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**Majorization, 4G Theorem and Schrödinger  
perturbations**

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## Majorization, 4G Theorem and Schrödinger perturbations

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### Abstract

Schrödinger perturbations of transition densities by singular potentials may fail to be comparable with the original transition density. A typical example is the transition density of a subordinator perturbed by any unbounded potential. In order to estimate such perturbations it is convenient to use an auxiliary transition density majorant and the 4G inequality, which is a modification of the 3G inequality, involving the original transition density and the majorant. We prove 4G inequality for the 1/2-stable and inverse Gaussian subordinators, discuss the corresponding class of admissible potentials and indicate estimates for the resulting transition densities of Schrödinger operators. The connection of transition densities to their generators is made via the weak-type notion of fundamental solution, and we prove a uniqueness result for fundamental solutions in the generality of strongly continuous operator semigroups.

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## 1 Introduction and Preliminaries

Schrödinger perturbation consists in adding to a given operator an operator of multiplication. On the level of inverse operators the addition results in perturbation series. We focus on transition densities  $p$  perturbed by non-negative functions  $q$ . Our main goal is to give pointwise estimates for the resulting perturbation series  $\tilde{p}$  under suitable integral conditions on  $p$  and  $q$ . For instance, bounded potentials  $q$  produce transition densities  $\tilde{p}$  comparable with the original  $p$  in finite time. In a series of recent papers, integral conditions leading to comparability of  $\tilde{p}$  and  $p$  were proposed which allow for explicit and rather singular potentials  $q$  if  $p$  satisfies the 3G Theorem [2, 4]. The integral conditions compare the second term in the perturbation series, that which is linear in  $q$ , with  $p$ , the first term of the series. The comparison is meant to prevent the instantaneous blowup and to control the long-time accumulation of mass. The first property is more crucial and gets secured by smallness conditions, like  $0 \leq \eta < 1$  below. The results are analogues of Gronwall inequality [3] and they utilize  $p$  as an approximate majorant for  $\tilde{p}$  in finite time [4]. Similar estimates for Green-type kernels were recently obtained in [8], [11], [9].

The 3G Theorem, which is related to the quasi-metric condition [8], is common for transition densities with power-type decay, e.g. the transition density of the fractional Laplacian. However, already the Gaussian kernel fails to

satisfy 3G. In [5] and [3] a more flexible majorization technique is proposed, motivated by earlier results of [17]. Namely, another transition density  $p^*$  serves as an approximate majorant for the perturbation series. Introducing  $p^*$  is not merely a technical device: for unbounded  $q$ ,  $\tilde{p}$  may fail to be comparable with  $p$  in finite time. As we show below, this is always the case if  $p$  is the transition density of a subordinator. Finding an appropriate  $p^*$  is essentially tantamount to estimating  $\tilde{p}$ , cf. (1.10), and may be hard, but occasionally  $p^*$  suffices that is a dilation of  $p$ . So is the case for the 1/2-stable subordinator, and the slightly more general inverse Gaussian subordinator, which are the focal examples in this work. The  $p^*$  majorization technique involves an integral smallness condition for  $p$ ,  $q$  and  $p^*$ , which is implied by the familiar Kato-type conditions, provided  $p$  and  $p^*$  satisfy 4G inequality. In this paper we prove a 4G inequality for the transition density of the inverse Gaussian subordinator, including the 1/2-stable subordinator, reveal a wide class of unbounded Schrödinger potentials admissible for  $p$ , and estimate Schrödinger perturbations series for this transition density using the framework of [5]. We thus extend the scope of the  $p^*$  majorization technique for Schrödinger perturbations beyond the transition densities of diffusion processes discussed in [5]. We expect 4G to be valid quite generally, but at present it is even open for the  $\alpha$ -stable subordinators with  $\alpha \neq 1/2$ . We note that the methods of [4], which make assumptions on potentials  $q$  in terms of *bridges* (see also [2]), fail for unbounded  $q$  in this case. In fact, if  $p$  is the transition density of a subordinator and  $q \geq 0$  is unbounded, then  $p$  and  $\tilde{p}$  are never comparable, which is proved in Section 3. The above case study and general results explain why we propose 4G and the framework of [5] as a viable general method to deal with unbounded Schrödinger perturbations of transition densities.

The structure of the paper is as follows. Below in this section we give notation and preliminaries. In Section 2 we present 4G inequality and applications to Kato-type perturbations for the 1/2-stable subordinator and the inverse Gaussian subordinator. In Section 3 we discuss unbounded perturbations  $q$ , when applied to subordinators. In Section 4 we discuss the notion of the fundamental solution and give application to Lévy-type generators.

Let  $X$  be an arbitrary set with a  $\sigma$ -algebra  $\mathcal{M}$  and a (non-negative)  $\sigma$ -finite measure  $m$  defined on  $\mathcal{M}$ . To simplify the notation we write  $dz$  for  $m(dz)$  in what follows. We also consider the Borel subsets  $\mathcal{B}$  of  $\mathbb{R}$ , and the Lebesgue measure,  $du$ , defined on  $\mathbb{R}$ . The *space-time*,  $\mathbb{R} \times X$ , is equipped with the  $\sigma$ -algebra  $\mathcal{B} \times \mathcal{M}$  and the product measure  $du dz = du m(dz)$ . We consider a *measurable transition density*  $p$  on space-time, i.e. we assume that  $p : \mathbb{R} \times X \times \mathbb{R} \times X \rightarrow [0, \infty]$  is  $\mathcal{B} \times \mathcal{M} \times \mathcal{B} \times \mathcal{M}$ -measurable and the following

Chapman-Kolmogorov equations hold for all  $x, y \in X$  and  $s < u < t$ :

$$\int_X p(s, x, u, z)p(u, z, t, y) dz = p(s, x, t, y). \quad (1.1)$$

All the functions considered below are assumed measurable on their respective domains. We consider (nonnegative and  $\mathcal{B} \times \mathcal{M}$ -measurable) function  $q: \mathbb{R} \times X \rightarrow [0, \infty]$ . The Schrödinger perturbation  $\tilde{p}$  of  $p$  by  $q$  is defined as

$$\tilde{p}(s, x, t, y) = \sum_{n=0}^{\infty} p_n(s, x, t, y), \quad (1.2)$$

where  $p_0(s, x, t, y) = p(s, x, t, y)$  and, for  $n = 1, 2, \dots$ ,

$$p_n(s, x, t, y) = \int_s^t \int_X p(s, x, u, z)q(u, z)p_{n-1}(u, z, t, y)dzdu. \quad (1.3)$$

