in \Omega, \\[2ex]
0 & > & x_n - g_j(x_1, \ldots, x_{n-1}) & > & -h_j & \Longrightarrow & x \notin \Omega.
\end{array}
$$

Let V_0 denote an open set such that $\overline{V_0} \subset \Omega$ and

$$\overline{\Omega} \subset \bigcup_{j=0}^{r} V_j.$$

Finally, we consider a partition of the unity $\{\varphi_j\}$, i.e. $\varphi_j \in C_0^\infty(V_j)$, $0 \le \varphi_j \le 1$, $\sum_{j=0}^r \varphi_j \equiv 1$ on $\overline{\Omega}$. Consider a fixed index $j \ge 1$. Let for $\delta \ll 1$

$$\sigma_j^\delta(x) := \begin{cases} \sigma(x + \delta e_n)\ \varphi^j\ (x), & x \in \Omega \cap V_j, \\ \quad 0, & x \in \Omega - V_j. \end{cases}$$

Note that $\sigma_j^\delta \equiv 0$ near the upper boundary part of V_j, the same is true near the "vertical boundary parts" which follows from the support properties of φ_j and appropriate choice of δ. If ω_ρ denotes a smoothing kernel, we let

$$\sigma^{\delta,\rho}(x) \;:=\; \omega_\rho * \Big[\varphi_0\sigma + \sum_{j=1}^{r} \;\sigma_j^\delta\Big](x),\;\; x \in \overline{\Omega}.$$

Assuming again the standard representation of the neighborhood V_j we get

$$\omega_\rho * \sigma_j^\delta(x) \;=\; \int_{\mathbb{R}^n} \omega_\rho(y-x)\;\sigma(y+\delta e_n)\;\varphi_j(y)\;dy,$$

and for ρ small enough depending on δ we see that for $y \in B_\rho(x), x \in \overline{\Omega}$, the point $y+\delta e_n$ belongs to Ω, and $\omega_\rho * \sigma_j^\delta(x)$ is well defined. Clearly $\omega_\rho * \sigma_j^\delta \in C^\infty(\overline{\Omega};\mathbb{R}^{nN})$ and

$$\omega_\rho * \sigma_j^\delta \underset{\rho\downarrow 0}{\longrightarrow} \sigma_j^\delta \;\text{ in } L^p(\Omega;\mathbb{R}^{nN}) \;\text{ for all } p<\infty,$$

moreover (see [A], 1.16 Lemma, p.18)

$$\sigma_j^\delta \underset{\delta\downarrow 0}{\longrightarrow} \sigma\varphi^j \;\text{ in } L^p(\Omega;\mathbb{R}^{nN}) \;\text{ for all } p<\infty.$$

We further have for $x \in \Omega$

$$\begin{aligned}
\operatorname{div}(\omega_\rho * \sigma_j^\delta)(x) &= \partial_{x_\alpha} \int \omega_\rho(y-x)\sigma_\alpha(y+\delta e_n)\varphi_j(y)\;dy \\
&= -\int (\partial_\alpha\omega_\rho)\;(y-x)\;\sigma_\alpha(y+\delta e_n)\;\varphi_j(y)\;dy \\
&= -\int (\partial_\alpha\omega_\rho)\;(y-x-\delta e_n)\;\sigma_\alpha(y)\;\varphi_j(y-\delta e_n)\;dy \\
&= -\int_{B_\rho(x+\delta e_n)} (\partial_\alpha\omega_\rho)\,(y-x-\delta e_n)\,\sigma_\alpha(y)\,\varphi_j(y-\delta e_n)\;dy,
\end{aligned}$$

and since it is sufficient to consider $x \in \Omega \cap V_j$, we see that the integration is performed over the ball $B_\rho(x+\delta e_n) \Subset \Omega$. Moreover, $y \longmapsto \omega_\rho(y-x-\delta e_n)$ has compact support in this ball, thus

