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Yana A. Butko (Kinderknecht), Martin Grothaus and Oleg G. Smolyanov

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Yana A. Butko (Kinderknecht)
Saarland University
Department of Mathematics
P.O. Box 15 11 50
66041 Saarbrücken
Germany
kinderknecht@math.uni-sb.de, yanabutko@yandex.ru

Martin Grothaus
Technical University of Kaiserslautern
Department of Mathematics
Postfach 3049
67653 Kaiserslautern
Germany
grothaus@mathematik.uni-kl.de

Oleg G. Smolyanov
Lomonosov Moscow State University
Department of Mechanics and Mathematics
119992, Vorob'evy gory, 1, Moscow
Russia
Smolyanov@yandex.ru

Edited by
FR 6.1 – Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-Mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

Abstract

This note is devoted to representation of some evolution semigroups. The semigroups are generated by pseudo-differential operators, which are obtained by different (parame-trized by a number τ) procedures of quantization from a certain class of functions (or symbols) defined on the phase space. This class contains functions which are second order polynomials with respect to the momentum variable and also some other functions. The considered semigroups are represented as limits of n-fold iterated integrals when n tends to infinity (such representations are called Feynman formulae). Some of these representations are constructed with the help of another pseudo-differential operators, obtained by the same procedure of quantization (such representations are called Hamiltonian Feynman formulae). Some representations are based on integral operators with elementary kernels (these ones are called Lagrangian Feynman formulae and are suitable for computations). A family of phase space Feynman pseudomeasures corresponding to different procedures of quantization is introduced. The considered evolution semigroups are represented also as phase space Feynman path integrals with respect to these Feynman pseudomeasures. The obtained Lagrangian Feynman formulae allow to calculate these phase space Feynman path integrals and to connect them with some functional integrals with respect to probability measures.

KEYWORDS Feynman formulae; Phase space Feynman path integrals, Hamiltonian Feynman path integrals, symplectic Feynman path integrals, Feynman–Kac formulae, functional integrals; Hamiltonian (symplectic) Feynman pseudomeasure, Chernoff theorem, pseudo-differential operators, approximations of semigroups, approximations of transition densities.

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Contents

1 Introduction

This paper is devoted to approximations of evolution semigroups $e^{-t\widehat{H}}$ generated by pseudo-differential operators $\widehat{H}$. The operators $\widehat{H}$ are obtained from a given function $(q,p) \mapsto H(q,p)$ (which is called a symbol of $\widehat{H}$) by some linear procedure (which is called a quantization). We consider a class of such procedures, parameterized by a number $\tau \in [0,1]$. This class includes *qp*-, *pq*- and Weyl quantizations. We obtain representations of the considered evolution semigroups by phase space Feynman path integrals which we define as limits of some usual integrals over finite dimensional spaces when the dimension of these spaces tends to infinity. Our approach is to approximate the semigroup $e^{-t\widehat{H}}$ (for a given procedure of quantization) by a family of pseudo-differential operators $\widehat{e^{-tH}}$ obtained by the same procedure of quantization from the function e^{-tH}. Note, that if the function H depends on both variables q and p, then $e^{-t\widehat{H}} \neq \widehat{e^{-tH}}$. Nevertheless, under certain conditions one succeeds to prove that

$$e^{-t\widehat{H}} = \lim_{n\to\infty} \left[\widehat{e^{-\frac{t}{n}H}}\right]^n. \tag{1.1}$$

The limit in the right hand side is the limit of n-fold iterated integrals over the phase space when n tends to infinity (such expressions are called Hamiltonian Feynman formulae). This limit can be interpreted as a phase space Feynman path integral with $\exp\left(-\int\limits_0^t H(q(s),p(s))ds\right)$ in the integrand.

On a heuristic level the same approach was used already in Berezin's papers [3], [4] for investigation of Schrödinger groups $e^{-it\widehat{H}}$. Berezin has assumed the identity

$$e^{-it\widehat{H}} = \lim_{n\to\infty} \left[\widehat{e^{-i\frac{t}{n}H}}\right]^n \tag{1.2}$$

and has interpreted the pre-limit expressions in the right hand side of the identity (1.2) as approximations to a phase space Feynman path integral. Moreover, Berezin has remarked that Feynman path integral is "very sensitive to the choice of approximations, and nonuniqueness appearing due to this dependence has the same character as nonuniqueness of quantization" (see [4]). In other words, Feynman path integral is different for different procedures of quantization. This difference may appear both in integrands and in the set of paths over which the integration takes place. Berezin has considered the case of Weyl quantization and his calculations have lead to a quit odd expression in the integrand of his Feynman path integral. The question, how to distinguish the procedure of quantization on the language of Feynman path integrals, remained open.

The rigorous justification of the above mentioned approach for approximation of evolution (semi)groups was first obtained only in 2002 in the paper [41]. The main technical tool suggested in [41] was the Chernoff Theorem (see Theorem 2.1 below, cf. [16]). It is a wide generalization of the classical Trotter's result used for rigorous handling of Feynman path integrals over paths in configuration space of a system (see, e.g., [31]). In the paper [41] the identity (1.2) has been established for τ-quantization of a class of functions $(q,p) \mapsto H(q,p)$ whose main ingredient is a function $(q,p) \mapsto h(q,p)$, which is Fourier transform of a finite σ-additive measure. This ingredient allows to use Parseval equality to succeed the proof. A scheme to construct a phase space Feynman path integral is also presented in [41] (however, quit independently on the established Hamiltonian Feynman formulae (1.2)).

Later on, evolution semigroups $e^{-t\widehat{H}}$ have been treated by the same approach in papers [13], [14], [7]. In [13] the identity (1.1) has been established for the case of qp-quantization of a function $(q,p) \mapsto H(q,p)$, which corresponds to a particle with variable mass in a potential field. The semigroup $e^{-t\widehat{H}}$ has been considered on the Banach space $C_\infty(\mathbb{R}^d)$ of continuous, vanishing at infinity functions. The scheme of [41] was adopted (for the case of qp-quantization) to interpret the obtained Hamiltonian Feynman formula (1.1) as a phase space Feynman path integral with respect to a Feynman type pseudomeasure. In [14] the identity (1.1) has been established for the semigroup $e^{-t\widehat{H}}$ on $C_\infty(\mathbb{R}^d)$ in the case of qp-quantization of a function $(q,p) \mapsto H(q,p)$, which is continuous and negative definite with respect to p, continuous and bounded with respect to q. This class of functions H contains, in particular, Hamilton functions of particles with variable mass in potential and magnetic fields and relativistic particles with variable mass. The semigroup $e^{-t\widehat{H}}$ (again on $C_\infty(\mathbb{R}^d)$) generated by τ-quantization of a function H, which is polynomial with respect to p with variable, depending on q coefficients, has been approximated in [7] by a family of pseudo-differential operators with some qp-symbols. The obtained Hamiltonian Feynman formula has been interpreted as a phase space Feynman path integral with respect to the Feynman pseudomeasure defined in [13].

This article continues the researches of [41], [13], [14], [7]. We consider Banach space $L^1(\mathbb{R}^d)$ and evolution semigroups $e^{-t\widehat{H}}$ generated by τ-quantization of a function $(q,p) \mapsto H(q,p)$, which is polynomial with respect to p with variable, depending on q coefficients. For all $\tau \in [0,1]$ we prove that the considered semigroups are being approximated as in (1.1) by families of pseudo-differential operators with τ-symbols e^{-tH}. We develop the scheme of [41] to construct a family of Feynman pseudomeasures Φ^τ, $\tau \in [0,1]$, and show that the limit in the right hand side of (1.1) for each $\tau \in [0,1]$ does coincide with a phase space Feynman path integral with respect to the corresponding pseudomeasure Φ^τ. For the case of *qp*-quantization we obtain the same result for a slightly more general class of functions H. We plan to obtain analogous formulae for Schrödinger groups e^{-itH} by the method of analytic continuation in our subsequent work. The considered semigroups $e^{-t\widehat{H}}$ are represented for all $\tau \in [0,1]$ also by some limits of integral operators with (more or less) elementary kernels (such representations are called Lagrangian Feynman formulae). These representations are suitable for direct calculations. Moreover, the pre-limit expressions in the obtained Lagrangian Feynman formulae coincide with some functional integrals with respect to probability measures corresponding to stochastic processes associated to the generators $\widehat{H}$. These different representations of the same semigroups allow to calculate some phase space Feynman path integrals and to connect them with stochastic analysis.

2 Notation and preliminaries

2.1 The Chernoff theorem and Feynman formulae

Let $(X, \|\cdot\|_X)$ be a Banach space, $\mathcal{L}(X)$ be the space of all continuous linear operators on X equipped with the strong operator topology, $\|\cdot\|$ denote the operator norm on $\mathcal{L}(X)$ and Id be the identity operator in X. If $\mathrm{Dom}(L) \subset X$ is a linear subspace and $L : \mathrm{Dom}(L) \to X$ is a linear operator, then $\mathrm{Dom}(L)$ denotes the domain of L. A one-parameter family $(T_t)_{t\geqslant 0}$ of bounded linear operators $T_t : X \to X$ is called a strongly continuous semigroup, if $T_0 = \mathrm{Id}$, $T_{s+t} = T_s \circ T_t$ for all $s, t \geqslant 0$ and $\lim_{t\to 0} \|T_t\varphi - \varphi\|_X = 0$ for all $\varphi \in X$. If $(T_t)_{t\geqslant 0}$ is a strongly continuous semigroup on a Banach space $(X, \|\cdot\|_X)$, then the generator L of $(T_t)_{t\geqslant 0}$ is defined by

$$L\varphi := \lim_{t\searrow 0} \frac{T_t\varphi - \varphi}{t}$$

with domain

$$\mathrm{Dom}(L) := \left\{ \varphi \in X \;\middle|\; \lim_{t\searrow 0} \frac{T_t\varphi - \varphi}{t} \quad \text{exists in } X \right\}.$$

Consider an evolution equation $\frac{\partial f}{\partial t} = Lf$. If L is the generator of a strongly continuous semigroup $(T_t)_{t\geqslant 0}$ on a Banach space $(X, \|\cdot\|_X)$, then the (mild) solution of the Cauchy problem for this equation with the initial value $f(0) = f_0 \in X$ is given by $f(t) = T_t f_0$ for all $f_0 \in X$. Therefore, solving the evolution equation $\frac{\partial f}{\partial t} = Lf$ means to construct a semigroup $(T_t)_{t\geqslant 0}$ with the given generator L. If the desired semigroup is not known explicitly it can be approximated. One of the tools to approximate semigroups is based on the Chernoff theorem [16] (here we present the version of Chernoff's theorem given in [41]).

