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TOWERS OF GL(r)-TYPE OF MODULAR CURVES

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Abstract. We construct Galois covers $X^{r,k}(N)$ over $\mathbb{P}^1/\mathbb{F}_q(T)$ with Galois groups close to $\mathrm{GL}(r, \mathbb{F}_q[T]/(N))$ ($r \geq 3$) and rationality and ramification properties similar to those of classical modular curves $X(N)$ over $\mathbb{P}^1/\mathbb{Q}$. As application we find plenty of good towers (with limsup $\frac{\text{number of rational points}}{\text{genus}} > 0$) of curves over the field $\mathbb{F}_{q^r}$ with q^r elements.

0. Introduction

(0.1) We let $\mathbb{F} = \mathbb{F}_q$ be the finite field with q elements, of characteristic p, $A = \mathbb{F}[T]$ (resp. $K = \mathbb{F}(T)$) the ring of polynomials (resp. field of rational functions) in an indeterminate T, and C_∞ the completed algebraic closure of $K_\infty = \mathbb{F}((1/T))$. The theory of Drinfeld A-modules of rank $r = 2$ provides modular curves $X(N)/C_\infty$ $(0 \neq N \in A)$, which are ramified coverings of the projective j-line $X(1) = \mathbb{P}^1/C_\infty$ with arithmetic-geometric properties similar to those of classical elliptic modular curves (see, e.g. [13]). Here j is the j-invariant which classifies rank-two Drinfeld A-modules, and $X(N)$ is connected, smooth, projective and galois over $\mathbb{P}^1$ with group

(0.1.1) $$G(N) = \{\gamma \in \mathrm{GL}(2, A/(N)) \mid \det(\gamma) \in \mathbb{F}^*\}/Z,$$

where Z is the group of $\mathbb{F}$-valued scalar matrices. In case N is nonconstant, the covering $X(N) \longrightarrow X(1) = \mathbb{P}^1$ is

- tamely ramified at *elliptic points* (those above $j = 0$) with cyclic ramification group of order $q + 1$;
- modestly ramified (second ramification groups are trivial) at *cusps* (points above $j = \infty$)

and unramified elsewhere.

Such curves, along with their relatives $X_0(N), X_1(N), \ldots$ (quotients of $X(N)$ by subgroups of $G(N)$; the family of these are labelled as *Drinfeld modular curves*) play a prominent role in the arithmetic of K, e.g., abelian and non-abelian class field theory [7], [6], Drinfeld modular forms [22], [11], [13], uniformization of elliptic curves [17].

(0.2) A more specific application is to the construction of good towers of algebraic curves over finite fields, that is series $(X_n/\mathbb{F})_{n \in \mathbb{N}}$ of curves whose ratio

$$\frac{|\{\mathbb{F}\text{−rational points of } X_n\}|}{\text{genus of } X_n}$$

has a positive limes superior. Using the known invariants and rationality properties of Hecke type Drinfeld modular curves $X_0(N)$, it has been shown in [18] that each tower $(X_0(N_n))_{n\in\mathbb{N}}$ with $\deg N_n \longrightarrow \infty$ and $(N_n, T) = 1$ after reduction modulo T becomes good in the above sense over the quadratic extension $\mathbb{F}^{(2)}$ of $\mathbb{F} = \mathbb{F}_T = A/(T)$, and even *optimal*, which means that the lim sup of the above ratio realizes the theoretical Drinfeld-Vladut upper bound $q - 1$. Similar properties were known earlier for some classical or Shimura modular curves [26], [34] or for Drinfeld modular curves $X_0(N)$ with prime conductor N [35]. (In fact, the relevant data already occur in [10].) The above results make use of the fact that "supersingular" points of classical or Drinfeld modular curves of Hecke type are rational over the quadratic extension $\mathbb{F}^{(2)}$ of their (relative) prime field [5], [18], which produces enough $\mathbb{F}^{(2)}$-rational points.

Unluckily the supersingular argument doesn't turn over neither to the prime field $\mathbb{F}$ itself nor to its extensions $\mathbb{F}^{(r)}$ of degree $r > 2$ (only the case of odd r is of interest; otherwise the above construction already yields optimal towers), due to the lack of modular curves adapted to the situation.

(0.3) Now it is known [15] that supersingular Drinfeld A-modules of arbitrary rank $r \geq 2$ may be defined over the r-th extension $\mathbb{F}^{(r)}$ of the prime field $\mathbb{F}$ and so define $\mathbb{F}^{(r)}$-rational points on suitable moduli schemes. Therefore it seems natural to construct convenient curves (with controlled genus) through the given supersingular points on the moduli scheme $M^r(N)$ of Drinfeld modules of rank $r \geq 3$. I learned of this brilliant simple idea through a talk by Alp Bassa in February 2013 [2]; apparently it motivated the authors of [3] to their construction of a tower of curves $(X_n^{s,t})_{n\in\mathbb{N}}$ (where $(s,t) = 1$ and $r = s + t$), which satisfies

(0.3.1)

$$\limsup_{n\to\infty} \frac{|\{\mathbb{F}^{(r)}\text{−rational points on } X_n^{s,t}\}|}{\text{genus } (X_n^{s,t})} \geq \frac{2(q^s-1)(q^t-1)}{q^s+q^t-2} =: C_q(s,t).$$

(Actually the paper [3] works purely algebraically with function fields instead of curves, and so its terminology differs strongly from ours.) The constant $C_q(s,t)$ collapses to the Drinfeld-Vladut bound $q-1$ for $s = t = 1$, is fairly close to the Drinfeld-Vladut bound $q^{r/2} - 1$ for $r = 3$, $(s,t) = (1,2)$ or $(2,1)$ and small values of q, and becomes worse for larger r or q. It is the best currently known lower estimate of the left hand side of (0.3.1) valid for arbitrary finite fields whose order is an r-th power.

One of the merits of Bassa-Beelen-Garcia-Stichtenoth's result is its explicit nature: The function fields of the curves $X_n^{s,t}$ are described recursively through relatively simple equations, which allows to estimate step by step the relevant quantities (ramification and genus, number of rational points). On the other hand, the conceptual meaning of their construction is not easy to understand. Also, it fails to "explain" the constant $C_q(s,t)$, and it doesn't allow deformations or the study of single curves.

(0.4) For this reason, but also for intrinsic interest motivated from the arithmetic of the field K, it is desirable to dispose of towers of modular-like curves $X^r(N)$ ($r \geq 2$) similar to those of (0.1), but with Galois group of GL(r)-type instead of GL(2)-type as in (0.1.1).

In the present work we construct such towers. Viz, we study certain curves $X^{r,k}(N)$ on the moduli schemes $M^r(N)$ of Drinfeld A-modules of rank r supplied with a structure of level $N \in A$, which turn out to have the wanted properties. Here $r \geq 3$, $1 \leq k < r$ is coprime with r, and $X^{r,k}(N)$ parametrizes the so-called (r,k)-sparse Drinfeld modules all of whose coefficients vanish except for the k-th. (The construction also works for $r = 2$, but then collapses to the now well-known theory of Drinfeld modular curves as referred to in (0.1).)

Analytically these curves are quotients by the congruence subgroup $\Gamma(N)$ of $\Gamma = \mathrm{GL}(r,A)$ of the one-dimensional subspace $\Omega^{r,k}$ of the Drinfeld symmetric space Ω^r. We show (precise definitions given later):

Theorem A: *Let (r,k) be integers with $1 \leq k < r$ and $(r,k) = 1$, $N \in A$, and let $X^{r,k}(N)$ be the modular curve that parametrizes k-sparse Drinfeld A-modules of rank r with a structure of level N.*

(i) *$X^{r,k}(N)$ is a connected, smooth, projective curve over C_∞ and is in fact defined over the finite abelian extension $K_+(N)$ of K.*

(ii) *$X^{r,k}(N)$ is a ramified Galois cover of $X^{r,k}(1) = \mathbb{P}^1$, the projective line with coordinate j, with group*

$$G(N) = \{\gamma \in \mathrm{GL}(r, A/(N) \mid \det(\gamma) \in \mathbb{F}^*\}/Z$$

(Z = group of $\mathbb{F}$-valued scalar matrices).

(iii) *Suppose that N is non-constant. Then $X^{r,k}(N)$ is ramified*
- *above $j = 0$ (i.e., at "elliptic points") with cyclic ramification groups of order $(q^r - 1)/(q-1)$; in particular, the ramification above $j = 0$ is tame;*
- *above $j = \infty$ (i.e, at "cusps"), with ramification groups conjugate to*

$$G_\infty(N) = \left\{ \left(\begin{array}{c|c} \alpha & \beta \\ \hline 0 & \delta \end{array} \right) \right\} /Z;$$

here α (resp. δ) runs through a fixed cyclic subgroup of order $(q^{r-k}-1)$ of $\mathrm{GL}(r-k,\mathbb{F})$ (resp. of order (q^k-1) of $\mathrm{GL}(k,\mathbb{F})$), 0 is the zero $k\times(r-k)$-matrix and β an arbitrary $(r-k)\times k$-matrix over $A/(N)$;

– *unramified elsewhere.*

(iv) *The ramification at cusps is modest, that is, with trivial second ramification groups.*

These properties suffice to determine the genus $g^{r,k}(N)$ of $X^{r,k}(N)$ by means of the Riemann-Hurwitz formula.

Now let $P(N) \hookrightarrow G(N)$ be the parabolic subgroup represented by matrices $\left(\begin{array}{c|c}\alpha & \beta\\ \hline 0 & \delta\end{array}\right)$ with an $(r-k)\times k$ block structure as in $G_\infty(N)$, but without restriction on the matrices α and δ, and define $X_0^{r,k}(N)$ as the quotient of $X^{r,k}(N)$ by $P(N)$. It is these curves which give rise to good towers; their properties are largely derived from those of $X^{r,k}(N)$.

Theorem B: *The curve $X_0^{r,k}(N)$ is defined over K and has good reduction at places $\mathfrak{p}$ of A with $\mathfrak{p} \not| N$.*

Again, knowing $g^{r,k}(N)$, the genus $g_0^{r,k}(N)$ of $X_0^{r,k}(N)$ may be calculated by applying the Riemann-Hurwitz formula to the cover $X^{r,k}(N)\longrightarrow X_0^{r,k}(N)$. But this is extremely laborious, as it requires to determine the sizes $|P(N)\cap{}^{\xi}G_\infty(N)|$ for all ξ from a system of representatives of $G(N)/G_\infty(N)$, and involves many case distinctions depending on the prime factorization of N. The interested reader may receive an impression in the relatively simple case of $r=2$, which is carried out in [10].

Here we give precise formulas (see Theorem 11.13) for $g_0^{r,r-1}(N))$, where $N=T^n$, and describe the asymptotic behavior of $g_0^{r,k}(T^n)$ for general k (Proposition 12.2). As $X_0^{r,k}(T^n)$ has good reduction at the place $\mathfrak{p}=(T-1)$, with many supersingular points, there result good towers of curves over $\mathbb{F}^{(r)}=\mathbb{F}_{\mathfrak{p}}^{(r)}$ with the same lower bound as in (0.3.1). More precisely, we get:

Theorem C: *Let $(N_n)_{n\in\mathbb{N}}$ be any series in A with $\deg N_n\longrightarrow\infty$ and N_n coprime with $\mathfrak{p}=(T-1)$, and suppose that $1\le k<r\ge 3$, $(r,k)=1$. Let $\overline{X}_0(N_n)$ be the reduction of $X_0^{r,k}(N_n)$ at the place $\mathfrak{p}$. Then $\overline{X}_0(N_n)$ is defined over $\mathbb{F}_{\mathfrak{p}}=\mathbb{F}$, and for the number of $\mathbb{F}^{(r)}$-rational points the estimate*

$$(0.4.1)\quad \limsup_{n\to\infty}\frac{|\{\mathbb{F}^{(r)}\text{−rational points on }\overline{X}_0(N_n)\}|}{g_0^{r,k}(N_n)}\ge C_q(r-k,k)$$

holds.

Presumably (working out the precise connection would require additional efforts) our tower $(X_0^{r,k}(N_n))_{n\in\mathbb{N}}$ reduced $(\mathrm{mod}\,\mathfrak{p})$ with $N_n = T^n$ is the same as Bassa-Beelen-Garcia-Stichtenoth's, so we present the geometry behind the algebraic construction of [3]. (By the way: this is why we chose here $\mathfrak{p} = (T-1)$ instead of $\mathfrak{p} = (T)$ as in (0.2); of course the choice of such a place of degree one is irrelevant.) But, in contrast with that paper, our approach is not limited to parameters $N = T^n$; in fact (0.4.1) holds for all towers or merely series of curves $X_0^{r,k}(N_n)$ as specified in Theorem C, and it leads to exact formulas for the genera, not only estimates. The constant $C_q(r-k,k)$ results in a natural way from the Riemann-Hurwitz formula and group theoretical data, but independently of the choice of our tower or series of curves.

The preparations and different proof steps for Theorem A form the contents of the first eight sections of the paper. Crucial points are the construction (4.12) of the fundamental domain $\mathcal{F} = \mathcal{F}^{r,k}$ for the group $\Gamma = \mathrm{GL}(r,A)$ on the subspace $\Omega^{r,k}$ of the Drinfeld symmetric space Ω^r and the elaboration of its connectedness properties (Theorems 6.9 and 8.2) and its behavior at the boundary (Theorem 8.4). Theorems B and C are then relatively easy consequences.

We now briefly describe the organization of the paper.

In **Section 1** we give the necessary background on Drinfeld modules that allows us to introduce and fix notation for the sequel. The contents is entirely known and essentially due to Drinfeld [7].

In **Section 2** we present identities for certain quantities related to finite or discrete infinite lattices in C_∞. All of this except for Proposition 2.9 is known.

In **Section 3** we discuss successive minimum bases for A-lattices in C_∞, which allows to define the fundamental domain $\mathfrak{G}^r \subset \Omega^r$ for Γ. This is a well-known topic in classical lattices, and has been introduced to the function field context by Taguchi [33].

The concept of (r,k)-sparse Drinfeld A-modules is introduced and discussed in **Section 4**. They define the Γ-stable analytic subspace $\Omega^{r,k}$ of dimension one of the Drinfeld symmetric space Ω^r. Proposition 4.9 allows to define the fundamental domain $\mathcal{F} = \mathcal{F}^{r,k}$ on $\Omega^{r,k}$, which is smooth by Theorem 4.15. The modular forms g and Δ and the invariant function j on $\Omega^{r,k}$ are introduced, as well as the spread function $\boldsymbol{\omega} \longmapsto s(\boldsymbol{\omega})$, some analogue of the (logarithm of the) "imaginary part" function on the complex upper half-plane.

In **Section 5** we define the uniformizer at infinity $\boldsymbol{\omega} \longmapsto t(\boldsymbol{\omega})$ and calculate the growth of t, Δ and j in terms of $s(\boldsymbol{\omega})$. This is needed to understand the geometry of $\mathcal{F}$ at infinity.

The connectedness properties of the analytic spaces $\mathcal{F}$ and $\Omega^{r,k}$ are established in **Section 6**. We use in a crucial way the surjective map $j : \mathcal{F} \longrightarrow \mathbb{A}^1(C_\infty) = C_\infty$ and a result of Kantor [27] about maximal subgroups of $\mathrm{GL}(r, \mathbb{F})$.

The affine modular curves $Y^{r,k}(N)$ and their compactifications $X^{r,k}(N)$ may be defined in two different ways:

- analytically as $\Gamma(N) \setminus \Omega^{r,k}$, where $\Gamma(N)$ is the congruence subgroup of level N of Γ;
- algebraically as the vanishing locus of the $r-2$ modular forms $g_1, \ldots, \hat{g}_k, \ldots g_{r-1}$ on the moduli scheme $M^r(N)$ of rank-r Drinfeld A-modules.

This is done, along with the necessary discussion, in **Section 7**.

In **Section 8** we show that the fundamental domain $\mathcal{F}$ determines a unique cusp, labelled "∞", of $X^{r,k}(N)$, with fixed group $G_\infty(N)$ and ramification filtration as described in Theorem A (Theorems 8.2 and 8.4). Again the argument is analytic: the growth of the uniformizer $t(\boldsymbol{\omega})$ plays an essential role.

It is now quite easy fo find the genera $g^{r,k}(N)$: see **Section 9**, Proposition 9.3.

Section 10 introduces the curves $X_0^{r,k}(N)$ and describes their rationality properties, which yields Theorem B.

Their genera $g_0^{r,k}(N)$ are calculated in **Section 11**, Theorem 11.13, in the special case where $k = r-1$ and $N = T^n$. Determining the precise value of $g_0^{r,k}(T^n)$ for general k is a considerable problem, due to the more complex geometry of the Grassmann manifold $P(N) \setminus G(N)$, and is left for further work.

The preceding suffices to establish Theorem C, which is done in **Section 12**.

The final **Section 13** is devoted to concluding remarks, comments, and suggestions for further research.

As has already been mentioned, the largest part of our proofs is related to analytic geometry over the complete valued field C_∞, and the relevant language is used without further definition or explication. This is why the reader should have some familiarity with rigid analytic geometry as presented e.g. in [20] or [8]; for an extended background see the book [4]. In view of the GAGA theorems valid in our framework (see [28], [29]), we usually don't distinguish between algebraic and analytic data/properties of projective varieties over C_∞, and we often use the underlying point set as a synonym for an analytic space.

I am grateful to Alp Bassa for pointing out the leading idea of [3], and to him as well as to Andreas Schweizer and Mihran Papikian for instructive conversations and communication about related topics. Thanks are also due to Gunter Malle for his hint to the paper [27].

Notation

The following notation is used throughout the paper. Necessary explanation is given in the text.

$\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $|X|$ = cardinality of the finite set X
$a|b$, $(a, b) = 1$, $\gcd(a, b)$ means a divides b, a and b are coprime, the greatest common divisor of $a, b \in \mathbb{N}$, respectively

$R^{m\times n}$	$=$	set of $m \times n$-matrices with coefficients in R
$\mathbb{F}$	$=$	finite field $\mathbb{F}_q$ with q elements, of characteristic p, with algebraic closure $\overline{\mathbb{F}}$
$\mathbb{F}^{(n)}$	$=$	the unique extension of $\mathbb{F}$ of degree n in $\overline{\mathbb{F}}$
A	$=$	$\mathbb{F}[T]$ the polynomial ring, with field of fractions $K = \mathbb{F}(T)$ and degree function $\deg: A \longrightarrow \mathbb{N}_0 \cup \{-\infty\}$
K_∞	$=$	$\mathbb{F}((1/T))$ completion of K at its infinite place, with absolute value "$\mid . \mid$", normalized by $\lvert T\rvert = q$
C_∞	$=$	completed algebraic closure of K_∞ with its extension of "$\mid . \mid$", with ring of integers $O_\infty = \{z \in C_\infty \mid \lvert z\rvert \leq 1\}$ and residue class field $\overline{\mathbb{F}}$. We write $z \longmapsto \overline{z}$ for the reduction map $O_\infty \longrightarrow \overline{\mathbb{F}}$ and $z \equiv w (\mathrm{mod}\ \infty)$ for $\overline{z} = \overline{w}$.
B_x	$=$	$\{z \in C_\infty \mid \lvert z\rvert \leq x\}$, $B_x^- = \{z \in C_\infty \mid \lvert z\rvert < x\}$ the "closed" and "open" ball of radius $x \in q^{\mathbb{Q}}$ around 0, thus $O_\infty = B_1$ with maximal ideal B_1^-

For a field L containing $\mathbb{F}$:

$L\{\tau\}$	$=$	twisted polynomial ring in the non-commutative variable τ with commutation rule $\tau a = a^q \tau$ $(a \in L)$
$L\{\{\tau\}\}$	$=$	formal power series ring in τ

Through $\tau^i \longmapsto X^{q^i}$, $L\{\tau\}$ is naturally identified with $\mathrm{End}_{L,\mathbb{F}}(\mathbb{G}_a) = \{\sum a_i X^{q^i} \mid a_i \in L\}$, the ring of $\mathbb{F}$-linear endomorphisms of the additive group scheme $\mathbb{G}_a/L$ (multiplication = composition).

M^r, $M^r(N)$	$=$	moduli schemes for Drinfeld A-modules of rank r,
$\phi_T = \sum_{0\leq i\leq r} g_i \tau^i$	$=$	$\sum_{0\leq i\leq r} g_i X^{q^i}$ the T-operator polynomial of the rank-r Drinfeld module ϕ, with kernel ${}_T\phi$

For $\mathbb{F}$- or A-lattices Λ in L, the exponential and logarithm functions are

$$\begin{array}{lcl}
e_\Lambda & = & \sum \alpha_i\tau^i,\ \log_\Lambda = \sum \beta_i \tau^i \in L\{\{\tau\}\},\ \text{with } \alpha_i=\alpha_i(\Lambda),\ \beta_i=\beta_i(\Lambda) \\
\Omega^r & = & \{\boldsymbol{\omega}=(\omega_1:\ldots:\omega_r)\in\mathbb{P}^{r-1}(C_\infty)\mid \omega_1,\ldots,\omega_r\ K_\infty\text{-linearly independent}\} \\
\Omega^r(L) & = & \text{set of } L\text{-rational points of } \Omega^r \\
\Omega^{r,k} & = & \{\boldsymbol{\omega}\in\Omega^r \mid \boldsymbol{\omega} \text{ is } k\text{-sparse}\} \\
\Lambda_{\boldsymbol{\omega}} & = & \sum\limits_{1\le i\le r} A\omega_i \text{ the lattice defined by } \boldsymbol{\omega}=(\omega_1,\ldots,\omega_r) \\
\phi^{\boldsymbol{\omega}} & = & \phi^{\Lambda_{\boldsymbol{\omega}}} \text{ the Drinfeld module associated with } \Lambda_{\boldsymbol{\omega}}
\end{array}$$

For fixed $2\le r\in\mathbb{N}$ (in most cases, we even assume $r\ge 3$):

$$\begin{array}{lcl}
\Gamma & = & \mathrm{GL}(r,A) \text{ with principal congruence subgroup } \Gamma(N),\ N\in A \\
Z & \cong & \mathbb{F}^* \text{ the group of } \mathbb{F}\text{-valued scalar matrices in } \Gamma \\
G(N) & = & \Gamma/\Gamma(N)\cdot Z \hookrightarrow \tilde{G}(N)=\mathrm{GL}(r,A/N)/Z
\end{array}$$

(We write A/N for the residue class ring $A/(N)$ and assume N non-constant where needed. The group Z is simultaneously regarded as a subgroup of Γ and of $\mathrm{GL}(r,A/N)$. Elements of Γ/Z or $\tilde{G}(N)$, i.e., classes of matrices modulo Z, are written as matrices.)
$\Gamma_0,\Gamma_s,\Gamma_\infty$ subgroups of Γ, with unipotent radicals $\Gamma_s^u,\Gamma_\infty^u$ (see (5.11))
$G_\infty(N)=$ image of Γ_∞ in $G(N)$

Let $1\le k<r\in\mathbb{N}$ be fixed with $(r,k)=1$.
We often specify an $r\times r$-matrix through a *block structure* $\left(\begin{array}{c|c}\alpha&\beta\\\hline 0&\delta\end{array}\right)$,
where α is an $(r-k)\times(r-k)$-matrix, β an $(r-k)\times k$-matrix, etc.
$\mathfrak{G}=\mathfrak{G}^r=$ fundamental domain of Γ on Ω^r
$\boldsymbol{\chi}$ resp. $\boldsymbol{\psi}$ elements of $\Omega^{r-k}(\mathbb{F}^{(r-k)})$ resp. $\Omega^k(\mathbb{F}^{(k)})$ fixed once for all,
$G_{\boldsymbol{\chi}}\subset\mathrm{GL}(r-k,\mathbb{F})$ resp. $G_{\boldsymbol{\psi}}\subset\mathrm{GL}(k,\mathbb{F})$ the associated Cartan subgroups
$\mathfrak{G}^{r,k}=\mathfrak{G}\cap\Omega^{r,k}$, $\mathcal{F}=\mathcal{F}^{r,k}\subset\mathfrak{G}^{r,k}$ the fundamental domain of Γ on $\Omega^{r,k}$
$s:\ \mathcal{F}\longrightarrow\mathbb{Q}$ the spread function, $s(\boldsymbol{\omega})=\log_q|\frac{\omega_1}{\omega_r}|$
$\mathcal{F}_{\le s},\mathcal{F}_s,\mathcal{F}_{\ge s},\mathcal{F}_+,\mathcal{F}_!$ subsets of $\mathcal{F}$ defined through conditions on $\boldsymbol{\omega}\in\mathcal{F}$

If the group G acts (from the left) on the set X and $Y\subset X$, $G\setminus X$ denotes the orbit space and $G\setminus Y$ the image of Y in $G\setminus X$. As usual, $G_x\subset G$ is the stabilizer of $x\in X$.
The primed sum $\sum'$ or product $\prod'$ is the sum or product over the non-zero elements of the corresponding index set.

1. Background on Drinfeld modules [7], [6].

(1.1) An A-field L is a field provided with a ring homomorphism $\gamma:A\longrightarrow L$. Its *$A$-characteristic* is the prime ideal $\mathrm{char}_A(L)=\ker(\gamma)=\mathfrak{p}$ if the latter is different from $\{0\}$, otherwise we write $\mathrm{char}_A(L)=\infty$. Thus L is an extension either of the finite L-field $\mathbb{F}_{\mathfrak{p}}=A/\mathfrak{p}$ or of

$K = \mathrm{Quot}(A)$.