The above is an explicit method of constructing of new transition densities. In particular,  $\tilde{p}$  satisfies the Chapman-Kolmogorov equations [2, Lemma 2]. Since  $q \geq 0$ , we trivially have  $\tilde{p} \geq p$ , and we focus on upper bounds for  $\tilde{p}$ . These may be obtained under suitable conditions on  $p_1$ . In [4] (see also [2], [13] and [18, Lemma 3.1]), the authors assume that for all  $s < t$ ,  $x, y \in X$ ,

$$\int_s^t \int_X p(s, x, u, z)q(u, z)p(u, z, t, y)dzdu \leq [\eta + Q(s, t)]p(s, x, t, y), \quad (1.4)$$

where  $0 \leq \eta < \infty$  and  $Q$  is superadditive:  $0 \leq Q(s, u) + Q(u, t) \leq Q(s, t)$ . The following sharp estimates follow: for all  $s < t$ ,  $x, y \in X$ ,

$$\tilde{p}(s, x, t, y) \leq p(s, x, t, y) \left( \frac{1}{1 - \eta} \right)^{1+Q(s,t)/\eta}, \quad (1.5)$$

provided  $0 < \eta < 1$ , and for  $\eta = 0$  we even have

$$\tilde{p}(s, x, t, y) \leq p(s, x, t, y)e^{Q(s,t)}. \quad (1.6)$$

The condition (1.4) may be considered as property of relative boundedness of  $q$ , or Miyadera-type condition for bridges [14, 2]. It is convenient to use (1.4), e.g., for the transition density of the isotropic  $\alpha$ -stable Lévy process because the so-called 3G inequality holds in this case:

$$p(s, x, u, z) \wedge p(u, z, t, y) \leq \text{const } p(s, x, t, y), \quad s < u < t, \quad x, y, z \in \mathbb{R}^d.$$

3G simplifies the verification of (1.4) and essentially specifies the acceptable growth of  $q$ , cf. [2, Corollary 11], [4, Section 4]. In general, however,

condition (1.4) may be troublesome. For instance the transition density of the Brownian motion fails to satisfy 3G and (1.4) may be difficult to verify. Moreover, as we see below, for some transition densities (1.4) holds for  $q(u, z) = q(z)$  only if  $q$  is bounded. This explains the need for modifications of [4]. The approach of [5] is based on the assumption that for all  $s < t$ ,  $x, y \in X$ ,

$$\int_s^t \int_X p(s, x, u, z) q(u, z) p^*(u, z, t, y) dz du \leq [\eta + Q(s, t)] p^*(s, x, t, y). \quad (1.7)$$

Here, furthermore, it is assumed that  $0 \leq \eta < \infty$ ,  $Q(s, t)$  is superadditive, right-continuous in  $s$  and left-continuous in  $t$ , and  $p^*$  is a majorizing transition density, i.e. there is a constant  $C \geq 1$  such that for all  $s < t$  and  $x, y \in X$ ,

$$p(s, x, t, y) \leq C p^*(s, x, t, y). \quad (1.8)$$

The above assumptions are abbreviated to  $q \in \mathcal{N}(p, p^*, C, \eta, Q)$ . By [5, Theorem 1.1], if  $q \in \mathcal{N}(p, p^*, C, \eta, Q)$  with  $\eta < 1$ , then for  $\varepsilon \in (0, 1 - \eta)$ ,

$$\tilde{p}(s, x, t, y) \leq p^*(s, x, t, y) \left( \frac{C}{1 - \eta - \varepsilon} \right)^{1 + \frac{Q(s, t)}{\varepsilon}}, \quad s < t, x, y \in X. \quad (1.9)$$

For instance  $p^*(s, x, t, y) = p(cs, x, ct, y)$  with  $c > 1$  is a convenient choice for the Gaussian kernel [5]. In principle, (1.7) relaxes (1.4) and allows for more functions  $q$ . This is seen in [5] and again in Section 3 below, where we consider applications to transition densities of subordinators. We should note that the flexibility comes at the expense of the sharpness of the resulting estimate, as seen from comparing (1.5) and (1.6) with (1.9). Also, the methods of [5] and the present paper are restricted to transition densities, while the methods of [4] handle general *forward* integral kernels. Last but not least, it may be cumbersome to point out  $p^*$  suitable for  $p$ , which essentially requires guessing the rate of inflation in  $\tilde{p}$  for a given class of perturbations  $q$ . In this connection we note that, trivially,

$$\int_s^t \int_X p(s, x, u, z) \eta q(u, z) \tilde{p}(u, z, t, y) dz du \leq \eta \tilde{p}(s, x, t, y). \quad (1.10)$$

Thus, for perturbations of  $p$  by  $\eta q \geq 0$  with  $0 \leq \eta < 1$  one may take  $p^* = \tilde{p}$ , which indicates that estimating  $\tilde{p}$  and finding an appropriate majorant  $p^*$  are related. Comparing to the approach of [4] we finally note that  $p^*$  should reflect the growth patterns of  $\tilde{p}$ , which  $p$  is not always able to do.

We say that  $q: \mathbb{R} \times X \rightarrow \mathbb{R}$  satisfies the parabolic Kato condition if

$$\lim_{h \rightarrow 0^+} \sup_{s \in \mathbb{R}, x \in X} \int_s^{s+h} \int_X p(s, x, u, z) q(u, z) dz du = 0, \quad (1.11)$$



and

$$\lim_{h \rightarrow 0^+} \sup_{t \in \mathbb{R}, y \in X} \int_{t-h}^t \int_X p(u, z, t, y) q(u, z) dz du = 0, \quad (1.12)$$

cf. [2, (29), (30)]. It is sometimes useful to strengthen (1.11) and (1.12) by adding possible time change (see [17]): we say that  $q: \mathbb{R} \times X \rightarrow \mathbb{R}$  belongs to the parabolic Kato class if for every  $c > 0$ ,

$$\lim_{h \rightarrow 0^+} \sup_{s \in \mathbb{R}, x \in X} \int_s^{s+h} \int_X p(cs, x, cu, z) |q(u, z)| dz du = 0, \quad (1.13)$$

and

$$\lim_{h \rightarrow 0^+} \sup_{t \in \mathbb{R}, y \in X} \int_{t-h}^t \int_X p(cu, z, ct, y) |q(u, z)| dz du = 0. \quad (1.14)$$

For time-independent  $q$ , i.e. when  $q(u, z) = q(z)$ , both parabolic Kato conditions are equivalent. For details of the relations between (1.4) and (1.11), (1.12) we refer the reader to [2]. A similar discussion can be carried out for (1.7), and if we specify  $p^*(s, x, t, y) = p(cs, x, ct, y)$ , then (1.14) will be involved.

Of particular interest is the special case of convolution semigroups of probability measures  $\{p_t\}_{t \geq 0}$  on  $\mathbb{R}^d$  defined by the generating (Lévy) triplets  $(A, b, \nu)$  [15], and generators

$$\begin{aligned} Lf(x) &= \frac{1}{2} \sum_{j,k=1}^d A_{j,k} \frac{\partial^2 f}{\partial x_j \partial x_k}(x) + \sum_{j=1}^d b_j \frac{\partial f}{\partial x_j}(x) \\ &\quad + \int_{\mathbb{R}^d} \left( f(x+y) - f(x) - \sum_{j=1}^d y_j \frac{\partial f}{\partial x_j}(x) 1_{|y| \leq 1}(y) \right) \nu(dy). \end{aligned} \quad (1.15)$$

Let  $P_t f(x) = \int_{\mathbb{R}^d} f(z+x) p_t(dz)$ ,  $t \geq 0$ . Recall that  $P = (P_t)_{t \geq 0}$  forms a strongly continuous semigroup on  $(C_0(\mathbb{R}^d), \|\cdot\|_\infty)$ , whose infinitesimal generator  $L$  coincides with (1.15) on  $C_0^2(\mathbb{R}^d)$ . For all  $s \in \mathbb{R}$ ,  $x \in \mathbb{R}^d$  and  $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$  (smooth compactly supported functions on space-time  $\mathbb{R} \times \mathbb{R}^d$ ) we have

$$\int_s^\infty \int_{\mathbb{R}^d} p_{u-s}(dz) \left[ \partial_u \phi(u, x+z) + L\phi(u, x+z) \right] dz du = -\phi(s, x). \quad (1.16)$$