Theorem 2.1 (Chernoff). *Let X be a Banach space, $F : [0,\infty) \to \mathcal{L}(X)$ be a (strongly) continuous mapping such that $F(0) = \mathrm{Id}$ and $\|F(t)\| \leqslant e^{at}$ for some $a \in [0,\infty)$ and all $t \geqslant 0$. Let D be a linear subspace of $\mathrm{Dom}(F'(0))$ such that the restriction of the operator $F'(0)$ to this subspace is closable. Let $(L, \mathrm{Dom}(L))$ be this closure. If $(L, \mathrm{Dom}(L))$ is the generator of a strongly continuous semigroup $(T_t)_{t\geqslant 0}$, then for any $t_0 > 0$ the sequence $(F(t/n))^n)_{n\in\mathbb{N}}$ converges to $(T_t)_{t\geqslant 0}$ as $n \to \infty$ in the strong operator topology, uniformly with respect to $t \in [0, t_0]$, i.e.*

$$T_t = \lim_{n\to\infty} \left[F(t/n)\right]^n . \tag{2.1}$$

Here the derivative at the origin of a function $F : [0,\varepsilon) \to L(X)$, $\varepsilon > 0$, is a linear mapping $F'(0) : \mathrm{Dom}(F'(0)) \to X$ such that

$$F'(0)g := \lim_{t\searrow 0} \frac{F(t)g - F(0)g}{t},$$

where $\mathrm{Dom}(F'(0))$ is the vector space of all elements $g \in X$ for which the above limit exists.
A family of operators $(F(t))_{t\geqslant 0}$ suitable for the formula (2.1) is called *Chernoff equivalent* to the semigroup $(T_t)_{t\geqslant 0}$, i.e. this family satisfies all the assertions of the Chernoff theorem with respect to this semigroup. In many cases the operators $F(t)$ are integral operators and, hence, we have a limit of iterated integrals on the right hand side of the equality (2.1). In this setting it is called *Feynman formula.*

Definition 2.2. *A Feynman formula is a representation of a solution of an initial (or initial-boundary) value problem for an evolution equation (or, equivalently, a representation of the semigroup solving the problem) by a limit of n-fold iterated integrals as $n \to \infty$.*

We use this notation since it was Richard Feynman ([19], [20]) who introduced a functional (path) integral as a limit of iterated finite dimensional integrals. The limits in Feynman formulae coincide with (or in some cases define) certain functional integrals with respect to probability measures or Feynman type pseudomeasures on a set of paths of a physical system. A representation of a solution of an initial (or initial-boundary) value problem for an evolution equation (or, equivalently, a representation of the semigroup resolving the problem) by a functional integral is usually called *Feynman–Kac formula.* Hence, the iterated integrals in a Feynman formula for some problem give approximations to a functional integral in the Feynman-Kac formula representing the solution of the same problem. These

approximations in many cases contain only elementary functions as integrands and, therefore, can be used for direct calculations and simulations.
The notion of a Feynman formula has been introduced in [41]. The method to obtain Feynman formulae has been developed in a series of papers [41]–[46]. This method is based on the Chernoff theorem and has been successfully applied recently to obtain Feynman formulae for different classes of problems for evolution equations on different geometric structures, see, e.g. [12], [8]–[14], [22], [32], [33], [35], [36], [37], [38], [39].
We call the identity (2.1) a *Lagrangian Feynman formula*, if the $F(t)$, $t > 0$, are integral operators with elementary kernels; if the $F(t)$ are pseudo-differential operators (the definition is given in Section 2.2), we speak of *Hamiltonian Feynman formulae*. This terminology is inspired by the fact that a Lagrangian Feynman formula gives approximations to a functional integral over a set of paths in the configuration space of a system (whose evolution is described by the semigroup $(T_t)_{t\geqslant 0}$), while a Hamiltonian Feynman formula corresponds to a functional integral over a set of paths in the phase space of some system.

2.2 Pseudo-differential operators, their symbols and tau-quantization

Let us consider a measurable function $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ and $\tau \in [0,1]$. We define a pseudo-differential operator (ΨDO) $\widehat{H}_\tau$ with τ-symbol H on a Banach space $(X, \|\cdot\|_X)$ of some functions on $\mathbb{R}^d$ by

$$\widehat{H}_\tau\varphi(q) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} e^{ip\cdot(q-q_1)} H(\tau q + (1-\tau)q_1, p)\varphi(q_1)\, dq_1\, dp, \quad q \in \mathbb{R}^d \tag{2.2}$$

where the domain $\mathrm{Dom}(\widehat{H}_\tau)$ is the set of all $\varphi \in X$ such that the right hand side of the formula (2.2) is well defined as an element of $(X, \|\cdot\|_X)$. We always assume that the set of smooth compactly supported functions $C_c^\infty(\mathbb{R}^d)$ belongs to the domain of the operator $\widehat{H}_\tau$.
The mapping $H \mapsto \widehat{H}_\tau$ from a space of functions on $\mathbb{R}^d \times \mathbb{R}^d$ into the space of linear operators on $(X, \|\cdot\|_X)$ is called the τ-quantization, the operator $\widehat{H}_\tau$ is called the τ-quantization of the function H. Note that if the symbol H is a sum of functions depending only on one of the variables q or p then the ΨDOs $\widehat{H}_\tau$ coincide for all $\tau \in [0,1]$. If $H(q,p) = qp = pq$, $q, p \in \mathbb{R}^1$ then $\widehat{H}_\tau\varphi(q) = -i\tau q\frac{\partial}{\partial q}\varphi(q) - i(1-\tau)\frac{\partial}{\partial q}(q\varphi(q))$. Therefore, different τ correspond to different orderings of non-commuting operators such that we have the "qp"-quantization for $\tau = 1$, the "pq"-quantization for $\tau = 0$ and the Weyl quantization for $\tau = 1/2$. A function H is usually considered as a Hamilton function of a classical system.

Then the operator $\widehat{H}_\tau$ is called the Hamiltonian of a quantum system obtained by τ-quantization of the classical system with the Hamilton function H.
In the sequel we use the following result (cf. [41][Lemma4]). The Schwartz space of smooth rapidly decreasing functions on $\mathbb{R}^d$ we denote by $S(\mathbb{R}^d)$.

Lemma 2.3. *Let $\tau = 1$. Let $f, g : \mathbb{R}^d \to \mathbb{C}$ be bounded continuous functions and $\lambda : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be the 1-symbol of ΨDO $\widehat{\lambda}_1$. Let $H(q,p) = f(q)g(p)\lambda(q,p)$, $q, p \in \mathbb{R}^d$. Then*

$$\widehat{H}_1\varphi = \left(\widehat{f} \circ \widehat{\lambda}_1 \circ \widehat{g}\right)\varphi$$

for all $\varphi \in S(\mathbb{R}^d) \cap \mathrm{Dom}(\widehat{H}_1) \cap \mathrm{Dom}(\widehat{f} \circ \widehat{\lambda}_1 \circ \widehat{g})$.

Proof. Let $\varphi \in S(\mathbb{R}^d) \cap \mathrm{Dom}(\widehat{H}_1) \cap \mathrm{Dom}(\widehat{f} \circ \widehat{\lambda}_1 \circ \widehat{g})$. Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ stand for Fourier transform and its inverse respectively. Then

$$\begin{aligned}\widehat{H}_1\varphi(q) &= (2\pi)^{-d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} e^{ip\cdot(q-q_1)}H(q,p)\varphi(q_1)\,dq_1\,dp = \\ &= (2\pi)^{-d/2}\int\limits_{\mathbb{R}^d} e^{ip\cdot q}f(q)g(p)\lambda(q,p)\mathcal{F}[\varphi](p)dp,\end{aligned}$$

and

$$\begin{aligned}\left(\widehat{f} \circ \widehat{\lambda}_1 \circ \widehat{g}\right)\varphi(q) &= \left(\widehat{f} \circ \widehat{\lambda}_1\right)\mathcal{F}^{-1}[g\mathcal{F}[\varphi]](q) = \\ &= f(q)(2\pi)^{-d/2}\int\limits_{\mathbb{R}^d} e^{ip\cdot q}\lambda(q,p)\mathcal{F}[\mathcal{F}^{-1}[g\mathcal{F}[\varphi]]](p)dp = \\ &= (2\pi)^{-d/2}\int\limits_{\mathbb{R}^d} e^{ip\cdot q}f(q)g(p)\lambda(q,p)\mathcal{F}[\varphi](p)dp.\end{aligned}$$

□

3 Feynman formulae for tau-quantization of some Lévy-Khintchine type Hamilton functions.

In the sequel we consider $\tau \in [0,1]$ and $(X, \|\cdot\|_X) := (L^1(\mathbb{R}^d), \|\cdot\|_1)$ the space of functions on $\mathbb{R}^d$ absolutely integrable with respect to the Lebesgue measure. Let us introduce two Hamilton functions h and H. Let the function $h : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be given by the formula

$$h(q,p) = c(q) + ib(q)\cdot p + p\cdot A(q)p, \quad q, p \in \mathbb{R}^d, \tag{3.1}$$

where for each $q \in \mathbb{R}^d$ we have $b(q) \in \mathbb{R}^d$, $c(q) \in \mathbb{R}$, $A(q)$ is a symmetric $d \times d$-matrix. Let us consider also a function $r : \mathbb{R}^d \to \mathbb{C}$ given by the formula

$$r(p) = \int\limits_{\mathbb{R}^d \setminus \{0\}} \left(1 - e^{iy \cdot p} + \frac{iy \cdot p}{1 + |y|^2}\right) N(dy), \tag{3.2}$$

where N is a Radon measure on $\mathbb{R}^d \setminus \{0\}$ with $\int\limits_{\mathbb{R}^d \setminus \{0\}} \frac{|y|^2}{1+|y|^2} N(dy) < \infty$. Note that we consider the case when N does not depend on $q \in \mathbb{R}^d$. We assume that for $q, p \in \mathbb{R}^d$ we have

$$\begin{aligned} H(q,p) &= h(q,p) + r(p) \qquad\qquad (3.3) \\ &= c(q) + ib(q) \cdot p + p \cdot A(q)p + \int\limits_{\mathbb{R}^d \setminus \{0\}} \left(1 - e^{iy \cdot p} + \frac{iy \cdot p}{1 + |y|^2}\right) N(dy). \end{aligned}$$

If $c \geqslant 0$ then the Hamilton function H is continuous negative definite with respect to the variable $p \in \mathbb{R}^d$ and the formula (3.3) is just the Lévy-Khintchine formula. We don't assume in the sequel that $c \geqslant 0$, that's why we call our symbol H a *Lévy-Khintchine type function*.