As usual, we identify $\mathrm{End}_{L,\mathbb{F}}(\mathbb{G}_a)$, the ring of $\mathbb{F}$-linear endomorphisms of the additive group scheme $\mathbb{G}_a$ over L, with the ring of $\mathbb{F}$-linear polynomials $\sum a_i X^{q^i}$ with coefficients in L, where multiplication is defined by the insertion of polynomials. Thus $\mathrm{End}_{L,\mathbb{F}}(\mathbb{G}_a)$ is the non-commutative polynomial ring $L\{\tau\}$ over L subject to the commutation rule $\tau a = a^q\tau$ for $a \in L$, where τ corresponds to the polynomial X^q.

A *Drinfeld-A-module of rank r* over L is the structure of A-module on $\mathbb{G}_a/L$ given by an $\mathbb{F}$-algebra homomorphism

$$\begin{aligned} \phi:\ \ A &\longrightarrow L\{\tau\}, \\ a &\longmapsto \phi_a \end{aligned}$$

where $\phi_T = \gamma(T)\tau^0 + g_1\tau + \cdots + g_r\tau^r = \gamma(T)X + \cdots + g_rX^{q^r}$ and $g_r \neq 0$. Hence specifying ϕ is the same as specifying $\phi_T \in L\{\tau\}$ of the shape above. The $g_i = g_i(\phi)$ are the *coefficients* of ϕ. We write $\Delta = \Delta(\phi)$ for the leading coefficient $g_r(\phi)$, which is called the *discriminant* of ϕ. A *structure of level* N on ϕ (where $0 \neq N \in A$ is coprime with $\mathrm{char}_A(L)$) is an isomorphism of A-modules $(A/N)^r \xrightarrow{\cong} {}_N\phi(L) = \{x \in L \mid \phi_N(x) = 0\}$. The definition extends to levels N possibly divisible by $\mathrm{char}_A(L)$, see [6] I, Sect. 6.

(1.2) Let $\Lambda \subset C_\infty$ be an A-lattice of rank r (an r-lattice for short), i.e., Λ is a free A-submodule of rank r and *discrete*, which means it has finite intersection with each ball B_x in C_∞. The exponential function of Λ is

$$e_\Lambda(z) = z\prod_{\lambda\in\Lambda}{}'(1 - z/\lambda), \tag{1.2.1}$$

which may be written as $\sum_{n\geq 0}\alpha_n z^{q^n}$ with suitable $\alpha_n \in C_\infty$. ($\prod'$ indicates the product over the non-zero $\lambda \in \Lambda$.) A well-known construction similar to the Weierstrass uniformization of elliptic curves allows to attach to Λ a Drinfeld A-module $\phi = \phi^\Lambda$ of rank r over C_∞, characterized by the functional equation

$$\phi_a(e_\Lambda(z)) = e_\Lambda(az) \tag{1.2.2}$$

for $a \in A$. This establishes a canonical and functorial $1-1$ correspondence $\Lambda \leftrightarrow \phi^\Lambda$ between r-lattices in C_∞ and rank-r Drinfeld modules over C_∞. The homothetic latttice $c\Lambda$, where $c \in C_\infty^*$,corresponds to the isomorphic Drinfeld module ϕ' given by $\phi'_T = T + c^{1-q}g_1\tau + c^{1-q^2}g_2\tau^2 + \ldots + c^{1-q^r}g_r\tau^r$. Moreover, the A-module ${}_N\phi(C_\infty)$ of N-torsion points of ϕ is canonically isomorphic with $N^{-1}\Lambda/\Lambda$.

(1.3) Let M^r/A (resp. $M^r(N)/A$) be the moduli scheme for rank-r Drinfeld modules (resp. for rank-r Drinfeld modules supplied with a level-N structure), see [7], [6]. Hence, for algebraically closed A-fields

L, the set of L-points $M^r(L)$ of M^r corresponds to the set of isomorphism classes of Drinfeld modules of rank r over L. As two such, ϕ and ϕ', given by

$$\phi_T = \gamma(T) + \sum_{1\leq i\leq r} g_i\tau^i,\ \phi'_T = \gamma(T) + \sum_{1\leq i\leq r} g'_i\tau^i,$$

are isomorphic if and only if there exists $c \in L^*$ such that $g'_i = c^{1-q^i}g_i$ for all i, we get after base extension with L

$$M^r \underset{A}{\times} L = \operatorname{Spec} L[g_1,\dots,g_{r-1},\Delta,\Delta^{-1}]/L^*, \tag{1.3.1}$$

where the multiplicative group $L^* = \mathbb{G}_m(L)$ acts through $c{*}g_i = c^{1-g^i}g_i$ ($c \in L^*$). Here we regard the g_i (where $g_r = \Delta$) as indeterminate coefficients of the universal Drinfeld module. For $N|N' \in A$, there are natural forgetful morphisms

$$M^r(N')\longrightarrow M^r(N)\longrightarrow M^r(1) = M^r.$$

(1.4) In order to describe $M^r \times C_\infty$, we consider the *Drinfeld symmetric space*

$$\Omega^r = \mathbb{P}^{r-1}(C_\infty)\setminus \bigcup_{H\ a\ K_\infty-rational\ hyperplane} H. \tag{1.4.1}$$

Thus $\boldsymbol{\omega} = (\omega_1,\dots,\omega_r) \in C_\infty^r$ defines a point $(\omega_1:\dots:\omega_r)$ in Ω^r if and only if the r entries ω_i are K_∞-linearly independent, that is, if and only if $\Lambda_{\boldsymbol{\omega}} = \sum_{1\leq i\leq r} A\omega_i$ is an r-lattice in C_∞. Note that the group $\mathrm{GL}(r, K_\infty)$ acts through matrix multiplication from the left on C_∞^r and $\mathbb{P}^{r-1}(C_\infty)$ and stabilizes Ω^r. Let $\tilde{\Omega}^r \subset C_\infty^r$ be the inverse image of Ω^r. The function

$$\begin{array}{rccl} g_i: & \tilde{\Omega}^r & \longrightarrow & C_\infty \\ & \boldsymbol{\omega} & \longmapsto & g_i(\boldsymbol{\omega}) = i\text{-th coefficient of the Drinfeld} \\ & & & \text{module } \phi^{\boldsymbol{\omega}} \text{ associated with } \Lambda_{\boldsymbol{\omega}}\ (1\leq i\leq r) \end{array}$$

is invariant under $\Gamma := \mathrm{GL}(r, A)$ and of weight $q^i - 1$, i.e.,

$$g_i(c\boldsymbol{\omega}) = c^{1-q^i}g_i(\boldsymbol{\omega}) \quad (c \in C_\infty^*). \tag{1.4.2}$$

Thus g_i is a *modular form* of weight q^i-1 for Γ, and the discriminant $\Delta = g_r$ is a nowhere vanishing modular form of weight $q^r - 1$.

If not specified otherwise, we normalize homogeneous coordinates $(\omega_1 : \dots : \omega_r)$ on Ω^r such that $\omega_r = 1$; then $\Lambda_{\boldsymbol{\omega}}$ is well-defined for $\boldsymbol{\omega} \in \Omega^r$, and we may regard the "forms" g_i as functions on Ω^r. The weight condition (1.4.2) together with the Γ-invariance translates to

$$g_i(\gamma\boldsymbol{\omega}) = \alpha(\gamma,\boldsymbol{\omega})^{q^i-1}g_i(\boldsymbol{\omega}) \tag{1.4.3}$$

with the automorphy factor $\alpha(\gamma,\boldsymbol{\omega}) := \sum_{1\leq j\leq r}\gamma_{r,j}\omega_j$ ($\gamma = (\gamma_{i,j}) \in \Gamma$, $\boldsymbol{\omega} = (\omega_1 : \dots : \omega_r = 1) \in \Omega^r$). The considerations of (1.2) and (1.3) give

$$\begin{array}{rcl} \Gamma \setminus \Omega^r & \xrightarrow{\cong} & M^r(C_\infty) \\ \boldsymbol{\omega} & \longmapsto & (g_1(\boldsymbol{\omega}) : \ldots : g_{r-1}(\boldsymbol{\omega}) : \Delta(\boldsymbol{\omega})), \end{array} \tag{1.4.4}$$

where the right hand side is the class of $(g_1, \ldots, g_{r-1}, \Delta)$ modulo the action of C_∞^* described in (1.3.1).

The Drinfeld space Ω^r has a natural structure of rigid analytic space over C_∞ (in fact, defined over K_∞), described in [7], [6] and [31], and compatible with the structures above: The g_i are holomorphic functions on Ω^r, and (1.4.4) is in fact an isomorphism of the quotient analytic space $\Gamma \setminus \Omega^r$ with the analytification of the variety $M^r \times C_\infty$.

(1.5) The C_∞-points of $M^r(N)$ may be analytically described in a similar way, with some complications arising from rationality questions.

Given a non-constant $N \in A$, let $K_+(N)$ be the "maximal real subfield of the N-th cyclotomic field extension $K(N)$ of K" (see [25] or [23] Sect. 3). This is the maximal abelian extension of conductor N of K in which the infinite place of K splits completely, and may be constructed via the N-torsion points of the Carlitz module (the rank-1 Drinfeld module ϕ defined by $\phi_T = T + \tau = TX + X^q$). We have

(1.5.1) $K \hookrightarrow K_+(N) \hookrightarrow K(N)$ with Galois groups

$$\mathrm{Gal}(K(N)|K) = (A/N)^*,\ \mathrm{Gal}(K_+(N)|K) = (A/N)^*/\mathbb{F}^*,$$

and precisely those primes $\mathfrak{p}$ of A that divide N are ramified in $K(N)|K$ (except for $q = 2$, $\deg N = 1$, in which case $K(N) = K$).

Let further $Z \cong \mathbb{F}^*$ be the subgroup of $\mathbb{F}$-valued scalar matrices in $\mathrm{GL}(r, A/N)$ and $\tilde{G}(N) = \mathrm{GL}(r, A/N)/Z$. Then $\tilde{G}(N)$ acts on $M^r(N)$, and in fact $\tilde{G}(N) \setminus M^r(N) \xrightarrow{\cong} M^r$. While $M^r(N)$ is an affine integral normal scheme flat over A ([7] Sect. 5), its base extension $M^r(N) \underset{A}{\times} C_\infty$ splits into a finite number of irreducible connected components, due to the fact that A is not integrally closed in (the function field of) $M^r(N)$. More precisely, associating with each rank-r Drinfeld A-module (with a structure of level N) its determinant module [1] yields a faithfully flat morphism "det": $M^r(N) \longrightarrow M^1(N) = \mathrm{Spec}\, B_+(N)$, where $B_+(N)$ is the ring of A-integers in $K_+(N)$, which is also the integral closure of A in $M^r(N)$. Therefore

$$M^r(N) \underset{A}{\times} C_\infty = \coprod_\sigma M^r(N) \underset{B_+(N),\sigma}{\times} C_\infty \tag{1.5.2}$$

splits into $[K_+(N) : K]$ many components naturally parametrized by the K-embeddings σ of $K_+(N)$ into C_∞. In classical terms, the variety $M^r(N)$ is "defined over $K_+(N)$". Let $G(N)$ be the subgroup $\{\gamma \in$

$\mathrm{GL}(r, A/N) \mid \det(\gamma) \in \mathbb{F}^*\}/Z$ of $\tilde{G}(N)$. The determinant map induces an isomorphism

$$\tilde{G}(N)/G(N) \xrightarrow{\cong} (A/N)^*/\mathbb{F}^* = \mathrm{Gal}(K_+(N)|K),$$

and $G(N)$ stabilizes the components in (1.5.2). Now fix one such embedding $\sigma = \mathrm{id}$, and regard

$$Y^r(N) := M^r(N) \underset{B_+(N)}{\times} C_\infty$$

via σ as a C_∞-variety. We find for the associated analytic spaces:

$$\Gamma(N) \setminus \Omega^r \xrightarrow{\cong} Y^r(N)(C_\infty) \tag{1.5.3}$$

with the congruence subgroup $\Gamma(N) = \{\gamma \in \Gamma \mid \gamma \equiv 1 \,(\mathrm{mod}\, N)\}$ of $\Gamma = \mathrm{GL}(r, A)$. Here to each $\boldsymbol{\omega} \in \Omega^r$ we attach the Drinfeld module $\phi^{\boldsymbol{\omega}}$ with lattice $\Lambda_{\boldsymbol{\omega}}$ and its canonical level-N structure, which up to isomorphism depends only on $\boldsymbol{\omega}$ modulo $\Gamma(N)$. Note that (1.4.4) and (1.5.3) are compatible, i.e., we have the commutative diagram

$$\begin{array}{ccc} \Gamma(N) \setminus \Omega^r & \xrightarrow{\cong} & Y^r(N) \\ \downarrow & & \downarrow \\ \Gamma \setminus \Omega^r & \xrightarrow{\cong} & Y^r = Y^r(1) \end{array} \tag{1.5.4}$$

with horizontal maps stemming from (1.5.3) and (1.4.4) and the forgetful vertical maps (we briefly write Y^r for $Y^r(C_\infty)$, etc.). Moreover, as Γ acts effectively via its quotient Γ/Z by the scalar subgroup $Z \cong \mathbb{F}^*$, the covering group in the left hand side is $\Gamma/\Gamma(N) \cdot Z$, which agrees through the natural map with the group $G(N)$ acting on the right hand side.

(1.6) Similar compatibilities exist for dividing out arbitrary congruence subgroups Γ' of Γ, i.e., groups that satisfy $\Gamma(N) \subset \Gamma' \subset \Gamma$ for some N. We finally note the following observation. Given any subgroup H of $\tilde{G}(N)$, the quotient scheme $M_H := H \setminus M^r(N)$ is a moduli scheme for Drinfeld modules with a certain level structure depending on H. The integral closure B_H of A in M_H (or the algebraic closure L_H of K in the function field of M_H) will be galois over A with group $\mathrm{Gal}(B_H|A) = \mathrm{Gal}(L_H|K) = (A/N)^*/\mathbb{F}^* \cdot \det(H)$. In particular we will have $B_H = A$, $L_H = K$ if $\det(H) = (A/N)^*/\mathbb{F}^*$, which holds for H's coming from Borel subgroups or parabolic subgroups of $\mathrm{GL}(r, A/N)$.

The interested reader will find more details about the analytic description of the moduli schemes $M^r(N)$ in [7], [6] and [13].

2. Some identities of lattice invariants [14], [19]

In the whole section, L is some field containing $\mathbb{F} = \mathbb{F}_q$. Recall that $L\{\tau\}$ denotes the non-commutative polynomial ring subject to $\tau a = a^q\tau$ for $a \in L$, identified via $\tau^i \leftrightarrow X^{q^i}$ with the ring of $\mathbb{F}$-linear polynomials with respect to the insertion of polynomials as multiplication.

Similarly, $L\{\{\tau\}\}$ is the ring of formal power series in τ. Given a finite-dimensional $\mathbb{F}$-subspace Λ of L (an $\mathbb{F}$-lattice for short), we put

$$e_\Lambda(X) = X\prod_{\lambda\in\Lambda}{}'(1 - X/\lambda) = \sum_{0\le i\le \dim\,\Lambda} \alpha_i X^{q^i} = \sum \alpha_i\tau^i. \tag{2.1}$$

It has a formal inverse in $L\{\{\tau\}\}$

$$\log_\Lambda(X) = \sum_{i\ge 0} \beta_i X^{q^i} = \sum \beta_i\tau^i, \tag{2.2}$$

whose coefficients β_i may be recursively determined through

$$\log_\Lambda \circ e_\Lambda = e_\Lambda \circ \log_\Lambda = 1,$$

i.e.,

$$\alpha_0 = \beta_0 = 1 \text{ and } \sum_{i+j=k} \beta_i\alpha_j^{q^i} = \sum_{i+j=k}\alpha_i\beta_j^{q^i} = 0 \text{ for } k > 0.$$

Furthermore, we have the important identity

$$\beta_i = -E_{q^i-1}(\Lambda) \tag{2.3}$$

with the *Eisenstein series*

$$E_k(\Lambda) = \sum_{\lambda\in\Lambda}{}'\lambda^{-k}.$$

(As usual, $\sum'$ denotes the sum over the non-zero elements of Λ, and $E_0(\Lambda) = -1$.) Similar formulas hold if $L = C_\infty$ and Λ is a possibly infinite-dimensional but discrete $\mathbb{F}$-subspace (finite intersection with each ball in C_∞). In this case, e_Λ is an infinite product with an expansion (2.1) as a power series in τ which defines an entire function on C_∞, and (2.3) holds with the corresponding infinite Eisenstein series.

Next, suppose that $\Lambda \in C_\infty$ is the A-lattice corresponding to the Drinfeld A-module ϕ of rank r, where

$$\phi_T = T + g_1\tau + \cdots + g_{r-1}\tau^{r-1} + g_r\tau^r,\ g_r \ne 0,\ g_0 = T.$$

Then

$$TE_{q^k-1} = \sum_{i+j=k} E_{q^i-1}g_j^{q^i} \tag{2.4}$$

holds for $k \ge 0$ (see, e.g., [14] 2.10).

2.5 Corollary. *Let $1 \le k \le r$ be given. Then the following are equivalent*

(a) $E_{q^i-1}(\Lambda) = 0$ *for $1 \le i < k$;*

(b) $g_i = 0$ *for* $1 \le i < k$.

In this case, $g_k = (T^{q^k} - T)E_{q^k-1}(\Lambda)$. $\square$

In the rest of this section we consider $\mathbb{F}$-lattices in its algebraic closure $\overline{\mathbb{F}}$. For $n \in \mathbb{N}$ let $\mathbb{F}^{(n)}$ denote the unique field extension of $\mathbb{F}$ of degree n in $\overline{\mathbb{F}}$. We further put

(2.6)

$$\Omega^r(\overline{\mathbb{F}}) = \mathbb{P}^{r-1}(\overline{\mathbb{F}}) \smallsetminus \bigcup_{H \text{ an } \mathbb{F}-\text{rational hyperplane}} H$$

$$= \{\boldsymbol{\omega} = (\omega_1 : \ldots : \omega_r) \in \mathbb{P}^{r-1}(\overline{\mathbb{F}}) \mid \text{the } \omega_i \text{ are } \mathbb{F}\text{-linearly independent}\}$$

and $\Omega^r(\mathbb{F}^{(n)})$ for the subset of points described by homogeneous coordinates in $\mathbb{F}^{(n)}$. (This is compatible with (1.4.1) and our general notation when considering $\overline{\mathbb{F}}$ as a subfield of C_∞.) Then, as is easily seen:

(2.7) $\Omega^r(\mathbb{F}^{(n)}) = \emptyset$ if $n < r$, and $\Omega^r(\mathbb{F}^{(r)})$ consists of one orbit under the action of $\mathrm{GL}(r, \mathbb{F})$ on $\Omega^r(\overline{\mathbb{F}})$.

As $\Omega^r(\overline{\mathbb{F}})$ classifies $\mathbb{F}$-lattices Λ of dimension r in $\overline{\mathbb{F}}$ supplied with an ordered basis, and up to $\overline{\mathbb{F}}$ -homotheties, we may regard the coefficients $\alpha_i(\Lambda)$, $\beta_i(\Lambda)$ of e_Λ, and the Eisenstein series $E_k(\Lambda)$, as "modular forms" on $\Omega^r(\overline{\mathbb{F}})$, and even as $\overline{\mathbb{F}}$-valued functions on $\Omega^r(\overline{\mathbb{F}})$ after normalizing $\omega_r = 1$ for $\boldsymbol{\omega} = (\omega_1 : \ldots : \omega_r) \in \Omega^r(\overline{\mathbb{F}})$ as in (1.4). We cite from [19] 1.13:

(2.8) $E_{q^i-1}(\boldsymbol{\omega}) = 0$ for $\boldsymbol{\omega} \in \Omega^r(\mathbb{F}^{(n)})$ and $n - r < i < n$.

Even more precisely:

2.9 Proposition. *Suppose* $r \le n$. *Then* $\Omega^r(\mathbb{F}^{(n)})$ *is the precise common vanishing locus of the* E_{q^i-1} $(n - r < i < n)$ *in* $\Omega^r(\overline{\mathbb{F}})$.

Proof. First note that the exponential functions e_Λ of lattices Λ homothetic with $\mathbb{F}^{(n)}$ are those of shape $1 - \alpha\tau^n$, where $0 \ne \alpha \in \overline{\mathbb{F}}$. Therefore, in view of (2.8), (2.3) and the computation rules in $\overline{\mathbb{F}}\{\tau\}$, we must show the following:

Let $\Lambda \subset \overline{\mathbb{F}}$ be an $\mathbb{F}$-lattice of dimension r that satiesfies $\beta_i(\Lambda) = 0$ for $n - r < i < n$. Then its exponential function e_Λ is a right divisor of some $1 - \alpha\tau^n$ in $\overline{\mathbb{F}}\{\tau\}$.

That is, given the vanishing of the β_i, we must show the existence of $u \in \overline{\mathbb{F}}\{\tau\}$ such that $u \circ e_\Lambda = 1 - \alpha\tau^n$, which is equivalent with $u = (1 - \alpha\tau^n)\log_\Lambda$ in $\overline{F}\{\{\tau\}\}$. Now the right hand side is

$$(1-\alpha\tau^n)(1+\beta_1\tau+\cdots+\beta_{n-r}\tau^{n-r}+\beta_n\tau^n+\cdots) \equiv 1+\beta_1\tau+\cdots+\beta_{n-r}\tau^{n-r} \pmod{\tau^n}.$$

That is, if such an u exists, it satisfies $u \equiv \log_\Lambda \pmod{\tau^n}$. Therefore we define u as the unique polynomial in $\overline{\mathbb{F}}\{\tau\}$ of degree $\deg_\tau u < n$ such that $u \equiv \log_\Lambda \pmod{\tau^n}$. The above shows $\deg_\tau u \leq n - r$ and $u \circ e_\Lambda \equiv 1 \pmod{\tau^n}$. As $\deg_\tau(u \circ e_\Lambda) = \deg_\tau(u) + \deg_\tau(e_\Lambda) \leq (n-r) + r = n$, we must have equality and $u \circ e_\Lambda = 1 - \alpha\tau^n$ for some $\alpha \neq 0$. $\square$

2.10 Corollary. *For $\boldsymbol{\omega} \in \Omega^r(\overline{\mathbb{F}})$ and the attached lattice $\Lambda = \Lambda_{\boldsymbol{\omega}} = \sum_{1 \leq i \leq r} \mathbb{F}_{\omega_i}$, the following are equivalent:*

(a) *$E_{q^i-1}(\boldsymbol{\omega}) = 0$ for all $0 < i < r$;*

(b) *$e_\Lambda(X) = X + \alpha X^{q^r}$ for some $0 \neq \alpha \in \overline{\mathbb{F}}$;*

(c) *$\boldsymbol{\omega} \in \Omega^r(\mathbb{F}^{(r)})$.* $\square$

(2.11) Next, choose an ordered $\mathbb{F}$-basis $\{\omega_1, \ldots, \omega_r\}$ of $\mathbb{F}^{(r)}$. The matrix representation of multiplication on $\mathbb{F}^{(r)}$ yields an embedding of $(\mathbb{F}^{(r)})^*$ into $\mathrm{GL}(r, \mathbb{F})$, the image of which we call a *Cartan subgroup* $\mathrm{Car} = \mathrm{Car}(r)$ of $\mathrm{GL}(r, \mathbb{F})$. Actually, Car agrees with the stabilizer $\mathrm{GL}(r, \mathbb{F})_{\boldsymbol{\omega}}$ of $\boldsymbol{\omega} = (\omega_1 : \ldots : \omega_r) \in \Omega^r(\mathbb{F}^{(r)})$. As $\mathrm{GL}(r, \mathbb{F})$ acts transitively on the set of bases as above, it also acts transitively on $\Omega^r(\mathbb{F}^{(r)})$ (as has already been stated in (2.7)).

3. Successive minimum bases.

An ordered A-basis $\{\omega_1, \ldots, \omega_r\}$ of the A-lattice Λ in C_∞ is a *successive minimum basis* (SMB) if for each $1 \leq i \leq r$ the vector ω_i has minimal length $|\omega_i|$ among all $\omega \in \Lambda$ not in the span $\sum_{1 \leq j < i} A\omega_j$ of $\{\omega_1, \ldots, \omega_{i-1}\}$.

3.1 Proposition.

(i) *Each A-lattice Λ possesses a SMB.*

(ii) *The sequence $|\omega_1| \leq |\omega_2| \leq \cdots \leq |\omega_r|$ of lengths of a SMB of Λ (the* successive minima *of Λ) is an invariant of Λ, i.e., is independent of the choice of the SMB $\{\omega_1, \ldots, \omega_r\}$.*

(iii) *Given a SMB $\{\omega_1, \ldots, \omega_r\}$ and $\omega = \sum a_i\omega_i$ with $a_i \in K_\infty$, we have*

$$|\omega| = \max_{1 \leq i \leq r} |a_i\omega_i|.$$

Proof. Easy and omitted. $\square$

3.2 Corollary. *Let $\mathcal{G} = \mathcal{G}^r$ be the subset $\{\boldsymbol{\omega} = (\omega_1 : \ldots : \omega_r) \in \Omega^r \mid \{\omega_r, \ldots, \omega_1\}$ is a SMB of $\sum A\omega_i\}$. Then each $\boldsymbol{\omega} \in \Omega^r$ is Γ-equivalent with at least one and at most finitely many elements of $\mathcal{G}$.*

Proof. The fact that $\Gamma\mathcal{G} = \Omega^r$ is immediate from (3.1) (i). Given $\boldsymbol{\omega} \in \mathcal{G}$, the condition $\gamma\boldsymbol{\omega} \in \mathcal{G}$ together with (3.1) (iii) implies that the coefficients of γ are bounded, and so γ runs through a finite subset of Γ. $\square$

3.3 Remarks. (i) In view of (3.2) we call elements $\boldsymbol{\omega} \in \mathcal{G}$ reduced and $\mathcal{G}$ a *fundamental domain* for the action of Γ on Ω^r. As uniqueness of the representative fails, this is much weaker than the classical notion of fundamental domain, but is the best we can achieve in our non-archimedean framework.