$$\begin{aligned}
&\operatorname{div}(\omega_\rho * \sigma_j^\delta)(x) \\
&= \int_{B_\rho(x+\delta e_n)} \omega_\rho(y-x-\delta e_n)\;\Big[\operatorname{div}\sigma(y)\varphi_j\,(y-\delta e_n) + \sigma(y)\nabla\varphi_j(y-\delta e_n)\Big]\;dy \\
&= \int_{\mathbb{R}^n} \omega_\rho(y-x)\;\Big[\operatorname{div}\sigma(y+\delta e_n)\,\varphi_j(y) + \sigma(y+\delta e_n)\nabla\varphi_j(y)\Big]\;dy
\end{aligned}$$

and as above

$$\operatorname{div}(\omega_\rho * \sigma_j^\delta) \underset{\rho\downarrow 0}{\longrightarrow} \operatorname{div}\sigma(\cdot+\delta e_n)\varphi_j + \sigma(\cdot+\delta e_n)\nabla\varphi_j$$

in $L^n(\Omega;\mathbb{R}^N)$. The right-hand side converges to

$$\operatorname{div}\sigma\;\varphi_j + \sigma\;\nabla\varphi_j$$

in $L^n(\Omega;\mathbb{R}^N)$ as $\delta \downarrow 0$. So, if we first fix a sequence $\delta_m \downarrow 0$, we find a sequence ρ_m depending on δ_m such that the convergence properties $i)$ and $ii)$ hold for $\sigma^m := \sigma^{\delta_m,\rho_m}$. The boundedness of $\|\sigma^m\|_{L^\infty(\Omega;\mathbb{R}^{nN})}$ implies $\sigma^m \overset{*}{\rightharpoondown} \widetilde{\sigma}$ in $L^\infty(\Omega;\mathbb{R}^{nN})$ for a subsequence and some tensor $\widetilde{\sigma} \in L^\infty(\Omega;\mathbb{R}^{nN})$ but $i)$ shows $\widetilde{\sigma} = \sigma$. It remains to prove $iv)$. Jensen's inequality applied to the measure $\omega_\rho(x-\cdot)\mathcal{L}^n$ gives

$$f^*\big(\sigma^{\delta,\rho}(x)\big) \;\le\; \int \omega_\rho(x-y) f^*\left(\varphi_0\sigma + \sum_{j=1}^r \sigma_j^\delta\right)(y)\;dy$$

and if we recall the definition of σ_j^δ we see that f^* is evaluated on the convex combination

$$\varphi_0(y)\sigma(y) + \sum_{j=1}^r \varphi_j(y)\;\sigma(\ldots)$$

where $\sigma(\ldots)$ has an obvious meaning for $j = 1,\ldots,r$. Our assumption $\sigma \in L(c)$ a.e. then implies

$$f^*\Big(\varphi_0\sigma + \sum_{j=1}^r \sigma_j^\delta\Big)(y) \le c,$$

i.e. $\sigma^{\delta,\rho} \in L(c)$. □

Let us now return to inequality (3.4). Lemma 3.2 gives the existence of a sequence $\{\tau_{m,k}\}_{m\in\mathbb{N}}$ in $C^\infty(\overline{\Omega};\mathbb{R}^{nN})$ with values in $L(c_k)$ and such that

$$\widetilde{l}(w,\tau_{m,k}) \underset{m\to\infty}{\longrightarrow} \widetilde{l}(w,\tau_k),$$

where

$$\int_\Omega f^*(\tau_{m,k})\;dx \underset{m\to\infty}{\longrightarrow} \int_\Omega f^*(\tau_k)\;dx$$

follows from $\tau_{m,k} \to \tau_k$ a.e. together with the level-set property, hence we deduce from (3.4) the inequality

$$\widetilde{J}[w] \;\le\; 3\varepsilon + \widetilde{l}(w,\tau_{m,k})$$

at least for $m \gg 1$, and (3.1) is established.