The identity is essentially a consequence of the fundamental theorem of calculus and is proved in the generality of strongly continuous operator semigroups

in Section 4. In particular we provide a uniqueness result for fundamental solutions. A special case of  $L$  is the Weyl derivative of order  $1/2$  on the real line:

$$\partial^{1/2}f(x) = \pi^{-1/2} \int_x^\infty f'(z)(z-x)^{-1/2} dz, \quad f \in C_c^1(\mathbb{R}). \quad (1.17)$$

We then have

$$p_t(dz) = (4\pi)^{-1/2} t z^{-3/2} \exp\{-t^2/(4z)\} \mathbf{1}_{z>0} dz, \quad (1.18)$$

the distribution of the  $1/2$ -stable subordinator [15] (also called the Lévy subordinator). More generally we let  $\lambda \geq 0$ ,  $\delta > 0$  and consider

$$p(t, z) = (4\pi)^{-1/2} \delta t z^{-3/2} \exp\left\{-\frac{(\delta t - 2\sqrt{\lambda}z)^2}{4z}\right\} \mathbf{1}_{z>0}, \quad z \in \mathbb{R}, \quad t > 0. \quad (1.19)$$

We note that  $p(t, x)$  is the density function of the inverse Gaussian subordinator i.e. the process  $\xi_t = \inf\{s > 0 : B_s + \gamma s = \sigma t\}$ , where  $B$  is the standard one-dimensional Brownian motion,  $\sigma = \delta/\sqrt{2}$  and  $\gamma = \sqrt{2\lambda}$  (cf. [1]). For  $f \in C_c^1(\mathbb{R})$  the corresponding generator is calculated as

$$Lf(x) = \frac{1/2}{\Gamma(1/2)} \int_x^\infty f'(z) \Gamma_\lambda(-1/2, z-x) dz. \quad (1.20)$$

Here  $\Gamma_\lambda(a, z) = \int_z^\infty e^{-\lambda y} y^{a-1} dy$  for  $\lambda, z > 0$ ,  $a \in \mathbb{R}$ , is the incomplete gamma function. For the readers's convenience we prove (1.17) and (1.20) in Section 4. The generator  $L$  is the pseudo-differential operator with the Fourier symbol  $u \mapsto \sigma(\sqrt{\gamma^2 + 2iu} - \gamma)$ , and the Laplace symbol  $u \mapsto \sigma(\sqrt{\gamma^2 + 2u} - \gamma)$ , see, e.g., [1].

## 2 4G inequality for the inverse Gaussian subordinator

Let  $\lambda \geq 0$ ,  $\delta > 0$  and let  $p$  be the density function given by (1.19). This density function may be obtained from the density function of the  $1/2$ -stable subordinator by the Esscher transform and time rescaling, see [15, Example 33.15] or [6, Sec. 4.4.2]. Namely, the Lévy measure  $\rho$  of the inverse Gaussian subordinator is obtained by the exponential tilting of the Lévy measure  $\nu$  of the  $1/2$ -stable subordinator:

$$\nu(dy) = \frac{1/2}{\Gamma(1/2)} y^{-3/2} \mathbf{1}_{y>0} dy \quad \text{and} \quad \rho(dy) = e^{-\lambda y} \nu(dy).$$

For  $c > 0$ ,  $0 \leq s \leq t$  and  $0 \leq x \leq y$  we define the transition density

$$p_c(s, x, t, y) := cp(c(t-s), c(y-x)), \quad (2.1)$$

where  $p$  is given by (1.19). If  $0 < a < b$ , then

$$p_b(s, x, t, y) \leq (b/a)^{1/2} p_a(s, x, t, y). \quad (2.2)$$

We observe that the 3G inequality does not hold for  $p_c$ . Indeed, if  $u - s = t - u = z - x = y - z = \theta$ , then

$$\begin{aligned} p_c(s, x, u, z) \wedge p_c(u, z, t, y) &= (4\pi)^{-1/2} \delta \sqrt{c} \theta^{-1/2} \exp \left\{ -c\theta(\delta - \sqrt{\lambda})^2/4 \right\}, \\ p_c(s, x, t, y) &= (4\pi)^{-1/2} \delta \sqrt{c} (2\theta)^{-1/2} \exp \left\{ -2c\theta(\delta - \sqrt{\lambda})^2/4 \right\}, \end{aligned}$$

and the second expression decays exponentially faster as  $\theta \rightarrow \infty$ .

We recall results of [5, Section 3] on the Gaussian kernel

$$g_c(s, \bar{x}, t, \bar{y}) := [4\pi(t-s)/c]^{-d/2} \exp \left\{ -|\bar{y} - \bar{x}|^2/[4(t-s)/c] \right\}, \quad (2.3)$$

where  $c > 0$ ,  $0 < s < t$ ,  $\bar{x}, \bar{y} \in \mathbb{R}^d$  and  $d \in \mathbb{N}$ . Namely, let

$$L(\alpha) = \max_{\tau \geq \alpha\sqrt{1/\alpha}} \left[ \ln(1+\tau) - \frac{\tau - \alpha}{1+\tau} \ln(\alpha\tau) \right],$$

and for  $0 < a < b$  denote  $M = \left(\frac{b}{b-a}\right)^{d/2} \exp \left[ \frac{d}{2} L\left(\frac{a}{b-a}\right) \right]$ . Then,

$$g_b(s, \bar{x}, u, \bar{z}) g_a(u, \bar{z}, t, \bar{y}) \leq M [g_{b-a}(s, \bar{x}, u, \bar{z}) \vee g_a(u, \bar{z}, t, \bar{y})] g_a(s, \bar{x}, t, \bar{y}), \quad (2.4)$$

where  $s < u < t$  and  $\bar{x}, \bar{z}, \bar{y} \in \mathbb{R}^d$  [5, Theorem 1.3 and Remark 3.2]. Moreover,  $M$  is the optimal constant in (2.4). This 4G inequality was used in [5] to obtain Gaussian estimates for fundamental solutions of Schrödinger perturbations of second order parabolic differential operators. In this section we prove the similar inequality for the transitional density  $p_c$  defined in (2.1).

**Theorem 2.1 (4G).** *Let  $0 < a < b$ . For all  $s < u < t$  and  $x < z < y$ ,*

$$p_b(s, x, u, z) p_a(u, z, t, y) \leq D \left[ p_{b-a}(s, x, u, z) \vee p_a(u, z, t, y) \right] p_a(s, x, t, y) \quad (2.5)$$

*holds with  $D = \left(\frac{b}{b-a}\right)^{3/2} \exp \left[ \frac{3}{2} L\left(\frac{a}{b-a}\right) \right]$ .*

*Proof.* We denote  $\bar{r} = (r, 0, 0) \in \mathbb{R}^3$  for  $r \in \mathbb{R}$ . For  $c > 0$ ,  $s < t$ ,  $x < y$ ,

$$p_c(s, x, t, y) = (4\pi\delta(t-s)/c) g_c(x, \delta\bar{s} - 2\sqrt{\lambda\bar{x}}, y, \delta\bar{t} - 2\sqrt{\lambda\bar{y}}).$$