To handle the proofs we need to assume (sometimes different) boundness and smoothness conditions on the symbol H. All the assumptions, we use in the sequel, are collected below.

Assumtion 3.1. (i) There exist constants $0 < a_0 \leqslant A_0 < +\infty$ such that for all $p \in \mathbb{R}^d$ and all $q \in \mathbb{R}^d$ the following inequalities hold

$$a_0 |p|^2 \leqslant p \cdot A(q)p \leqslant A_0 |p|^2.$$

(ii) The coefficients A, b, c with all their derivatives up to the 4th order are continuous and bounded.

(iii) The coefficients A, b, c are infinite differentiable and bounded with all their derivatives.

(iv) The function $H(q, \cdot)$ is of class $C^\infty(\mathbb{R}^d)$ for each $q \in \mathbb{R}^d$.

Consider an operator $\widehat{H}_\tau$ with the τ-symbol H for $\tau \in [0,1]$ in $L^1(\mathbb{R}^d)$, i.e. for any function $\varphi \in \mathrm{Dom}(\widehat{H}_\tau) \subset L^1(\mathbb{R}^d)$

$$\widehat{H}_\tau \varphi(q) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} e^{ip \cdot (q - q_1)} H(\tau q + (1-\tau) q_1, p) \varphi(q_1)\, dq_1\, dp, \quad q \in \mathbb{R}^d. \tag{3.4}$$

Remark 3.2. Note that (due to [7] and [14]) the operator $\widehat{H}_\tau$ with τ-symbol H given by the formula (3.3) for $A \in C^2(\mathbb{R}^d)$, $b \in C^1(\mathbb{R}^d)$, $c \in C(\mathbb{R}^d)$ and each $\tau \in [0,1]$ can be extended to the set $C_b^2(\mathbb{R}^d)$ of twice continuously differentiable functions bounded with all their derivatives by

$$\begin{aligned}\widehat{H}_\tau\varphi(q) = &- \operatorname{tr}(A(q)\operatorname{Hess}\varphi(q)) + [b(q) - 2(1-\tau)\operatorname{div} A(q)]\cdot\nabla\varphi(q) + \\ &+ [c(q) + (1-\tau)\operatorname{div} b(q) - (1-\tau)^2\operatorname{tr}(\operatorname{Hess} A(q))]\varphi(q) + \\ &+ \int_{y\neq 0}\left(\varphi(q+y) - \varphi(q) - \frac{y\cdot\nabla\varphi(q)}{1+|y|^2}\right)N(dy),\end{aligned} \tag{3.5}$$

i.e. $\widehat{H}_\tau$ is a sum of a second order differential operator with continuous coefficients and an integro-differential operator generating a Lévy process.

Assumtion 3.3. We assume that the coefficients A, b, c, N are such that the closure $(L^\tau, \operatorname{Dom}(L^\tau))$ of a ψDO $(\widehat{H}_\tau, C_c^\infty(\mathbb{R}^d))$ with the τ-symbol H as in (3.3) generates a strongly continuous semigroup $(T_t^\tau)_{t\geqslant 0}$ on the space X.

Remark 3.4. Due to the representation (3.5) it is clear that the requirement $C_c^\infty(\mathbb{R}^d) \subset \operatorname{Dom}(L^\tau)$ is equivalent to the requirement $C_c^\infty(\mathbb{R}^d) \subset \operatorname{Dom}(\widehat{r})$. The last one holds for example if the measure N has a compact support. In the case $c = 0$, $N = 0$ the explicit conditions on A, b to fulfill the Assumption 3.3 are given in [47]. Then the case with nonzero coefficients c and N can be proceeded by the technique of relatively bounded perturbations of generators (see Def.4.4.1 and Th.4.4.3 in [25]).

Consider a family $(F_\tau(t))_{t\geqslant 0}$ of ψDOs with the τ-symbol $e^{-tH(q,p)}$ in the space X, i.e. for any $\varphi \in \operatorname{Dom}(F_\tau(t))$

$$F_\tau(t)\varphi(q) = (2\pi)^{-d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} e^{ip\cdot(q-q_1)}e^{-tH(\tau q+(1-\tau)q_1,p)}\varphi(q_1)dq_1dp. \tag{3.6}$$

Lemma 3.5. *Under Assumption 3.1(i),(ii) for any $\varphi \in C_c^\infty(\mathbb{R}^d)$ and any $\tau \in [0,1]$ we have $F_\tau(t)\varphi \in X$. For all $t \geqslant 0$ the operators $F_\tau(t)$ can be extended to bounded mappings on the space X and there exists a constant $k \geqslant 0$ such that for all $t \geqslant 0$ holds the estimate:*

$$\|F_\tau(t)\| \leqslant e^{tk}. \tag{3.7}$$

Proof. Using the inequalities of Assumption 3.1(i) and the fact, that the real part of each continuous negative definite function is nonnegative (see inequalities (3.123) and (3.117) in [25]), we obtain the estimate

$$\sup_{q\in\mathbb{R}^d}|e^{-tH(q,p)}| \leqslant e^{-ta_0p^2}\exp\left(-t\min_{q\in\mathbb{R}^d}c(q)\right). \tag{3.8}$$

Hence, the function $f_{t,q} = (2\pi)^{-d/2}e^{-tH(q,\cdot)} \in L^1(\mathbb{R}^d)$ for each $q \in \mathbb{R}^d$ and $t \geqslant 0$. Moreover, $f_{t,q}(0) = (2\pi)^{-d/2}e^{-tc(q)}$. Therefore, the inverse Fourier transform of $f_{t,q}$ has the view $e^{-tc(q)}P_t^q$, where for each $q \in \mathbb{R}^d$ and $t \geqslant 0$ the function $P_t^q \in C_\infty(\mathbb{R}^d)$ is a density of a probability measure. This follows from the Bochner Theorem and the fact that Fourier transform maps $L^1(\mathbb{R}^d)$ into $C_\infty(\mathbb{R}^d)$.
Consider first the case $\tau = 0$. Then for each $\varphi \in C_c^\infty(\mathbb{R}^d)$ by Fubini–Tonelly Theorem we have

$$\begin{aligned} F_0(t)\varphi(q) &= \frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} e^{ip\cdot(q-q_1)}e^{-tH(q_1,p)}\varphi(q_1)dq_1dp \\ &= \int\limits_{\mathbb{R}^d} \varphi(q_1)e^{-tc(q_1)}P_t^{q_1}(q-q_1)dq_1. \end{aligned}$$

Again by Fubini–Tonelli Theorem for each $\varphi \in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned} &\|F_0(t)\varphi\|_1 \\ &= \left\| \int\limits_{\mathbb{R}^d} \varphi(q_1)e^{-tc(q_1)}P_t^{q_1}(q-q_1)dq_1 \right\|_1 \leqslant \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} |\varphi(q_1)|e^{-tc(q_1)}P_t^{q_1}(q-q_1)dq_1dq \\ &= \int\limits_{\mathbb{R}^d} |\varphi(q_1)|e^{-tc(q_1)}\left[\int\limits_{\mathbb{R}^d} P_t^{q_1}(q-q_1)dq\right]dq_1 \leqslant \exp\left(-t\min_{x\in\mathbb{R}^d} c(x)\right)\|\varphi\|_1. \end{aligned}$$

Therefore, for any $\varphi \in C_c^\infty(\mathbb{R}^d)$ we have $F_0(t)\varphi \in L^1(\mathbb{R}^d)$ and $F_0(t)$ is a bounded operator from $C_c^\infty(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$. Then due to the B.L.T. Theorem the operator $F_0(t)$ can be extended to a bounded operator on $L^1(\mathbb{R}^d)$ with the same norm. Hence, the lemma is true for $\tau = 0$. Let us now prove the lemma for the case $\tau \in (0, 1]$.
Let us now consider the function H given by (3.3) as a sum of functions h and r (see formulas (3.1), (3.2), (3.3)). Under Assumption 3.1(i),(ii) consider a family $(G_{A,b,c}^\theta(t))_{t\geqslant 0}$ of operators on $L^1(\mathbb{R}^d)$ defined for each fixed $\theta \in (0,1]$ by the formula

$$\begin{aligned} G_{A,b,c}^\theta(t)\varphi(q) &= \frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} e^{ip\cdot(q-q_1)}e^{-th(\theta q,p)}\varphi(q_1)dq_1dp = \frac{e^{-tc(\theta q)}}{(4\pi t)^{d/2}(\det A(\theta q))^{1/2}} \\ &\times \int\limits_{\mathbb{R}^d} \exp\left(-\frac{(q-q_1-tb(\theta q))\cdot A^{-1}(\theta q)(q-q_1-tb(\theta q))}{4t}\right)\varphi(q_1)dq_1, \quad q \in \mathbb{R}^d. \end{aligned} \tag{3.9}$$

Each $G_{A,b,c}^\theta(t)$ is a integral operator with the kernel

$$g_t^x(z) = (4\pi t)^{-d/2}(\det A(x))^{-1/2}e^{-tc(x)}\exp\left\{-\frac{(z-tb(x))\cdot A^{-1}(x)(z-tb(x))}{4t}\right\}, \tag{3.10}$$

for $x = \theta q$ and $z = q - q_1$. Note, that g_t^x is an inverse Fourier transform of the function $(2\pi)^{-d/2} e^{-th(x,\cdot)}$. Due to [35] there is a constant $k < \infty$ such that for $\theta = 1$ the estimate $\|G^1_{A,b,c}(t)\| \leqslant e^{kt}$ holds. For each fixed $\theta \in (0,1]$ the operator $G^\theta_{A,b,c}(t)$ equals the operator $G^1_{A_\theta,b_\theta,c_\theta}(t)$ with new coefficients $A_\theta(q) = A(\theta q)$, $b_\theta(q) = b(\theta q)$, $c_\theta(q) = c(\theta q)$ which remain as smooth and bounded as the original A, b and c are. Therefore, the estimate $\|G^\theta_{A,b,c}(t)\| \leqslant e^{kt}$ still holds for each $\theta \in (0,1]$ and the same k. Hence, by Fubini–Tonelli Theorem for $\varphi \in C_c^\infty(\mathbb{R}^d)$ we have

$$
\begin{aligned}
F_\tau(t)\varphi(q) &= (2\pi)^{-d} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} e^{ip\cdot(q-q_1)} e^{-tH(\tau q + (1-\tau)q_1, p)} \varphi(q_1) dq_1 dp \\
&= (2\pi)^{-d} \int\limits_{\mathbb{R}^d} \varphi(q_1) \left[\int\limits_{\mathbb{R}^d} e^{ip\cdot(q-q_1)} e^{-th(\tau q + (1-\tau)q_1, p)} e^{-tr(p)} dp \right] dq_1 \\
&= \int\limits_{\mathbb{R}^d} \varphi(q_1) \left[g_t^{\tau q + (1-\tau)q_1} * \mu_t \right] (q - q_1) dq_1. \qquad (3.11)
\end{aligned}
$$