Extend next the function $w \in BV(\Omega;\mathbb{R}^N)$ via u_0 to a function $\widehat{w}$ defined on $\widehat{\Omega}$ and consider a tensor $\sigma \in C_0^\infty(\widehat{\Omega};\mathbb{R}^{nN})$, $\sigma(x) \in L(c)$ for some $c \in \mathbb{R}$. Then

$$\int_{\widehat{\Omega}} \operatorname{div}\sigma \cdot (u_0 - \widehat{w})\;dx - \int_\Omega f^*(\sigma)\;dx + \int_\Omega \sigma : \nabla u_0\;dx \;=\; \widetilde{l}(w,\sigma_{|\Omega}),$$

hence

$$\widetilde{J}[w] \;\geq\; \sup\Bigg\{\int_{\widehat{\Omega}} \operatorname{div}\sigma\cdot(u_0-\widehat{w})\,dx - \int_\Omega f^*(\sigma)\,dx + \int_\Omega \sigma : \nabla u_0\,dx :$$
$$\sigma\in C_0^\infty(\widehat{\Omega};\mathbb{R}^{nN}),\sigma\in L(c) \text{ for some } c\in\mathbb{R}\Bigg\}.$$

Conversely, fix $\tau\in C^\infty(\overline{\Omega};\mathbb{R}^{nN})$, $\tau\in L(c)$ for some c, such that

$$\widetilde{J}[w] \;\leq\; \widetilde{l}(w,\tau)+\varepsilon.$$

A modification of Lemma 3.2 yields a sequence $\{\tau_m\}\in C_0^\infty(\widehat{\Omega};\mathbb{R}^{nN})$ such that $\tau_m\in L(c)$ and

$$\left\{\begin{array}{rcll} \tau_m & \to & \tau & \text{in } L^t(\Omega;\mathbb{R}^{nN}) \text{ and a.e. for all } t<\infty,\\ \operatorname{div}\tau_m & \to & \operatorname{div}\tau & \text{in any space } L^s(\Omega;\mathbb{R}^N),\ s<\infty,\\ \tau & \overset{*}{\rightharpoondown} & \tau & \text{in } L^\infty(\Omega;\mathbb{R}^{nN}).\end{array}\right.$$

To be precise, we use the notation from the proof of Lemma 3.2, in particular, we recall the definition of τ_j^δ, $j=1,\dots,r$, which are now tensors of class C^∞. Clearly the definition of $\tau_j^\delta(x)$ makes sense for points x such that $x+\delta e_n\in\overline{\Omega}$, i.e.

$$-\delta \;\leq\; x_n - g_j(x_1,\dots,x_{n-1}),$$

so that

$$\tau_j^\delta\in C^\infty\Big(V_j\cap\big[-\delta\leq x_n-g_j(x_1,\dots,x_{n-1})\big]\Big).$$

Let

$$\Psi_\delta^j(x) \;:=\; \left\{\begin{array}{ll} 1, & x\in V_j,\ x_n-g_j(x_1,\dots,x_{n-1}) \;\geq\; 0,\\ 0, & x\in V_j,\ x_n-g_j(x_1,\dots,x_{n-1}) \;\leq\; -\frac{\delta}{4},\end{array}\right.$$

with suitable functions $\Psi_\delta^j\in C_0^\infty(\mathbb{R}^n)$, $0\leq\Psi_\delta^j\leq 1$. The function $\Psi_\delta^j\,\tau_j^\delta$ is of class $C_0^\infty(\widehat{\Omega};\mathbb{R}^{nN})$ and $\Psi_\delta^j\,\tau_j^\delta \;=\; \tau_j^\delta$ on Ω. Finally we let

$$\tau^\delta \;:=\; \varphi_0\tau+\sum_{j=1}^r\ \Psi_\delta^j\,\tau_j^\delta\in C_0^\infty(\widehat{\Omega};\mathbb{R}^{nN}).$$