By (2.4) for all  $s < u < t$  and  $x < z < y$  we have

$$\begin{aligned} & p_b(s, x, u, z)p_a(u, z, t, y) \\ &= \frac{(4\pi\delta)^2(u-s)(t-u)}{ab} g_b(x, \delta\bar{s} - 2\sqrt{\lambda\bar{x}}, z, \delta\bar{u} - 2\sqrt{\lambda\bar{z}}) g_a(z, \delta\bar{u} - 2\sqrt{\lambda\bar{z}}, y, \delta\bar{t} - 2\sqrt{\lambda\bar{y}}) \\ &\leq \frac{(4\pi\delta)^2(u-s)(t-u)}{ab} D g_a(x, \delta\bar{s} - 2\sqrt{\lambda\bar{x}}, y, \delta\bar{t} - 2\sqrt{\lambda\bar{y}}) \times \\ &\quad \times [g_{b-a}(x, \delta\bar{s} - 2\sqrt{\lambda\bar{x}}, z, \delta\bar{u} - 2\sqrt{\lambda\bar{z}}) \vee g_a(z, \delta\bar{u} - 2\sqrt{\lambda\bar{z}}, y, \delta\bar{t} - 2\sqrt{\lambda\bar{y}})] \\ &\leq D \left[ \left( \frac{t-u}{t-s} \frac{b-a}{b} \right) p_{b-a}(s, x, u, z) \vee \left( \frac{u-s}{t-s} \frac{a}{b} \right) p_a(u, z, t, y) \right] p_a(s, x, t, y) \\ &\leq D \left[ p_{b-a}(s, x, u, z) \vee p_a(u, z, t, y) \right] p_a(s, x, t, y). \end{aligned}$$

□

Recall that the generator  $L$  of the process  $\xi_t$  with transition density  $p$  is given by (1.20). Here is a connection of  $\tilde{p}$  with  $L + q$ .

**Lemma 2.2.** *If  $q \in \mathcal{N}(p, p_a, (1/a)^{1/2}, Q, \eta)$ , where  $0 < a < 1$ ,  $\eta \in [0, 1)$  and  $p, p_a$  are given by (1.19) and (2.1) with  $c = a$ , respectively, then*

$$\int_s^\infty \int_{\mathbb{R}} \tilde{p}(s, x, u, z) \left[ \partial_u \phi(u, z) + L\phi(u, z) + q(u, z)\phi(u, z) \right] dz du = -\phi(s, x),$$

for  $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$ .

*Proof.* By (1.16) we have

$$\int_s^\infty \int_{\mathbb{R}} p(s, x, u, z) \left[ \partial_u \phi(u, z) + L\phi(u, z) \right] dz du = -\phi(s, x).$$

Then the argument is similar to that of [4, Lemma 4].

□

By Lemma 2.2 and Chapman-Kolmogorov, for  $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R})$  we obtain

$$\begin{aligned} & \int_s^t \int_{\mathbb{R}} \tilde{p}(s, x, u, z) \left[ \partial_u \phi(u, z) + L\phi(u, z) + q(u, z)\phi(u, z) \right] dz du \\ &= \int_{\mathbb{R}} \tilde{p}(s, x, t, z) \phi(t, w) dw - \phi(s, x), \quad s < t, \quad x \in \mathbb{R}, \end{aligned}$$

and by choosing  $\phi$  constant in time on  $(s, t)$ , for  $\varphi \in C_c^\infty(\mathbb{R})$  we get

$$\int_{\mathbb{R}} \tilde{p}(s, x, t, z) \varphi(z) dz - \varphi(x) = \int_s^t \int_{\mathbb{R}} \tilde{p}(s, x, u, z) \left[ L\varphi(z) + q(u, z)\varphi(z) \right] dz du.$$

Similar relationships hold for strongly continuous operator semigroups. A detailed discussion of the connection between  $\tilde{p}$  and  $L + q$  is given in Section 4. We now investigate the class  $\mathcal{N}(p, p^*, \eta, Q, C)$ , where  $p(s, x, t, y) = p_b(s, x, t, y)$ ,  $p^*(s, x, t, y) = p_a(s, x, t, y)$ ,  $C = (b/a)^{1/2}$  and  $0 < a < b$ . We only need to propose conditions that lead to (1.7). We start with a direct consequence of Theorem 2.1.

**Corollary 2.3.** *Let  $q: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and assume that for all  $s < t$ ,  $x < y$ ,*

$$D \int_s^t \int_{\mathbb{R}} \left[ p_{b-a}(s, x, u, z) + p_a(u, z, t, y) \right] |q(u, z)| dz du \leq \eta + Q(s, t).$$

*Then  $|q| \in \mathcal{N}(p_b, p_a, (b/a)^{1/2}, \eta, Q)$ .*

For  $V: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and  $c, h > 0$  we let

$$\begin{aligned} N_h^c(V) &= \sup_{s,x} \int_s^{s+h} \int_{\mathbb{R}} p_c(s, x, u, z) |V(u, z)| dz du + \sup_{t,y} \int_{t-h}^t \int_{\mathbb{R}} p_c(u, z, t, y) |V(u, z)| dz du. \end{aligned}$$

**Proposition 2.4.** *Let  $0 < a < b$  and  $D' = \left(\frac{b-a}{a} \vee \frac{a}{b-a}\right)^{1/2} D$ . If  $q: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is such that*

$$N_h^{(b-a) \wedge a}(q) \leq \eta/D', \quad (2.6)$$

*for some  $0 < h \leq \infty$ , then*

$$|q| \in \mathcal{N}(p_b, p_a, (b/a)^{1/2}, \eta, Q),$$

*where  $Q(s, t) = \eta(t - s)/h$ .*

*Proof.* Follow [5, p. 165]. □

The condition  $\lim_{h \rightarrow 0} N_h^c(q) = 0$  for all  $c > 0$ , defines the parabolic Kato condition, cf. Section 1, and if it is satisfied, then Proposition 2.4 applies. A thorough discussion of the Kato condition for arbitrary Lévy processes on  $\mathbb{R}^d$  is given in the forthcoming paper [10]. For the considered inverse Gaussian

subordinator (1.19), including the 1/2-stable subordinator, if  $q(u, z) = q(z)$  is time-homogeneous, then the Kato condition is equivalent to

$$\lim_{r \rightarrow 0^+} \sup_{x \in \mathbb{R}} \int_{x-r}^{x+r} |q(z)| |z - x|^{-1/2} dz = 0.$$

We refer the reader to [10] for this result. In the remainder of this section we only consider the case  $\lambda = 0$ , i.e. the 1/2-stable subordinator, with emphasis on honest constants in estimates which are directly available in this case.