Here the function in the squared brackets for each fixed $q, q_1 \in \mathbb{R}^d$ and $\tau \in (0,1]$ is an inverse Fourier transform of the product $e^{-th(\tau q + (1-\tau)q_1, \cdot)} \cdot (2\pi)^{-d/2} e^{-tr(\cdot)}$, i.e. a convolution of a function $g_t^{\tau q + (1-\tau)q_1}$ given by the formula (3.10) and a probability measure μ_t corresponding to the Lévy process with the symbol r. And this function is taken at the point $(q - q_1)$. Hence, with $y := q + \frac{1-\tau}{\tau} q_1$ and $x := q_1/\tau$

$$
\begin{aligned}
\left\| F_\tau(t)\varphi \right\|_1 &= \int\limits_{\mathbb{R}^d} \left| \int\limits_{\mathbb{R}^d} \varphi(q_1) \left[g_t^{\tau q + (1-\tau)q_1} * \mu_t \right] (q - q_1) dq_1 \right| dq \\
&\leqslant \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} |\varphi(q_1)| \left[g_t^{\tau q + (1-\tau)q_1} * \mu_t \right] (q - q_1) dq_1 dq = \\
&= \int\limits_{\mathbb{R}^d} \left[\int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} |\varphi(q_1)| g_t^{\tau q + (1-\tau)q_1} (q - q_1 - z) dq_1 dq \right] \mu_t(dz) \leqslant \\
&\leqslant \int\limits_{\mathbb{R}^d} \mu_t(dz) \cdot \sup_{z \in \mathbb{R}^d} \left[\tau^d \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} g_t^{\tau y} (y - z - x) |\varphi(\tau x)| dx dy \right] \\
&= \tau^d \sup_{z \in \mathbb{R}^d} \int\limits_{\mathbb{R}^d} G^\tau_{A,b,c}(t) |\varphi_\tau| (y - z) dy,
\end{aligned}
$$

where $\varphi_\tau(q) := \varphi(\tau q)$ and the operator $G^\tau_{A,b,c}$ is given by the formula (3.9) for each $\tau \in (0,1]$. Therefore, due to the estimate $\|G^\tau_{A,b,c}(t)\| \leqslant e^{kt}$ for each $\varphi \in C_c^\infty(\mathbb{R}^d)$

we have

$$\begin{aligned}
&\big\|F_\tau(t)\varphi\big\|_1 \\
&\leqslant \tau^d \sup_{z\in\mathbb{R}^d} \int\limits_{\mathbb{R}^d} G^\tau_{A,b,c}(t)|\varphi_\tau|(y-z)dy \leqslant \tau^d e^{kt}\|\varphi_\tau\|_1 = \tau^d e^{kt}\int\limits_{\mathbb{R}^d}|\varphi(\tau q)|dq = e^{kt}\|\varphi\|_1.
\end{aligned}$$

Once again by the B.L.T. Theorem the estimate $\big\|F_\tau(t)\varphi\big\|_1 \leqslant e^{kt}\|\varphi\|_1$ is true for all $\varphi\in L^1(\mathbb{R}^d)$. □

Remark 3.6. Due to results of [14] in the case $\tau=1$ the statement of the Lemma is also valid in the space $X = C_\infty(\mathbb{R}^d)$ with $k=0$. Therefore, in the case $\tau=1$ by Riesz–Thorin theorem the estimate (3.7) holds also in all spaces $L_{\mathrm{p}}(\mathbb{R}^d)$, $\mathrm{p}\geqslant 1$ (with some other constants k).

Remark 3.7. As it follows from the representation (3.11), the operators $F_\tau(t)$ can be considered as integral operators acting on $\varphi\in C_c^\infty(\mathbb{R}^d)$ as

$$F_\tau(t)\varphi(q) = \int\limits_{\mathbb{R}^d}\varphi(q_1)\big[g_t^{\tau q+(1-\tau)q_1} * \mu_t\big](q-q_1)dq_1, \quad q\in\mathbb{R}^d. \tag{3.12}$$

Hence, this representation can be used to construct a Lagrangian Feynman formula.

Lemma 3.8. *Let $N=0$ in the formula (3.2), i.e. $H=h$. Under Assumption 3.1(i),(ii) and Assumption 3.3 for any $\varphi\in C_c^\infty(\mathbb{R}^d)$, any $\tau\in[0,1]$ and any $t_0\geqslant 0$ we have*

$$\lim_{t\searrow 0}\left\|\frac{F_\tau(t)\varphi-\varphi}{t}+\widehat{H}_\tau\varphi\right\|_1 = 0 \quad \textit{and} \quad \lim_{t\to t_0}\|F_\tau(t)\varphi - F_\tau(t_0)\varphi\|_1 = 0.$$

Proof. Let $\varphi\in C_c^\infty(\mathbb{R}^d)\subset \mathrm{Dom}(\widehat{H}_\tau)$, $t>0$. By Taylor's formula with $\theta\in(0,1)$ we have

$$\begin{aligned}
&\left\|\frac{F_\tau(t)\varphi-\varphi}{t}+\widehat{H}_\tau\varphi\right\|_1 \\
&= t\int\limits_{\mathbb{R}^d}\left|(2\pi)^{-d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} e^{ip\cdot(q-q_1)}h^2(\tau q+(1-\tau)q_1,p)e^{-\theta t h(\tau q+(1-\tau)q_1,p)}\varphi(q_1)dq_1dp\right|dq.
\end{aligned}$$

Here $p\mapsto h^2(\tau q+(1-\tau)q_1,p)$ is a 4th order polynomial with bounded coefficients continuously depending on q and q_1. Let us present the calculations for the case

$d = 1$ and $b = 0$, $c = 0$ for simplicity. General case can be handled similarly.

$$
\begin{aligned}
&\left\| \frac{F_\tau(t)\varphi - \varphi}{t} + \widehat{H}_\tau \varphi \right\|_1 \\
&= \frac{t}{2\pi} \int\limits_{\mathbb{R}} \left| \int\limits_{\mathbb{R}^2} e^{ip\cdot(q-q_1)} A^2(\tau q + (1-\tau)q_1) p^4 e^{-\theta t A(\tau q + (1-\tau)q_1)p^2} \varphi(q_1) dq_1 dp \right| dq \\
&= \frac{t}{2\pi} \int\limits_{\mathbb{R}} \left| \int\limits_{\mathbb{R}^2} \partial^4_{q_1} \left[e^{ip\cdot(q-q_1)} \right] \left[A^2(\tau q + (1-\tau)q_1) e^{-\theta t A(\tau q + (1-\tau)q_1)p^2} \varphi(q_1) \right] dq_1 dp \right| dq \\
&= \frac{t}{2\pi} \int\limits_{\mathbb{R}} \left| \int\limits_{\mathbb{R}^2} \left[e^{ip\cdot(q-q_1)} \right] \partial^4_{q_1} \left[A^2(\tau q + (1-\tau)q_1) e^{-\theta t A(\tau q + (1-\tau)q_1)p^2} \varphi(q_1) \right] dq_1 dp \right| dq.
\end{aligned}
\tag{3.13}
$$

Consider first the case $\tau = 1$. Then

$$
\partial^4_{q_1} \left[A^2(\tau q + (1-\tau)q_1) e^{-\theta t A(\tau q + (1-\tau)q_1)p^2} \varphi(q_1) \right] = A^2(q) e^{-\theta t A(q)p^2} \varphi^{(4)}(q_1)
$$

and by the Fubini–Tonelli theorem

$$
\begin{aligned}
&\left\| \frac{F_1(t)\varphi - \varphi}{t} + \widehat{H}_1 \varphi \right\|_1 \\
&= t \int\limits_{\mathbb{R}} \left| (2\pi)^{-1} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \left[e^{ip\cdot(q-q_1)} \right] \cdot \left[A^2(q) e^{-\theta t A(q)p^2} \varphi^{(4)}(q_1) \right] dq_1 dp \right| dq \\
&= t \int\limits_{\mathbb{R}} \left| A^2(q) \int\limits_{\mathbb{R}} (4\pi\theta t A(q))^{-1/2} e^{-\frac{(q-q_1)^2}{4\theta t A(q)}} \varphi^{(4)}(q_1) dq_1 \right| dq \\
&\leqslant t A_0^2 \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} (4\pi\theta t a_0)^{-1/2} e^{-\frac{(q-q_1)^2}{4\theta t A_0}} |\varphi^{(4)}(q_1)| dq_1 dq = t(A_0^{5/2} a_0^{-1/2}) \|\varphi^{(4)}\|_1.
\end{aligned}
$$

Consider now the case when $\tau \in [0, 1)$. Then

$$
\begin{aligned}
&\partial^4_{q_1} \left[A^2(\tau q + (1-\tau)q_1) e^{-\theta t A(\tau q + (1-\tau)q_1)p^2} \varphi(q_1) \right] \\
&= e^{-\theta t A(\tau q + (1-\tau)q_1)p^2} \sum_{k=0}^{4} (\theta t p^2)^k \psi_k(\tau q + (1-\tau)q_1, q_1),
\end{aligned}
$$

where the functions $(x, y) \mapsto \psi_k(x, y)$ are linear combinations of the products $A^k(x)(A^2)^{(m)}(x)\varphi^{(n)}(y)$ with $m, n = 0, \ldots, 4$. Hence, $\psi_k(x, \cdot) \in C_c(\mathbb{R})$ and $\psi_k(\cdot, y) \in$