Then, as $\delta\downarrow 0$, the desired convergence properties of τ^δ on Ω are immediate, moreover (observe $\varphi_0+\sum_{j=1}^r\Psi_\delta^j\,\varphi_j\leq 1$ on $\widehat{\Omega}$, $f^*(0)=0$) $f^*(\tau^\delta)\leq\varphi_0 f^*(\tau)+\sum_{j=1}^r\Psi_\delta^j\varphi_j f^*(\tau(\dots))\leq c$ on $\widehat{\Omega}$. As a consequence we end up with the formula

$$\widetilde{J}[w] \;=\; \sup\Bigg\{\int_{\widehat{\Omega}} \operatorname{div}\sigma\cdot(u_0-\widehat{w})\,dx - \int_\Omega f^*(\sigma)\,dx + \int_\Omega \sigma : \nabla u_0\,dx :$$
$$\sigma\in C_0^\infty(\widehat{\Omega};\mathbb{R}^{nN}),\ \sigma\in L(c) \text{ for some } c\in\mathbb{R}\Bigg\},$$

and we may use integration by parts to get for any $w \in BV(\Omega;\mathbb{R}^N)$

$$
\begin{aligned}
\widetilde{J}[w] &= \sup\Bigg\{\int_{\overline{\Omega}} \sigma : \nabla\widehat{w} - \int_\Omega f^*(\sigma) : \\
&\qquad \sigma \in C_0^\infty(\widehat{\Omega};\mathbb{R}^{nN}),\ \sigma \in L(c) \text{ for some } c \in \mathbb{R}\Bigg\}.
\end{aligned}
\tag{3.5}
$$

Clearly, a smoothing argument shows, that in (3.5) the space $C_0^\infty(\widehat{\Omega};\mathbb{R}^{nN})$ can be replaced by $C_0^0(\widehat{\Omega};\mathbb{R}^{nN})$. Let us fix $w \in BV(\Omega;\mathbb{R}^N)$ and a tensor $\sigma \in C_0^0(\widehat{\Omega};\mathbb{R}^{nN})$ with the property $\sigma \in L(c)$. Then $f^* \circ \sigma$ is of class $L^1(\widehat{\Omega})$. Conversely, let us assume that $f^* \circ \sigma$ is integrable on $\widehat{\Omega}$ implying $\sigma(x) \in \mathrm{dom}\ f^*$. Then, using the arguments presented after the proof of Lemma 3.1, we can construct tensors $\sigma_k := (1-\lambda_k)Q_0 + \lambda_k\sigma$ as before which we multiply by some function $\eta \in C_0^0(\widehat{\Omega})$, $0 \le \eta \le 1$, $\eta \equiv 1$ on a neighborhood of $\overline{\Omega}$. Then the tensors $\tau_k := \eta\sigma_k$ are of class $C_0^0(\widehat{\Omega};\mathbb{R}^N)$ and

$$
f^*(\tau_k) \;=\; f^*\big((1-\eta)\,0 + \eta\,\sigma_k\big) \;\le\; \eta\, f^*(\sigma_k) \;\le\; c_k
$$

provided $\sigma_k \in L(c_k)$. We have

$$
\int_{\overline{\Omega}} \tau_k : \nabla\widehat{w} \;=\; \int_{\overline{\Omega}} \sigma_k : \nabla\widehat{w} \underset{k\to\infty}{\longrightarrow} \int_{\overline{\Omega}} \sigma : \nabla\widehat{w},
$$

on Ω we estimate

$$
\begin{aligned}
0 \;\le\; f^*(\tau_k) \;&=\; f^*(\sigma_k) \;\le\; (1-\lambda_k)f^*(Q_0) + \lambda_k f^*(\sigma) \\
&\le\; \max\,\{f^*(Q_0), f^*(\sigma)\} \in L^1(\Omega),
\end{aligned}
$$

hence

$$
\int_\Omega f^*(\tau_k)\ dx \;\longrightarrow\; \int_\Omega f^*(\sigma)\ dx
$$

follows from dominated convergence and $f^*(\tau_k) \to f^*(\sigma)$ a.e. on Ω. Altogether we find