*Example.* Let  $r > 2$  and  $f: \mathbb{R} \rightarrow [0, \infty] \in L^r(\mathbb{R})$ . We consider  $q(u, z) = f(z)$ . Observe that for  $s < u$ ,

$$\int_{\mathbb{R}} p_c^\delta(s, x, u, z) dz = \frac{c'_\delta}{c^{\delta-1}} (u - s)^{-2(\delta-1)}, \quad \delta \geq 1,$$

where  $c'_\delta = (4\pi)^{-\delta/2} (4/\delta)^{3\delta/2-1} \Gamma(3\delta/2 - 1) \leq [(4\pi)^{-1/2} (6/e)^{3/2}]^{\delta-1}$ . By Hölder's inequality, for  $h > 0$ ,

$$\begin{aligned} \sup_{s,x} \int_s^{s+h} \int_{\mathbb{R}} p_c(s, x, u, z) q(u, z) dz du \\ \leq \sup_{s,x} \int_s^{s+h} (u - s)^{-2/r} du \frac{(c'_{r/(r-1)})^{(r-1)/r}}{c^{1/r}} \|f\|_r \\ = h^{1-2/r} \left[ (c'_{r/(r-1)})^{(r-1)/r} c^{-1/r} \|f\|_r / (1 - 2/r) \right]. \end{aligned}$$

Thus for every  $c > 0$ ,

$$N_h^c(q) \leq h^{1-2/r} 2 \left[ \frac{(c'_{r/(r-1)})^{(r-1)/r}}{(1 - 2/r) c^{1/r}} \|f\|_r \right] \rightarrow 0, \quad \text{if } h \rightarrow 0^+. \quad (2.7)$$

Notice also that  $(c'_{r/(r-1)})^{(r-1)/r} \leq [(4\pi)^{-1/2} (6/e)^{3/2}]^{1/r}$ . Finally, by Proposition 2.4 we obtain that for all  $0 < a < b$ , we have

$$q \in \mathcal{N}(p_b, p_a, (b/a)^{1/2}, \eta, Q),$$

with arbitrary  $\eta > 0$  and  $Q(s, t) = \eta(t - s)/h$ , provided  $h$  satisfies

$$h^{1-2/r} \frac{2D \left( \frac{b-a}{a} \vee \frac{a}{b-a} \right)^{1/2}}{(1 - 2/r)} \left[ \frac{(4\pi)^{-1/2} (6/e)^{3/2}}{(b-a) \wedge a} \right]^{1/r} \|f\|_r = \eta.$$

Indeed, (2.7) implies (2.6).

We keep investigating the class  $\mathcal{N}(p_b, p_a, (b/a)^{1/2}, \eta, Q)$  by estimating  $N_h^c(q)$  for time-independent  $q$ . We prove an auxiliary lemma in a general case of  $\alpha$ -stable subordinator,  $\alpha \in (0, 1)$ . Let  $U: \mathbb{R} \rightarrow \mathbb{R}$  and

$$I_\delta(U) = \sup_{x \in \mathbb{R}} \int_{|x-z| < \delta} \frac{|U(z)|}{|x-z|^{1-\alpha}} dz, \quad \delta > 0.$$

**Lemma 2.5.** *For all  $c, r, \tau > 0$  and  $0 < \alpha < 1$ ,*

$$\sup_{s \in \mathbb{R}, x \in \mathbb{R}} \int_s^{s+\tau} \int_{\mathbb{R}} p_c(s, x, u, z) |U(z)| dz du \leq \left( \frac{1}{c^{1-\alpha} \Gamma(\alpha)} + \frac{\tau}{r^\alpha} \right) I_r(U).$$

Here  $p_c(s, x, t, y) := c p(c(t-s), c(y-x)) = p(c^{1-\alpha}s, x, c^{1-\alpha}t, y)$  and  $p$  is the transition density of the  $\alpha$ -stable subordinator.

*Proof.* Let  $c > 0$  and  $k(x) = \int_0^\tau p_c(0, 0, u, |x|) du$ ,  $K(x) = \int_0^\infty p_c(0, 0, u, |x|) du = |x|^{\alpha-1} / (c^{1-\alpha} \Gamma(\alpha))$  in [5, Lemma 4.2]. Observe that  $c_1 = \tau$  and  $c_2 = r^\alpha / (c^{1-\alpha} \Gamma(\alpha))$ .  $\square$

A direct consequence is that for any  $\alpha$ -stable subordinator for all  $s < t$ ,  $x < t$  and  $h > 0$  we have

$$\begin{aligned} & \int_s^t \int_{\mathbb{R}} \left[ p_{b-a}(s, x, u, z) + p_a(u, z, t, y) \right] |q(z)| dz du p_a(s, x, t, y) \\ &= I_{h^{1/\alpha}}(q) \left[ \frac{1}{\Gamma(\alpha)} \frac{a^{1-\alpha} + (b-a)^{1-\alpha}}{[a(b-a)]^{1-\alpha}} + \frac{2(t-s)}{h} \right] p_a(s, x, t, y). \end{aligned}$$

For  $\alpha = 1/2$  we may use Theorem 2.1 to get for all  $s < t$ ,  $x < y$  and  $h > 0$ ,

$$\begin{aligned} & \int_s^t \int_{\mathbb{R}} p_b(s, x, u, z) |q(z)| p_a(u, z, t, y) dz du \\ &= DI_{h^2}(q) \left[ \frac{1}{\Gamma(1/2)} \frac{\sqrt{a} + \sqrt{b-a}}{\sqrt{a(b-a)}} + \frac{2(t-s)}{h} \right] p_a(s, x, t, y). \end{aligned}$$

**Corollary 2.6.** *Let  $q: \mathbb{R} \rightarrow \mathbb{R}$  be such that  $I_{h^2}(q) < \infty$  for some  $h > 0$ . Then  $|q| \in \mathcal{N}(p_b, p_a, (b/a)^{1/2}, \eta, Q)$  with*

$$\begin{aligned} \eta &= DI_{h^2}(q) \left( \sqrt{a} + \sqrt{b-a} \right) / \left( \Gamma(1/2) \sqrt{a(b-a)} \right), \\ Q(s, t) &= 2DI_{h^2}(q)(t-s)/h. \end{aligned}$$

### 3 Relative boundedness for subordinators with transition density

In this section we consider an arbitrary transition density  $p$  of a subordinator. Thus,  $p$  is space-time homogeneous,  $p(s, x, t, y) = 0$  whenever  $s \geq t$  or  $y \leq x$ , and  $p(s, x, t, y) > 0$  otherwise. We first discuss time-independent functions  $q$ , aiming at the condition (1.4).

We denote, as usual,  $\|f\|_\infty = \text{ess sup}_{x \in \mathbb{R}} |f(x)|$ . Let functions  $(\phi_j)_{j \in \mathbb{N}}$  be an approximation to identity in  $L^1(\mathbb{R})$ , that is real-valued on  $\mathbb{R}$  with the following properties:

$$\phi_j \geq 0 \quad \text{and} \quad \int_{\mathbb{R}} \phi_j(z) dz = 1, \quad (3.1)$$

$$\forall \delta > 0 \exists j_0 \in \mathbb{N} \forall j \geq j_0 \quad \text{supp}(\phi_j) \subset (-\delta, \delta). \quad (3.2)$$

**Lemma 3.1.** *Let  $f \in L^1_{loc}(\mathbb{R})$ . If  $\sup_{n \in \mathbb{N}} \|\phi_n * f\|_\infty < \infty$ , then  $f \in L^\infty(\mathbb{R})$ .*

*Proof.* We see that  $\phi_n * f$  is well defined. Let  $0 < \delta < R$  and  $M = \sup_{n \in \mathbb{N}} \|\phi_n * f\|_\infty$ . Choose  $j_0 \in \mathbb{N}$  according to (3.2). Since the functions  $f \mathbf{1}_{|z| < R} * \phi_n$  converge to  $f \mathbf{1}_{|x| < R} \in L^1(\mathbb{R})$  in the  $L^1$  norm, a subsequence  $f \mathbf{1}_{|z| < R} * \phi_{n_k}$  converges almost surely to  $f \mathbf{1}_{|x| < R}$ . For  $n_k \geq j_0$ ,

$$f \mathbf{1}_{|z| < R} * \phi_{n_k}(x) = f * \phi_{n_k}(x), \quad \text{if } |x| < R - \delta.$$