(ii) The condition defining $\mathcal{G}$ is equivalent with the validity of inequalities for absolute values of linear combinations of the coordinates ω_i of $\boldsymbol{\omega}$. Therefore $\mathcal{G}$ is an admissible open analytic subspace of Ω^r, which among others implies that the local rings of $\boldsymbol{\omega}$ in $\mathcal{G}$ and in Ω^r agree.

(iii)The reverse order for SMB in the definition of $\mathcal{G}$ is chosen in order to be compatible with the classical setting of modular forms and intuition derived therefrom.

In what follows, we let Λ be an A-lattice provided with a SMB $\{\omega_1, \dots, \omega_r\}$ and with attached Drinfeld module $\phi = \phi^\Lambda$, $\phi_T = T + \sum_{1 \leq i \leq r} g_i \tau^i$. The elements

$$\mu_i = e_\Lambda(\frac{\omega_i}{T}) \tag{3.3.1}$$

are T-division points (i.e., $\phi_T(\mu_i) = 0$) and in fact form an $\mathbb{F}$-basis of the A/T-module (i.e., $\mathbb{F}$-vector space) ${}_T\phi$ of T-division points of ϕ.

3.4 Lemma. *For $1 \leq i < r$ we have $|\mu_i| \leq |\mu_{i+1}|$, with equality if and only if $|\omega_i| = |\omega_{i+1}|$.*

Proof.

$$|\mu_i| = |e_\Lambda(\frac{\omega_i}{T})| = |\frac{\omega_i}{T}| \prod_{\substack{\lambda \in \Lambda \\ |T\lambda| \leq |\omega_i|}}{}' \; |1 - \frac{\omega_i}{T\lambda}|$$

and similarly

$$|\mu_{i+1}| = |\frac{\omega_{i+1}}{T}| \prod_{\substack{\lambda \in \Lambda \\ |T\lambda| \leq |\omega_i|}}{}' \; |1 - \frac{\omega_{i+1}}{T\lambda}| \prod_{\substack{\lambda \in \Lambda \\ |\omega_i| < |\lambda| \leq |\omega_{i+1}|}} |1 - \frac{\omega_{i+1}}{T\lambda}|.$$

If $|T\lambda| < |\omega_i|$ then $|1 - \frac{\omega_i}{T\lambda}| = |\frac{\omega_i}{T\lambda}| \leq |\frac{\omega_{i+1}}{T\lambda}| = |1 - \frac{\omega_{i+1}}{T\lambda}|$. If $|T\lambda| = |\omega_i|$ then $|1 - \frac{\omega_i}{T\lambda}| \leq 1$ and in fact $|1 - \frac{\omega_i}{T\lambda}| = 1$, as follows from $\lambda \in \sum_{j<i} A\omega_j$ and the SMB property (3.1) (iii).

Furthermore, $|1 - \frac{\omega_{i+1}}{T\lambda}| \geq 1$ for $|\omega_i| < |T\lambda| \leq |\omega_{i+1}|$. Therefore all the factors of $|\mu_{i+1}|$ are larger or equal to the corresponding factors of $|\mu_i|$, which shows $|\mu_i| \leq |\mu_{i+1}|$, with equality if and only if $|\omega_i| = |\omega_{i+1}|$. □

Next, we write ϕ_T as an $\mathbb{F}$-linear polynomial

$$\phi_T(X) = TX + \sum_{1 \leq i \leq r} g_i X^{q^i} = \Delta \cdot \prod_{z \in {}_\phi T} (X - z), \text{ where } \Delta = g_r \neq 0.$$

3.5 Proposition. *Suppose that $g_i = 0$ for some $i < r$. Then $|\mu_i| = |\mu_{i+1}|$.*

Proof. By the preceding lemma, $\{\mu_1, \dots, \mu_r\}$ is a SMB of the $\mathbb{F}$-vector space ${}_\phi T$. Thus, by means of the sequence $|\mu_1| \leq \cdots \leq |\mu_r|$, we can count the number of elements of ${}_T\phi$ of a given length. Now consider the Newton polygon NP of $\phi_T(X)$ as described e.g. in [30] II Sect. 6. According to the shape of $\phi_T(X)$, it is composed of pieces of horizontal width $q-1, q(q-1), \ldots, q^{r-1}(q-1)$, and by the central property of Newton polygons (*loc. cit.* Theorem 6.2), we have: Two neighboring pieces of sizes $q^{i-1}(q-1), q^i(q-1)$ belong to the same segment of NP (i.e., they don't form a break point of NP) if and only if $|\mu_i| = |\mu_{i+1}|$. But the former property holds if $g_i = 0$. $\square$

3.6 Corollary. *If Λ and ϕ are such that $g_i = 0$ for some $1 \leq i < r$ and $\{\omega_1, \ldots, \omega_r\}$ is a SMB of Λ, then $|\omega_i| = |\omega_{i+1}|$.*

Proof. This follows from (3.4) and (3.5). $\square$

4. Sparse Drinfeld modules.

4.1 Definition. *The Drinfeld A-module ϕ of rank r is k-*sparse *(or briefly (r,k)-*sparse*) if all the coefficients g_j of ϕ_T vanish except for $j = k$ or r, that is, if*

$$\phi_T = T + g_k\tau^k + g_r\tau^r =: T + g\tau^k + \Delta\tau^r.$$

If $\phi = \phi^\Lambda$ with some r-lattice $\Lambda = \Lambda_{\boldsymbol{\omega}}$, we also call Λ and $\boldsymbol{\omega}$ k-sparse or (r,k)-sparse.

Remark. The k-sparsity condition may be rephrased as the following condition on Eisenstein series, as is easy to verify from (2.4):

(4.1.1) Some $\boldsymbol{\omega} \in \Omega^r$ (or its attached data $\Lambda_{\boldsymbol{\omega}}$, $\phi^{\boldsymbol{\omega}}$) is k-sparse if and only if for all i, $1 \leq i < r$, the following holds:

$$E_{q^i-1}(\boldsymbol{\omega}) = \begin{matrix} 0 & \text{if } i \not\equiv 0 \,(\mathrm{mod}\, k) \\ c_{i,k}(E_{q^k-1}(\boldsymbol{\omega}))^{(q^i-1)/(q^k-1)}, & \text{if } i \equiv \,(\mathrm{mod}\, k) \end{matrix}$$

with some constant $c_{i,k} \in K$ of absolute value 1 (which may easily worked out if necessary).

As k-sparsity is defined through the vanishing of the $r-2$ modular $g_1, \ldots, \hat{g}_k, \ldots, g_{r-1}$ (as usual, $(\hat{*})$ means omitting $(*)$), the set

$$\Omega^{r,k} := \{\boldsymbol{\omega} \in \Omega^r \mid \boldsymbol{\omega} \text{ is } k\text{-sparse}\} \tag{4.2}$$

is a closed analytic subspace of Ω^r. The next sections will be devoted to studying its properties. First note:

(4.3) $\Omega^{r,k}$ is stable under the action of Γ on Ω^r, and the restrictions g of g_k and Δ of g_r to $\Omega^{r,k}$ are holomorphic functions on $\Omega^{r,k}$, where Δ nowhere vanishes.

We used the following result from elementary number theory, the proof of which we leave as an exercise.

4.4 Lemma. *Let q be a prime power. For natural numbers k, r we have: $(q^k-1)|(q^r-1) \Leftrightarrow k|r$ and $\gcd(q^k-1, q^r-1) = q^d-1$, where $d = \gcd(k,r)$.* □

We now make the assumption, valid for the rest of the paper.

4.5 Assumption. The rank r is larger or equal to 3, and k satisfies $1 \leq k < r$ and $(k,r) = 1$.

As (r,k)-sparsity is an empty condition and everything that follows is tautological or well-known for $r = 1, 2$, the requirement on r is no restriction. Also the condition on k is merely a normalization of our situation: If otherwise $d = \gcd(k,r) > 1$, we could replace $\mathbb{F} = \mathbb{F}_q$ by $\mathbb{F}' = \mathbb{F}^{(d)}$ and A by $A' = \mathbb{F}^{(d)}[T]$; then an (r,k)-sparse Drinfeld A-module was in fact an (r',k')-sparse Drinfeld A'-module, where $k' = k/d$, $r' = r/d$, etc.

Under this assumption, we have:

4.6 Proposition. *The function $j := \frac{g^{(q^r-1)/(q-1)}}{\Delta^{(q^k-1)/(q-1)}}$ is invariant under Γ and identifies the quotient analytic space $\Gamma \setminus \Omega^{r,k}$ with $\mathbb{A}^1(C_\infty) = C_\infty$.*

Proof. Both the numerator and the denominator have weight $(q^k-1)(q^r-1)/(q-1)$, hence j is invariant under Γ. From (4.5) and (4.4) we see that the exponents $(q^r-1)/(q-1)$ and $(q^k-1)/(q-1)$ are coprime, which implies that two (r,k)-sparse Drinfeld modules $\phi^{\boldsymbol{\omega}}$, $\phi^{\boldsymbol{\omega}'}$ are isomorphic if and only if $j(\boldsymbol{\omega}) = j'(\boldsymbol{\omega}')$. This shows bijectivity; it is obvious that j is analytic. □

(4.7) Let now $\boldsymbol{\omega} = (\omega_1 : \ldots : \omega_r) \in \mathcal{G}^{r,k} := \mathcal{G}^r \cap \Omega^{r,k}$ be reduced, with lattice $\Lambda_{\boldsymbol{\omega}} = \sum_{1 \leq i \leq r} A\omega_i$ and Drinfeld module $\phi = \phi^{\boldsymbol{\omega}}$, and $\mu_i = e_\Lambda(\frac{\omega_i}{T})$ as in (3.3.1). From (3.6) we see that

$$(4.7.1) \qquad |\omega_1| = \ldots = |\omega_{r-k}| \geq |w_{r-k+1}| = \ldots = |\omega_r|,$$

i.e., there is at most one proper inequality. We define the *spread* of $\boldsymbol{\omega}$ by

$$(4.7.2) \qquad s(\boldsymbol{\omega}) := \log_q |\frac{\omega_1}{\omega_r}|.$$

As we shall see, it is an analogue of the function $\log \circ \operatorname{im}$ (“im” = imaginary part) on the standard fundamental domain for $\mathrm{SL}(2,\mathbb{Z})$.

(4.8) It seems natural to use $\mathcal{G}^{r,k}$ as a fundamental domain for Γ on $\Omega^{r,k}$, but we can do better. We first define two maps $R : \mathcal{G}^{r,k} \longrightarrow \Omega^k(\overline{\mathbb{F}})$ and $S : \mathcal{G}^{r,k} \longrightarrow \Omega^{r-k}(\overline{\mathbb{F}})$ as follows (note that $\Omega^1 = \mathbb{P}^0$ consists of one point):

$$(4.8.1)\qquad \begin{array}{rccl} R: & \mathcal{G}^{r,k} & \longrightarrow & \Omega^k(\overline{\mathbb{F}}), \\ & \boldsymbol{\omega} & \longmapsto & \text{reduction of } (\omega_{r-k+1}:\ldots:\omega_r) \text{ modulo } \infty \end{array}$$

which is well-defined in view of (4.7.1), as the residue class field of the ring of integers O_∞ of C_∞ is naturally identified with $\overline{\mathbb{F}}$. The definition of S is more involved. From (4.7.1) and (3.4) we see that $|\mu_1| = \ldots = |\mu_{r-k}|$, so we may consider the reduction $\overline{\boldsymbol{\mu}}$ of $(\mu_1 : \ldots : \mu_{r-k})$ in $\mathbb{P}^{r-k-1}(\overline{\mathbb{F}})$. In the next proposition we show that $\overline{\boldsymbol{\mu}}$ actually lies in $\Omega^{r-k}(\overline{\mathbb{F}})$, and S will be the mapping

$$(4.8.2)\qquad \begin{array}{rccl} S: & \mathcal{G}^{r,k} & \longrightarrow & \Omega^{r-k}(\overline{\mathbb{F}}), \\ & \boldsymbol{\omega} & \longmapsto & \overline{\boldsymbol{\mu}} \end{array}.$$

4.9 Proposition.

(i) *S takes its values in $\Omega^{r-k}(\overline{\mathbb{F}})$.*

Suppose that $s(\boldsymbol{\omega}) > 0$. Then

(ii) *$R(\boldsymbol{\omega}) \in \Omega^k(\mathbb{F}^{(k)})$ and*

(iii) *$S(\boldsymbol{\omega}) \in \Omega^{r-k}(\mathbb{F}^{(r-k)})$.*

Proof. We first notice that changing homogeneous coordinates $(\omega_1 : \ldots : \omega_r) \rightsquigarrow (c\omega_1 : \ldots : c\omega_r)$ with $c \in C_\infty^*$ gives $(\mu_1 : \ldots : \mu_r) \rightsquigarrow (c\mu_1 : \ldots : c\mu_r)$, and doesn't affect the assertions. We may therefore conveniently normalize the coordinates.

(ii) Suppose $\omega_r = 1$ (our usual assumption), $\boldsymbol{\omega} = (\omega_1 : \ldots : 1)$. From (2.5) we have that $E_{q^i-1}(\boldsymbol{\omega}) = 0$ for $1 \leq i < k$. But, due to (3.1) (iii),

$$E_{q^i-1}(\boldsymbol{\omega}) = \sum_{a_1,\ldots,a_r \in A}{}'(a_1\omega_1 + \cdots + a_r\omega_r)^{1-q^i} \equiv \sum_{a_{r-k+1},\ldots,a_r \in \mathbb{F}}{}'\ (a_{r-k+1}\omega_{r-k+1} + \cdots + a_r\omega_r)^{1-q^i},$$

where "$\equiv$" denotes congruence modulo ∞. (The omitted terms are less than 1 in absolute value.) Therefore $R(\boldsymbol{\omega})$ lies in fact in $\Omega^k(\mathbb{F}^{(k)})$ by Corollary 2.10.

For the proof of (i) and (iii) we assume $\boldsymbol{\omega}$ normalized such that $\mu_1 = 1$, so

$$(*)\qquad 1 = |\mu_1| = \ldots = |\mu_{r-k}| \geq |\mu_{r-k+1}| = \ldots = |\mu_r|.$$

Let $\phi = \phi^{\boldsymbol{\omega}}$ with T-division polynomial $\phi_T(X)$, and write

(4.9.1)

$$f(X) = \Delta^{-1}\phi_T(X) = \frac{T}{\Delta}X + \frac{q}{\Delta}X^{q^k} + X^{q^r} = \prod(X - \mu) \in O_\infty[X],$$

where μ runs through ${}_T\phi$, the $\mathbb{F}$-span of $\{\mu_1, \ldots, \mu_r\}$.

If equality holds in $(*)$, then $|T/\Delta| = 1$, the reduction $\overline{f}$ of f in $\overline{\mathbb{F}}[X]$ is separable, with the $\mathbb{F}$-linear combinations of the $\overline{\mu}_i$ (= reduction of

μ_i) as its zeroes. Therefore the $\overline{\mu}_i$ $(1 \leq i \leq r)$ are $\mathbb{F}$-linearly independent, which shows (i) in this case. If we have strict inequality in $(*)$ then $\overline{f}(X) = \overline{(g/\Delta)}X^{q^k} + X^{q^r} = h(X)^{q^k}$ with the $\mathbb{F}$-linear polynomial $h(X) = \overline{(g/\Delta)}^{q^{-k}}X + X^{q^{r-k}} \in \overline{\mathbb{F}}[X]$. The $\overline{\mu}_i$ $(1 \leq i \leq r-k)$ form a basis of the $\mathbb{F}$-vector space $\mathrm{Ker}(h) = \sum_{1 \leq i \leq r-k} \mathbb{F}\overline{\mu}_i$, thus (i), and $\overline{\boldsymbol{\mu}} = (\overline{\mu}_1 : \ldots : \overline{\mu}_{r-k})$ even belongs to $\Omega^{r-k}(\mathbb{F}^{(r-k)})$ by Corollary 2.10. $\square$

(4.10) Let $\gamma \in \Gamma = \mathrm{GL}(r, A)$ be such that $\gamma(\mathcal{G}^{r,k}) \cap \mathcal{G}^{r,k} \neq \emptyset$ and $\boldsymbol{\omega} \in \gamma(\mathcal{G}^{r,k}) \cap \mathcal{G}^{r,k}$.

Suppose that $\boxed{s(\boldsymbol{\omega}) > 0}$. As the application of γ to $\boldsymbol{\omega}$ corresponds to a base change in $\Lambda_{\boldsymbol{\omega}} = \sum_{1 \leq i \leq r} A\omega_i$, which maps the short vectors ω_i $(r-k < i \leq r)$ to $\mathbb{F}$-linear combinations of these (and similar restrictions for the images of the long vectors ω_i $(1 \leq i \leq r-k)$), we find that γ has the shape

$$\left(\begin{array}{c|c} \alpha & \beta \\ \hline 0 & \delta \end{array}\right) \tag{4.10.1}$$

where $\alpha \in \mathrm{GL}(r, \mathbb{F})$, $\delta \in \mathrm{GL}(k, \mathbb{F})$, 0 is the $k \times (r-k)$ zero matrix, and the $(r-k) \times k$-matrix β has entries $b \in A$ such that $\deg b \leq s(\boldsymbol{\omega})$.

If otherwise $\boxed{s(\boldsymbol{\omega}) = 0}$ then γ belongs to $\mathrm{GL}(r, \mathbb{F})$.

On the other hand, if these conditions on γ are satisfied, then γ maps $\boldsymbol{\omega} \in \mathcal{G}^{r,k}$ to another element $\boldsymbol{\omega}' \in \mathcal{G}^{r,k}$.

(4.11) Decompose $\mathcal{G}^{r,k}$ into subspaces $\mathcal{G}_0$ and $\mathcal{G}_+$, where

$$\mathcal{G}_0 = \{\boldsymbol{\omega} \in \mathcal{G}^{r,k} \mid s(\boldsymbol{\omega}) = 0\},\ \mathcal{G}_+ = \{\boldsymbol{\omega} \in \mathcal{G}^{r,k} \mid s(\boldsymbol{\omega}) > 0\}.$$

From (4.9), the restriction

$$(S \times R)_+ :\ \mathcal{G}_+ \longrightarrow \Omega^{r-k}(\mathbb{F}^{(r-k)}) \times \Omega^k(\mathbb{F}^{(k)}) \tag{4.11.1}$$

of $S \times R$ to $\mathcal{G}_+$ is well-defined and compatible with the natural actions (see (4.10.1)) of the subgroup $H := \mathrm{GL}(r-k, \mathbb{F}) \times \mathrm{GL}(k, \mathbb{F})$ of Γ. As H acts transitively on the target, $(S \times R)_+$ is onto. Furthermore, it is constant on connected components of the analytic space $\mathcal{G}_+$, since its image is finite. Therefore the fibers of $(S \times R)_+$ are open subspaces of $\mathcal{G}_+$.

Now choose and fix points

$\boldsymbol{\chi} = (\chi_1 : \ldots : \chi_{r-k}) \in \Omega^{r-k}(\mathbb{F}^{(r-k)})$ and $\boldsymbol{\psi} = (\psi_{r-k+1} : \ldots : \psi_r) \in \Omega^k(\mathbb{F}^{(k)})$,

and set

(4.12)

$$\begin{aligned} \mathcal{F}^{r,k} &:= \{\boldsymbol{\omega} \in \mathcal{G}^{r,k} \mid s(\boldsymbol{\omega}) = 0 \text{ or } (S \times R)_+(\boldsymbol{\omega}) = (\boldsymbol{\chi}, \boldsymbol{\psi})\} \\ &= \{\boldsymbol{\omega} \in \Omega^{r,k} \mid \boldsymbol{\omega} \text{ reduced and, in case } s(\boldsymbol{\omega}) > 0,\ S(\boldsymbol{\omega}) = \boldsymbol{\chi},\ R(\boldsymbol{\omega}) = \boldsymbol{\psi}\}. \end{aligned}$$

Then $\mathcal{F}^{r,k}$ is a non-empty open subspace of $\mathcal{G}^{r,k}$, and each element $\boldsymbol{\omega}$ of $\Omega^{r,k}$ is Γ-equivalent with at least one and at most finitely many elements of $\mathcal{F}^{r,k}$, that is, $\mathcal{F}^{r,k}$ is a fundamental domain for the action of Γ on $\Omega^{r,k}$. In view of the transitivity of the group H on $\Omega^{r-k}(\mathbb{F}^{(r-k)}) \times \Omega^{k}(\mathbb{F}^{(k)})$, the choices of $\boldsymbol{\chi}$ and $\boldsymbol{\psi}$ are unimportant. (To some extent their choice corresponds to the choice of a fourth root of unity in $\mathbb{C}$ in the classical context.) It will turn out later (Theorem 8.2) that $\mathcal{F}^{r,k}$ (and not $\mathcal{G}^{r,k}$) is the "correct" fundamental domain in that it determines a unique "cusp".

(4.13) Note that $\Gamma = \mathrm{GL}(r, A)$ acts effectively on Ω^r and $\Omega^{r,k}$ through its quotient group $\overline{\Gamma} = \mathrm{PGL}(r, A) = \Gamma/Z$. The j-invariant of (4.6) induces a map

$$\mathcal{F}^{r,k} \longrightarrow \Gamma \backslash \mathcal{F}^{r,k} = \Gamma \backslash \Omega^{r,k} \xrightarrow{\cong} C_\infty,$$

which is locally finite and étale away of fixed points of $\overline{\Gamma}$.

4.14 Proposition. *For $\boldsymbol{\omega} \in \mathcal{F}^{r,k}$ the following conditions are equivalent:*

(a) *$\boldsymbol{\omega}$ is a fixed point of a non-trivial element of $\overline{\Gamma}$;*
(b) *$j(\boldsymbol{\omega}) = 0$;*
(c) *$\boldsymbol{\omega} \in \Omega^r(\mathbb{F}^{(r)})$.*

If these are satisfied, then $s(\boldsymbol{\omega}) = 0$ and the fixed group $\Gamma_{\boldsymbol{\omega}}$ is the Cartan subgroup $G_{\boldsymbol{\omega}} \subset \mathrm{GL}(r, \mathbb{F}) \subset \Gamma$ as defined in (2.11). Points $\boldsymbol{\omega} \in \mathcal{F}^{r,k}$ or, more generally, of $\Omega^{r,k}$ with $j(\boldsymbol{\omega}) = 0$ are called elliptic.

Proof. First note that each $\gamma \in \Gamma$ that fixes $\boldsymbol{\omega}$ induces an automorphism of $\phi^{\boldsymbol{\omega}}$, which yields an embedding of $\Gamma_{\boldsymbol{\omega}}$ into $\mathrm{Aut}(\phi^{\boldsymbol{\omega}})$ (in fact, an identity). As $\phi_T^{\boldsymbol{\omega}} = T + g(\boldsymbol{\omega})\tau^k + \Delta(\boldsymbol{\omega})\tau^r$, we have

$$\mathrm{Aut}(\phi^{\boldsymbol{\omega}}) = \begin{cases} \mathbb{F}^* & \Leftrightarrow \quad g(\boldsymbol{\omega}) \neq 0 \quad \Leftrightarrow \quad j(\boldsymbol{\omega}) \neq 0 \\ \mathbb{F}^{(r)*} & \Leftrightarrow \quad g(\boldsymbol{\omega}) = 0 \quad \Leftrightarrow \quad j(\boldsymbol{\omega}) = 0, \end{cases}$$

which yields (a) $\Rightarrow$ (b). For (b) $\Rightarrow$ (c) we note that $j(\boldsymbol{\omega}) = 0$ implies $E_{q^n-1}(\boldsymbol{\omega}) = 0$ for $1 \leq i < r$, by (2.5). If $\boxed{s(\boldsymbol{\omega}) > 0}$ then

$$E_{q^i-1}(\boldsymbol{\omega}) \equiv \sum_{a_{r-k+1},\ldots,a_r \in \mathbb{F}}{}' (a_{r-k+1}\omega_{r-k+1} + \cdots + a_r\omega_r)^{1-q^i} \pmod{\infty}$$

as in the proof of (4.9), which is $\not\equiv 0 \pmod{\infty}$ for $i = k$. Hence this case cannot occur, and we have $\boxed{s(\boldsymbol{\omega}) = 0}$. Then similarly

$$E_{q^i-1}(\boldsymbol{\omega}) \equiv E_{q^i-1}(\overline{\boldsymbol{\omega}}) = \sum_{a_1,\ldots,a_r \in \mathbb{F}}{}' (a_1\overline{\omega}_1 + \ldots + a_r\overline{\omega}_r)^{1-q^i} \; (1 \leq i < r)$$

for the reduction $\overline{\boldsymbol{\omega}} = (\overline{\omega}_1 : \ldots : \overline{\omega}_r) \in \Omega^r(\overline{\mathbb{F}})$ of $\boldsymbol{\omega}$. By (2.10) $\overline{\boldsymbol{\omega}} \in \Omega^r(\mathbb{F}^{(r)})$, i.e., $\gamma\overline{\boldsymbol{\omega}} = \overline{\boldsymbol{\omega}}$ for $\gamma \in \mathrm{GL}(r, \mathbb{F})_{\overline{\boldsymbol{\omega}}} \subset \Gamma$. Now regarding $\overline{\boldsymbol{\omega}} \in \Omega^r(\mathbb{F}^{(r)}) \hookrightarrow \Omega^r$ as an approximative solution of the equation $\gamma\boldsymbol{\omega} = \boldsymbol{\omega}$, the uniqueness part of Hensel's lemma implies $\boldsymbol{\omega} = \overline{\boldsymbol{\omega}}$. The implication (c) $\Rightarrow$ (a) is trivial, since $\boldsymbol{\omega} \in \Omega^r(\mathbb{F}^{(r)})$ has the stabilizer $\mathrm{GL}(r,\mathbb{F})_{\boldsymbol{\omega}} \supsetneq Z$. $\square$

4.15 Proposition. *The analytic space $\Omega^{r,k}$ is smooth.*

Proof. (i) Smoothness in points $\boldsymbol{\omega}$ with $j(\boldsymbol{\omega}) \neq 0$ follows from (4.13) and (4.14). We are therefore reduced to showing smoothness in $\boldsymbol{\omega}$ with $j(\boldsymbol{\omega}) = 0$, without restriction, $\boldsymbol{\omega} \in \Omega^r(\mathbb{F}^{(r)})$, $\boldsymbol{\omega} = (\omega_1 : \ldots : \omega_r)$, $\omega_i \in \mathbb{F}^{(r)}$.