$C_b(\mathbb{R})$ for all $x, y \in \mathbb{R}$. Therefore, with the change of variables $\sqrt{\theta t}p = \rho$, $\frac{q-q_1}{\sqrt{\theta t}} = y$ we have

$$\left\| \frac{F_\tau(t)\varphi - \varphi}{t} + \widehat{H}_\tau \varphi \right\|_1$$
$$= \frac{t}{2\pi} \int\limits_{\mathbb{R}} \left| \int\limits_{\mathbb{R}^2} e^{ip\cdot(q-q_1)} e^{-\theta t A(\tau q + (1-\tau)q_1)p^2} \sum_{k=0}^{4} (\theta t p^2)^k \psi_k(\tau q + (1-\tau)q_1, q_1) dq_1 dp \right| dq$$
$$= \frac{t}{2\pi} \int\limits_{\mathbb{R}} \left| \int\limits_{\mathbb{R}^2} e^{i\rho\cdot y} e^{-A(q-\sqrt{\theta t}(1-\tau)y)\rho^2} \sum_{k=0}^{4} (\rho^2)^k \psi_k(q - \sqrt{\theta t}(1-\tau)y, q - \sqrt{\theta t}y) d\rho dy \right| dq$$
$$= t \int\limits_{\mathbb{R}} \Bigg| \int\limits_{\mathbb{R}} \sum_{k=0}^{4} (-1)^k \, \partial_\xi^{2k} \left[\frac{\exp\left\{ -\frac{\xi^2}{4A(q-\sqrt{\theta t}(1-\tau)y)} \right\}}{(4\pi A(q - \sqrt{\theta t}(1-\tau)y))^{1/2}} \right] \Bigg|_{\xi=y}$$
$$\times \psi_k(q - \sqrt{\theta t}(1-\tau)y, q - \sqrt{\theta t}y) dy \Bigg| dq$$
$$\leqslant t \int\limits_{\mathbb{R}} (4\pi a_0)^{-1/2} e^{-\frac{y^2}{4A_0}} C_5(1+y^8) \sum_{k=0}^{4} C_k \int\limits_{\mathbb{R}} |\varphi^{(k)}(q - \sqrt{\theta t}y)| dq dy \leqslant t \sum_{k=0}^{4} C'_k \|\varphi^{(k)}\|_1$$

with some positive constants $C_k < \infty$ and $C'_k < \infty$. Analogously, for $\varphi \in C_c^\infty(\mathbb{R}^d)$ by Taylor's formula with $\theta \in (0,1)$ and $t, t_0 \geqslant 0$, $t \to t_0$ we have

$$\|F_\tau(t)\varphi - F_\tau(t_0)\varphi\|_1 = |t - t_0|$$
$$\times \int\limits_{\mathbb{R}^d} \left| (2\pi)^{-d} \int\limits_{\mathbb{R}^{2d}} e^{ip\cdot(q-q_1)} h(\tau q + (1-\tau)q_1, p) e^{-[t_0 + \theta(t-t_0)] h(\tau q + (1-\tau)q_1, p)} \varphi(q_1) dq_1 dp \right| dq.$$

Once again let us present the calculations for the case $d = 1$ and $b = 0$, $c = 0$ for simplicity. For any fixed $t_0 > 0$ take $t \in (t_0/2, 2t_0)$. Hence, $\alpha(t) := t_0 + \theta(t - t_0) \in$

$(t_0/2, 2t_0)$ and

$$\|F_\tau(t)\varphi - F_\tau(t_0)\varphi\|_1 = |t-t_0|$$
$$\times \int\limits_{\mathbb{R}} \left| (2\pi)^{-1} \int\limits_{\mathbb{R}^2} e^{ip\cdot(q-q_1)} A(\tau q + (1-\tau)q_1) p^2 e^{-\alpha(t)A(\tau q+(1-\tau)q_1)p^2} \varphi(q_1) dq_1 dp \right| dq$$
$$= |t-t_0|$$
$$\times \int\limits_{\mathbb{R}} \left| \int\limits_{\mathbb{R}} (4\pi\alpha(t)A(\tau q + (1-\tau)q_1))^{-1/2} \partial_\xi^2 \left[e^{-\frac{\xi^2}{4\alpha(t)A(\tau q+(1-\tau)q_1)}} \right] \Big|_{\xi=q-q_1} \varphi(q_1) dq_1 \right| dq$$
$$\leqslant |t-t_0| \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} (2\pi t_0 a_0)^{-1/2} e^{-\frac{(q-q_1)^2}{8t_0 A_0}} C(t_0)(1+(q-q_1)^2)|\varphi(q_1)| dq_1 dq$$
$$\leqslant |t-t_0| C'(t_0) \|\varphi\|_1$$

with some positive constants $C < \infty$ and $C' < \infty$ depending only on t_0. In the case $t_0 = 0$ we have $F(t_0) = \mathrm{Id}$ and we proceed as before

$$\|F_\tau(t)\varphi - \varphi\|_1$$
$$= t \int\limits_{\mathbb{R}} \left| (2\pi)^{-1} \int\limits_{\mathbb{R}^2} e^{ip\cdot(q-q_1)} A(\tau q + (1-\tau)q_1) p^2 e^{-\theta t A(\tau q+(1-\tau)q_1)p^2} \varphi(q_1) dq_1 dp \right| dq$$
$$= t \int\limits_{\mathbb{R}} \left| (2\pi)^{-1} \int\limits_{\mathbb{R}^2} \partial_{q_1}^2 \left[e^{ip\cdot(q-q_1)} \right] A(\tau q + (1-\tau)q_1) e^{-\theta t A(\tau q+(1-\tau)q_1)p^2} \varphi(q_1) dq_1 dp \right| dq$$
$$= t \int\limits_{\mathbb{R}} \left| (2\pi)^{-1} \int\limits_{\mathbb{R}^2} e^{ip\cdot(q-q_1)} \partial_{q_1}^2 \left[A(\tau q + (1-\tau)q_1) e^{-\theta t A(\tau q+(1-\tau)q_1)p^2} \varphi(q_1) \right] dq_1 dp \right| dq$$
$$\leqslant t \sum_{k=0}^{2} C_k \|\varphi^{(k)}\|_1,$$

where the integrals in the penultimate line can be handled as in (3.13). A 3-ε–argument concludes the proof of $\lim\limits_{t\to t_0} \|F_\tau(t)\varphi - F_\tau(t_0)\varphi\|_1 = 0$ for all $\varphi \in L^1(\mathbb{R}^d)$. □

Theorem 3.9. *Let $L^1(\mathbb{R}^d)$, $\tau \in [0,1]$ and $H = h$, where h is given by the formula (3.1). Under Assumption 3.1(i),(ii) and Assumption 3.3 the family $(F_\tau(t))_{t\geqslant 0}$ given by the formula (3.6) is Chernoff equivalent to the semigroup $(T_t^\tau)_{t\geqslant 0}$, generated by the closure $(L^\tau, \mathrm{Dom}(L^\tau))$ of a ψDO $(\widehat{H}_\tau, C_c^\infty(\mathbb{R}^d))$ with the τ-symbol $H(q,p)$.*

Therefore, the Feynman formula

$$(T_t^\tau)\varphi = \lim_{n\to\infty} (F_\tau(t/n))^n \varphi \tag{3.14}$$

holds in $L^1(\mathbb{R}^d)$ *locally uniformly with respect to* $t \in [0,\infty)$*. Moreover, this Feynman formula* (3.14) *converts into the Lagrangian one:*

$$(T_t^\tau)\varphi(q_0) = \lim_{n\to\infty} \int\limits_{\mathbb{R}^d} \cdots \int\limits_{\mathbb{R}^d} \varphi(q_n) \prod_{k=1}^n g_{t/n}^{\tau q_{k-1}+(1-\tau)q_k}(q_{k-1}-q_k)dq_1\dots dq_n, \tag{3.15}$$

where Gaussian type density $g_t^x(z)$ *is given by the formula* (3.10)*. Additionally, under Assumption 3.1(iii) we have* $F_\tau(t) : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ *and the Feynman formula* (3.14) *with* $\varphi \in S(\mathbb{R}^d)$ *converts also into the Hamiltonian one:*

$$\begin{aligned}(T_t^\tau)\varphi(q_0) = \lim_{n\to\infty} (2\pi)^{-nd} \int\limits_{\mathbb{R}^{2nd}} \exp\left(i \sum_{k=1}^n p_k \cdot (q_{k+1}-q_k) \right) \\ \times \exp\left(-\frac{t}{n} \sum_{k=1}^n H(\tau q_{k+1} + (1-\tau)q_k, p_k) \right) \varphi(q_1)dq_1dp_1\dots dq_ndp_n,\end{aligned} \tag{3.16}$$

where $q_{n+1} := q_0$ *for all* $n \in \mathbb{N}$ *in the pre-limit expressions in the right hand side.*

This Theorem follows immediately from two previous Lemmas, Remark 3.7 and the Chernoff Theorem 2.1.

Remark 3.10. If we consider the case $H(q,p) = p \cdot Ap + c(q)$, $q, p \in \mathbb{R}^d$, where the matrix A doesn't depend on q (it is the Hamilton function of a particle with constant mass in a potential field c), then ψDOs $\widehat{H}_\tau$ (and hence the semigroups $(T_t^\tau)_{t\geqslant 0}$) do coincide for all $\tau \in [0,1]$. However, the families $(F_\tau(t))_{t\geqslant 0}$, given by (3.6), are different since they are ψDOs whose τ-symbols $e^{-t[p\cdot Ap + c(q)]}$ nontrivially depend on both variables q and p. Nevertheless, one can easily show, that $\|F_{\tau_1}(t)\varphi - F_{\tau_2}(t)\varphi\|_1 = C(\tau_1,\tau_2)t$ with some constant C depending only on τ_1 and τ_2 (see [34] for further discussion).