$$
\int_{\overline{\Omega}} \sigma : \nabla\widehat{w} - \int_\Omega f^*(\sigma)\ dx = \lim_{k\to\infty}\left\{\int_{\overline{\Omega}} \tau_k : \nabla\widehat{w} - \int_\Omega f^*(\tau_k)\ dx\right\},
$$

and (3.5) implies

$$
\widetilde{J}[w] = \sup\Big\{\int_{\widehat{\Omega}} \mathbf{1}_{\overline{\Omega}}\, \sigma : \nabla\widehat{w} - \int_{\Omega} f^*(\sigma)\ dx :\ \sigma \in C_0^0(\widehat{\Omega};\mathbb{R}^{nN}),\ f^*\circ\sigma \in L^1(\widehat{\Omega})\Big\}
$$

$$
\tag{3.6} \geq \sup\Big\{\int_{\widehat{\Omega}} \mathbf{1}_{\overline{\Omega}}\, \sigma : \nabla\widehat{w} - \int_{\widehat{\Omega}} f^*(\sigma)\ dx :\ \sigma \in C_0^0(\widehat{\Omega};\mathbb{R}^{nN}),\ f^*\circ\sigma \in L^1(\widehat{\Omega})\Big\},
$$

the inequality being a consequence of $f^* \geq 0$. Consider a tensor σ which realizes the first supremum up to given $\varepsilon > 0$. Then we let $\eta_k \in C_0^0(\widehat{\Omega})$, $0 \leq \eta_k \leq 1$, such that $\eta_k \equiv 1$ on $\overline{\Omega}$ and $\eta_k \longrightarrow \mathbf{1}_{\overline{\Omega}}$. Let $\sigma_k := \eta_k\sigma$. Observing

$$
\int_{\widehat{\Omega}} \mathbf{1}_{\overline{\Omega}}\, \sigma_k : \nabla\widehat{w} = \int_{\widehat{\Omega}} \mathbf{1}_{\overline{\Omega}}\, \sigma : \nabla\widehat{w}, \quad \int_{\widehat{\Omega}} f^*(\sigma_k)\ dx \longrightarrow \int_{\Omega} f^*(\sigma)\ dx
$$

(note: $0 \leq f^*(\sigma_k) \leq \eta_k f^*(\sigma) \leq f^*(\sigma)$, $\sigma_k \to \mathbf{1}_{\overline{\Omega}}\sigma$) we get

$$
\int_{\widehat{\Omega}} \mathbf{1}_{\overline{\Omega}}\, \sigma : \nabla\widehat{w} - \int_{\Omega} f^*(\sigma)\ dx \ \leq\ \varepsilon + \int_{\widehat{\Omega}} \mathbf{1}_{\overline{\Omega}}\, \sigma_k : \nabla\widehat{w} - \int_{\widehat{\Omega}} f^*(\sigma_k)\ dx
$$

for $k \gg 1$, and the suprema in (3.6) coincide. This leads to the final representation formula

$$
\tag{3.7} \widetilde{J}[w] = \sup\Big\{\int_{\widehat{\Omega}} \mathbf{1}_{\overline{\Omega}}\, \sigma : \nabla\widehat{w} - \int_{\widehat{\Omega}} f^*(\sigma)\ dx :\ \sigma \in C_0^0(\widehat{\Omega};\mathbb{R}^{N}),\ f^*\circ\sigma \in L^1(\widehat{\Omega})\Big\}
$$

being valid for any $w \in BV(\Omega;\mathbb{R}^N)$. Now the right-hand side of (3.7) can be identified with [DT], Proposition 1.2: the right-hand side equals

$$
\int_{\overline{\Omega}} f(\nabla^a\widehat{w})\ dx + \int_{\overline{\Omega}} f_\infty\left(\frac{\nabla^s\widehat{w}}{|\nabla^s\widehat{w}|}\right)\ d|\nabla^s\widehat{w}|,
$$

the first integral being equal to $\int_\Omega f(\nabla^a w)\ dx$ and the second integral may be decomposed as $\int_\Omega \ldots + \int_{\partial\Omega} \ldots$, the boundary integral being given by (see [AFP], Theorem 3.77, p.171)

$$
\int_{\partial\Omega} f_\infty\Big(-(w-u_0)\otimes\nu\Big)\ d\,\mathcal{H}^{n-1},
$$

hence $\widetilde{J}[w] = K[w]$ for all $w \in BV(\Omega;\mathbb{R}^N)$. $\square$