(ii) Let D_j be the differential operator $\frac{\partial}{\partial \omega_j}$ $(1 \leq j \leq r)$. We must show that the $(r-2) \times r$-matrix $(D_j g_i)(\boldsymbol{\omega})$, where $1 \leq i \leq r-1$, $i \neq k$, and $1 \leq j \leq r$ has the maximal possible rank $r-2$ in $\boldsymbol{\omega} \in \Omega^r(\mathbb{F}^{(r)})$.

(iii) Put for simplicity $e_i(\boldsymbol{\omega})$ for the Eisenstein series $E_{q^i-1}(\boldsymbol{\omega})$. Applying D_j to the equation (see (2.4))

$$T e_s = \sum_{t+u=s} e_t g_u^{q^t}$$

and using $D_j(g_u^{q^t}) = 0$ for $t > 0$ and $e_t(\boldsymbol{\omega}) = 0 = g_t(\boldsymbol{\omega})$ for $1 \leq t < r$, we find

$$(D_j g_i)(\boldsymbol{\omega}) = (T^{q^i} - T)(D_j e_i)(\boldsymbol{\omega}).$$

We are thus entitled to replace the g_i by the Eisenstein series e_i in the matrix in (ii).

(iv) Now for our $\boldsymbol{\omega}$,

$$\begin{aligned} (D_j e_i)(\boldsymbol{\omega}) &= \sum_{a_1,\ldots,a_r \in A}{}' \frac{a_j}{(a_1\omega_1 + \ldots + a_r\omega_r)^{q^i}} \\ &\equiv \sum_{a_1,\ldots,a_r \in \mathbb{F}}{}' \frac{a_j}{(a_1\omega_1 + \ldots + a_r\omega_r)^{q^i}} = M(\lambda_j)^{q^i}. \end{aligned}$$

Here $\lambda_j : \mathbb{F}^{(r)} \longrightarrow \mathbb{F}$ is the $\mathbb{F}$-linear map $(a_1\omega_1 + \ldots + a_r\omega_r) \longmapsto a_j$ and

$$M(\lambda) := \sum_{x \in \mathbb{F}^{(r)}}{}' \frac{\lambda(x)}{x}$$

for any $\lambda \in \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^{(r)}, \mathbb{F})$.

(v) Consider

$$\begin{aligned} M : \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^{(r)}, \mathbb{F}) &\longrightarrow \mathbb{F}^{(r)} \, . \\ \lambda &\longmapsto M(\lambda) \end{aligned}$$

It is additive and satisfies $M(\lambda \circ c) = cM(\lambda)$ for $c \in \mathbb{F}^{(r)}$, and is therefore $\mathbb{F}^{(r)}$-linear for the $\mathbb{F}^{(r)}$-structure $c \cdot \lambda := \lambda \circ c$ on $\mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^{(r)}, \mathbb{F})$.

(vi) Let λ be the trace map on $\mathbb{F}^{(r)}$: $\lambda(x) = x + x^q + \ldots + x^{q^{r-1}}$. Then

$$M(\lambda) = \sum_{x\in\mathbb{F}^{(r)}}{}' 1 + \sum{}' x^{q-1} + \ldots + \sum{}' x^{q^{r-1}-1}.$$

As the maps $x \longmapsto x^{q^i-1}$ for $1 \leq i < r$ are non-trivial characters, their sums vanish, and we get $M(\lambda) = \sum{}'1 = -1 \neq 0$. We conclude that M is bijective and the $M_j := M(\lambda_j)$ $(1 \leq j \leq r)$ are $\mathbb{F}$-linearly independent.

(vii) Consider the $r\times r$-matrix $(M_j^{q^{i-1}})_{1\leq i,j\leq r}$. It is a Moore matrix (see [23] 1.3, Lemma 1.3.3), and is non-singular in view of the independence of the M_j. Hence the $(r-2)\times r$-matrix obtained by omitting the first and the $(k+1)$-th rows has full rank $r-2$.

(viii) In view of (iv) and (vii) we have the wanted rank property for the reduction modulo ∞ of the matrix $((D_j e_j)(\boldsymbol{\omega}))$. As a consequence of the Nakayama lemma, the property holds for $(D_j(e_i)(\boldsymbol{\omega}))$ itself. □

5. Growth of modular forms along $\mathcal{F}^{r,k}$.

In this section, $\mathcal{F} = \mathcal{F}^{r,k}$ as before. Coordinates of points $\boldsymbol{\omega} = (\omega_1 : \cdots : \omega_r)$ are normalized such that $\omega_r = 1$. We introduce a uniformizer t "at infinity" of $\mathcal{F}$ and express the absolute values of $g(\boldsymbol{\omega})$, $\Delta(\boldsymbol{\omega})$, $j(\boldsymbol{\omega})$, $t(\boldsymbol{\omega})$ in terms of the spread $s(\boldsymbol{\omega})$ of $\boldsymbol{\omega} \in \mathcal{F}$. This is used in later sections to establish connectedness properties of $\mathcal{F}$.

5.1 Definition. *For non-negative $s \in \mathbb{Q}$ we define subsets (in fact, admissible open subspaces) of $\mathcal{F}$ through conditions on $s(\boldsymbol{\omega})$ as follows:*
(5.1.1)
$\mathcal{F}_{\leq s} = \{\boldsymbol{\omega} \in \mathcal{F} \mid s(\boldsymbol{\omega}) \leq s\}$, $\mathcal{F}_{\geq s} = \{\boldsymbol{\omega} \in \mathcal{F} \mid s(\boldsymbol{\omega}) \geq s\}$, $\mathcal{F}_s = \mathcal{F}_{\leq s} \cap \mathcal{F}_{\geq s}$

We further let $\mathcal{F}_+ = \bigcup_{s>0} \mathcal{F}_{\geq s} = \mathcal{F} \cap \mathcal{G}_+$.

(5.1.2) Given $\boldsymbol{\omega} = (\omega_1 : \ldots : 1) \in \Omega^r$, write $\boldsymbol{\omega} = (\boldsymbol{\omega}^{(1)}, \boldsymbol{\omega}^{(2)})$ with

$$\boldsymbol{\omega}^{(1)} = (\omega_1, \ldots, \omega_{r-k}),\ \boldsymbol{\omega}^{(2)} = (\omega_{r-k+1}, \ldots, 1),$$

and let $\Lambda_{\boldsymbol{\omega}}^{(2)}$ be the sublattice $\sum_{r-k<i\leq r} A\omega_i$ of $\Lambda_{\boldsymbol{\omega}}$ with exponential function $e_{\boldsymbol{\omega}}^{(2)}$.

(5.1.3) Define $t : \Omega^r \longrightarrow C_\infty$ by $t(\boldsymbol{\omega}) = (e_{\boldsymbol{\omega}}^{(2)}(\omega_1 + \omega_2 + \ldots + \omega_{r-k}))^{-1}$.

The following properties are easily verified:

(5.1.4) $t(\boldsymbol{\omega})$ is well-defined (as $\omega_1 + \ldots + \omega_{r-k} \notin \Lambda_{\boldsymbol{\omega}}^{(2)}$ and thus $e_{\boldsymbol{\omega}}^{(2)}(\omega_1 + \ldots + \omega_{r-k}) \neq 0$);

(5.1.5) t is holomorphic on Ω^r;

(5.1.6) let $\gamma \in \Gamma$ have an $(r-k)\times k$ block structure $\gamma = \left(\begin{array}{c|c} 1 & \beta \\ \hline 0 & \delta \end{array}\right)$, i.e., 1 ist the $(r-k)\times(r-k)$ unit matrix, 0 the $k \times (r-k)$ zero matrix, and β, δ are A-matrices of the right format. Then

$$t(\gamma\boldsymbol{\omega}) = \alpha(\gamma, \boldsymbol{\omega})t(\boldsymbol{\omega})$$

with the automorphy factor $\alpha(\gamma, \boldsymbol{\omega})$ of (1.4.3). In particular,

$$t(\gamma\boldsymbol{\omega}) = t(\boldsymbol{\omega}) \tag{5.1.7}$$

if δ is the $k \times k$ unit matrix.

We briefly recall the standard argument for (5.1.6). We have $\gamma\boldsymbol{\omega} = \gamma(\boldsymbol{\omega}^{(1)}, \boldsymbol{\omega}^{(2)}) = \alpha^{-1}(\boldsymbol{\omega}^{(1)}+\beta\boldsymbol{\omega}^{(2)}, \delta\boldsymbol{\omega}^{(2)}) = (\alpha^{-1}\boldsymbol{\omega}^{(1)}+\alpha^{-1}\beta\boldsymbol{\omega}^{(2)}, \alpha^{-1}\delta\boldsymbol{\omega}^{(2)})$ (the factor $\alpha^{-1} = \alpha(\gamma, \boldsymbol{\omega})^{-1}$ serves to normalize the last coordinate, $\beta\boldsymbol{\omega}^{(2)}$ and $\delta\boldsymbol{\omega}^{(2)}$ are matrix products). As $\delta\boldsymbol{\omega}^{(2)}$ generates the same lattice $\Lambda_{\boldsymbol{\omega}}^{(2)}$ as $\boldsymbol{\omega}^{(2)}$, $e_{\boldsymbol{\omega}}^{(2)}$ is $\Lambda_{\boldsymbol{\omega}}^{(2)}$-periodic, and the formula $e_{\alpha\Lambda}(\alpha z) = \alpha e_{\Lambda}(z)$ holds for lattices Λ and in particular for $\Lambda_{\boldsymbol{\omega}}^{(2)}$, we get

$$t(\gamma\boldsymbol{\omega}) = (\alpha^{-1}e_{\boldsymbol{\omega}}^{(2)}(\omega_1 + \ldots + \omega_{r-k}))^{-1} = \alpha t(\boldsymbol{\omega})$$

as wanted.

(5.2) Suppose now that $\boldsymbol{\omega} = (\boldsymbol{\omega}^{(1)}, \boldsymbol{\omega}^{(2)}) \in \mathcal{F}$. The entries of $\boldsymbol{\omega}^{(2)}$ have absolute value 1, those of $\boldsymbol{\omega}^{(1)}$ absolute value $q^{s(\boldsymbol{\omega})}$. Write $\omega = \omega_1 + \cdots + \omega_{r-k}$. Then $|\omega| = |\omega_1| = q^{s(\boldsymbol{\omega})}$ and

$$|t(\boldsymbol{\omega})^{-1}| = |e_{\boldsymbol{\omega}}^{(2)}(\omega)| = |\omega| \prod_{a_{r-k+1},\ldots,a_r \in A}{}' |1 - \frac{\omega}{a_{r-k+1}\omega_{r-k+1} + \ldots + a_r\omega_r}|.$$

The factor $(1 - \frac{\omega}{a_{r-k+1}\omega_{r-k+1}+\ldots+a_r\omega_r})$ has absolute value 1 if $|\omega| \le |a_{r-k+1}\omega_{r-k+1} + \cdots + a_r\omega_r| = \max\limits_{k-r<i\le r}|a_i|$ and $\frac{|\omega|}{\max|a_i|}$ otherwise.

Hence, from counting the number of k-tuples $(a_{r-k+1}, \ldots, a_r) \in A^k$ with a given value of $\max\limits_{k-r<i\le r}|a_i| < |\omega| = q^{s(\boldsymbol{\omega})}$, we find after an (omitted) elementary calculation the following result.

5.3 Proposition. *Let $d \in \mathbb{N}_0$ be such that $d \le s(\boldsymbol{\omega}) < d+1$. Then*

$$-\log_q |t(\boldsymbol{\omega})| = (s(\boldsymbol{\omega}) - d)q^{k(d+1)} + q^k \frac{q^{kd} - 1}{q^k - 1}.$$

In particular, $|t(\boldsymbol{\omega})|$ depends only on $s(\boldsymbol{\omega})$, and it decays superexponentially with growing $s(\boldsymbol{\omega})$. □

Remark. Note that the preceding formula depends only on the facts (a) $\boldsymbol{\omega} = (\omega_1 : \ldots : \omega_{r-1} : 1)$ is reduced, (b) $|\omega_1| = \ldots = |\omega_{r-k}| = q^s$ and (c) $|\omega_{r-k+1}| = \ldots = |\omega_r| = 1$, where $d \le s < d+1$. Therefore the same formula holds for such $\boldsymbol{\omega} \in \mathcal{G}^r$ even if they fail to belong to $\mathcal{F} = \mathcal{F}^{r,k}$.

(5.4) As usual, we let $\phi^{(\boldsymbol{\omega})}$ be the Drinfeld module attached to $\boldsymbol{\omega} \in \mathcal{F}$. Recall that with $g = g(\boldsymbol{\omega})$, $\Delta = \Delta(\boldsymbol{\omega})$

$$\Delta^{-1}\phi_T(X) = T/\Delta\, X + g/\Delta\, X^{q^k} + X^{q^r} = \prod_{\mu \in {}_T\phi} (X - \mu),$$

so

$$\Delta = T {\prod_{\mu \in {}_T\phi}}' \mu^{-1}$$

and

$$|\Delta| = q \cdot |\mu_r|^{1-q^k} |\mu_1|^{q^k - q^r}, \tag{5.4.1}$$

as there are $q^k - 1$ "short" roots μ of length $|\mu_r|$ and $q^r - q^k$ "long" roots μ of length $|\mu_1|$. Here $\{\mu_1, \dots, \mu_r\}$ is the distinguished basis (3.3.1) of ${}_T\phi$. Hence to find $|\Delta(\boldsymbol{\omega})|$ for $\boldsymbol{\omega} \in \mathcal{F}$, we need to find $|\mu_1(\boldsymbol{\omega})|$ and $|\mu_r(\boldsymbol{\omega})|$. Furthermore, as g/Δ up to sign equals the $(q^r - q^k)$-th elementary symmetric polynomial in the μ, we have in case $s(\boldsymbol{\omega}) > 0$

$$|g/\Delta| = \prod_{\mu \text{ a long root}} |\mu| = |\mu_1|^{q^r - q^k},$$

as the product of $q^r - q^k$ roots μ is strictly smaller than that value if at least one short root appears as a factor. Therefore, for $s(\omega) > 0$:

$$|g| = \Delta |\mu_1|^{q^r - q^k} = q|\mu_r|^{1-q^k}. \tag{5.4.2}$$

5.5 Lemma. In the given situation ($\boldsymbol{\omega} = (\boldsymbol{\omega}^{(1)}, \boldsymbol{\omega}^{(2)}) \in \mathcal{F}$, $d \le s(\boldsymbol{\omega}) < d+1$ with $d \in \mathbb{N}_0$) the following hold:

$$\begin{aligned} \log_q |\mu_1(\boldsymbol{\omega})| &= (s(\boldsymbol{\omega}) - d)q^{kd} + q^k \frac{q^{k(d-1)} - 1}{q^k - 1} \\ |\mu_r(\boldsymbol{\omega})| &= q^{-1} \end{aligned}$$

Proof. With $\Lambda = \Lambda_{\boldsymbol{\omega}} = \sum_{1 \le i‘r} A\omega_i$ we have

$$\begin{aligned} |\mu_r| &= |e_\Lambda(\frac{\omega_r}{T})| = |\frac{\omega_r}{T}| = q^{-1} \quad \text{and} \\ |\mu_1| &= |e_\Lambda(\frac{\omega_1}{T})| = |\frac{\omega_1}{T}| {\prod_{\substack{\lambda \in \Lambda \\ |T\lambda| \le |\omega_1|}}}' |1 - \frac{\omega_1}{T\lambda}| = |\frac{\omega_1}{T}| {\prod_{\substack{\lambda \in \Lambda \\ |T\lambda| < |\omega_1|}}}' |1 - \frac{\omega_1}{T\lambda}| \end{aligned}$$

The λ figuring in the right hand product are the $\lambda = \sum_{r-k<i\le r} a_i\omega_i$ with $a_i \in A$ and $q \cdot \max |a_i| < |\omega_1| = q^{s(\boldsymbol{\omega})}$, and then

$$|1 - \frac{\omega_1}{T\lambda}| = |\frac{\omega_1}{T\lambda}| = q^{s(\boldsymbol{\omega})-1}/\max|a_i|.$$

With a similar calculation as in (5.3) we find the result. □

5.6 Proposition. *With the same assumptions as before, we have for $\boldsymbol{\omega} \in \mathcal{F}$:*

(i) $\log_q |\Delta(\boldsymbol{\omega})| = q^k + (q^k - q^r)[s(\boldsymbol{\omega}) - d)q^{kd} + q^k \frac{q^{k(d-1)}-1}{q^k-1}]$;

(ii) *if $s(\boldsymbol{\omega}) > 0$ then $g(\boldsymbol{\omega})| = q^{q^k}$, i.e., $\log_q |g(\boldsymbol{\omega})| = q^k$.*

Proof. This follows from the formulas in (5.4) and (5.5). □

Remark. The argument of (5.4) shows that $|g(\boldsymbol{\omega})|$ is bounded on the whole of $\mathcal{F}$ by q^{q^k}; this can also be seen from the relation (2.5) between $g = g_k$ and the Eisenstein series E_{q^k-1}.

Now the j-invariant $j(\boldsymbol{\omega}) = j(\phi^{\boldsymbol{\omega}})$ is defined as

$$j(\boldsymbol{\omega}) = g^{r'}/\Delta^{k'} \text{ with } r' = (q^r-1)/(q-1),\ k' = (q^k-1)/(q-1).$$

Hence (5.6) and comparison with (5.3) yields the basic relation between the functions j and t.

5.7 Corollary. *For $\boldsymbol{\omega} \in \mathcal{F}_+$ (i.e., $s(\boldsymbol{\omega}) > 0$), $d \leq s(\boldsymbol{\omega}) < d+1$ with $d \in \mathbb{N}_0$, we have*

(i) $\log_q |j(\boldsymbol{\omega})| = \frac{q^{r-k}-1}{q-1} q^{k(d+1)} + m(s(\boldsymbol{\omega}) - d)q^{k(d+1)}$, *where* $m = (q^{r-k}-1)(q^k-1)/(q-1)$;

(ii) $-\log_q |t(\boldsymbol{\omega})|^m = \log |j(\boldsymbol{\omega})| - (q^r - q^k)/(q-1)$. □

5.8 Corollary. *Both $|t(\boldsymbol{\omega})|$ and $|j(\boldsymbol{\omega})|$ as functions on $\mathcal{F}_+$ depend only on $s(\boldsymbol{\omega})$. As a function of $s(\boldsymbol{\omega})$, $|t(\boldsymbol{\omega})|$ (resp. $|j(\boldsymbol{\omega})|$) is strictly monotonically decreasing (resp. increasing). The product $t(\boldsymbol{\omega})^m j(\boldsymbol{\omega})$ with m as above has constant absolute value $q^{(q^r-q^k)/(q-1)}$ on $\mathcal{F}_+$.* □

(5.9) By definition $\mathcal{F} = \mathcal{F}_0 \dot\cup \mathcal{F}_+$, and it follows from (5.3) and (5.6) that the functions t and Δ have properties similar to those of (5.8) on all of $\mathcal{F}$. Viz,

$$(5.9.1) \qquad |t(\boldsymbol{\omega})| = 1,\ \log_q |\Delta(\boldsymbol{\omega})| = q^r \text{ for } \omega \in \mathcal{F}_0.$$

In contrast, g and j (having zeroes on $\mathcal{F}_0$) are only bounded above by

$$(5.9.2) \qquad \log_q |g(\boldsymbol{\omega})| \leq q^k,\ \log_q |j(\boldsymbol{\omega})| \leq (q^r - q^k)/(q-1),\ \boldsymbol{\omega} \in \mathcal{F}_0;$$

these bounds are attained, and are also the limits of $\log_q |g(\boldsymbol{\omega})|$ resp. $\log_q |j(\boldsymbol{\omega})|$ for $s(\boldsymbol{\omega}) \longrightarrow 0$.

5.10 Lemma. *The following are equivalent for $\boldsymbol{\omega} \in \mathcal{F}_0$: (a) $\log_q |g(\boldsymbol{\omega})| = q^k$; (b) $\log_q |j(\boldsymbol{\omega})| = (q^r - q^k)/(q-1)$; (c) $|E_{q^k-1}(\boldsymbol{\omega})| = 1$; (d) the reduction $\overline{\boldsymbol{\omega}}$ of $\boldsymbol{\omega}$ in $\Omega^r(\overline{\mathbb{F}})$ does not belong to $\Omega^r(\mathbb{F}^{(r)})$.*

Proof. (a) ⇔ (b) is immediate, (a) ⇔ (c) follows from (2.5) and (a) ⇔ (d) from (2.10) and the congruence

$$E_{q^k-1}(\boldsymbol{\omega}) \equiv \sum_{a_1,\ldots,a_r \in \mathbb{F}}{}' (a_1\omega_1 + \cdots + a_r\omega_r)^{1-q^k} \pmod{\infty}$$

for $\boldsymbol{\omega} \in \mathcal{F}_0$. □

We define the subspace

$$
(5.10.1)\qquad \begin{aligned} \mathcal{F}_! &= \{\boldsymbol{\omega} \in \mathcal{F} \mid \log_q |g(\boldsymbol{\omega})| \geq q^k\} \\ &= \mathcal{F}_+ \cup \{\boldsymbol{\omega} \in \mathcal{F}_0 \mid \text{the conditions of (5.10) hold}\} \end{aligned}
$$

Then $j(\mathcal{F}_!) = \{z \in C_\infty \mid \log_q |z| \geq \rho\}$ with the constant

$$\rho = (q^r - q^k)/(q-1);$$

that is, the reciprocal $1/j$ maps $\mathcal{F}_!$ onto the pointed ball $B_{q^{-\rho}} \setminus \{0\}$, where $B_x = \{z \in C_\infty \mid |z| \leq x\}$. From the commutative diagram

(5.10.2)

$$
\begin{tikzcd}
\mathcal{F} \arrow[r] & \Gamma \setminus \mathcal{F} \arrow[r, "\cong"] & \Gamma \setminus \Omega^{r,k} \arrow[r, "\cong", "1/j"'] & \mathbb{P}^1(C_\infty) \setminus \{0\} \\
\mathcal{F}_! \arrow[r] \arrow[u, hook] & \Gamma \setminus \mathcal{F}_! \arrow[rr, "\cong"] \arrow[u, hook] & & B_{q^{-\rho}} \setminus \{0\} \arrow[u, hook]
\end{tikzcd}
$$

we see that the restriction $1/j : \mathcal{F}_! \longrightarrow B_{q^{-\rho}} \setminus \{0\}$ is locally finite and étale, since Γ has no fixed points on $\mathcal{F}_!$. Similarly, $1/j$ maps each $\mathcal{F}_{\geq s}$ $(s > 0)$ onto some pointed ball of radius depending on s and $\mathcal{F}_+$ to the "interior" $B^-_{q^{-\rho}} \setminus \{0\} = \{z \in C_\infty \mid 0 < |z| < q^{-\rho}\}$ of $B_{q^{-\rho}} \setminus \{0\}$.

(5.11) Recall that we have fixed $\boldsymbol{\chi} \in \Omega^{r-k}(\mathbb{F}^{(r-k)})$ and $\boldsymbol{\psi} \in \Omega^k(\mathbb{F}^{(k)})$ in order to define $\mathcal{F} = \mathcal{F}^{(r,k)}$ as a subspace of $\mathcal{G}^{r,k}$, see (4.12). Let $G_{\boldsymbol{\chi}} \subset \mathrm{GL}(r-k, \mathbb{F})$ and $G_{\boldsymbol{\psi}} \subset \mathrm{GL}(k, \mathbb{F})$ be the corresponding fixed groups (Cartan subgroups). It follows from (4.10) that

$$
(5.11.1)\qquad \Gamma_\infty := \{\gamma \in \Gamma \mid \gamma(\mathcal{F}_+) \cap \mathcal{F}_+ \neq \emptyset\}
$$

consists of the matrices with $(r-k) \times k$ block structure $\left(\begin{array}{c|c} \alpha & \beta \\ \hline 0 & \delta \end{array}\right)$, where $\alpha \in G_{\boldsymbol{\chi}}$, $\delta \in G_{\boldsymbol{\chi}}$, and β is an arbitrary $(r-k) \times k$ matrix with entries in A. Similarly,

(5.11.2)

$$
\begin{aligned}
\Gamma_s &= \{\gamma \in \Gamma \mid \gamma(\mathcal{F}_s) \cap \mathcal{F}_s \neq \emptyset\} = \{\gamma \in \Gamma \mid \gamma(\mathcal{F}_s) = \mathcal{F}_s\} \\
&= \left\{ \left(\begin{array}{c|c} \alpha & \beta \\ \hline 0 & \delta \end{array}\right) \mid \alpha \in G_{\boldsymbol{\chi}},\, \delta \in G_{\boldsymbol{\psi}},\ \begin{array}{l} \text{all entries } b \text{ of } \beta \\ \text{satisfy } \deg b \leq s \end{array} \right\}
\end{aligned}
$$

for $0 < s \in \mathbb{Q}$ and $\Gamma_0 = \mathrm{GL}(r, \mathbb{F})$. These groups are semidirect products

$$\Gamma_\infty = (G_{\boldsymbol{\chi}} \times G_{\boldsymbol{\psi}}) \ltimes \Gamma^u_\infty,\ \Gamma_s = (G_{\boldsymbol{\chi}} \times G_{\boldsymbol{\psi}}) \ltimes \Gamma^u_s \quad (s > 0)$$

with their respective unipotent radicals (= p-Sylow subgroups) Γ^u_*, $* \in \mathbb{Q}_{>0} \cup \{\infty\}$.