Remark 3.11. Let $\tau = 1$. Under Assumptions 3.1(i),(ii) and Assumption (3.3) with $N = 0$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$ the function $T_t^\tau\varphi$ can be represented also by a Feynman–Kac formula (cf. [21] §2.1, [26] §5.7):

$$T_t^\tau \varphi(q_0) = \mathbb{E}_b^{q_0}\left[\exp\left(\int\limits_0^t c(\xi_s)ds \right) \varphi(\xi_t) \right], \tag{3.17}$$

where $\mathbb{E}_b^{q_0}$ is the expectation of a (starting at q_0) diffusion process $(\xi_t)_{t\geqslant 0}$ with variable diffusion matrix A and drift b. Therefore, the Lagrangian Feynman formula (3.15) gives (suitable for direct calculations) approximations of a functional integral in the Feynman–Kac formula (3.17). Moreover, due to the representation (3.17) for the case $b = 0$ one can expect (compare with the formula (3) in [30]) that the expression in the right hand side of the Lagrangian Feynman formula (3.15) does coincide with the following functional integral

$$\begin{aligned} T_t^\tau \varphi(q_0) =& \mathbb{E}^{q_0}\Bigg[\exp\Bigg(\int_0^t c(X_s)ds\Bigg)\exp\Bigg(\frac{1}{2}\int_0^t A^{-1}(X_s)b(X_s)\cdot dX_s\Bigg)\times \\ &\times\exp\Bigg(-\frac{1}{4}\int_0^t A^{-1}(X_s)b(X_s)\cdot b(X_s)ds\Bigg)\varphi(X_t)\Bigg], \end{aligned} \tag{3.18}$$

where $\mathbb{E}^{q_0}$ is the expectation of a diffusion process $(X_t)_{t\geqslant 0}$ with variable diffusion matrix A and without any drift, a stochastic integral $\int_0^t A^{-1}(X_\tau)b(X_\tau)\cdot dX_\tau$ is an Itô integral. Since the functional integrals in formulae (3.17) and (3.18) coincide, one obtains the analogue of the Girsanov–Cameron–Martin–Reimer–Maruyama formula for the case of diffusion processes with variable diffusion matrices. Due to Remark 3.2 the similar results are then valid for all $\tau \in [0, 1]$.

Lemma 3.12. *Consider the general case of symbol $H = h+r$ given by the formula (3.3) and $\tau = 1$. Under Assumption 3.1(i),(iii),(iv) and Assumption 3.3 for any $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $t_0 \geqslant 0$ we have*

$$\lim_{t\searrow 0}\left\|\frac{F_\tau(t)\varphi-\varphi}{t}+\widehat{H}_\tau\varphi\right\|_1 = 0 \quad \textit{and} \quad \lim_{t\to t_0}\|F_\tau(t)\varphi - F_\tau(t_0)\varphi\|_1 = 0.$$

Proof. Fix $t_0 \geqslant 0$ and let $t_0 \in [0, t_0+1]$. By Taylor's formula with θ in between t and t_0, by the Fubini–Tonelli theorem, by Lemma 2.3 and Lemma 3.5 with a probability measure $\mu_\theta = (2\pi)^{-d/2}\mathcal{F}^{-1}[e^{-\theta r(p)}]$, for each $\varphi \in C_c^\infty(\mathbb{R}^d)$ we have

$$
\begin{aligned}
\|F_1(t)\varphi - F_1(t_0)\varphi\|_1 &= \left\| \frac{t-t_0}{(2\pi)^d} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} e^{ip\cdot(q-q_1)} H(q,p) e^{-\theta H(q,p)} \varphi(q_1)dq_1dp \right\|_1 \\
&= |t-t_0| \left\| \left(\widehat{He^{-\theta H}}\right)_1 \varphi \right\|_1 \leqslant |t-t_0| \left[\left\| \left(\widehat{he^{-\theta H}}\right)_1 \varphi \right\|_1 + \left\| \left(\widehat{re^{-\theta H}}\right)_1 \varphi \right\|_1 \right] \\
&= |t-t_0| \left[\left\| \left(\widehat{he^{-\theta h}}\right)_1 \circ \left(\widehat{e^{-\theta r}}\right) \varphi \right\|_1 + \left\| \left(\widehat{e^{-\theta H}}\right)_1 \circ \widehat{r}\,\varphi \right\|_1 \right] \\
&\leqslant |t-t_0| \left[\left\| \left(\widehat{he^{-\theta h}}\right)_1 (\mu_\theta * \varphi) \right\|_1 + \|F_1(\theta)\|\, \|\widehat{r}\,\varphi\|_1 \right] \\
&\leqslant |t-t_0| \left[\sum_{k=0}^{2} C_k(t_0) \|(\mu_\theta * \varphi)^{(k)}\|_1 + e^{k\theta} \|\widehat{r}\,\varphi\|_1 \right] \\
&\leqslant |t-t_0| \left[\sum_{k=0}^{2} C_k(t_0)\mu_\theta(\mathbb{R}^d) \|\varphi^{(k)}\|_1 + e^{k\theta} \|\widehat{r}\,\varphi\|_1 \right] \\
&= |t-t_0| K(t_0,\varphi)
\end{aligned} \tag{3.19}
$$

with some constants $C_k(t_0) < \infty$ depending only on t_0 and $K(t_0,\varphi) < \infty$ depending only on t_0 and φ. These constants $C_k(t_0)$ arise from the calculations with the operator $\left(\widehat{he^{-\theta h}}\right)_1$ obtained in the Lemma 3.8. Note, that all calculations in Lemma 3.8 remain true for any $\varphi \in S(\mathbb{R}^d)$. Moreover, by Assumption 3.1(iv) we have $r \in C^\infty(\mathbb{R}^d)$ and, as a negative definite function, r grows at infinity with all its derivatives not faster than a polynomial (cf. Lemma 3.6.22 and Theo.3.7.13 in [25]). Therefore, $\widehat{r^m}\varphi$, $\left(\widehat{r^m e^{-\theta r}}\right)\varphi \in S(\mathbb{R}^d)$ for any $\varphi \in C_c^\infty(\mathbb{R}^d) \subset S(\mathbb{R}^d)$ and any $m \in \mathbb{N}\cup\{0\}$. In the same way by Lemma 2.3 and Lemma 3.8 with $[0,1] \ni t \to 0$ and $\theta \in (0,t)$ we obtain

$$
\begin{aligned}
&\left\| \frac{F_1(t)\varphi - \varphi}{t} + \widehat{H}_1\varphi \right\|_1 = t \left\| \left(\widehat{H^2e^{-\theta H}}\right)_1 \varphi \right\|_1 \leqslant \\
&\leqslant t \left\| \left(\widehat{h^2e^{-\theta h}}\right)_1 \circ \left(\widehat{e^{-\theta r}}\right)\varphi + 2\left(\widehat{he^{-\theta h}}\right)_1 \circ \left(\widehat{re^{-\theta r}}\right)\varphi + \left(\widehat{e^{-\theta h}}\right)_1 \circ \left(\widehat{r^2e^{-\theta r}}\right)\varphi \right\|_1 = \\
&= t \left\| \left(\widehat{h^2e^{-\theta h}}\right)_1 (\mu_\theta * \varphi) + 2\left(\widehat{he^{-\theta h}}\right)_1 (\mu_\theta * [\widehat{r}\varphi]) + F_1(\theta)(\mu_\theta * [\widehat{r^2}\varphi]) \right\|_1 \leqslant \\
&\leqslant t\mu_\theta(\mathbb{R}^d) \left[\sum_{k=0}^{4} C_k \|\varphi^{(k)}\|_1 + 2\sum_{k=0}^{2} C'_k \|(\widehat{r}\varphi)^{(k)}\|_1 + e^{k\theta} \left\| \widehat{(r^2)}\varphi \right\|_1 \right] = tK'(\varphi).
\end{aligned}
$$

□

Theorem 3.13. *Let $\tau = 1$. Under Assumptions 3.1(i),(iii),(iv) and Assumption 3.3 the family $(F_\tau(t))_{t\geqslant 0}$ given by the formula* (3.6) *is Chernoff equivalent to the* *semigroup* $(T_t^\tau)_{t\geqslant 0}$*, generated by the closure* $(L^\tau, \mathrm{Dom}(L^\tau))$ *of a* ψDO $(\widehat{H}_\tau, C_c^\infty(\mathbb{R}^d))$ *with the* τ*-symbol* H *as in* (3.3)*. Therefore, the Feynman formula*

$$(T_t^\tau)\varphi = \lim_{n\to\infty} (F_\tau(t/n))^n \varphi$$

holds in $L^1(\mathbb{R}^d)$ *locally uniformly with respect to* $t \in [0,\infty)$*. The obtained Feynman formula converts also into a Lagrangian Feynman formula*

$$\begin{aligned}
&(T_t^\tau)\varphi(q_0) = \qquad (3.20)\\
&= \lim_{n\to\infty} \int\limits_{\mathbb{R}^{2nd}} e^{-\sum\limits_{k=1}^{n} \frac{A^{-1}(q_{k-1})(q_k - q_{k-1} + z_k - b(q_{k-1})t/n)\cdot(q_k - q_{k-1} + z_k - b(q_{k-1})t/n)}{4t/n}}\\
&\times \prod_{k=1}^{n} \left((2\pi t/n)^d \det A(q_{k-1})\right)^{-1/2} e^{-\frac{t}{n}\sum\limits_{k=1}^{n} c(q_{k-1})} \varphi(q_n) dq_1\, \mu_{t/n}(dz_1)\dots dq_n\, \mu_{t/n}(dz_n)
\end{aligned}$$

and with $\varphi \in S(\mathbb{R}^d)$ *into a Hamiltonian Feynman formula*

$$\begin{aligned}
(T_t^\tau)\varphi(q_0) = \lim_{n\to\infty} \frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^{2nd}} \exp\left(i \sum_{k=1}^{n} p_k\cdot(q_{k+1} - q_k)\right) \qquad (3.21)\\
\times \exp\left(-\frac{t}{n}\sum_{k=1}^{n} H(q_{k+1}, p_k)\right) \varphi(q_1) dq_1 dp_1 \dots dq_n dp_n,
\end{aligned}$$

where $q_{n+1} := q_0$ *in the pre-limit expressions for each* $n \in \mathbb{N}$*.*

Proof. The statement of the Theorem is a straightforward consequence of Chernoff Theorem 2.1, Lemma 3.5 and Lemma 3.8. Lagrangian Feynman formula is obtained with the help of representation (3.12). Under Assumptions 3.1 (i), (iii), (iv) we have $F_\tau(t) : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ (due to Lemma 3.3 in [14]). Therefore, all expressions in the right hand side of the Hamiltonian Feynman formula are well defined. □

4 The construction of phase space Feynman path integrals via a family of Hamiltonian Feynman pseudomeasures for $\tau \in [0,1]$.

A Feynman pseudomeasure on a (usually infinite dimensional) vector space is a continuous linear functional on a locally convex space of some functions defined on this vector space. The value of this functional on a function belonging to its domain is called Feynman integral with respect to this Feynman pseudomeasure.