4 A Uniqueness Theorem

In [B1] (compare also [B2]) the following class of variational integrands with linear growth is discussed: there exist positive constants ν_1, ν_2, ν_3 such that for any $Z \in \mathbb{R}^{nN}$

i) $f \in C^2(\mathbb{R}^{nN})$;

ii) $|\nabla f(Z)| \leq \nu_1$;

iii) for any $Y \in \mathbb{R}^{nN}$ we have

$$\nu_2 \left(1+|Z|^2\right)^{-\frac{3}{2}} |Y|^2 \;\leq\; D^2 f(Z)(Y,Y) \;\leq\; \nu_3 \left(1+|Z|^2\right)^{-\frac{1}{2}} |Y|^2 .$$

Note that this provides a generalization of the minimal surface case where one benefits from the additional geometric structure. It turns out that just conditions *i)*—*iii)* (plus some natural boundedness assumption) lead to a solution u^* such that

$$u^* \in \mathcal{M}' \;:=\; \Big\{ u \in \mathcal{M} : \, u \in W^1_{1,loc}\big(\Omega; \mathbb{R}^N\big)\Big\} \;=\; \mathcal{M} \cap W^1_1\big(\Omega; \mathbb{R}^N\big).$$

Here we give a clear interpretation of the set $\mathcal{M}'$. To this purpose consider the variational problem

$$(\mathcal{P}') \;\; \int_\Omega f(\nabla w)\, dx + \int_{\partial\Omega} f_\infty\big((u_0 - u)\otimes \nu\big)\, d\mathcal{H}^{n-1} \;\to\; \min \text{ in } W^1_1\big(\Omega; \mathbb{R}^N\big).$$

Then the set $\mathcal{M}'$ precisely gives the solutions of problem $(\mathcal{P}')$ and these solutions are unique up to a constant.

THEOREM 4.1 *Suppose that the variational integrand f satisfies our general assumption and assume there exists $u^* \in \mathcal{M}'$. Then we have*

i) The elements of $\mathcal{M}'$ are solutions of problem $(\mathcal{P}')$ and vice versa.

ii) The set $\mathcal{M}'$ is uniquely determined up to constants.

Proof. ad *i)*. On account of the K-minimizing property of $u^* \in \mathcal{M}'$ and since $\nabla^s u^* \equiv 0$, the representation of K clearly implies that $u^* \in \mathcal{M}'$ is a solution of $(\mathcal{P}')$. Conversely, consider a solution v^* of problem $(\mathcal{P}')$ and a J-minimizing sequence $\{u_m\}$ from $u_0 + \overset{\circ}{W}{}^1_1(\Omega; \mathbb{R}^N)$. The minimality of v^* gives

$$K[v^*] \;=\; \int_\Omega f(\nabla v^*)\, dx + \int_{\partial\Omega} f_\infty\big((u_0 - v^*)\otimes\nu\big)\, d\mathcal{H}^{n-1} \;\leq\; \int_\Omega f(\nabla u_m)\, dx,$$

and i) follows from Theorem 1.2, ii) iii).

ad ii): to prove uniqueness up to a constant, we just observe that f_∞ is convex, whereas f is strictly convex. This immediately gives $\nabla u^* = \nabla u^{**}$ almost everywhere and for any two generalized minimizers u^*, $u^{**} \in \mathcal{M}'$, hence Theorem 4.1. $\square$

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