It follows from (5.10.2) and (5.11.2) that $1/j$ provides Galois coverings

$$
(5.11.3)\quad 1/j : \mathcal{F}_s \longrightarrow \Gamma_s \setminus \mathcal{F}_s \xrightarrow{\cong} \{z \in C_\infty \mid \log_q |z| = -\rho(s)\}\ (s > 0)
$$

with group Γ_s/Z, where the logarithmic diameter $-\rho(s)$ is given by (5.7) (i). For $s = 0$ we have the covering, ramified above $z = 0$

$$(5.11.4) \qquad j : \mathcal{F}_0 \longrightarrow \Gamma_0 \setminus \mathcal{F}_0 \xrightarrow{\cong} \{z \in C_\infty \mid \log_q |z| \le \rho\} = B_{q^\rho}$$

with Galois group $\Gamma_0/Z = \mathrm{PGL}(r, \mathbb{F})$. The inverse image of the circumference $\{z \in C_\infty \mid \log_q |z| = \rho\}$ is $\mathcal{F}_0 \cap \mathcal{F}_!$.

6. Connectedness properties.

Notation is as before: $r \ge 3$ and k are natural numbers with $k < r$ and $(r, k) = 1$, and $0 \ne N$ is some element of A, without restriction non-constant. We let $d \in \mathbb{N}$ be the degree of N. In this section we show that $\Gamma(N) \setminus \Omega^{r,k}$ is connected as analytic space. Before, we study similar connectedness properties of $\mathcal{F} = \mathcal{F}^{r,k}$ and its subspaces.

(6.1) Given N, let
(6.1.1)

$$\begin{aligned} \Gamma_\infty(N) &= \Gamma_\infty \cap \Gamma(N) = \{\left(\begin{array}{c|c} 1 & \beta \\ \hline 0 & 1 \end{array}\right) \in \Gamma \mid \beta \in (NA)^{(r-k)\times k}\} \\ &\subset \Gamma_\infty^u = \{\left(\begin{array}{c|c} 1 & \beta \\ \hline 0 & 1 \end{array}\right) \mid \beta \in A^{(r-k)\times k}\}. \end{aligned}$$

We write D for the diagonal subgroup of Γ_∞/Z, i.e., $D = (G_{\boldsymbol{\chi}} \times G_{\boldsymbol{\psi}})/Z$, which by (4.4) is cyclic of order $m = (q^{r-k} - 1)(q^k - 1)/(q - 1)$.

(6.2) The map $1/j$ provides an isomorphism

$$(6.2.1) \qquad \Gamma_\infty \setminus \mathcal{F}_+ \xrightarrow{\cong} B^-$$

with the interior B^- of the pointed ball $B = B_{q^{-\rho}} \setminus \{0\}$, see (5.10.2). In particular, $\Gamma_\infty \setminus \mathcal{F}_+$ is connected. The map $\Gamma_\infty^u \setminus \mathcal{F}_+ \longrightarrow \Gamma_\infty \setminus \mathcal{F}_+$ is an étale Galois cover with group D. By (5.7) (ii) the function $t^m \cdot j$ on $\Gamma_\infty \setminus \mathcal{F}_+$ has constant absolute value; as t is Γ_∞^u-invariant, this implies that

$$(6.2.2) \qquad t : \Gamma_\infty^u \setminus \mathcal{F}_+ \xrightarrow{\cong} (B')^-$$

is a uniformizer onto the interior $(B')^-$ of $B' = \{z \in C_\infty \mid 0 < |z| \le q^{\rho'}\}$, where $\rho' = q^k/(q^k - 1)$ comes out by (5.3). So $\Gamma_\infty^u \setminus \mathcal{F}$ is connected, too.

(6.3) Let a group G act strictly continuously on an analytic space X such that the quotient $Y = G \setminus X$ exists and is connected, and let $X = \bigcup_{i \in I} X_i$ be the decomposition into connected components, with some index set I. Then G permutes the X_i transitively, and we have $G_i \setminus X_i \xrightarrow{\cong} G \setminus X = Y$ with the stabilizer group G_i of X_i. Hence the quotient map restricted to each X_i is surjective onto Y.

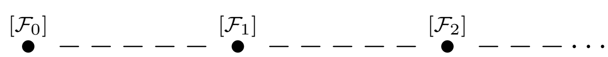
(6.4) The natural map from Γ_{d-1}^u to $\Gamma_\infty^u/\Gamma_\infty(N)$ is an isomorphism if $d > 1$; for $d = 1$ we use $(\Gamma_0 \cap \Gamma_1)^u$ in place of Γ_{d-1}^u. For the rest of this section we abbreviate $\Theta := \Gamma_{d-1}^u$ if $d > 1$; $\Theta = (\Gamma_0 \cap \Gamma_1)^u = \{\left(\begin{array}{c|c} 1 & \beta \\ \hline 0 & 1 \end{array}\right) \mid \beta \in \mathbb{F}^{(r-k)\times k}\}$ if $d = 1$. It acts on the analytic spaces $\Theta\mathcal{F}_+ = \bigcup_{\gamma\in\Theta} \gamma\mathcal{F}_+$ and $\Gamma_\infty(N) \setminus \Theta\mathcal{F}_+$, and the natural map

$$\Gamma_\infty(N) \setminus \Theta\mathcal{F}_+ \longrightarrow \Theta \setminus (\Gamma_\infty(N) \setminus \Theta\mathcal{F}_+) = \Gamma_\infty^u \setminus \mathcal{F}_+ \tag{6.4.2}$$

is the associated quotient map. By (6.2.2) and (6.3), each connected component U of $\Gamma_\infty(N) \setminus \Theta\mathcal{F}_+$ maps surjectively onto $\Gamma_\infty^u \setminus \mathcal{F}_+$.

But how do these components look like?

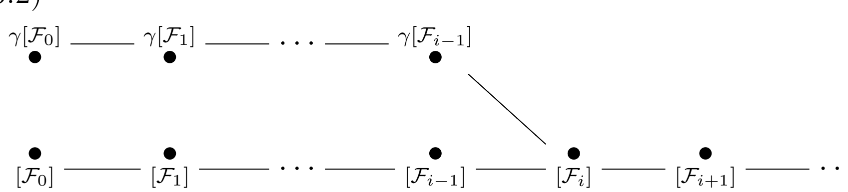
(6.5) Let $\lambda : \Omega^r \longrightarrow I(K_\infty^r)(\mathbb{Q})$ be the *building map*, described in [6] III, Sections 2 - 4, onto the rational points of the Bruhat-Tits building $I(K_\infty^r)$ of the group $\mathrm{PGL}(r, K_\infty)$. From the description *loc. cit.* it is clear that $\lambda(\mathcal{F})$ is a straight line

(6.5.1) 

i.e., a graph which is an infinite half-line with vertices $[\mathcal{F}_i]$ indexed by the $\mathcal{F}_i$ $(i \in \mathbb{N}_0)$. (More precisely: $\lambda(\mathcal{F})$ is the set of points with rational barycentric coordinates in the realization of such a graph. In the sequel we abuse language and suppress that distinction.) In fact, the map $s : \mathcal{F} \longrightarrow \mathbb{Q}_{\geq 0}$ factors over λ, and we may use s as a coordinate on $\lambda(\mathcal{F})$. Note that $\lambda(\mathcal{F}_+) = \lambda(\mathcal{F}) \smallsetminus \{[\mathcal{F}_0]\}$.

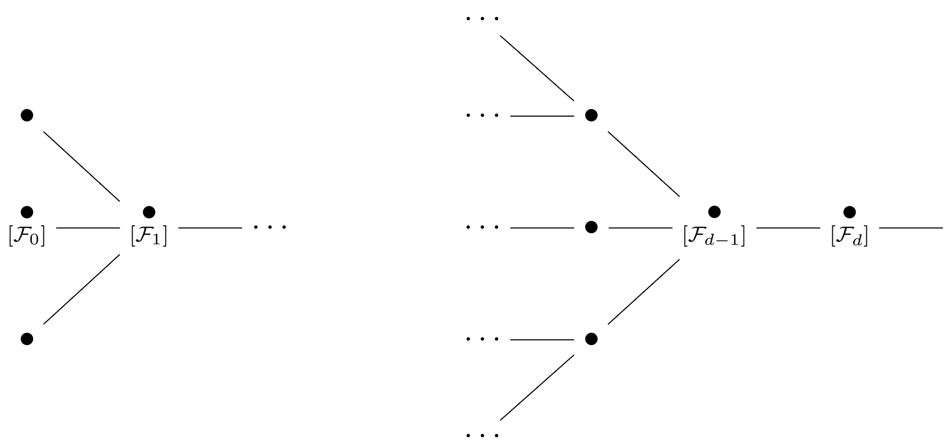
Let $\gamma \in \Gamma_i \smallsetminus \Gamma_{i-1}$ $(i \in \mathbb{N})$. As λ is Γ-equivariant and γ fixes $\mathcal{F}_i, \mathcal{F}_{i+1}, \ldots$ but not $\mathcal{F}_{i-1}$, $\lambda(\gamma\mathcal{F}) = \gamma(\lambda(\mathcal{F}))$ differs from $\lambda(\mathcal{F})$ in a path of length i, that is, looks like

(6.5.2)



As a result, the image $\lambda(\Theta\mathcal{F})$ is the graph $\Theta\lambda(\mathcal{F})$ of shape

(6.5.3)



an infinite tree composed of one infinite half-line $[\mathcal{F}_{d-1}], [\mathcal{F}_d], [\mathcal{F}_{d+1}], \ldots$ and a finite number of paths of length $d-1$ emanating from $[\mathcal{F}_{d-1}]$ "to the left". In particular, each infinite half-line in $\Theta\lambda(\mathcal{F})$ agrees with $\lambda(\mathcal{F})$ for all but finitely many vertices.

(6.5.4) We note that the equivalence relation given by the action of $\Gamma_\infty(N)$ is trivial on $\Theta\lambda(\mathcal{F})$, since for $1 \neq \gamma \in \Gamma_\infty(N)$ and a vertex v of $\Theta\lambda(\mathcal{F})$, either $\gamma v = v$ or $\gamma v \notin \Theta\lambda(\mathcal{F})$.

(6.5.5) By (6.4.2) and (6.5.3), for each connected component U of $\Gamma_\infty(N)\backslash\Theta\mathcal{F}_+$ the set $\lambda(U)$ (which is well-defined by the remark (6.5.4)) contains a maximal half-line (minus its endpoint) of $\Theta\lambda(\mathcal{F})$. Applying some $\gamma \in \Theta$ if necessary, we may achieve that this half-line equals $\lambda(\mathcal{F}_+) = \lambda(\mathcal{F})\smallsetminus\{[\mathcal{F}_0]\}$.

6.6 Proposition. *For $d \leq n \in \mathbb{N}$, the analytic space $\Gamma_\infty(N) \setminus \mathcal{F}_n$ is connected.*

Proof. (i)By (6.2.2) and the formula (5.3) for $|t|$, $\Gamma^u_\infty \setminus \mathcal{F}_n = \Gamma^u_n \setminus \mathcal{F}_n$ is via t isomorphic with a circumference $\{z \in C_\infty \mid |z| = c\}$ with a constant c, and is therefore connected. Hence Γ^u_n acts transitively on $\pi_0(\mathcal{F}_n) \xrightarrow{\cong} \pi_0(\overline{\mathcal{F}}_n)$, where $\pi_0(\cdots)$ is the set of connected components and $\overline{\mathcal{F}}_n$ the canonical reduction of $\mathcal{F}_n$.

(ii) For $n \geq 2$, the operation of Γ^u_n on $\overline{\mathcal{F}}_n$ is through its quotient Γ^u_{n-1} (and for $n=1$ through $\Gamma^u_1/(\Gamma^u_1 \cap \Gamma_0)$; we omit that case in what follows). That is, only the leading terms of β in $\gamma = \left(\begin{array}{c|c} 1 & \beta \\ \hline 0 & 1 \end{array}\right)$ account for the action of γ on $\overline{\mathcal{F}}_n$, and thus on $\pi_0(\mathcal{F}_n)$.

(iii) For $N \in \mathbb{N}$ such that $\deg N = d \leq n$, the group $\Gamma_\infty(N) \cap \Gamma^u_n$ maps surjectively onto $\Gamma_n^u/\Gamma_{n-1}^u$. Therefore $\pi_0(\Gamma_\infty(N)\setminus\mathcal{F}_n)=\pi_0(\Gamma_n^u\setminus\mathcal{F}_n)$ consists of one element. □

6.7 Corollary. *The analytic space $\Gamma(N)\backslash\Theta\mathcal{F}_+$ is connected. Similarly, $\Gamma(N)\setminus\mathcal{F}_{\geq s}$ is connected for $s\geq d$.*

Proof. We have $\Gamma(N)\setminus\Theta\mathcal{F}_+=\Gamma_\infty(N)\setminus\Theta\mathcal{F}_+$. For $n\geq d$, each connected component meets $\Gamma_\infty(N)\setminus\Theta\mathcal{F}_n=\Gamma_\infty(N)\setminus\mathcal{F}_n$, as follows from (6.5.5). But $\Gamma_\infty(N)\setminus\mathcal{F}_n$ is connected by the last proposition. Further, for $s\geq d$, $\Gamma(N)\setminus\Theta\mathcal{F}_{\geq s}=\Gamma_\infty(N)\backslash\mathcal{F}_{\geq s}$, and the same argument applies. □

6.8 Proposition. *Let N,d,Θ be as before. There exists an admissible open subspace W of $\Gamma(N)\backslash\Omega^{r,k}$ that satisfies*

(a) *W contains $\Gamma(N)\backslash\Theta\mathcal{F}$;*
(b) *W is connected.*

Proof. (i) We have $\{\gamma\in\Gamma\mid\gamma\Gamma_0\mathcal{F}_{\leq 1/2}\cap\Gamma_0\mathcal{F}_{\leq 1/2}\}=\Gamma_0$. Therefore the j-function describes the quotient modulo Γ_0 of $\Gamma_0\mathcal{F}_{\leq 1/2}$:

$$\Gamma_0\mathcal{F}_{\leq 1/2}\longrightarrow\Gamma_0\backslash\Gamma_0\mathcal{F}_{\leq 1/2}\xrightarrow[j]{\cong}B\,.$$

Here the right hand side B is a ball B_x of radius x strictly larger than q^ρ, and is connected. By (6.3) each connected component U of $\Gamma_0\mathcal{F}_{\leq 1/2}$ meets $\mathcal{F}_0=\Gamma_0\mathcal{F}_0$, the inverse image of B_{q^ρ}. We choose U as some component which intersects nontrivially with $\mathcal{F}_+$. Then $W:=\Gamma(N)\backslash(\Theta\mathcal{F}\cup U)$ trivially satisfies (a); thus we have to show (b), which in view of (6.7) will follow from the connectedness of $\Gamma(N)\backslash(\mathcal{F}_0\cup U)=\mathcal{F}_0\cup U$.

(ii)Consider the canonical reduction $X:=\overline{\mathcal{F}}_0$ of $\mathcal{F}_0$. It is the affine subscheme of $\Omega^r(\overline{\mathbb{F}})$ defined by the vanishing of the $r-2$ forms $\alpha_1,\ldots,\hat{\alpha}_k,\ldots,\alpha_{r-1}$ (as usual, we use the set of geometric points to describe the scheme $\Omega^r/\mathbb{F}$). More precisely, $\overline{\boldsymbol{\omega}}=(\overline{\boldsymbol{\omega}}_1:\ldots:\overline{\boldsymbol{\omega}}_r)\in\Omega^r(\overline{\mathbb{F}})$ defines the $\mathbb{F}$-lattice $\Lambda=\Lambda_{\overline{\boldsymbol{\omega}}}=\sum\mathbb{F}\overline{\omega}_i$, the α_i are the coefficients of $e_\Lambda(z)=z+\sum_{1\leq i\leq r}\alpha_i(\overline{\boldsymbol{\omega}})z^{q^i}$, and the reduction mapping from $\mathcal{F}_0$ to X is

(6.8.1)
$$\boldsymbol{\omega}=(\omega_1:\ldots:\omega_r)\longmapsto\overline{\boldsymbol{\omega}}=(\overline{\omega}_1:\ldots:\overline{\omega}_r)=((\overline{T\mu_1}):\ldots:(\overline{T\mu_r})).$$

Here the μ_i are the basis vectors of ${}_T\phi^{\boldsymbol{\omega}}$ as in (3.3.1), which satisfy

$$\mu_i=\frac{\omega_i}{T}(1+u_i)\text{ with }|u_i|<1,$$

therefore the last equality in (6.8.1). For a given $\boldsymbol{\omega}=(\omega_1:\ldots:\omega_r)\in\Omega^r$ with $|\omega_i|=1$ for all i and Drinfeld module $\phi=\phi^{\boldsymbol{\omega}}$,

$$\phi_T(X)=TX+\sum_{1\leq i\leq r}g_i(\boldsymbol{\omega})X^{q^i}=\Delta(\boldsymbol{\omega})\prod_\mu(X-\mu),$$

where μ runs through the $\mathbb{F}$-span ${}_T\Phi$ of $\mu_1, \ldots, \mu_r$. Therefore, with $\Delta = \Delta(\boldsymbol{\omega})$,

$$\prod_\mu (X - T\mu) = \frac{T^{q^r}}{\Delta} X + \Delta^{-1} \sum_{1 \le i \le r} T^{q^r - q^i} g_i(\boldsymbol{\omega}) X^{q^i},$$

where $|T^{q^r}/\Delta| = 1 = |g_r(\boldsymbol{\omega})/\Delta|$ and $|T^{q^r-q^i} g_i(\boldsymbol{\omega})/\Delta| \le 1$ for $1 \le i \le r-1$. Hence the condition $g_i(\boldsymbol{\omega}) = 0$ $(1 \le i < r,\ i \ne k)$ reduces to $\alpha_i(\overline{\boldsymbol{\varpi}}) = 0$ for the same i, as stated above.

(iii) $G := \Gamma_0 = \mathrm{GL}(r, \mathbb{F})$ acts on $\Omega^r(\overline{\mathbb{F}})$ and stabilizes X. As $\Omega^r(\overline{\mathbb{F}})$ classifies classes of $\mathbb{F}$-lattices Λ of dimension r (plus the choice of an ordered basis), X classifies those which are k-sparse, i.e., satisfy $\alpha_i(\Lambda) = 0$ for $1 \le i < r$, $i \ne k$. With the reduced version of the j-invariant:

$$j_{\mathrm{red}} := \alpha_k^{r'}/\alpha_r^{k'},\ r' = (q^r - 1)/(q-1),\ k' = (q^k - 1)/(q-1),$$

we find

$$j_{\mathrm{red}} :\ G\backslash X \xrightarrow{\cong} \mathbb{A}^1(\overline{\mathbb{F}}) = \mathbb{F}.$$

Let $X = \bigcup_{i \in I} X_i$ be the splitting into connected components $I =$ some index set), which are transitively permuted by G. As $\Omega^r(\mathbb{F}^{(r)}) \subset X$, each X_i contains some $\boldsymbol{\omega} \in \Omega^r(\mathbb{F}^{(r)})$. Thus the Cartan subgroup $G_{\boldsymbol{\omega}}$ of G is contained in the stabilizer G_i of X_i, for each i.

(iv) Let $X^{\mathrm{Zar}} = \bigcup X_i^{\mathrm{Zar}}$ be the Zariski closure of X (resp. X_i) in $\mathbb{P}^{r-1}(\overline{\mathbb{F}})$. (The X_i^{Zar} need not be disjoint.) Let $\mathbf{s} := (s_1 : \ldots : s_r)$ be an $\overline{\mathbb{F}}$-point of some $X_i^{\mathrm{Zar}} \smallsetminus X_i$. Then the $\mathbb{F}$-lattice $\Lambda_{\mathbf{s}} = \sum \mathbb{F}s_i$ has rank k, and the forms α_i $(1 \le i < k)$ vanish on $\mathbf{s}$. As such $\mathbf{s}$ are all G-conjugate, we may assume $\mathbf{s} = (0 : \ldots 0 : s_{r-k+1} : \ldots : s_r)$, where $\mathbf{s}^{(2)} = (s_{r-k+1} : \ldots : s_r) \in \Omega^k(\mathbb{F}^{(k)})$ (see (2.10)). The stabilizer $G_{\mathbf{s}}$ of $\mathbf{s}$ consists of the matrices $\left(\begin{array}{c|c} \alpha & 0 \\ \hline \beta & \delta \end{array}\right) \in G$, where $\alpha \in \mathrm{GL}(r-k, \mathbb{F})$, δ belongs to the Cartan group $G_{\mathbf{s}^{(2)}} \subset \mathrm{GL}(k, \mathbb{F})$, and $\beta \in \mathbb{F}^{k \times (r-k)}$.

(v) The stabilizer of the connected component of X^{Zar} determined by X_i^{Zar} contains G_{s} and the Cartan subgroup $G_{\boldsymbol{\omega}}$ (if $\boldsymbol{\omega} \in X_i$). (In the terminology of [27], $G_{\boldsymbol{\omega}}$ is the group generated by a Singer cycle.) As $G_{\mathbf{s}}$ apparently doesn't normalize any subgroup of G of shape $\mathrm{GL}(r/r', \mathbb{F}^{(r')})$ with $1 < r'|r$, naturally embedded in G, the result of [27] implies that $\langle G_{\boldsymbol{\omega}}, G_{\mathbf{s}} \rangle = G$. Therefore X^{Zar} is connected.

(vi)The preceding argument shows more precisely: $X \cup \{\mathbf{s}\}$ is connected for each boundary point $\mathbf{s} \in X^{\mathrm{Zar}} \backslash X$. Hence such an $\mathbf{s}$ belongs to each of the X_i^{Zar}.

(vii) As U is connected but not contained in $\mathcal{F}_0$, it determines upon reduction at least one boundary $\mathbf{s}$ of $X = \overline{\mathcal{F}}_0$. Hence by (vi), the reduction of $\mathcal{F}_0 \cup U$, and thus $\mathcal{F}_0 \cup U$ itself, is connected, which had to be shown. $\square$

The next result, crucial for this paper, is now an easy consequence.

6.9 Theorem. *For each $0 \neq N \in A$, the analytic space $\Gamma(N) \setminus \Omega^{r,k}$ is connected.*

Proof. For $N \in \mathbb{F}$, $\Gamma(N) = \Gamma$ and $\Gamma\backslash\Omega^{r,k} \xrightarrow{\cong} \mathbb{A}^1(C_\infty) = C_\infty$ through the j-function, and is connected. Thus assume N non-constant. The group Γ acts transitively on the connected components of $\Gamma(N)\backslash\Omega^{r,k}$. Let U be the component that encloses the subspace W as in Proposition 6.8, with stabilizer group Γ_U. Then $\Gamma_0 = \mathrm{GL}(r, \mathbb{F}) \subset \Gamma_U$ and also $\Gamma_\infty \subset \Gamma_U$. Now Γ_0 contains all the permutation matrices; therefore Γ_U contains all the stricly upper and lower triangular matrices (i.e., triangular matrices with 1's on the diagonal). As these, by the elementary divisor theorem, generate $\mathrm{SL}(r, A)$, we find that

$$\Gamma_U \supset \Gamma_0 \cdot \mathrm{SL}(r, A) = \mathrm{GL}(r, A) = \Gamma,$$

and so $\Gamma_U = \Gamma$ and $U = \Gamma(N)\backslash\Omega^{r,k}$. $\square$

7. The modular curves $X^{r,k}$ and $X^{r,k}(N)$.

We maintain the notation and assumptions of the last section.

(7.1) As the natural map $\Gamma(N)\backslash\Omega^{r,k} \longrightarrow \Gamma\backslash\Omega^{r,k} \xrightarrow{\cong} \mathbb{A}^1(C_\infty)$ is finite and the analytic space $\Gamma(N)\backslash\Omega^{r,k}$ is one-dimensional, smooth and connected, there exists a unique (up to unique isomorphism) smooth connected affine algebraic curve $\tilde{Y}^{r,k}(N)$ over C_∞ with $\Gamma(N)\backslash\Omega^{r,k}$ as its set of C_∞-points, and such that the above map comes from a morphism of algebraic curves. This follows from the GAGA-type theorems of Kiehl ([28], [29]). By abuse of language we write $\tilde{Y}^{r,k}(N) = \Gamma(N)\backslash\Omega^{r,k}$. Further, we let $\tilde{X}^{r,k}(N)$ be the "compactification" of $\tilde{Y}^{r,k}(N)$, i.e., the smooth projective model of $\tilde{Y}^{r,k}(N)$. Points of $\tilde{X}^{r,k}(N)$ not in $\tilde{Y}^{r,k}(N)$ are labelled as "cusps". Then the meromorphic functions on $\tilde{X}^{r,k}(N)$ (in the algebraic sense) are the meromorphic functions on the analytic space $\Gamma(N)\backslash\Omega^{r,k}$ that satisfy extra meromorphy conditions around the cusps (see Lemma 7.4).