If the considered vector space is itself a set of functions taking values in classical configuration or phase space then the corresponding Feynman integral is called configuration or phase space Feynman path integral.

There are many approaches for giving a mathematically rigorous meaning to phase space Feynman path integrals. Some phase space Feynman path integrals are defined via the Fourier transform and via Parseval's equality (see [40], cf. [2]; see [38], [18], [15] and references therein); some are defined via an analytic continuation of a Gaussian measure on the set of paths in a phase space [40], some — via regularization procedures, e.g., as limits of integrals with respect to Gaussian measures with a diverging diffusion constant [17]; the integrands of some phase space Feynman path integrals are realized as Hida distributions in the setting of White Noise Analysis [5], [6]. A variety of approaches treats Feynman path integrals as limits of integrals over some finite dimensional subspaces of paths when the dimension tends to infinity. Such path integrals are sometimes called sequential and are most convenient for direct calculations. The general definition of a sequential Feynman pseudomeasure (Feynman path integral) in an abstract space (on a set of paths in a phase space, in particular) can be found in [40]. Some concrete realizations are e.g. presented in [41], [13], [1], [24], [29], [28], [27], [23].

Let us outline some general concepts of the construction of Feynman pseudomeasure, in particular phase space Feynman path integrals. One of the most convenient definitions of a Feynman pseudomeasure on a conceptual level is via its Fourier transform. However, the most convenient for calculations is the sequential approach, which treats Feynman path integrals as limits of finite dimensional integrals with unboundedly growing dimension. Let X be a locally convex space and X^* be the set of all continuous linear functionals on X. So, let E be a real vector space and for all $x \in E$ and any linear functional g on E let $\phi_g(x) = e^{ig(x)}$. Let F_E be a locally convex set of some complex valued functions on E. Elements of the set $F(E)^*$ are called *$F(E)^*$-distributions on E* or just *distributions on E* (if we don't specify the space $F(E)^*$ exactly). Let G be a vector space of some linear functionals on E distinguishing elements of E and let $\phi_g \in F_E$ for all $g \in G$. Then *G-Fourier transform* of an element $\eta \in F_E^*$ is a function on G denoted by $\widetilde{\eta}$ or $\mathcal{F}[\eta]$ and defined by the formula

$$\widetilde{\eta}(g) \equiv \mathcal{F}[\eta](g) := \eta(\phi_g).$$

If a set $\{\phi_g : g \in G\}$ is total in F_E (i.e. its linear span is dense in F_E) then any element η is uniquely defined by its Fourier transform.

Definition 4.1. *Let b be a quadratic functional on G, $a \in E$ and $\alpha \in \mathbb{C}$. Then Feynman α-pseudomeasure on E with correlation functional b and mean a is a* *distribution $\Phi_{b,a,\alpha}$ on E whose Fourier transform is given by the formula*

$$\mathcal{F}[\Phi_{b,a,\alpha}](g) = \exp\left(\frac{\alpha b(g)}{2} + ig(a)\right).$$

If $\alpha = -1$ and $b(x) \geqslant 0$ for all $x \in G$ then Feynman α-pseudomeasure is a Gaussian G-cylindrical measure on E (which however can be not $\sigma-$additive). If $\alpha = i$ then we have a "standard" Feynman pseudomeasure which is usually used for solving Schrödinger type equations. In the sequel we will consider only these "standard" Feynman $i-$pseudomeasures with $a = 0$.

Definition 4.2 (Hamiltonian (or phase space) Feynman pseudomeasure). *Let $E = Q \times P$, where Q and P are locally convex spaces, $Q = P^*$, $P = Q^*$ (as vector spaces) with the duality $\langle \cdot, \cdot \rangle$; the space $G = P \times Q$ is identified with the space of all linear functionals on E in the following way: for any $g = (p_g, q_g) \in G$ and $x = (q,p) \in E$ let $g(x) = \langle q, p_g\rangle + \langle q_g, p\rangle$. Then* ***Hamiltonian (or symplectic, or phase space) Feynman pseudomeasure*** *on E is a Feynman i-pseudomeasure Φ on E whose correlation functional b is given by the formula $b(p_g, q_g) = 2\langle q_g, p_g\rangle$ and mean $a = 0$, i.e.*

$$\mathcal{F}[\Phi](g) = \exp\big(i\langle q_g, p_g\rangle)\big).$$

Definition 4.3. *Assume that there exists a linear injective mapping $B : G \to E$ such that $b(g) = g(B(g))$ for all $g \in G$ (B is called* correlation operator *of Feynman pseudomeasure). Let* $\mathrm{Dom}(B^{-1})$ *be the domain of B^{-1}. A function* $\mathrm{Dom}(B^{-1}) \ni x \mapsto f(x) = e^{\frac{\alpha^{-1}B^{-1}(x)(x)}{2}}$ *is called the* ***generalized density*** *of Feynman $\alpha-$pseudomeasure (cf. [42]).*

Example 4.4. (i) If $E = \mathbb{R}^d = G$ then the Feynman i-pseudomeasure on E with correlation operator B can be identified with a complex-valued measure (with unbounded variation) on a δ-ring of bounded Borel subsets of $\mathbb{R}^d$ whose density with respect to the Lebesgue measure is $f(x) = e^{-\frac{i}{2}(B^{-1}x,x)}$. In this case the generalized density coincides with the density in usual sense.
(ii) If we consider the Hamiltonian Feynman pseudomeasure on $E = Q \times P$ then take $B : (p,q) \in G \subset E^* \to (q,p) \in E$. Then we have $g(B(g)) = g(B(p_g, q_g)) = g(q_g, p_g) = 2\langle q_g, p_g\rangle = b(g)$. Moreover, $B^{-1} : E \to E^*$ is defined by the formula $B^{-1}(q,p) = (p,q)$ and hence the generalized density of the Hamiltonian Feynman pseudomeasure is given by the formula $f(q,p) = \exp\{i\langle q,p\rangle\}$.

The concepts given above allow to introduce the following definition of a Feynman pseudomeasure in the frame of sequential approach (in the sequel we assume any standard regularization of oscillating integrals, e.g., $\int\limits_E f(z)dz = \lim\limits_{\varepsilon\to 0} \int\limits_E f(z)e^{-\varepsilon|z|^2}dz$).

Definition 4.5 (Sequential Feynman pseudomeasure). *Let $\{E_n\}_{n\in\mathbb{N}}$ be an increasing sequence of finite dimensional subsets of* $\mathrm{Dom}(B^{-1})$*. Then the value of a sequential Feynman $\alpha-$pseudomeasure $\Phi_{B,\alpha}^{\{E_n\}}$ (with mean $a=0$) associated with the sequence $\{E_n\}_{n\in\mathbb{N}}$ on a function $\psi : E \to \mathbb{C}$ (this value is called sequential Feynman integral of ψ) is defined by the formula*

$$\Phi_{B,\alpha}^{\{E_n\}}(\psi) = \lim_{n\to\infty} \left(\int\limits_{E_n} e^{\frac{\alpha^{-1}B^{-1}(x)(x)}{2}} dx \right)^{-1} \int\limits_{E_n} \psi(x) e^{\frac{\alpha^{-1}B^{-1}(x)(x)}{2}} dx,$$

where one integrates with respect to the Lebesgue measure on E_n, if the limit in the r.h.s. exists.

The fact that a function ψ belongs to the domain of the functional $\Phi^{\{E_n\}}$ depends only on restrictions of this function to the subspaces E_n. In the particular case of Hamiltonian Feynman pseudomeasure Definition 4.5 can be read as follows:

Definition 4.6. *Let $\{E_n = Q_n \times P_n\}_{n\in\mathbb{N}}$ be an increasing sequence of finite dimensional vector subspaces of $E = Q \times P$, where Q_n and P_n are vector subspaces of Q and P respectively. The value $\Phi_{\{E_n\}}(G)$ of the Hamiltonian Feynman pseudomeasure $\Phi_{\{E_n\}}$, associated with the sequence $\{E_n\}_{n\in\mathbb{N}}$, on a function $\psi : E \to \mathbb{C}$, i.e. a Feynman path integral of ψ, is defined by the formula*

$$\Phi_{\{E_n\}}(\psi) = \lim_{n\to\infty} \Big(\int\limits_{E_n} e^{i\langle p,q\rangle}\, dq\, dp \Big)^{-1} \int\limits_{E_n} \psi(q,p) e^{i\langle p,q\rangle}\, dq\, dp, \tag{4.1}$$

if this limit exists. In this formula (as well as before) all integrals must be considered in a suitably regularized sense.

In the sequel we present a construction of the Hamiltonian Feynman pseudomeasure for a particular family of spaces $E_t^{x,\tau}$ with $\tau \in [0,1]$, cf. [41], [7]. For any $t > 0$ let $PC([0,t],\mathbb{R}^d)$ be the vector space of all functions on $[0,t]$ taking values in $\mathbb{R}^d$ whose distributional derivatives are measures with finite support. Let $PC^l([0,t],\mathbb{R}^d)$ denote the space of all left continuous functions from $PC([0,t],\mathbb{R}^d)$. Let $PC^\tau([0,t],\mathbb{R}^d)$ be the collection of functions f from $PC([0,t],\mathbb{R}^d)$ such that for all $s \in (0,t)$

$$f(s) = \tau f(s+0) + (1-\tau) f(s-0). \tag{4.2}$$

For each $x \in \mathbb{R}^d$ let

$$Q_t^{x,\tau} = \{f \in PC^\tau([0,t],\mathbb{R}^d) : \ f(0) = \lim_{t\to+0} f(t),\ f(t) = x\},$$

$$P_t = \{f \in PC^l([0,t],\mathbb{R}^d) : \ f(0) = \lim_{t\to+0} f(t)\}$$

and $E_t^{x,\tau} = Q_t^{x,\tau} \times P_t$. The spaces $Q_t^{x,\tau}$ and P_t are taken in duality by the form:

$$\langle q(\cdot), p(\cdot)\rangle \mapsto \int_0^t p(s)\dot{q}(s)\,ds,$$

where $\dot{q}(s)\,ds$ denotes the measure which is the distributional derivative of $q(\cdot)$. We will consider the elements of $E_t^{x,\tau}$ as functions taking values in $\mathbf{E} = \mathbf{Q} \times \mathbf{P} = \mathbb{R}^d \times \mathbb{R}^d$.