Thus for almost all  $|x| < R - \delta$ ,

$$|f(x)| = \lim_{k \rightarrow \infty} |f * \phi_{n_k}| \leq M.$$

Therefore  $|f(x)| \leq M$  for almost all  $x \in \mathbb{R}$ . □

**Lemma 3.2.** *Let  $h > 0$ . Assume that for all  $0 < t - s \leq h$  and  $x \in \mathbb{R}$ ,*

$$\int_s^t \int_{\mathbb{R}} p(s, x, u, z) |q(z)| dz du \leq M.$$

*Then  $q \in L^1_{loc}(\mathbb{R})$ .*

*Proof.* Let  $\varphi \in C_0(\mathbb{R})$  be such that  $\varphi \geq 0$ ,  $\varphi = 1$  on  $[0, 1/2]$  and  $\int_{\mathbb{R}} \varphi(x) dx = 1$ . For arbitrary fixed  $x_0 \in \mathbb{R}$  we have

$$\begin{aligned} M &\geq \int_s^t \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x_0 - x) p(s, x, u, z) dx |q(z)| dz du \\ &= \int_s^t \int_{\mathbb{R}} T_{u-s} \varphi(x_0 - z) |q(z)| dz du \geq (\varepsilon/2) \int_{x_0-1/2}^{x_0} |q(z)| dz, \end{aligned}$$

where  $0 < \varepsilon \leq h$  is such that  $\|T_u \varphi - \varphi\|_\infty \leq 1/2$  for  $u \leq \varepsilon$ . □



Lemma 3.2 generalizes to arbitrary Lévy processes in  $\mathbb{R}^d$ , see [10].

**Theorem 3.3.** *Let  $q: \mathbb{R} \rightarrow \mathbb{R}$ . Assume that for some  $s < t$ ,*

$$\sup_{\substack{x < y \\ y - x \leq 1}} \int_s^t \int_{\mathbb{R}} \frac{p(s, x, u, z)p(u, z, t, y)}{p(s, x, t, y)} |q(z)| dz du < \infty.$$

*Then  $q \in L^\infty(\mathbb{R})$ .*

*Proof.* By the assumption there is  $M' > 0$  such that for some fixed  $s < t$ ,

$$\int_s^t \int_{\mathbb{R}} \frac{p(s, x, u, z)p(u, z, t, y)}{p(s, x, t, y)} |q(z)| dz du \leq M', \quad x < y.$$

By Lemma 3.2,  $q \in L^1_{loc}(\mathbb{R})$ . For  $s < t$  and  $n \in \mathbb{N}$ , we let

$$\phi_n(z) = \frac{1}{t-s} \int_s^t \frac{p(s, -1/n, u, -z)p(u, -z, t, 1/n)}{p(s, -1/n, t, 1/n)} du, \quad |z| < 1/n,$$

and  $\phi_n(z) = 0$  for  $|z| \geq 1/n$ . Obviously  $\phi_n$  satisfies conditions (3.1) and (3.2). Furthermore, for all  $x \in \mathbb{R}$ ,

$$\phi_n * |q|(x) = \frac{1}{t-s} \int_s^t \int_{\mathbb{R}} \frac{p(s, x - 1/n, u, z)p(u, z, t, x + 1/n)}{p(s, x - 1/n, t, x + 1/n)} |q(z)| dudz.$$

Thus,  $\sup_{n \in \mathbb{N}} \|\phi_n * q\|_\infty \leq M'/(t-s) = M < \infty$ . Lemma 3.1 ends the proof.  $\square$

**Corollary 3.4.** (i) *A time-independent  $q \geq 0$  satisfies (1.4) if and only if  $q \in L^\infty(\mathbb{R})$ .*

(ii) *Let  $q: \mathbb{R} \rightarrow [0, \infty]$ . Assume that there are  $s < t$  and  $C \geq 0$  such that  $\tilde{p}(s, x, t, y) \leq C p(s, x, t, y)$ , for all  $x < y$ . Then  $q \in L^\infty(\mathbb{R})$ .*

If we allow  $q$  to depend on time, obviously the statements of the corollary are no longer valid. Indeed, let  $q(u, z) = u_+^{-1/2}$ , where  $u_+ = u \vee 0$ . Then for all  $s < t$  and  $x < y$ ,

$$\int_s^t \int_{\mathbb{R}} p(s, x, u, z) |q(u, z)| p(u, z, t, y) dz du \leq 2(t_+ - s_+)^{1/2} p(s, x, t, y).$$

We see that such unbounded time-dependent  $q$  belongs to the Kato class of every transition density  $p$ .

Corollary 3.4 means that the methods of [4] fail to deliver interesting perturbation results for transition densities of subordinators. On the contrary, as we see in Section 2, methods based on auxiliary semigroup majorants and 4G have the potential to handle such situations.

The next example builds on the ideas proposed in [13, Example 4].

*Example.* Consider the second term  $p_1$  of the perturbation series (1.2) for  $\tilde{p}$ . Let  $q(u, z) \geq 0$  be such that

$$\sup_{\substack{s \leq u \leq t \\ x \leq z \leq y}} q(u, z) \leq \eta / (t - s),$$

for some  $\eta \geq 0$  and for all  $s < t$ ,  $x < y$  such that  $(s, x), (t, y) \in F := \{(u, z) : q(u, z) > 0\}$ . Then for all  $s < t$  and  $x < y$ ,

$$p_1(s, x, t, y) \leq \eta p(s, x, t, y). \quad (3.3)$$

For the proof we consider a Borel non-decreasing function  $\omega : [s, t] \rightarrow \mathbb{R}$ ,  $s < t$ , such that  $\omega(s) = x < y = \omega(t)$  let  $T(\omega) = \{u : s \leq u \leq t, (u, \omega(u)) \in F\}$ . If  $T(\omega)$  is empty, then

$$\int_s^t q(u, \omega(u)) du = 0 \leq \eta.$$

Otherwise we consider  $\sigma = \inf\{u : u \in T(\omega)\}$  and  $\tau = \sup\{u : u \in T(\omega)\}$  and there are  $s_n \leq t_n$  such that  $(s_n, \omega(s_n)), (t_n, \omega(t_n)) \in F$ ,  $s_n \downarrow \sigma$  and  $t_n \uparrow \tau$ , hence

$$\begin{aligned} \int_s^t q(u, \omega(u)) du &= \int_\sigma^\tau q(u, \omega(u)) du = \lim_{n \rightarrow \infty} \int_{s_n}^{t_n} q(u, \omega(u)) du \\ &\leq \lim_{n \rightarrow \infty} (t_n - s_n) \sup_{\substack{s_n \leq u \leq t_n \\ \omega(s_n) \leq z \leq \omega(t_n)}} q(u, z) \leq \eta. \end{aligned}$$

Finally, let  $\{Y_u\}_{u \geq 0}$  be the subordinator. Given  $s < t$ ,  $x < y$  we denote by  $\{Z_u\}_{s \leq u \leq t}$  the bridge corresponding to  $\{Y_u\}_{u \geq 0}$ , which starts from  $x$  at time  $s$  and reaches  $y$  at time  $t$ . Since the trajectories of  $\{Z_u\}_{u \geq 0}$  are almost surely non-decreasing we have for all  $s < t$ ,  $x < y$ ,

$$p_1(s, x, t, y) / p(s, x, t, y) = \mathbb{E}_{s,x}^{t,y} \left[ \int_s^t q(u, Z_u) \right] du \leq \mathbb{E}_{s,x}^{t,y} [\eta] = \eta,$$

as claimed.