We have the commutative diagram similar to (1.5.4):

$$\begin{tikzcd} \Gamma(N) \setminus \Omega^{r,k} = \widetilde{Y}^{r,k}(N) \arrow[rr, hook] \arrow[d] & & \tilde{X}^{r,k}(N) \arrow[d] \\ \Gamma \setminus \Omega^{r,k} = \widetilde{Y}^{r,k}(1) \arrow[r, "\cong", "j"'] & \mathbb{A}^1(C_\infty) \arrow[r, hook] & \mathbb{P}^1(C_\infty), \end{tikzcd} \tag{7.1.1}$$

where the group $G(N) = \Gamma/\Gamma(N) \cdot Z$ (cf. (1.5)) acts on the objects in the upper row and the vertical maps are the quotient maps. Ramification occurs above $j = 0$, with ramification group $G_{\boldsymbol{\omega}}/Z$, where $G_{\boldsymbol{\omega}}$ is a Cartan subgroup of $\mathrm{GL}(r, \mathbb{F})$, and possibly above $j = \infty$, i.e., at cusps. This follows from (4.14); the situation at cusps will be studied in the next section.

(7.2) We first describe the function field of $\tilde{X}^{r,k}(N)$. Let $\mathbf{u} = (u_1, \dots, u_n)$ be an element of $(N^{-1}A)^r$ not in A^r, and consider the function on Ω^r:

$$e_{\mathbf{u}}(\boldsymbol{\omega}) := e_{\boldsymbol{\omega}}(\mathbf{u}\boldsymbol{\omega}), \tag{7.2.1}$$

where $e_{\boldsymbol{\omega}} = e_{\Lambda_{\boldsymbol{\omega}}}$ and $\mathbf{u}\boldsymbol{\omega} = \sum_{1 \leq i \leq r} u_i \omega_i$; as usual we have normalized $\omega_r = 1$. The following properties are easily verified:

(7.2.2) $e_{\mathbf{u}}$ depends only on $\mathbf{u}$ modulo A^r; therefore we let $\mathbf{u}$ run through $(N^{-1}A/A)^r$;

(7.2.3) $e_{c\mathbf{u}}(\boldsymbol{\omega}) = c e_{\mathbf{u}}(\boldsymbol{\omega})$, $c \in \mathbb{F}^*$;

(7.2.4) $e_{\mathbf{u}}$ is holomorphic and vanishes nowhere on Ω^r;

(7.2.5) $\phi_N^{\boldsymbol{\omega}}(e_{\mathbf{u}}(\boldsymbol{\omega})) = 0$;

(7.2.6) $e_{\mathbf{u}}(\gamma\boldsymbol{\omega}) = \alpha(\gamma, \boldsymbol{\omega})^{-1} e_{\mathbf{u}\gamma}(\boldsymbol{\omega})$, $\gamma \in \Gamma$.

Here γ acts as a matrix from the right on $\mathbf{u}$, and $\alpha(\gamma, \boldsymbol{\omega})$ is the factor of automorphy introduced in (1.4.3). In particular, $e_{\mathbf{u}}$ behaves like a modular form of weight -1 for $\Gamma(N)$, since $\mathbf{u}\gamma = \mathbf{u}$ for $\gamma \in \Gamma(N)$;

(7.2.7) The reciprocal $e_{\mathbf{u}}(\boldsymbol{\omega})^{-1}$ equals the partial Eisenstein series

$$E_{\mathbf{u}}(\boldsymbol{\omega}) = \sum_{\substack{\mathbf{a} \in K^r \\ \mathbf{a} \equiv \mathbf{u} \pmod{A^r}}} (\mathbf{a}\boldsymbol{\omega})^{-1}$$

of weight 1. (The two-lines proof of (7.2.7) is left as an exercise.)

(7.3) The functions $e_{\mathbf{u}}$ may be used to construct modular functions on Ω^r for $\Gamma(N)$; viz,

$$f_{\mathbf{u}}^{(i)}(\boldsymbol{\omega}) := e_{\mathbf{u}}^{q^i - 1}(\boldsymbol{\omega}) g_i(\boldsymbol{\omega}) \tag{7.3.1}$$

is holomorphic on Ω^r and invariant under $\Gamma(N)$ (and is similar to classical Fricke functions). We define the function $f_{\mathbf{u}}$ as the restriction of

$f_{\mathbf{u}}^{(k)}$ to $\Omega^{r,k}$.

(7.3.2) $$f_{\mathbf{u}}(\boldsymbol{\omega}) = e_{\mathbf{u}}^{q^k-1}(\boldsymbol{\omega})g_k(\boldsymbol{\omega}) \quad (\boldsymbol{\omega} \in \Omega^{r,k}).$$

7.4 Lemma. *Given $0 \neq \mathbf{u} \in (N^{-1}A/A)^r$, there exists a constant $C > 0$ and an exponent $M \in \mathbb{N}$ such that for all $\boldsymbol{\omega} \in \mathcal{F}_+$ the estimate*

$$|f_{\mathbf{u}}(\boldsymbol{\omega})| \leq C|j(\boldsymbol{\omega})|^M$$

holds.

Proof. (Sketch) We have calculated $|g| = |g_k|$ and $|j|$ in (5.6) and (5.7), and may easily estimate $e_{\mathbf{u}}(\boldsymbol{\omega}) = \mathbf{u}\boldsymbol{\omega}\prod'_{\mathbf{a}\in A^r}(1-\frac{\mathbf{u}\boldsymbol{\omega}}{\mathbf{a}\boldsymbol{\omega}})$ in a similar fashion, as we know the absolute values of the factors for given $\boldsymbol{\omega} \in \mathcal{F}_+$. Details are left to the reader. □

Remark. It would be elementary though laborious to write down an exact formula à la (5.7) for $|e_{\mathbf{u}}(\boldsymbol{\omega})|$ and $|f_{\mathbf{u}}(\boldsymbol{\omega})|$, $\boldsymbol{\omega} \in \mathcal{F}_+$.

The lemma guarantees that $f_{\mathbf{u}}$ is meromorphic at cusps of $\tilde{X}^{r,k}(N)$ and thus an element of the field $\mathcal{K}^{r,k}(N)$ of meromorphic functions on $\tilde{X}^{r,k}(N)$.

7.5 Theorem. *The function field $\mathcal{K}^{r,k}(N)$ of $\tilde{X}^{r,k}(N)$ is generated over $C_\infty(j)$ by the functions $f_{\mathbf{u}}$ ($0 \neq \mathbf{u} \in (N^{-1}A/A)^r$).*

Proof. By Galois theory, it will suffice to show: If $\gamma \in \Gamma$ fixes all the $f_{\mathbf{u}}$, then $\gamma \in \Gamma(N) \cdot Z$.

Thus suppose $f_{\mathbf{u}} \circ \gamma = f_{\mathbf{u}}$. Then $e_{\mathbf{u}}^{q^k-1} = e_{\mathbf{u}\gamma}^{q^k-1}$, that is,

$(*)$ $$e_{\mathbf{u}\gamma} = e \cdot e_{\mathbf{u}}$$

with some $c \in (\mathbb{F}^{(k)})^*$, since $\Gamma(N)\backslash\Omega^{r,k}$ is connected. This holds in particular at the elliptic points $\boldsymbol{\omega} \in \Omega^{r,k}(\mathbb{F}^{(r)})$. At such a point, the Drinfeld A-module $\phi^{\boldsymbol{\omega}}$ has shape

$$\phi_T^{\boldsymbol{\omega}} = T + \Delta\tau^r$$

and may be seen as a rank-one Drinfeld $A^{(r)}$-module for $A^{(r)} = \mathbb{F}^{(r)}[T]$. After scaling the lattice $\Lambda_{\boldsymbol{\omega}}$, we may assume that $\Delta = 1$, that is, $\phi^{\boldsymbol{\omega}}$ is the Carlitz module. It is well-known [25] that the field extension generated by the N-torsion points of the Carlitz module contains no proper constant field extension (here: of $\mathbb{F}^{(r)}$). Therefore $c \in \mathbb{F}^{(k)} \cap \mathbb{F}^{(r)} = \mathbb{F}$, which implies $\mathbf{u}\gamma = c\mathbf{u}$ with some $c = c(\gamma, \mathbf{u}) \in \mathbb{F}^*$. It is easily seen that, given γ, $c(\gamma, \mathbf{u})$ is independent of $\mathbf{u}$, and thus $\gamma \in \Gamma(N) \cdot Z$. □

(7.6) We define the closed subscheme $M^{r,k}(N)/A$ of $M^r(N)/A$ as the vanishing locus of the modular forms $g_1, \ldots \hat{g}_k, \ldots, g_r$. It is the moduli scheme for (r,k)-sparse Drinfeld A-modules supplied with a structure of level N, and is easily seen to be flat over A. As the restriction of the determinant map $M^r(N) \longrightarrow M^1(N) = \operatorname{Spec} B_+(N)$ (see (1.5)) to $M^{r,k}(N)$ is still faithfully flat, $M^{r,k}(N)$ is a flat $B_+(N)$-scheme. Similar to (1.5.2)

$$(7.6.1) \qquad M^{r,k}(N) \underset{A}{\times} C_\infty = \coprod_\sigma M^{r,k}(N) \underset{B_+(N),\sigma}{\times} C_\infty,$$

where σ runs through the K-embeddings of $K_+(N)$ into C_∞. As in (1.5.3) we fix one such $\sigma = \mathrm{id}$ and define

$$Y^{r,k}(N) := M^{r,k}(N) \underset{B_+(N)}{\times} C_\infty.$$

Then

$$(7.6.2) \qquad \tilde{Y}^{r,k}(N) = \Gamma(N)\backslash\Omega^{r,k} \xrightarrow{\cong} Y^{r,k}(N)$$

induced from $\boldsymbol{\omega} \longmapsto \phi^{\boldsymbol{\omega}}$. Note that we have *defined* $\tilde{Y}^{r,k}(N)$ as the unique algebraic curve characterized through its C_∞-points, while $Y^{r,k}(N)$ is the distinguished component of (7.6.1). As by GAGA affine curves over C_∞ are determined through their analytifications, we need no longer distinguish between $\tilde{Y}^{r,k}(N)$ and $Y^{r,k}(N)$ (resp. between $\tilde{X}^{r,k}(N)$ and the projective model $X^{r,k}(N)$ of $Y^{r,k}(N)$). Henceforth we omit the tildes and regard (7.6.2) as an identification.

Since $Y^{r,k}(N)$ is smooth and connected, the above implies that the integral closure of A in its function field $\mathcal{K}^{r,k}(N)$ equals $B_+(N)$, hence the variety $Y^{r,k}(N)$ and its projective model $X^{r,k}(N)$ are "defined over $K_+(N)$".

The natural action of $\tilde{G}(N) = \mathrm{GL}(r, A/N)/Z$ on $M^r(N)$ restricts to $M^{r,k}(N)$ and gives $\tilde{G}(N)\backslash M^{r,k}(N) \xrightarrow{\cong} M^{r,k}(1)$. As in (1.5.2) the subgroup $G(N)$ stabilizes components of (7.6.1), and the identifications

$$(7.6.3) \qquad \begin{array}{ccc} \Gamma(N)\backslash\Omega^{r,k} & \xrightarrow{\cong} & Y^{r,k}(N) \\ \downarrow & & \downarrow \\ \Gamma\backslash\Omega^{r,k} & \xrightarrow{\cong} & Y^{r,k}(1) \end{array}$$

are compatible with the actions of the group $G(N) = \Gamma/\Gamma(N) \cdot Z$.

(7.7) The $K_+(N)$-structure of $X^{r,k}(N)$ may be described as follows: Let $K^{r,k}(N)$ be the subfield of $\mathcal{K}^{r,k}(N)$ generated over K by j and the $f_{\mathbf{u}}$ ($0 \neq \mathbf{u} \in ((N^{-1}A)/A)^r$; actually j may be omitted from this system of generators), which by the preceding equals the function field of the integral scheme $M^{r,k}(N)$. The group $G(N)$ acts on the generators $f_{\mathbf{u}}$ by

$$(*) \qquad f_{\mathbf{u}} \circ \gamma = f_{\mathbf{u}\gamma},$$

as follows from the formulas in (7.2) and (7.3). The natural action of $\tilde{G}(N)$ on $M^{r,k}(N)$ and thus on $K^{r,k}(N)$ is given by the same formula $(*)$, but for the larger group $\tilde{G}(N)$. The fixed field of $G(N)$ in $K^{r,k}(N)$ is $K_+(N)(j)$, the function field of $X^{r,k} = X^{r,k}(1)$ over $K_+(N)$.

We have thus defined the curves that appear in Theorem A, and established assertions (i), (ii) and a part of (iii) of that theorem. It remains to discuss the behavior at cusps of the coverings (7.6.3).

8. Ramification at the cusps.

Continuing with the setting of the last sections, we find that our fundamental domain $\mathcal{F} = \mathcal{F}^{r,k}$ determines a unique cusp, also labelled "∞", of $X^{r,k}(N)$, and we describe the ramification filtration on the fixed group.

(8.1) The geometry around $\infty \in X^{r,k} \xrightarrow{\cong} \mathbb{P}^1(C_\infty)$ of the natural map $X^{r,k}(N) \longrightarrow X^{r,k}$ is described by the commutative diagram

$$\begin{array}{ccccc} \Gamma_\infty(N)\backslash\mathcal{F}_{\geq s} & \hookrightarrow & Y^{r,k}(N) & \hookrightarrow & X^{r,k}(N) \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma_\infty\backslash\mathcal{F}_{\geq s} & \hookrightarrow & Y^{r,k} & \hookrightarrow & X^{r,k} \end{array} \tag{8.1.1}$$

for sufficiently large $s \in \mathbb{Q}$. Here the left hand side inclusions are open embeddings of analytic spaces. From now on, we assume that $\boxed{s \geq d = \deg N}$.

(8.1.2) Let $G_\infty(N)$ (resp. $U(N)$) be the image of Γ_∞ (resp. of Γ_∞^u) in $G(N) = \Gamma/\Gamma(N) \cdot Z$. Then $G_\infty(N) = D \ltimes U(N)$ with the group D of (6.1.1), and $U(N)$ consists of the matrices with block structure $\left(\begin{array}{c|c} 1 & \mathbf{u} \\ \hline 0 & 1 \end{array}\right)$, $\mathbf{u} \in (A/N)^{(r-k)\times k}$. (Essentially it is the group labelled Θ in Section 6.) In what follows we omit reference to N, i.e., $G = G(N)$, $G_\infty = G_\infty(N)$, $U = U(N)$.

The canonical projection $\Gamma_\infty(N)\backslash\mathcal{F}_{\geq s} \longrightarrow \Gamma_\infty\backslash\mathcal{F}_{\geq s}$ is an étale Galois cover with group G_∞; it factors

$$\Gamma_\infty(N)\backslash\mathcal{F}_{\geq s} \longrightarrow \Gamma_\infty^u\backslash\mathcal{F}_{\geq s} \longrightarrow \Gamma_\infty\backslash\mathcal{F}_{\geq s}, \tag{8.1.3}$$

where the group is U in the first and D in the second step. All three spaces are connected, as follows from the existence of the uniformizers $1/j$ for $\Gamma_\infty\backslash\mathcal{F}_{\geq s}$ and t for $\Gamma_\infty^u\backslash\mathcal{F}_{\geq s}$, and from (6.7) for $\Gamma_\infty(N)\backslash\mathcal{F}_{\geq s}$. Therefore the maps in (8.1.3) are completely ramified above the missing point corresponding to $(1/j) = 0$ of the pointed ball $\Gamma_\infty\backslash\mathcal{F}_{\geq s}$, and there exists a unique cusp of $X^{r,k}(N)$ belonging to $\mathcal{F}$. We summarize:

8.2 Theorem. *There exists a unique cusp, labelled* $\infty \in X^{r,k}(N)$*, of* $X^{r,k}(N)$ *such that the analytic space* $\Gamma_\infty(N)\backslash\mathcal{F}_{\geq s}$ *in* (8.1.1) *is a pointed neighborhood of* ∞*. The fixed group of* ∞ *in the ramified covering* $X^{r,k}(N)\longrightarrow X^{r,k}$ *is the group* $G_\infty = G_\infty(N)$ *defined in* (8.1.2)*.*

Proof. The first part has been shown. The second assertion follows from (5.11.1). □

Remark. We have defined G_∞ independently of the stated fixed point property. The theorem justifies the notation G_∞.

(8.3) We recall some well-known definitions and facts about higher ramification. For more details, see [32], Chapter IV. Suppose that $\varphi : X\longrightarrow Y$ is a ramified Galois covering of (smooth, connected, projective) algebraic curves over the algebraically closed field of characteristic $p > 0$, with Galois group G. For a closed point $x \in X$ with uniformizer $\pi = \pi_x$ and $i \in \mathbb{N}_0$, let G_x be the fixed group and

(8.3.1) $$G_{x,i} := \{\sigma \in G \mid \sigma \text{ acts trivially modulo } \pi^{i+1}\}.$$

Then $G_{x,i}$ is a normal subgroup of G_x independent of the choice of π, and $G_x = G_{x,0} \supset G_{x,1} \supset G_{x,2} \supset \cdots$ is the *ramification filtration* of G_x. Furthermore: $G_{x,0}/G_{x,1}$ is a cyclic group of order coprime with p, and $G_{x,1}$ is a p-group. The function

$$\begin{array}{rcl} i_x : G_x\smallsetminus\{1\} & \longrightarrow & \mathbb{N} \\ \sigma & \longmapsto & i_x(\sigma) := \sup\{i \mid \sigma \in G_{x,i}\} + 1 \end{array}$$

is the *ramification function*. For $x \in X$ we define the *ramification number*

$$a_x := \sum_{1\neq\sigma\in G_x} i_x(\sigma).$$

Note that $a_x = 0 \Leftrightarrow G_x \neq \{1\} \Leftrightarrow x$ is ramified in φ. We call φ *modestly ramified* at $x \in X$ (or x modestly ramified in φ) if $G_{x,2} = \{1\}$. If f is modestly ramified at x, then

(8.3.2) $i_x(\sigma) = 1$ for $\sigma \in G_x\smallsetminus G_{x,1}$ and $i_x(\sigma) = 2$ if $\sigma \in G_{x,1}\smallsetminus\{1\}$,

thus

$$a_x = |G_x| + |G_{x,1}| - 2,$$

where $G_{x,1}$ is the (unique) p-Sylow subgroup of G_x.

8.4 Theorem. *The natural map* $\varphi^{r,k}(N) : X^{r,k}(N)\longrightarrow X^{r,k}$ *is modestly ramified at cusps of* $X^{r,k}(N)$*.*

Proof. (i) As the Galois group $G = G(N)$ acts transitively on the cusps, it suffices to consider the ramification filtration at ∞. Since U is the p-Sylow subgroup of G_∞, we must show that the ramification function i_∞ on $G_\infty\smallsetminus\{1\}$ has constant value 2 on $U\smallsetminus\{1\}$. This means that for $1 \neq \sigma \in U$ the function $\pi_\infty \circ \sigma - \pi_\infty$ has precise vanishing order 2 at ∞ for one (and thus for each) uniformizer π_∞ of $X^{r,k}(N)$ at ∞.

(ii) Let $\mathbf{a} = (a_1, \ldots, a_{r-k})$ be an arbitrary non-zero vector in $\mathbb{F}^{r-k}$, and define the function $t_{N,\mathbf{a}}$ on $\mathcal{F}$ by

$$t_{N,\mathbf{a}}(\boldsymbol{\omega}) := e^{(2)}_{\boldsymbol{\omega}}(N^{-1}(\mathbf{a}\boldsymbol{\omega}^{(1)}))^{-1}, \tag{8.4.1}$$

where as usual, $\boldsymbol{\omega} = (\boldsymbol{\omega}^{(1)}, \boldsymbol{\omega}^{(2)})$, $e^{(2)}_{\boldsymbol{\omega}}$ is the exponential function of $\Lambda^{(2)}_{\boldsymbol{\omega}} = \sum_{r-k<i\leq r} A\omega_i$, and $\mathbf{a}\boldsymbol{\omega}^{(1)} = \sum_{1\leq i\leq r-k} a_i\omega_i$. It is invariant under $\Gamma_\infty(N)$, and Proposition 5.3 (plus the following remark) together with $|\mathbf{a}\boldsymbol{\omega}^{(1)}| = |\omega_1| = q^{s(\boldsymbol{\omega})}$ yields

$$|t_{N,\mathbf{a}}(\boldsymbol{\omega})|^{|N|^k} = C|t(\boldsymbol{\omega})| \tag{8.4.2}$$

with $C = q^{q^k-1}$, as long as $\boldsymbol{\omega} \in \mathcal{F}_{\geq s}$. That is, $t_{N,\mathbf{a}}$ "behaves like an $|N|^k$-th root of t".

(iii) As the group Γ^u_∞ is abelian, we write it additively, i.e., identify it with the group $A^{(r-k)\times k}$ through $\left(\begin{array}{c|c} 1 & \beta \\ \hline 0 & 1 \end{array}\right) \longmapsto \beta$. Similarly,

$$\begin{array}{rcl} U = U(N) = (\Gamma^u_\infty\Gamma(N)\cdot Z)/\Gamma(N)\cdot Z & \xrightarrow{\cong} & (A/N)^{(r-k)\times k} \\ \left(\begin{array}{c|c} 1 & \mathbf{u} \\ \hline 0 & 1 \end{array}\right) & \longmapsto & \mathbf{u} = \begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{r-k} \end{pmatrix} \end{array}$$

with the row vectors $\mathbf{u}_i$ of $\mathbf{u}$.

(iv) Let $\alpha_{\mathbf{a}}$ be the surjective A/N-linear map

$$\begin{array}{rcl} \alpha_{\mathbf{a}} : U = (A/N)^{(r-k)\times k} & \longrightarrow & (A/N)^k \\ \mathbf{u} & \longmapsto & \sum_{1\leq i\leq r-k} a_i\mathbf{u}_i \end{array}.$$

Then $t_{N,\mathbf{a}}$, which is invariant under $\Gamma_\infty(N) = NA^{(r-k)\times k}$ by (ii), is in fact invariant under $\ker(\alpha_a)$ (i.e., under the inverse image $K_{\mathbf{a}}$ of $\ker(\alpha_{\mathbf{a}})$ in $A^{(r-k)\times k}$), for each $0 \neq \mathbf{a} \in \mathbb{F}^{r-k}$.

(v) Consider the factorization of the first arrow in (8.1.3):

$$\begin{array}{ccccc} \Gamma_\infty(N)\backslash\mathcal{F}_{\geq s} & \longrightarrow & K_{\mathbf{a}}\backslash\mathcal{F}_{\geq s} & \longrightarrow & \Gamma^u_\infty\backslash\mathcal{F}_{\geq s} \\ \| & & & & \| \\ NA^{(r-k)\times k}\backslash\mathcal{F}_{\geq s} & & & & A^{(r-k)\times k}\backslash\mathcal{F}_{\geq s} \end{array}. \tag{8.4.3}$$

The degree of the right hand mapping is $[(A/N)^{(r-k)\times k} : \ker(\alpha_{\mathbf{a}})] = |(A/N)^k| = |N|^k$; thus (8.4.2) shows that $t_{N,\mathbf{a}}$ is a uniformizer around infinity of $K_{\mathbf{a}}\backslash\mathcal{F}_{\geq s}$.

(vi) Let $\mathbf{u}$ be an element of $K_{\mathbf{a}}$ not in $NA^{(r-k)\times k}$, and let $\sigma_{\mathbf{u}} \in \ker(\alpha_{\mathbf{a}}) \subset (A/N)^{(r-k)\times k} = U$ be the corresponding Galois operator. Then

$$\begin{array}{l} (t_{N,\mathbf{a}}\circ\sigma)(\boldsymbol{\omega}) - t_{N,\mathbf{a}}(\boldsymbol{\omega}) = \\ (e^{(2)}_{\boldsymbol{\omega}}(N^{-1}\mathbf{a}(\boldsymbol{\omega}^{(1)} + \mathbf{u}\boldsymbol{\omega}^{(2)})))^{-1} - (e^{(2)}_{\boldsymbol{\omega}}(N\mathbf{a}^{-1}\boldsymbol{\omega}^{(1)}))^{-1}, \end{array} \tag{$*$}$$

where $\mathbf{u}\boldsymbol{\omega}^{(2)}$ is the matrix product, $\boldsymbol{\omega}^{(2)}$ regarded as a $(k \times 1)$-matrix, i.e., $\mathbf{u}\boldsymbol{\omega}^{(2)}$ is the $(r-k)$-vector with entries

$$\mathbf{u}_1\boldsymbol{\omega}^{(2)} = \sum_{r-k \leq i \leq r} u_{1,i}\omega_i, \ldots, \mathbf{u}_{r-k}\boldsymbol{\omega}^{(2)} = \sum_{r-k < i \leq r} u_{r-k,i}\omega_i.$$

From the additivity of $e^{(2)}_{\boldsymbol{\omega}}$ we find

$$\begin{aligned}(*) &= [e^{(2)}_{\boldsymbol{\omega}}(N^{-1}\mathbf{a}\boldsymbol{\omega}^{(1)}) + e^{(2)}_{\boldsymbol{\omega}}(N^{-1}\mathbf{a}(\mathbf{u}\boldsymbol{\omega}^{(2)}))]^{-1} - (e^{(2)}_{\boldsymbol{\omega}}(N^{-1}\mathbf{a}\boldsymbol{\omega}^{(1)}))^{-1} \\ &= (y+z)^{-1} - y^{-1},\end{aligned}$$

where we have put $y(\boldsymbol{\omega}) = e^{(2)}_{\boldsymbol{\omega}}(N^{-1}\mathbf{a}\boldsymbol{\omega}^{(1)})$, $z(\boldsymbol{\omega}) = e^{(2)}_{\boldsymbol{\omega}}(N^{-1}\mathbf{a}(\mathbf{u}\boldsymbol{\omega}^{(2)}))$. Now $z(\boldsymbol{\omega})$ has constant absolute value $\neq 0$ on $\mathcal{F}_{\geq s}$, as it depends only on $\boldsymbol{\omega}^{(2)}$, whose entries satisfy $|\omega_{r-k+1}| = \cdots = |\omega_r| = 1$. (Actually $z(\omega)$ is a non-trivial N-division point of the rank-k Drinfeld module associated with $\boldsymbol{\omega}^{(2)}$, and we could easily calculate the precise value of $|z(\boldsymbol{\omega})|$.) Therefore we may write the quantity $(*)$ in a neighborhood of ∞ (i.e., for small values of $y^{-1}(\boldsymbol{\omega}) = t_{N,\mathbf{a}}(\boldsymbol{\omega})$ or, what is the same, for large values of $s(\boldsymbol{\omega}) = \log_q |\omega_1|$) in the form

$$(*) = y^{-2}(-z + \frac{z^2}{y} - \frac{z^3}{y^2} + \ldots) = t^2_{N,\mathbf{a}} \text{ times a unit around infinity.}$$

That is, we are done if $K_{\mathbf{a}}$ agrees with $\Gamma_\infty(N)$, i.e., if $\ker(\alpha_{\mathbf{a}}) = \{0\}$, which happens if and only if $k = r-1$.