Let $t_0 = 0$ and for any $n \in \mathbb{N}$ and any $k \in \mathbb{N}$, $k \leqslant n$, let $t_k = \frac{k}{n}t$. Let $F_n \subset PC([0,t], \mathbb{R}^d)$ be the space of functions, the restrictions of which to any interval (t_{k-1}, t_k) are constant functions. Let $Q_n^\tau = F_n \cap Q_t^{x,\tau}$, $P_n = F_n \cap P_t$. Let J_n^τ be the mapping of $E_n^\tau = Q_n^\tau \times P_n$ to $(\mathbb{R}^d \times \mathbb{R}^d)^n$, defined by

$$\begin{aligned} J_n^\tau(q,p) = &\big(q(\tfrac{t}{n} - 0), p(\tfrac{t}{n}), \ldots, q(\tfrac{(n-1)t}{n} - 0), p(\tfrac{(n-1)t}{n}), q(\tfrac{nt}{n} - 0), p(\tfrac{nt}{n})\big) \equiv \\ &\equiv \big(q_1, p_1, \ldots.., q_n, p_n\big). \end{aligned}$$

The map J_n^τ is a one-to-one correspondence of E_n^τ and $(\mathbb{R}^d \times \mathbb{R}^d)^n$. Therefore, in this particular case Definition 4.6 can be rewritten in the following way:

Definition 4.7. *The Hamiltonian (or phase space) Feynman path integral*

$$\Phi_x^\tau(\psi) \equiv \int_{E_t^{x,\tau}} \psi(q,p)\Phi_x^\tau(dq,dp) \equiv \int_{E_t^{x,\tau}} \psi(q(s),p(s))e^{i\int_0^t p(s)\dot{q}(s)ds} \prod_{\tau=0}^{t} dq(s)\,dp(s)$$

of a function $\psi : Q_t^{x,\tau} \times P_t \to \mathbb{R}$ is defined as a limit:

$$\begin{aligned} \Phi_x^\tau(\psi) = \lim_{n\to\infty} \frac{1}{(2\pi)^{dn}} &\int_{(\mathbb{R}^d\times\mathbb{R}^d)^n} \psi((J_n^\tau)^{-1}(q_1, p_1, \ldots.., q_n, p_n))\times \\ &\times \exp\left[i\sum_{k=1}^n p_k \cdot (q_{k+1} - q_k)\right] dq_1\,dp_1 \ldots\ldots dq_n\,dp_n, \end{aligned} \tag{4.3}$$

where $q_{n+1} := x$ in each pre-limit expression.

Remark 4.8. The generalized density of the pseudomeasure Φ_x^τ can be defined through the formula

$$\begin{aligned} &\int_{E_t^{x,\tau}} \psi(q(s),p(s))\Phi_x^\tau(dq\,dp) := \\ &\lim_{n\to\infty} C_n \int_{Q_n^\tau\times P_n} \psi(q(s),p(s)) \exp\left[i\int_0^t p(s)\dot{q}(s)\,ds\right] \nu_n(dq)\,\nu_n(dp), \end{aligned}$$

where $(C_n)^{-1} = \int_{Q_n^\tau \times P_n} \exp\left[i \int_0^t p(s)\dot{q}(s)\, ds\right] \nu_n(dq)\, \nu_n(dp)$ and ν_n is Lebesgue measure.

Remark 4.9. The construction described above is a modification and an extension of constructions introduced in [41], [33], [13] and [7].

5 Phase space Feynman path integrals for evolution semigroups generated by τ-quatization of some Lévy-Khintchine type Hamilton functions for $\tau \in [0, 1]$

In this section we show that Hamiltonian Feynman formulae obtained above can be interpreted as some phase space Feynman path integrals. Therefore, the corresponding phase space Feynman path integrals do exist and coincide with some functional integrals with respect to countably additive (mainly probability) measures associated with some Feller type semigroups.

Theorem 5.1. *Let $\tau \in [0, 1]$ and $H = h$, where h is given by the formula* (3.1)*. Let Assumption 3.1(i),(iii) and Assumption 3.3 fulfil. Let $(T_t^\tau)_{t\geqslant 0}$ be the semigroup generated by the closure $(L^\tau, \mathrm{Dom}(L^\tau))$ of a ψDO $(\widehat{H}_\tau, C_c^\infty(\mathbb{R}^d))$ with the τ-symbol H. Then the Hamiltonian Feynman formula* (3.16) *can be interpreted as a phase space Feynman path integral*

$$T_t^\tau \varphi(x) = \int\limits_{E_t^{x,\tau}} e^{-\int\limits_0^t H(q(s),p(s))ds} \varphi(q(0)) \Phi_x^\tau(dqdp). \tag{5.1}$$

Proof. Indeed, using Definition 4.7 we get

$$(T_t^\tau \varphi)(x) = \lim_{n\to\infty} (2\pi)^{-dn} \int\limits_{(\mathbb{R}^d)^{2n}} \exp\left(i \sum_{k=1}^n p_k \cdot (q_{k+1} - q_k)\right) \times$$
$$\times \exp\left(-\frac{t}{n} \sum_{k=1}^n H(\tau q_{k+1} + (1-\tau) q_k, p_k)\right) \varphi(q_1)\, dq_1\, dp_1 \cdots dq_n\, dp_n =$$
$$= \int\limits_{E_t^{x,\tau}} e^{-\int\limits_0^t H(q(s),p(s))ds} \varphi(q(0)) \Phi_x^\tau(dqdp),$$

where, in each pre-limit expression in the Hamilton formula, we have $q_{n+1} := x$; moreover, $q_1 = q(t/n - 0) \to q(0)$ as $n \to \infty$ due to the definition of the space $Q_t^{x,\tau}$

and

$$\frac{t}{n}\sum_{k=1}^{n} H(\tau q_{k+1}+(1-\tau)q_k, p_k) = \sum_{k=1}^{n} H(q(t_k), p(t_k))(t_k - t_{k-1}) \to \int_0^t H(q(s), p(s))ds$$

since any path $(q(s), p(s)) \in E_t^{x,\tau}$ is piecewise continuous and has a finite number of jumps, H is a continuous function. □

Remark 5.2. Note, that the integrand in the Feynman path integral (5.1) is the same for all $\tau \in [0,1]$, only the space $E_t^{x,\tau}$, defining the sequential pseudomeasure Φ_x^τ is different; this space contains those paths $q(s)$ which are "τ-continuous" (see the formula (4.2) as the definition).

Remark 5.3. Under Assumptions 3.1(i),(ii), Assumption 3.3 and due to the formula (3.5) (i.e. Lemma 2.1 in [7]) we see that $\widehat{H}_\tau\varphi(q) = \widehat{H}_1^\tau(q,D)\varphi(q)$, where $\widehat{H}_1^\tau(q,D)\varphi(q)$ is a pseudo-differential operator with 1-symbol

$$H^\tau(q,p) = c_\tau(q) + ib_\tau(q)\cdot p + p\cdot A(q)p$$

and we have

$$b_\tau(q) = b(q) - 2(1-\tau)\operatorname{div} A(q),$$
$$c_\tau(q) = c(q) + (1-\tau)\operatorname{div} b(q) - (1-\tau)^2\operatorname{tr}(\operatorname{Hess} A(q)).$$

Therefore, due to Theorem 3.9 and Theorem 5.1 there is a kind of "change of variable formula" for $H(q,p) = c(q) + ib(q)\cdot p + p\cdot A(q)p$:

$$T_t^\tau\varphi(x) = \int_{E_t^{x,1}} \exp\left[-\int_0^t H^\tau(q(s),p(s))ds\right]\varphi(q(0))\,\Phi_x^1(dq\,dp) =$$

$$= \int_{E_t^{x,\tau}} \exp\left[-\int_0^t H(q(s),p(s))ds\right]\varphi(q(0))\Phi_x^\tau(dqdp).$$

i.e.,

$$T_t^\tau\varphi(x)=\int\limits_{E_t^x}\exp\left[-\int\limits_0^t p(s)\cdot A(q(s))p(s)\,ds\right]$$

$$\times\exp\left[-\int\limits_0^t\Big[c(q(s))+(1-\tau)\operatorname{div}b(q(s))-(1-\tau)^2\operatorname{tr}(\operatorname{Hess}A(q(s)))\Big]ds\right]\times$$

$$\times\exp\left[-i\int\limits_0^t\Big[(b(q(s))-2(1-\tau)\operatorname{div}A(q(s)))\cdot p(s)\Big]ds\right]\varphi(q(0))\Phi_x^1(dq\,dp)=$$

$$=\int\limits_{E_t^{x,\tau}}e^{-\int\limits_0^t p(s)\cdot A(q(s))p(s)\,ds-i\int\limits_0^t b(q(s))\cdot p(s)\,ds-\int\limits_0^t c(q(s))ds}\varphi(q(0))\Phi_x^\tau(dqdp).$$

Due to Definition 4.7 the Hamiltonian Feynman formula (3.21) can be interpreted as a Hamiltonian Feynman path integral with respect to the Feynman pseudomeasure Φ_x^1. Therefore the following theorem is true (cf. [7]).

Theorem 5.4. *Let $\tau=1$. Under Assumptions 3.1(i),(iii),(iv) and Assumption 3.3 the semigroup $(T_t^\tau)_{t\geqslant 0}$, generated by the closure $(L^\tau,\operatorname{Dom}(L^\tau))$ of a ψDO $(\widehat{H}_\tau, C_c^\infty(\mathbb{R}^d))$ with the τ-symbol H as in (3.3) can be represented by a Hamiltonian Feynman path integral with respect to the Feynman pseudomeasure Φ_x^1:*

$$T_t\varphi(x)=\int\limits_{E_t^{x,1}}e^{-\int\limits_0^t H(q(s),p(s))ds}\varphi(q(0))\,\Phi_x^1(dq\,dp).\tag{5.2}$$

Remark 5.5. Note, that our definition of the space $E_t^{x,1}$ differs from the definition of the space E_t^x in the paper [7]. Hence the corresponding sequential Feynman pseudomeasures used above and in [7] are also different. This leads to different Feynman path integrals representing the considered evolution semigroup (cf. with Theorem 3.5 in [7]).

Remark 5.6. Lagrangian Feynman formula (3.15) (resp. (3.20)) actually provides a tool to compute Feynman path integral (5.1) (resp. (5.2)). The limits in both Lagrangian formulas coincide with functional integrals over some probability measures.

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