Typical applications are  $q(u, z) = \eta z 1_{(0,1/u)}(z)$ , cf. [13, Example 4], and  $q(u, z) = \eta z^2 1_F(u, z)$ , where  $F = \bigcup_{n=1}^{\infty} (1/(n+1), n) \times (n-1, n)$ . Both functions tend to infinity when time goes to zero and the space variable grows correspondingly.

In the next example we show that the estimate (3.3) may not be improved.

*Example.* We concentrate on  $q(u, z) = \eta z 1_{(0, 1/u)}(z)$ ,  $\eta > 0$ . Let  $\nu < \eta$ . We claim that there is no superadditive  $Q$  such that

$$p_1(s, x, t, y) \leq [\nu + Q(s, t)]p(s, x, t, y). \quad (3.4)$$

Indeed, it is clear from [5, Lemma 5.3] that we may assume that  $Q$  is regular superadditive. Thus there is  $t$  such that  $[\nu + Q(0, t)] < (\nu + \eta)/2$ . On the other hand for  $x = (1 + \nu/\eta)/(2t) < y = 1/t$  we have

$$\begin{aligned} p_1(s, x, t, y) &= \int_0^t \int_x^y p(s, x, u, z) q(u, z) p(u, z, t, y) dz du \\ &\geq \eta x \int_0^t \int_x^y p(s, x, u, z) p(u, z, t, y) dz du \\ &\geq \eta x t p(s, x, t, y) = [(\eta + \nu)/2] p(s, x, t, y), \end{aligned}$$

which is a contradiction.

## 4 Appendix: Fundamental solutions

In this section we prove (1.16) and its analogues in the setting of general semigroup theory. We consider a Banach space  $(Y, \|\cdot\|)$ . Let  $T = (T_t)_{t \geq 0}$  be a strongly continuous semigroup on  $Y$ . Let  $L$  be the corresponding infinitesimal generator with domain  $D(L)$  [16, IX].

**Theorem 4.1.** *Let  $\xi: \mathbb{R} \rightarrow D(L)$  be such that*

$$t \mapsto \xi(t) \text{ is differentiable in } (Y, \|\cdot\|), \quad (4.1)$$

$$t \mapsto \xi'(t) \text{ is continuous in } (Y, \|\cdot\|), \quad (4.2)$$

$$t \mapsto L\xi(t) \text{ is continuous in } (Y, \|\cdot\|), \quad (4.3)$$

$$t \mapsto \xi(t) \text{ has compact support in } \mathbb{R}. \quad (4.4)$$

*Then*

$$\int_s^\infty T_{u-s} [\xi'(u) + L\xi(u)] du = -\xi(s), \quad s \in \mathbb{R}, \quad (4.5)$$

*where the integral is the Riemann type integral of a Banach space valued function.*

Theorem 4.1 applies, e.g., to  $\xi(t) = f(t)\xi_0$  with  $\xi_0 \in D(L)$  and  $f \in C_c^1(\mathbb{R})$ . We may summarize (4.5) by saying that  $(T_t)_{t \geq 0}$  is the *fundamental solution* of  $\partial_t + L$ .

Theorem 4.1 follows from two auxiliary lemmas.

**Lemma 4.2.** *If  $\xi$  satisfies (4.1), then  $t \mapsto T_t \xi(t)$  is differentiable in  $(Y, \|\cdot\|)$  and*

$$\frac{d}{dt} T_t \xi(t) = T_t \xi'(t) + T_t L \xi(t), \quad t \geq 0.$$

For  $t = 0$  the derivative is understood as the right-hand derivative. The lemma is a version of the differentiation rule for products.

*Proof of Lemma 4.2.* Let  $h \neq 0$  ( $h > 0$  if  $t = 0$ ) and  $h \rightarrow 0$ . Clearly,

$$\begin{aligned} & \frac{T_{t+h} \xi(t+h) - T_t \xi(t)}{h} \\ &= T_{t+h} \xi'(t) + T_{t+h} \left( \frac{\xi(t+h) - \xi(t)}{h} - \xi'(t) \right) + \left( \frac{T_{t+h} - T_t}{h} \right) \xi(t). \end{aligned}$$

For some  $M, \omega \geq 0$ , we have  $\|T_t\| \leq M e^{\omega t}$ ,  $t \geq 0$  [16]. The lemma follows because

$$\left\| T_{t+h} \left( \frac{\xi(t+h) - \xi(t)}{h} - \xi'(t) \right) \right\| \leq M e^{\omega(t+h)} \left\| \frac{\xi(t+h) - \xi(t)}{h} - \xi'(t) \right\| \rightarrow 0.$$

□

Let  $a, b \in \mathbb{R}$ ,  $a < b$ . We write  $\xi \in C^1([a, b], Y)$  if  $\xi: [a, b] \rightarrow Y$  and (4.1) and (4.2) hold, with one-sided derivatives at the endpoints  $a$  and  $b$ . Here is the fundamental theorem of calculus for Riemann type Banach space integrals (see [7, Lemma 1.1.4] or [12, Lemma 2.3.24]).

**Lemma 4.3.** *If  $\psi \in C^1([a, b], Y)$ , then  $\int_a^b \frac{d}{du} [\psi(u)] du = \psi(b) - \psi(a)$ .*

*Proof of Theorem 4.1.* Let  $s \in \mathbb{R}$ . By Lemma 4.2, assumptions (4.2), (4.3) and (4.4), and by Lemma 4.3, we obtain the result:

$$\int_0^\infty T_u \left[ \xi'(u+s) + L \xi(u+s) \right] du = \int_0^\infty \frac{d}{du} [T_u \xi(u+s)] du = -\xi(s). \quad (4.6)$$

In fact, if  $s$  is fixed, the assumptions on  $\xi(t)$  only need to hold in  $[s, \infty)$ . □

We shall give a partial converse to Theorem 4.1 by showing that the infinitesimal generator of  $T$  is the only operator  $L$  making (4.5) true. This addresses question posed by Zhen-Qing Chen.

**Theorem 4.4.** *Let  $A$  be a linear operator on a linear space  $D(A) \subset Y$ . Assume  $\xi: \mathbb{R} \rightarrow D(A)$  is such that*

$$t \mapsto \xi(t) \text{ is differentiable in } (Y, \|\cdot\|), \quad (4.7)$$

$$t \mapsto \xi'(t) \text{ is continuous in } (Y, \|\cdot\|), \quad (4.8)$$

$$t \mapsto A\xi(t) \text{ is continuous in } (Y, \|\cdot\|), \quad (4.9)$$

$$t \mapsto \xi(t) \text{ has compact support in } \mathbb{R}, \quad (4.10)$$

$$\int_s^\infty T_{u-s} [\xi'(u) + A\xi(u)] du = -\xi(s), \quad s \in \mathbb{R}. \quad (4.11)$$

Then  $\xi(t) \in D(L)$  and  $L\xi(t) = A\xi(t)$  for all  $t \in \mathbb{R}$ .