(vii) For the general case, we have another look to (8.4.3). The total group of this Galois cover is $U = U(N) = (A/N)^{(r-k)\times k}$, with subgroup $\ker(\alpha_{\mathbf{a}}) = \mathrm{Gal}(\Gamma_\infty(N)\backslash\mathcal{F}_{\geq s} \mid K_{\mathbf{a}}\backslash\mathcal{F}_{\geq s})$ and quotient group $\overline{U} = U/\ker(\alpha_{\mathbf{a}}) \xrightarrow{\cong} (A/N)^k$ as the group of the second covering. Let $\sigma \longmapsto \overline{\sigma}$ denote the canonical map from U to $\overline{U}$, $i = i_u$ the ramification function on U, $\overline{i} = i_{\overline{u}}$ the ramification function on $\overline{U}$. The formula [32] IV Sect. 1, Proposition 3 now reads

$$(8.4.4) \qquad |\ker(\alpha_{\mathbf{a}})|\overline{i}(\overline{\sigma}) = \sum_{\sigma \to \overline{\sigma}} i(\sigma)$$

for each $0 \neq \overline{\sigma} \in \overline{U}$, where the sum is over all $\sigma \in U$ that project to $\overline{\sigma}$. As $\overline{i}(\overline{\sigma}) = 2$ by (vi) and $i(\sigma) \geq 2$ for all the $|\ker(\alpha_{\mathbf{a}})| = |N|^{(r-k-1)\times k}$ many $\sigma \longrightarrow \overline{\sigma}$ (since U is a p-group and thus agrees with its first ramification group), (8.4.4) implies $i(\sigma) = 2$. This holds for all $0 \neq \sigma \in U$ for which there exists an $\mathbf{a} \in \mathbb{F}^{r-k}$ such that $\sigma \notin \ker(\alpha_{\mathbf{a}})$. Now if $\sigma = \begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{r-k} \end{pmatrix}$ with some $\mathbf{u}_j \neq 0$ then $\mathbf{a} = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 at the j-th place satisfies $\alpha_{\mathbf{a}}(\sigma) = \mathbf{u}_j \neq 0$, thus in fact $i(\sigma) = 2$ for all $0 \neq \sigma \in U$, which had to be shown. $\square$

Remark. Note that the quotient D of G_∞ by $G_{\infty,1} = U$ is cyclic of p-coprime order $m = (q^{r-k}-1)(q^k-1)/(q-1)$, in coherence with the requirements for the ramification filtration.

8.5 Corollary.

(i) *The number of cusps of* $X^{r,k}(N)$ *is given by*

$$|\{\textit{cusps of } X^{r,k}(N)\}| = \frac{|G(N)|}{m|N|^{(r-k)k}},$$

where $|N| = q^{\deg N} = q^d$ *and* $m = (q^{r-k}-1)(q^k-1)/(q-1)$.

(ii) *The ramification number* a_∞ *(see* (8.3.1)*) of the cusp* ∞ *(and thus* a_x *for arbitrary cusps* x*) in the map* $\varphi^{r,k}(N) : X^{r,k}(N) \longrightarrow X^{r,k}$ *is given by* $a_\infty = (m+1)|N|^{(r-k)k} - 2$.

Proof. (i) comes from (8.2), as $G(N)$ acts transitively on the set of cusps with fixed group $G_\infty(N)$ of ∞. (ii) is a rephrasing of (8.3.2) once we take the theorem into account. $\square$

9. The genus of $X^{r,k}(N)$.

We are now in a position to calculate the genus $g^{r,k}(N)$ of $X^{r,k}(N)$ by applying the Riemann-Hurwitz formula to the natural map $\varphi^{r,k}(N)$: $X^{r,k}(N) \longrightarrow X^{r,k}(1) = X^{r,k} \xrightarrow{\cong} \mathbb{P}^1$.

(9.1) The *Riemann-Hurwitz formula* RHF (see [32] VI Sect. 4 or [24] IV Sect. 2). In the setting of (8.3), let $g(X)$ and $g(Y)$ be the genera of X and Y, respectively, with Euler-Poincaré characteristics $e(X) = 2 - 2g(X)$ and $e(Y) = 2 - 2e(Y)$. Then

$$e(X) = |G|e(Y) - \sum_{x \in X} a_x. \tag{9.1.1}$$

Remark. As the EP characteristic $e(\ .\ )$ behaves smoother than $g(\ .\ )$, we prefer $e(\ .\ )$ for calculations.

(9.2) The data needed to evaluate (9.1.1) for $\varphi^{r,k}(N)$ are:

(9.2.1) $e(X^{r,k}) = 2$, i.e., $g(X^{r,k}) = g(\mathbb{P}^1) = 0$;

(9.2.2) $G(N) = \{\gamma \in \mathrm{GL}(r, A/N) \mid \det(\gamma) \in \mathbb{F}^*\}/Z$, thus $|G(N)| = |\mathrm{SL}(r, A/N)|$, which is easy to evaluate once the prime decomposition of N is given; for example, if N is prime of degree d then

$$|G(N)| = (|N|-1)^{-1} \prod_{0 \le i < r} (|N|^r - |N|^i),\ |N| = q^d.$$

By Proposition 4.14, $\varphi^{r,k}(N)$ is unramified off elliptic points and cusps.

(9.2.3) As $G(N)$ acts transitively on elliptic points of $X^{r,k}(N)$ with stabilizers of cardinality $(q^r-1)/(q-1)$, there are $|G(N)|(q-1)/(q^r-1)$ many elliptic points x, each with ramification number $a_x = (q^r-1)/(q-1)-1 = (q^r-q)/(q-1)$. The total contribution of elliptic points to (9.1.1) is

$$A_{\text{ell}} := \sum_{x\in X^{r,k}(N) \text{ elliptic}} a_x = \frac{|G(N)|(q-1)}{(q^r-1)}(\frac{q^r-q}{q-1}).$$

(9.2.4) The cuspidal contribution to (9.1.1) is by Corollary 8.5

$$A_{\text{cusp}} := \sum_{x \text{ cusp of} X^{r,k}(N)} a_x = \frac{|G(N)|}{m|N|^{(r-k)k}}((m+1)|N|^{(r-k)k}-2),$$
$$m = (q^{r-k}-1)(q^k-1)/(q-1),\ |N| = q^{\deg N} = q^d.$$

Plugging in, we find:

9.3 Proposition. *The genus $g^{r,k}(N)$ of $X^{r,k}(N)$ is given by*

$$g^{r,k}(N) = 1 - |G(N)| + \frac{1}{2}(A_{\text{ell}} + A_{\text{cusp}})$$

with $|G(N)|$, A_{ell}, A_{cusp} as described in (9.2).

9.4 Example. Let $r=3$ and $k=1$ or 2. Then

(9.4.1) $$g^{3,k}(N) = 1 + \frac{1}{2}|\mathrm{SL}(3,A/N)|(\frac{q^2+q}{q^2+q+1} + \frac{|N|^2q^2-2}{|N|^2(q^2-1)} - 2).$$

Suppose moreover that $d = \deg N = 1$. Then $|\mathrm{SL}(3,A/N)| = (q^3-1)(q^2-1)q^3$, and the above yields

(9.4.2) $$g^{3,k}(N) = 1 + q(q^4-q^3)/2 - q^2 + 1).$$

10. The modular curve $X_0^{r,k}(N)$.

(10.1) As before, fix some non-constant $N \in A$, and let H be a subgroup of $\tilde{G}(N) = \mathrm{GL}(r,A/N)/Z$. We define the moduli scheme $M_H^{r,k}$ as the quotient

(10.1.1) $$M_H^{r,k} = H\backslash M^{r,k}(N).$$

It is a flat, generically smooth A-scheme, which classifies (r,k)-sparse Drinfeld A-modules with a structure of level H. In particular, its C_∞-points correspond bijectively to isomorphism classes of such objects over C_∞. Let $\det(H) \subset (A/N)^*/\mathbb{F}^*$ be the image of H under the determinant map, with fixed field $K_H := K_+(N)^{\det(H)}$ (see (1.5.1)) and B_H the ring of A-integers in K_H. The map $\det : M^{r,k}(N)\longrightarrow M^1(N) =$

Spec $B_+(N)$ induces a faithfully flat map $M_H^{r,k} \longrightarrow \operatorname{Spec} B_H$, and as in (1.5.2) and (7.6.1),

$$M_H^{r,k} \underset{A}{\times} C_\infty = \coprod_\sigma M_H^{r,k} \underset{B_H,\sigma}{\times} C_\infty, \tag{10.1.2}$$

where σ runs through the K-embeddings of K_H into C_∞. Again as in (7.6), we fix one such embedding $\sigma = \mathrm{id}$ and define

$$Y_H^{r,k} := M_H^{r,k} \underset{B_H}{\times} C_\infty, \tag{10.1.3}$$

a smooth connected affine curve over C_∞, and $X_H^{r,k}$ as its smooth compactification. Then

$$\Gamma_H \backslash \Omega^{r,k} \xrightarrow{\cong} Y_H^{r,k}(C_\infty) \tag{10.1.4}$$

induced from $\boldsymbol{\omega} \longmapsto \phi^{\boldsymbol{\omega}}$ with $\Gamma_H := \{\gamma \in \Gamma \mid \gamma \in H \text{ modulo } N\}$. Both $Y_H^{r,k}$ and $X_H^{r,k}$ are defined over K_H, with the fixed field $K^{r,k}(N)^H$ of H as function field (see (7.7)). It follows from general principles of moduli schemes but may easily be seen directly that the fibers of $M_H^{r,k}$ over primes $\mathfrak{p}$ of B_H with $\mathfrak{p} \nmid N$ are smooth; in other words, $Y_H^{r,k}$ and $X_H^{r,k}$ have good reduction at such $\mathfrak{p}$.

(10.2) We are mainly interested in the case where H is the parabolic subgroup $P = P(N)$ of $\tilde{G}(N)$ of matrices with an $(r-k) \times k$ block structure $\left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right)$, which encompasses $G_\infty(N)$. Then $\det(P) = (A/N)^*/Z$, so $K_P = K$, and $X_P^{r,k}$ is defined over K. The scheme $M_P^{r,k}$ classifies triples (ϕ, u, ϕ), where ϕ is an (r,k)-sparse Drinfeld A-module with an isogeny $u : \phi \longrightarrow \phi'$ whose kernel is isomorphic with $(A/N)^k$. (Strictly speaking, this moduli interpretation is valid only above Spec $A[N^{-1}]$; for A-characteristics $\mathfrak{p}$ of ϕ dividing N the definition of the level structure on ϕ is a bit more involved: see [7] Sect. 5. We will however not need this.) As $M_P^{r,k}$ is fully determined through the data r, k and N, we write

(10.2.1) $M_0^{r,k}(N)$, $Y_0^{r,k}(N)$, $X_0^{r,k}(N)$ for $M_P^{r,k}$, $Y_P^{r,k}$, $X_P^{r,k}$, respectively. Of course, these collapse to the "classical" Drinfeld modular curves of Hecke type [12] if $(r,k) = (2,1)$ (the case we have excluded from our current considerations).

(10.3) Let $\mathfrak{p}$ be a prime of degree one of A with $\mathfrak{p} \nmid N$. Then $\mathbb{F} \xrightarrow{\cong} \mathbb{F}_\mathfrak{p} := A/\mathfrak{p}$, and the reduced curve $\overline{X_0^{r,k}(N)}^{(\mathfrak{p})}$ (i.e., the projectivization of the special fiber $M_0^{r,k}(N) \underset{A}{\times} \mathbb{F}_\mathfrak{p}$ is smooth and projective over $\mathbb{F}$ of genus $g_0^{r,k}(N) = $ genus of $X_0^{r,k}(N)/K$. For what follows, we abbreviate

$$\overline{X}_0(N) \text{ for } \overline{X_0^{r,k}(N)}^{(\mathfrak{p})}, \tag{10.3.1}$$

which is a (non-Galois) covering of $\overline{X} = \overline{X^{r,k}}^{(\mathfrak{p})}$ = projective j-line over $\mathbb{F}_{\mathfrak{p}} = \mathbb{F}$, of degree $[\Gamma : \Gamma_P] = [\tilde{G}(N) : P(N)] = [G(N) : G(N) \cap P(N)]$.

In accordance with Proposition 4.14, a geometric point $x \in \overline{X}(\overline{\mathbb{F}})$ is called *elliptic* if it corresponds to a Drinfeld module ϕ that satisfies one of the conditions (easily verified to be equivalent):

(10.3.2) (a) $\mathrm{Aut}(\phi) \supsetneq \mathbb{F}^*$; (b) $\mathrm{Aut}(\phi) \cong (\mathbb{F}^{(r)})^*$; (c) $j(x) = j(\phi) = 0$; (d) $\phi_T(X) = \overline{T}X + \Delta X^{q^r}$. (Here $\overline{T}$ is the image of T in $A/\mathfrak{p} = \mathbb{F}_{\mathfrak{p}}$.) Similarly, some $x \in \overline{X}_0(N)(\overline{\mathbb{F}})$ is elliptic if it lies above $j = 0$. It follows that an elliptic $x \in \overline{X}_0(N)(\overline{\mathbb{F}})$ is ramified above $\overline{X}$ with ramification index $(q^r - 1)/(q - 1)$. Hence the number of elliptic points is

(10.3.3) $|\{\text{elliptic points in } \overline{X}_0(N)(\overline{\mathbb{F}})\}| = [\Gamma : \Gamma_{P(N)}](q-1)/(q^r-1)$.

10.4 Proposition. *Each elliptic point of $\overline{X_0}(N)$ is defined over $\mathbb{F}^{(r)}$, the field extension of $\mathbb{F} = \mathbb{F}_{\mathfrak{p}}$ of degree r.*

Proof. In the given situation, the elliptic Drinfeld A-modules are all supersingular and thus defined over $\mathbb{F}^{(r)}$ [15]. The result then follows from a standard argument, see. e.g. the proof of Proposition 9.1 in [18], which applies in a slightly modified form. □

Hence we have a large supply of $\mathbb{F}^{(r)}$-rational points on $\overline{X}_0(N)/\mathbb{F}_{\mathfrak{p}}$, and such curves are good candidates to produce large ratios

number of rational points/genus.

Note that Theorem B is automatic from the construction of $X_0^{r,k}(N)$ and the preceding study of $X^{r,k}(N)$. Thus our remaining task in order to find good towers of curves as described in the introduction and to prove Theorem C is to determine or at least to estimate the genera $g_0^{r,k}(N)$ of $X_0^{r,k}(N)$.

11. The genus of $X_0^{r,r-1}(T^n)$.

In this section we make very specific choices for our parameters N and k, namely: $N = T^n \in A$, and $K = r - 1$. For these choices we obtain explicit albeit unpleasant formulas for the genera of the corresponding curves $X_0^{r,r-1}(T^n)$. These are calculated by applying the RH formula and the known value of $g^{r,r-1}(T^n)$ to the ramified covering $\varphi_0^{r,r-1}(T^n) : X^{r,r-1}(T^n) \longrightarrow X_0^{r,r-1}(T^n)$. In the actual calculation we restrict to the case $r = 3$; the case $r > 3$ is more complex concerning computations and presentation but uses essentially the same arguments; we give the result without full proof details in Theorem 11.13. In contrast, determining $g_0^{r,k}(T^n)$ for general k with $(r, k) = 1$ requires even more efforts and technical tools. In this case, postponed to the next section, we restrict to merely presenting the asymptotic behavior of $g_0^{r,k}(T^n)$ for $n\longrightarrow\infty$.

(11.1) In what follows (up to (11.12)), we assume $\boxed{r=3}$ and use the following simplified notation (which partially conflicts with notation used earlier).

(11.1.1)

$$\begin{array}{rcl}
N &=& T^n,\ n\in\mathbb{N},\ R=A/T^n\\
G &=& G(N)=\{\gamma\in \mathrm{GL}(3,R)\mid \det(\gamma)\in\mathbb{F}^*\}/Z\\
 && \text{(where as usual } Z \text{ is the group of } \mathbb{F}\text{-valued scalar matrices)}\\
P &=& P(N)\cap G(N)=\left\{\text{matrices of shape } \left(\begin{array}{c|cc} * & * & * \\ \hline 0 & * & * \\ 0 & * & * \end{array}\right)\right\}/Z\\
C &=& G_\infty(N)=\left\{\left(\begin{array}{c|cc} a & * & * \\ \hline 0 & & \\ 0 & & \gamma \end{array}\right) \mid \begin{array}{l} a\in\mathbb{F}^*\\ \gamma\in\mathrm{Car}\end{array}\right\}/Z, \text{ where } \gamma \text{ runs through}\\
 && \text{a fixed Cartan subgroup Car of } \mathrm{GL}(2,\mathbb{F})\\
U &=& U(N)=\text{ the } p\text{-Sylow subgroup of } C=\left\{\left(\begin{array}{c|cc} a & * & * \\ \hline 0 & a & 0 \\ 0 & 0 & a \end{array}\right)\mid a\in\mathbb{F}^*\right\}/Z
\end{array}$$

(An asterisk $*$ stands for an arbitrary element of R.)

$X=X^{3,2}(N)$, $X_0=X_0^{3,2}(N)$, $\varphi=\varphi^{3,2}(N):\ X\longrightarrow X^{3,2}\xrightarrow{\cong}\mathbb{P}^1$, $\varphi_0=\varphi_0^{3,2}(N):\ X\longrightarrow X_0$.

All these depend on n, we have

$$\text{(11.1.2)}\qquad \begin{array}{rcl}
|R| &=& |T^n|=q^n\\
|G| &=& |\mathrm{SL}(3,R)|=(q^3-1)(q^2-1)q^{8n-5}\\
|P| &=& |\mathrm{GL}(2,R)||R|^2=(q^2-1)(q-1)q^{6n-3}\\
|C| &=& (q^2-1)q^{2n}\\
|U| &=& q^{2n}.
\end{array}$$

(11.2) The cusps of X are in canonical bijection with G/C: If $\{\xi\}$ is a system of representatives for G/C, then

$$\begin{array}{ccc}
G/C & \xrightarrow{\cong} & \{\text{cusps of } X\}\\
\xi & \longmapsto & \xi(\infty)
\end{array}.$$

If $Q=\xi(\infty)$ then the ramification number a_Q in $\varphi_0:\ X\longrightarrow X_0$ is

$$\text{(11.2.1)}\qquad a_Q=|P\cap{}^\xi C|+|P\cap{}^\xi U|-2,$$

where ${}^\xi C=\xi C\xi^{-1}$ and ${}^\xi U=\xi U\xi^{-1}$, as follows from the definition of a_Q and Theorem 8.4.

Fix systems of representatives $\{x\}$ for G/P and $\{y\}$ for P/C, respectively. Then

(11.2.2) $\{xy\}$ is a system of representatives for G/C.

11.3 Lemma. $^{y}C = \left\{ \left(\begin{array}{c|cc} a & * & * \\ \hline 0 & & \\ 0 & & \gamma \end{array} \right) \mid \begin{array}{c} a \in \mathbb{F}^* \\ \gamma \in \mathrm{Car}' \end{array} \right\} /Z$, *where* γ *runs through a conjugate* Car$'$ *of* Car *in* GL$(2, R)$.

Proof. Obvious. □

As each such Car$'$ projects onto a Cartan subgroup of GL$(2,\mathbb{F})$ under Car$' \hookrightarrow$ GL$(2,R) \longrightarrow$ GL$(2,\mathbb{F})$, we also term such Car$'$ as Cartan subgroups of GL$(2,R)$. The following calculations (see Properties 11.6 and 11.8) will show that $|P \cap {}^{\xi}C|$ (and thus its p-part $|P \cap {}^{\xi}U|$) depends only on the x-part of $\xi = xy$. Therefore, to evaluate (11.2.1) we need to determine $|P \cap {}^{x}C|$ for a suitably chosen system $\{x\}$.

In the following calculations in G, we usually write matrices and perform calculations with them but mean the corresponding elements of G, that is, classes modulo Z.

11.4 Lemma. *The following set* RS *is a system of representatives for* $G/P:$ $RS = RS(1) \cup RS(2) \cup RS(3)$, *where*

$$\begin{aligned} RS(1) &= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ v & 1 & 0 \\ w & 0 & 1 \end{pmatrix} \mid v, w \in R \right\} \\ RS(2) &= \left\{ \begin{pmatrix} u & 1 & 0 \\ 1 & 0 & 0 \\ w & 0 & 1 \end{pmatrix} \mid u \in TR,\ w \in R \right\} \\ RS(3) &= \left\{ \begin{pmatrix} u & 1 & 0 \\ v & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \mid u, v \in TR \right\}. \end{aligned}$$

Elements of $RS(i)$ *are called representatives of type* i $(i = 1,2,3)$.

Proof. As $G/P \xrightarrow{\cong} \mathrm{GL}(3,R)/\tilde{P} \xrightarrow{\cong} \mathbb{P}^2(R)$, where $\tilde{P}$ is the parabolic subgroup $\left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\}$ of GL$(3,R)$, the assertion is well-known and easy to show. The second bijection maps the class of a matrix in GL$(3,R)$ to the element of $\mathbb{P}^2(R)$ determined by the first column. □

(11.5) We let $s:\ R \longrightarrow \{0, 1, \ldots, n\}$ be the truncated valuation on R, $s(a) = i$ if $a \in T^iR \setminus T^{i+1}R$ for $i < n$, $s(0) = n$. For $x \in RS(1)$ with coordinates v, w, we define $s(x) := \min\{s(v), s(w)\}$. Then

$$(11.5.1)\qquad \begin{array}{rcll} |\{x \in RS(1) \mid s(x) = s\} & = & (q^2-1)q^{2(n-1-s)} & 0 \le s < n \\ & = & 1 & s = n \end{array}.$$

11.6 Proposition.

(i) *For $x \in RS(1)$ with $s(x) = s$, we have*

$$\begin{array}{rclcrcll} |P \cap {}^xC| & = & (q-1)q^n & \textit{and thus} & |P \cap {}^xU| & = & q^n, & s = 0 \\ & = & q^{n+2s} & & \text{''} & & q^{n+2s}, & 1 \le s < \frac{n}{2} \\ & & q^{2n} & & \text{''} & & q^{2n}, & \frac{n}{2} \le s < n \\ & & (q^2-1)q^{2n} & & \text{''} & & q^{2n}, & s = n. \end{array}$$

(ii) *The formulas of (i) remain true if C is replaced by ${}^yC = yCy^{-1}$, $y \in P$.*

Proof. (i) Let $x = \begin{pmatrix} 1 & 0 & 0 \\ v & 1 & 0 \\ w & 0 & 1 \end{pmatrix} \in RS(1)$ and $g = \left(\begin{array}{c|cc} a & b & c \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{γ}} \\ 0 & & \end{array}\right)$ represent an element of C, that is, $a \in \mathbb{F}^*$, $b, c \in R$, $\gamma \in \mathrm{Car} \subset \mathrm{GL}(2, \mathbb{F})$. A small calculation shows that ${}^xg = xgx^{-1}$ lies in P if and only if the following condition holds:

$$(*)\qquad (a - (vb + wc))\begin{pmatrix} v \\ w \end{pmatrix} - \gamma \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } R^2.$$

Suppose that $\boxed{s = 0}$. Considered $(\mathrm{mod}\, T))$, $(*)$ means that $\binom{v}{w}$ is an eigenvector for γ with eigenvalue $a - (vb - wc)$. That is, γ is the scalar matrix corresponding to a', the unique element of $\mathbb{F}$ that satisfies $a' \equiv a - (vb + wc)(\,\mathrm{mod}\, T)$. Then $(*)$ becomes $(a - a' - (vb + wc))\binom{u}{v} = \binom{0}{0}$. As $\binom{u}{v} \not\equiv \binom{0}{0}$ $(\mathrm{mod}\, T)$, we find $a' = a - (vb + wc)$ belongs to $\mathbb{F}^*$, and in this case γ is the scalar matrix a'. If v is a unit in R then a, a', c may be chosen freely and we may solve for b in order that $(*)$ holds; if otherwise w is a unit then we may freely choose a, a', b and solve for c. This yields

$$|P \cap {}^xC| = (q-1)^2 q^n/(q-1) = (q-1)q^n$$

(where the denominator $q - 1$ takes care for the group Z that must be divided out).

Suppose that $\boxed{0 < s < n}$. Then $(vb + wc)\binom{v}{w} \equiv \binom{0}{0}$ $(\mathrm{mod}\, T^{2s})$, and $(*)$ yields

$$a\begin{pmatrix} v \\ w \end{pmatrix} - \gamma \begin{pmatrix} v \\ w \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ (\mathrm{mod}\, T^{s+1}),$$

which is possible only for $\gamma \equiv \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} (\operatorname{mod} T^{s+1})$, so $\gamma = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. Then $(*)$ becomes

$$(*') \qquad (vb + wc)\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ independently of } a.$$

Now suppose $\boxed{0 < s < n/2}$. Then $(*')$ is equivalent with $s(vb + wc) \geq n - s$. If $s(v) = s(x)$ then c may be freely chosen, and by Lemma 11.7 there are precisely q^{2s} many $b \in R$ such that $s(vb + w') \geq r - s$, with $w' = w \cdot c$. If otherwise $s(w) = s(x)$, b may be chosen freely, and there are q^{2s} many choices for c. We thus find $|P \cap {}^xC| = (q-1)q^{r+2s}/(q-1) = q^{r+2s}$.

If $\boxed{n/2 \leq s < n}$ then a, b and c may be chosen freely, so $|P \cap {}^xC| = (q-1)q^{2r}/(q-1) = q^{2r}$.

The case $\boxed{s = n}$, i.e., $x = 1$ is trivial.

(ii) A closer look to the counting arguments used in (i) (together with (11.3)) shows that these also apply to the case where the entry γ of g runs through an arbitrary Cartan subgroup in $\mathrm{GL}(2, R)$, i.e., a $\mathrm{GL}(2, R)$-conjugate of $\mathrm{Car} \hookrightarrow \mathrm{GL}(2, \mathbb{F}) \hookrightarrow \mathrm{GL}(2, R)$. □

11.7 Lemma. *Suppose $0 < s < n - s < n$, and let $v, w' \in R$ be given with $s(v) = s \leq s(w')$. Then there are precisely q^{2s} elements $b \in R$ such that $s(vb+w') \geq n-s$.* □

Proof. Easy and omitted. □

11.8 Proposition.

(i) *Let $x \in RS(2) \cup RS(3)$. Then $|P \cap {}^xC| = (q-1)q^n$ and thus $|P \cap {}^xU| = q^n$.*

(ii) *The assertion of (i) remains true if C is replaced by ${}^yC$, $y \in P$.*

Proof. (i) Let $x = \begin{pmatrix} u & 1 & 0 \\ 1 & 0 & 0 \\ w & 0 & 1 \end{pmatrix} \in RS(2)$, $g = \left(\begin{array}{c|cc} a & b & c \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{γ}} \\ 0 & & \end{array}\right) \in C$ be as in the proof of (11.6). Calculation yields:

$$(*) \qquad {}^xg \in P \Leftrightarrow b = 0 \text{ and } \gamma \text{ a scalar } a' \in \mathbb{R}^*.$$

Therefore $a, a' \in \mathbb{F}^*$and $c \in R$ may be arbitrarily chosen in order that ${}^xg \in P$. That is,

$$|P \cap {}^xC| = (q-1)^2 q^n/(q-1) = (q-1)q^n.$$

For $x = \begin{pmatrix} u & 1 & 0 \\ v & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \in RS(3)$ we find the same condition $(*)$, which yields the same value for $|P \cap {}^xC|$.

(ii) As in (11.6) the above arguments turn over to ${}^yC$. $\square$

(11.9) We now have control of the ramification numbers $a_Q = a_Q(\varphi_0)$ of the cusps Q. Namely,

$$(11.9.1) \qquad \sum_{Q \text{ cusp of } X} a_Q = [P:C] \sum_{x \in RS} (|P \cap {}^xC| + |P \cap {}^xU| - 2)$$

by (11.2.1), where the ingredients are given by (11.6) and (11.8).

We define for later use:

$$(11.9.2) \qquad \mathcal{T} = \mathcal{T}(3,n) = \sum_{x \in RS} (|P \cap {}^xC| + |P \cap {}^xU|)$$

11.10 Proposition. *The quantity $\mathcal{T}$ is given by*

$$\mathcal{T} = q^{2(n+1)} + (q+1)q^{3n} + 2q^{2n}(q^{2[n/2]} - 1) + 2[\frac{n-1}{2}](q^2-1)q^{3n-2},$$

where $[\ .\ ]$ *ist the Gauß bracket.*

Proof. This follows from inserting the values of $|P \cap {}^xC|$ and $|P \cap {}^xU|$ given by (11.6) and (11.8) and summing up. We omit the complicated but elementary calculations. $\square$

Due to the next observation, there are no non-cuspidal ramification contributions.

11.11 Proposition. *The map* $\varphi_0 = \varphi_0^{3,2}(T^n) : X^{3,2}(T^n) \longrightarrow X_0^{3,2}(T^n)$ *is unramified off cusps.*

Proof. The only possible ramification may appear at elliptic points. Let $Q_0 \in X = X^{3,2}(T^n)$ be one such, with ramification group $G_{Q_0} = \mathrm{Car}/Z \subset G$ with respect to φ, where $\mathrm{Car} = \mathrm{Car}(3) \cong (\mathbb{F}^{(3)})^*$ is a Cartan subgroup of $\mathrm{GL}(3,\mathbb{F})$. Then $G/G_{Q_0} \xrightarrow{\cong} \{\text{elliptic points of } X\}$. For $x \in G$ and $Q = x(Q_0)$, the ramification group w.r.t. φ_0 is $P \cap {}^xG_{Q_0} := P_Q$. As the natural map $\pi : G \longrightarrow \mathrm{GL}(3, R/T)/Z = \mathrm{PGL}(3,\mathbb{F})$ has a p-group as kernel, π maps P_Q isomorphically onto its image in $\mathrm{PGL}(3,\mathbb{F})$. But the intersection of a Cartan group with a parabolic group is trivial in $\mathrm{PGL}(3,\mathbb{F})$; hence $P_Q = \{1\}$ and Q is unramified in φ_0. $\square$

Remark. The above statement "$\varphi_0^{r,k}(N) : X^{r,k}(N) \longrightarrow X_0^{r,k}(N)$ is unramified off cusps" holds more generally for all (r,k) and all $N \in A$ that satisfy: $\gcd\{r, \deg \mathfrak{p} \mid \mathfrak{p} \text{ a prime divisor of } N\} = 1$. Otherwise there will occur some elliptic ramification.

(11.12) Next, we apply the RHF twice: to $\varphi : X \longrightarrow X^{3,2}(1) = \mathbb{P}^1$ and to $\varphi_0 : X \longrightarrow X_0$, and find for the Euler-Poincaré characteristics

$$\begin{aligned} e(X) &= 2|G| - \frac{|G|}{|C|}(|C| + |U| - 2) - |G|\frac{q^2 - q}{q^2 + q + 1} \text{ (by (9.2))} \\ &= |P|e(X_0) - \frac{|P|}{|C|}\mathcal{T} + 2\frac{|G|}{|C|} \text{ (by (11.9) and (11.11)).} \end{aligned}$$

Solving and inserting the cardinalities, we may determine $e(X_0)$ and $g(X_0) = 1 - e(X_0)/2$: see Theorem 11.13.

Now we suppress the restriction to $r = 3$ and consider the ramified covering $\varphi_0^{r,r-1}(T^n) : X^{r,r-1}(T^n) \longrightarrow X_0^{r,r-1}(T^n)$ for arbitrary $r \geq 3$. All the definitions, calculations and assertions from (11.1) to (11.2) generalize. This finally yields:

11.13 Theorem. *The Euler-Poincaré characteristic of $X_0^{r,r-1}(T^n)$ is for $r \geq 3$ given by*
(11.13.1)

$$e(X_0^{r,r-1}(T^n)) = -\frac{q^{r-1} + q - 2}{(q^{r-1} - 1)(q - 1)}q^{(r-1)(n-1)} + \frac{\mathcal{T}(r,n)}{(q^{r-1} - 1)q^{(r-1)n}}.$$

Here $\mathcal{T}(3,n)$ is as in (11.10)*, and for $r \geq 4$:*
(11.13.2)

$$\begin{aligned} \mathcal{T}(r,n) &= q^{(r-1)(n+1)} + \tfrac{q^{r-1}-1}{q-1}q^{(2r-3)n-r+3} + 2q^{(r-1)n}(q^{(r-1)[\frac{n}{2}]} - 1) \\ &\quad +2(q^{r-1} - 1)(\tfrac{q^{(r-3)\frac{n-1}{2}}-1}{q^{r-3}-1})q^{rn-2+[\frac{n}{2}](r-3)}. \end{aligned}$$

Accordingly, the genus is
(11.13.3)

$$\begin{aligned} g_0^{r,r-1}(T^n) &= g(X_0^{r,r-1}(T^n)) \\ &= 1 + \frac{q^{r-1} + q - 2}{2(q^{r-1} - 1)(q - 1)}q^{(r-1)(n-1)} - \frac{\mathcal{T}(r,n)}{2(q^{r-1} - 1)q^{(r-1)n}}. \end{aligned}$$

11.14 Remarks. (i) The special role of $r = 3$ in $\mathcal{T}(r,n)$ vanishes if we evaluate the factor $\frac{q^{(r-3)[\frac{n-1}{2}]}-1}{q^{r-3}-1}$ in the last summand of $\mathcal{T}(r,n)$ by de l'Hôpital's rule, regarding q as a variable. That factor actually comes from a geometric series in q^{r-3}, which collapses to the number $[\frac{n-1}{2}]$ of its terms for $r = 3$.

(ii) There is no reason *a priori* that (r,k)-sparse Drinfeld modules or their moduli schemes $M^{r,k}(N)$ resp. $M_0^{r,k}(N)$ should be related to $(r, r-k)$-sparse Drinfeld modules or their moduli schemes. However, our purely group theoretic description (9.3) of $g^{r,k}(N)$ shows that it remains unchanged under replacing k by $r - k$. It is easy to see that similarly $g_0^{r,k}(N) = g_0^{r,r-k}(N)$ for arbitrary data, r, k and N.

11.15 Table. The first few values of $g_0^{r,r-1}(T^n) = g_0^{r,1}(T^n)$ are given in the table

n	$g_0^{r,r-1}(T^n)$
1	0
2	0
3	$q^{r-2}(q^{r-1}+q-2)/2$
4	$q^{r-1}(q^{2r-3}+q^{2r-4}+q^{r-1}-2q^{r-3}-1)/2$

11.16 Lemma. *The quantity* $\mathcal{T}(r,n)$ *satisfies* $\lim_{n\to\infty} q^{-(r-1)(2n-1)}\mathcal{T}(r,n) = 0$.

Proof. It suffices to show the assertion for each of the four summands of $\mathcal{T}(r,n)$ in (11.13.2) or (11.10). This is obvious for the first one, and follows by small calculations for the three other terms. $\square$

The lemma means that the term containing $\mathcal{T}(r,n)$ in (11.13.3) has strictly smaller magnitude compared to the principal term. Therefore:

11.17 Corollary. $g_0^{r,r-1}(T^n) \sim \frac{q^{r-1}+q-2}{2(q^{r-1})(q-1)}q^{(r-1)(n-1)}$ *as n tends to infinity.* $\square$

Here $f \sim g$ means asymptotic equivalence of functions f, g on $\mathbb{N}$:

$$f \sim g :\Leftrightarrow \lim_{n\to\infty} f(n)/g(n) = 1.$$

12. Proof of Theorem C.

We present lower estimates for $g_0^{r,k}(N)$ for general data r, k, N, from which Theorem C will follow.

(12.1) The quantity

$$\epsilon^{r,k}(N) := [G(N) : G(N) \cap P(N)] \tag{12.1.1}$$

plays an important role in our formulas. We collect without proofs a number of easily established properties.

(12.12) The natural map induces a bijection

$$G(N)/G(N) \cap P(N) \xrightarrow{\cong} \tilde{G}(N)/P(N).$$

(12.1.3) The residue class set $\tilde{G}(N)/P(N)$ is in natural bijection with $\mathrm{Grass}^{r,r-k}(A/N)$, the set of free direct summands of dimension $r-k$ of the A/N-module $(A/N)^r$, through the map that associates with each $\gamma \in \tilde{G}(N)$ the submodule generated by the first $r-k$ columns.

(12.1.4) $\epsilon(N)$ is weakly multiplicative, that is $\epsilon^{r,k}(N_1N_2) = \epsilon^{r,k}(N_1) \cdot \epsilon^{r,k}(N_2)$ if $(N_1, N_2) = 1$.

(12.1.5) If $(N) = \mathfrak{p}^n$ with a prime $\mathfrak{p}$ of A then

$$\epsilon^{r,k}(N) = \epsilon^{r,k}(\mathfrak{p}) q^{(r-k)k(n-1)\deg \mathfrak{p}},$$

where

$$\epsilon^{r,k}(\mathfrak{p}) = |\mathrm{Grass}^{r,r-k}(\mathbb{F}_{\mathfrak{p}})| = \frac{\mathrm{GL}(r, \mathbb{F}_{\mathfrak{p}})}{|\mathrm{GL}(r-k, \mathbb{F}_{\mathfrak{p}})|\,|\mathrm{GL}(k, \mathbb{F}_{\mathfrak{p}})|} q^{-(r-k)k \deg \mathfrak{p}}.$$

Here $\mathbb{F}_{\mathfrak{p}} = A/\mathfrak{p} = \mathbb{F}^{(\deg \mathfrak{p})}$.

The preceding suffices to determine $\epsilon^{r,k}(N)$ in all cases. For $N = T^n$ we find

(12.1.6) $$\epsilon^{r,k}(T^n) = \frac{\mathrm{GL}(r, \mathbb{F})}{|\mathrm{GL}(r-k, \mathbb{F})|\ |\mathrm{GL}(k, \mathbb{F})|} q^{(r-k)k(n-2)}.$$

For such general k, the exact evaluation of the ramification numbers in $\varphi_0^{r,k}(T^n) : X^{r,k}(T^n) \longrightarrow X_0^{r,k}(T^n)$ has not yet been carried out. However, a crude estimate (omitted) shows that similar to (11.16) their contribution to the RHF is of smaller magnitude compared to the principal term. Therefore, arguing as in (11.12), we find

12.2 Proposition. $g_0^{r,k}(T^n) \sim \frac{q^{r-k}+q^k-2}{2(q^{r-k}-1)(q^k-1)} \epsilon^{r,k}(T^n)(\frac{q-1}{q^r-1})$ *as* $n \to \infty$. □

We could now use (11.17) and (12.2) to deduce asymptotic lower estimates of the ratio of numbers of $\mathbb{F}^{(r)}$-rational points/genera for the attached curves $X_0^{r,k}(T^n)$. That estimate doesn't however depend on the special towers $X_0^{r,k}(T^n)$ but is valid in much greater generality.

(12.3) For given coprime natural numbers $r-k$ and k summing up to r, we define

(12.3.1) $$C_q(r-k, k) := \frac{2(q^{r-k}-1)(q^k-1)}{q^{r-k}+q^k-2},$$

the harmonic mean of $(q^{r-k}-1)$ and (q^k-1). It occurs naturally in our estimate.

(12.4) We again use the reasoning of (11.12). Fix notation as in (11.1): $e(X)$, $e(X_0)$ and $g(X)$, $g(X_0)$ are the Euler-Poincaré characteristics and the genera of $X = X^{r-k}(N)$ and $X_0 = X_0^{r,k}(N)$, respectively, where $N \in A$ is arbitrary non-constant,

(12.4.1) $G = G(N)$, $P = P(N) \cap G(N)$, $C = G_\infty(N)$, $U =$ the p-Sylow subgroup of C,

$$\begin{aligned} A_{\mathrm{cusp}} &= A_{\mathrm{cusp}}(\varphi_0) = \sum_{Q \text{ cusp of } X} a_Q(\varphi_0), \text{ the sum of the cuspidal} \\ &\quad \text{ramification numbers in the ramified covering } \varphi_0 : X \longrightarrow X_0, \\ A'_{\mathrm{cusp}} &= \sum_Q (a_Q(\varphi_0) + 2) = A_{\mathrm{cusp}} + 2|G|/|C|, \\ A_{\mathrm{ell}} &= A_{\mathrm{ell}}(\varphi_0) = \sum_{Q \text{ elliptic point of } X} a_Q(\varphi_0) \\ &\quad \text{(which in most cases vanishes, see remark after (11.11)).} \end{aligned}$$

Then by the RHF,

$$\begin{aligned} e(X) &= 2|G| - \frac{|G|}{|C|}(|C| + |U| - 2) - |G|(\frac{q^r - q}{q-1})(\frac{q-1}{q^r-1}) \\ &= |P|e(X_0) - A_{\mathrm{cusp}} - A_{\mathrm{ell}}. \end{aligned}$$

Equating the right hand sides and cancelling, we find

$$|G| - |G\frac{|U|}{|C|} - |G|(\frac{q^r - q}{q^r - 1}) = |P|e(X_0) - A'_{\mathrm{cusp}} - A_{\mathrm{ell}},$$

thus

$$(12.4.2) \qquad |G|(1 - \frac{|U|}{|C|} - \frac{q^r - q}{q^r - 1}) = |P|e(X_0) - A'_{\mathrm{cusp}} - A_{\mathrm{ell}}.$$

Now $[C : U] = (q^{r-k} - 1)(q^k - 1)/(q - 1) =: m$, so the factor on the left hand side is

(12.4.3)

$$\begin{aligned} 1 - \frac{|U|}{|C|} - \frac{q^r - q}{q^r - 1} = \frac{q-1}{q^r - 1} - \frac{1}{m} &= (\frac{q-1}{q^r-1})(\frac{-q^{r-k} - q^k + 2}{(q^{r-k} - 1)(q^k - 1)}) \\ &= -2(\frac{q-1}{q^r-1})C_q(r-k,k)^{-1}, \end{aligned}$$

and (12.4.2) becomes

$$e(X_0) = -2(\frac{q-1}{q^r-1})C_q(r-k,k)^{-1}\frac{|G|}{|P|} + \frac{A'_{\mathrm{cusp}}}{|P|} + \frac{A_{\mathrm{ell}}}{|P|},$$

thus finally, as $|G|/|P| = \epsilon^{r,k}(N)$:

$$(12.4.4) \quad g(X_0) = 1 + \frac{q-1}{q^r-1}C_q(r-k,k)^{-1}\epsilon^{r,k}(N) - \frac{1}{2|P|}(A'_{\mathrm{cusp}} + A_{\mathrm{ell}})$$

Note that $A'_{\mathrm{cusp}} > 0$, $A_{\mathrm{ell}} \geq 0$, so $B := \frac{1}{2|P|}(A'_{\mathrm{cusp}} + A_{\mathrm{ell}}) > 0$.

(12.5) Now we assume that N is coprime with $\mathfrak{p} = (T - 1)$ and

$g(X_0) = g_0^{r,k}(N) > 0$, so that the considerations of Section 10 apply. For $\overline{X}_0(N) = \overline{X}_0^{r,k}(N)^{(\mathfrak{p})}$ we find

(12.5.1)

$$\frac{|\{\mathbb{F}^{(r)} - \text{rational points of } \overline{X}_0(N)\}|}{\text{genus of } \overline{X}_0(N)} \geq \frac{|\{\text{elliptic points of } \overline{X}_0(N)\}|}{g(X_0^{r,k}(N))}$$
$$= \frac{[G(N) : G(N) \cap P(N)]}{g(X_0^{r,k}(N))}\left(\frac{q-1}{q^r-1}\right) = \left(\frac{q-1}{q^r-1}\right)\frac{\epsilon^{r,k}(N)}{1 + \frac{q-1}{q^r-1}C_q(r-k,k)^{-1}\epsilon^{r,k}(N) - B}\ .$$

$$> \left(\frac{q-1}{q^r-1}\right)\frac{\epsilon^{r,k}(N)}{1 + (\frac{q-1}{q^r-1})C_q(r-k,k)^{-1}\epsilon^{r,k}(N)}$$

Now assume that N tends to infinity, that is N is an element of a series $(N_n)_{n\in\mathbb{N}}$ with deg $N_n \longrightarrow \infty$. Then $\epsilon^{r,k}(N) \longrightarrow \infty$, too, we may suppress the constant summand 1 in the limit, and thus

$$(12.5.2) \quad \limsup_{\deg N \to \infty} \frac{|\{\mathbb{F}^{(r)} - \text{rational points of } \overline{X}_0(N)\}|}{\text{genus of } \overline{X}_0(N)} \geq C_q(r-k,k).$$

This holds for each series $(N_n)_{n\in\mathbb{N}}$ of elements N_n of A coprime with $\mathfrak{p} = (T-1)$ and deg $N_n \longrightarrow \infty$.

Theorem C is established.

12.6 Remarks.

(i) (12.5.2) and Theorem C are valid for arbitrary *series* $(N_n)_{n\in\mathbb{N}}$ with deg $N_n \longrightarrow \infty$; so we need not assume that the $X_0^{r,k}(N_n)$ form a *tower*, i.e., conditions like $N_n | N_{n+1}$.

(ii) In establishing (12.5.2) we didn't need to evaluate A_{cusp}, A_{ell}; we just used non-negativity. Any evaluation of these will result in a sharpening of the inequality (12.5.1) on a finite level.

(iii) Suppose there exists a series $(N_n)_{n\in\mathbb{N}}$ for which the analogue of (11.16) fails, i.e., where the quantity $\frac{1}{2}A'_{\text{cusp}}/|P|$ as a function of n has the same magnitude as the principal term in (12.4.4). (The magnitude cannot be strictly larger than the principal term, and $\frac{1}{2}A_{\text{ell}}/|P|$ is smaller anyway.) Then $g(X_0^{r,k}(N_n))$ would grow slower, and we had a strictly sharper lower estimate than (12.5.2). By analogy with (11.16) and (12.2), this appears highly unlikely.

13. Concluding remarks.

In this final section we address some open questions that naturally arise from the present work, and propose topics of future research.

(13.1) In order to construct the curves that occur in Theorem C, viz

$\overline{X}_0(N) =$ reduction of $X_0^{r,k}(N)$ at the place $\mathfrak{p}$ of A of degree one, we had chosen the parabolic subgroup $P = \left\{ \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array} \right) \right\}$ of $\tilde{G}(N)$ and put $X_0^{r,k}(N) = P \backslash X^{r,k}(N)$ (see Section 10). The fact that $G_\infty(N) \subset P$ is helpful for the calculation of $g_0^{r,k}(N)$ performed in a special case in Section 11, but is not crucial for the construction of good towers as in Theorem C. Instead of P we could use any subgroup H of $\tilde{G}(N)$ with $\det(H) = (A/N)^*/\mathbb{F}^*$, for example other parabolic groups (i.e., groups H encompassing a Borel subgroup of $\tilde{G}(N)$), in particular the Borel subgroup of upper triangular matrices in $\tilde{G}(N)$. Then again, $X_H^{r,k}(N) := H \setminus X^{r,k}(N)$ is defined over K with good reduction at places $\mathfrak{p} \not| N$, and the reasoning of Section 12 yields the statement of Theorem C even for that larger class of curves. We preferred to state Theorem C as in the introduction for the $X_0^{r,k}(N)$ only, for reasons of presentation and to avoid technicalities.

(13.2) While that generalization - replacing $X_0^{r,k}(N)$ by $X_H^{r,k}(N)$ as above - doesn't give better constants in the asymptotic estimate (0.4.1), it enhances our supply of curves which are candidates for a large ratio number of rational points/genus on a finite level.

As many of the optimal or record curves in [9] are modular of some sort (elliptic, Shimura, Drinfeld ...) or derived from such curves, it is quite likely that intelligent choices of the parameters (r, k), N and H will give rise to new record curves. Therefore we propose to perform a systematic study of the $X_H^{r,k}(N)$, which essentially means to calculate their genera. To do so, we must generalize the results of section 11 from

- $k = r - 1$ to arbitrary k, notably $k \approx r/2$;
- $N = T^n$ to arbitrary elements N of A;
- $H = P$ to arbitrary subgroups H as in (13.1).

There will result awfully complicated formulas, but which will allow a computer-aided search for interesting curves.

(13.3) The curves $X^{r,k}(N)$ are defined as the projective models of $Y^{r,k}(N) = \Gamma(N) \setminus \Omega^{r,k}$, where only the latter have an immediate modular interpretation. In our work we by-passed any modular interpretation of the cusps. It is desirable to work this out and to describe the local geometry of $X^{r,k}(N)$ at its intersection with the boundary of $\Gamma(N) \setminus \Omega^r$.

(13.4) The reduction $\overline{\mathcal{F}}$ of $\mathcal{F}_0$ modulo ∞ (see the proof of Proposition 6.8) is some local version of $X^{r,k}(N)$, but interesting in its own as a (not necessarily connnected) affine curve over $\mathbb{F}$. Describe its connected components and their projective models!

(13.5) What is the motivic significance of the curves $X_H^{r,k}(N)$? More specifically, is there a comprehensive characterization of the abelian varieties that occur as isogeny factors of the Jacobian of $X_H^{r,k}(N)$ similar to the case $X_0^{2,1}(N)$ dealt with in [17] Sect. 8?

(13.6) Instead of the vanishing locus $\Omega^{r,k}$ of $g_1, \ldots, \hat{g}_k, \ldots, g_{r-1}$ in Ω^r one might consider the vanishing locus of less than $r-2$ of the forms g_i (or of other modular forms on Ω^r). The associated subvarieties of $\Gamma(N) \setminus \Omega^r$ can be studied with methods similar to those of the present paper.

(13.7) It would be interesting to describe the image $\lambda(\Omega^{r,k})$ of $\Omega^{r,k}$ (or of more general vanishing loci as in (13.6)) under the building map $\lambda : \Omega^r \longrightarrow I(K_\infty^r)(\mathbb{Q})$ of (6.5). We expect that $\lambda(\Omega^{r,k})$ is a sub-tree $\mathcal{T}^{r,k}$ of the simplicial complex $I(K_\infty^r)$, and the graph $\Gamma(N) \setminus \mathcal{T}^{r,k}$ should provide a combinatorial picture of the canonical reduction at ∞ of $\Gamma(N) \setminus \Omega^{r,k}$. Presumably, this allows a study of $\Gamma(N) \setminus \mathcal{T}^{r,k}$ similar to that performed in [16] for the case $(r,k) = (2,1)$.

The preceding isn't but a brief extract of a long list of natural questions that come up in connection with the moduli of sparse Drinfeld modules and the curves $X^{r,k}(N)$.

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