*Proof.* Let  $t \in \mathbb{R}$  and  $h > 0$ . By (4.11),

$$\int_{t+h}^\infty T_{u-t} [\xi'(u) + A\xi(u)] du = \int_{t+h}^\infty T_{u-(t+h)} T_h [\xi'(u) + A\xi(u)] du = -T_h \xi(t+h).$$

Subtracting this from (4.11) with  $s = t$  we get

$$\int_t^{t+h} T_{u-t} [\xi'(u) + A\xi(u)] du = T_h \xi(t+h) - \xi(t).$$

We get

$$\left( \frac{T_h - I}{h} \right) \xi(t) = \frac{1}{h} \int_t^{t+h} T_{u-t} [\xi'(u) + A\xi(u)] du - T_h \left( \frac{\xi(t+h) - \xi(t)}{h} \right).$$

By (4.7)–(4.9) the limit on the right hand side exists as  $h \rightarrow 0^+$  and equals

$$L\xi(t) = T_0 (\xi'(t) + A\xi(t)) - T_0 \xi'(t) = A\xi(t).$$

In fact, the assumptions (4.7)–(4.11) only need to hold on  $[t, t+\varepsilon)$ ,  $\varepsilon > 0$ .  $\square$

*Remark 4.5.* We call  $\xi$  satisfying (4.7)–(4.10) a *path for  $A$* . Define

$$D(A, T) = \{\xi(t): t \in \mathbb{R}, \xi \text{ is a path for } A \text{ satisfying (4.11)}\}.$$

If  $A$  is the infinitesimal generator of a strongly continuous semigroup  $S = (S_t)_{t \geq 0}$  on  $Y$  and  $D(A, T)$  contains the cores of  $L$  and  $A$ , then  $L \equiv A$  and  $T \equiv S$ . Indeed, by the comment following Theorem 4.1, for the infinitesimal generator  $L$  of  $T = (T_t)_{t \geq 0}$  we have  $D(L, T) = D(L)$ . Theorem 4.4 means that  $D(A, T) \subseteq D(A) \cap D(L)$ , and  $A = L$  on  $D(A, T)$ . This identifies  $L$  with  $A$  and  $T$  with  $S$ .

We now focus on Lévy semigroups and generators discussed in Introduction.

*Proof of (1.16).* Recall that  $C_c^\infty(\mathbb{R}^d) \subset C_0^2(\mathbb{R}^d) \subset D(L)$ . We shall verify the assumptions of Theorem 4.1 for  $\xi(t) = \phi(t, \cdot)$ . It suffices to justify (4.3). Recall that (1.15) holds for  $f \in C_0^2(\mathbb{R}^d)$  and  $L$  is continuous from  $C_0^2(\mathbb{R}^d)$  to  $C_0(\mathbb{R}^d)$  [15, p. 211]. We note that  $t \mapsto \phi(t, \cdot)$  is continuous in  $C_0^2(\mathbb{R}^d)$ . Therefore  $t \mapsto L\phi(t, \cdot)$  is continuous in  $(C_0(\mathbb{R}^d), \|\cdot\|_\infty)$ . By Theorem 4.1,

$$-\xi(s) = \int_s^\infty P_{u-s} \left[ \xi'(u) + L\xi(u) \right] du$$

in  $C_0(\mathbb{R}^d)$ . Recall that the Riemann integrals converge in norm. Evaluation at a point is continuous on  $(C_0(\mathbb{R}^d), \|\cdot\|_\infty)$ , therefore the above identity holds pointwise, i.e. (1.16) holds. The integral in (1.16) may be interpreted as absolutely Lebesgue integral on  $\mathbb{R} \times \mathbb{R}^d$ .  $\square$

**Theorem 4.6 (Uniqueness).** *Let  $C_c^\infty(\mathbb{R}^d)$  be a core of a closed linear operator  $A$  with domain  $D(A) \subset (C_0(\mathbb{R}^d), \|\cdot\|_\infty)$ . If for all  $s \in \mathbb{R}$ ,  $x \in \mathbb{R}^d$  and  $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$ ,*

$$\int_s^\infty \int_{\mathbb{R}^d} p_{u-s}(dz) \left[ \partial_u \phi(u, x+z) + A\phi(u, x+z) \right] dz du = -\phi(s, x), \quad (4.12)$$

then  $A \equiv L$ .

*Proof.* For  $\varphi \in C_c^\infty(\mathbb{R}^d)$  and  $f \in C_c^1(\mathbb{R}^d)$  we let  $\xi(t) = f(t)\varphi$ . Then  $\xi$  is a path for  $A$  and  $\zeta(t) := \int_t^\infty P_{u-t} [\xi'(u) + A\xi(u)] du \in C_0(\mathbb{R}^d)$  converges in norm. By continuity of evaluations and (4.12) with  $\phi(t, x) = f(t)\varphi(x)$  we have  $\zeta(t)(x) = -\xi(t)(x)$ ,  $t \in \mathbb{R}$ ,  $x \in \mathbb{R}^d$ . By Theorem 4.4,  $A = L$  on the common core  $C_c^\infty(\mathbb{R}^d)$ . This ends the proof.  $\square$

*Remark 4.7.* If the Lévy process  $\{X_t\}$  has (transition) density function, i.e.  $p_t(dy) = p(t, y)dy$  for  $t > 0$ , then (1.16) reads as

$$\int_s^\infty \int_{\mathbb{R}^d} p(u-s, z-x) \left[ \partial_u \phi(u, z) + L\phi(u, z) \right] dz du = -\phi(s, x), \quad s \in \mathbb{R}, \quad x \in \mathbb{R}^d.$$

We shall focus on the case when  $d = 1$  and  $\{X_t\}$  is a subordinator i.e. nondecreasing Lévy process. The Lévy measure  $\nu$  of  $X_t$  is concentrated on  $(0, \infty)$ . Since  $\int (x \wedge 1)\nu(dx) < \infty$  and  $L$  is a closed operator, (1.15) may be rearranged: we obtain  $C_0^1(\mathbb{R}) \subset D(L)$  and

$$Lf(x) = b \frac{df}{dx}(x) + \int_0^\infty \left( f(x+y) - f(x) \right) \nu(dy), \quad f \in C_0^1(\mathbb{R}).$$

Here  $b \geq 0$  is the drift coefficient. Furthermore, for  $f \in C_c^1(\mathbb{R})$  we obtain

$$\begin{aligned} \int_0^\infty (f(x+y) - f(x)) \nu(dy) &= \int_0^\infty \int_0^y f'(x+z) dz \nu(dy) \\ &= \int_0^\infty f'(x+z) \left( \int_z^\infty \nu(dy) \right) dz. \end{aligned}$$

Let  $\bar{\nu}(z) = \int_z^\infty \nu(dy)$ . We thus have

$$Lf(x) = b f'(x) + \int_x^\infty f'(z) \bar{\nu}(z-x) dz, \quad f \in C_c^1(\mathbb{R}). \quad (4.13)$$

*Example.* Let  $\alpha \in (0, 1)$  and  $\{X_t\}$  be the  $\alpha$ -stable subordinator, i.e.

$$b = 0 \quad \text{and} \quad \nu(dy) = \frac{\alpha}{\Gamma(1-\alpha)} y^{-\alpha-1} \mathbf{1}_{y>0} dy.$$

We see that the generator of  $\{X_t\}$  coincides on  $C_c^1(\mathbb{R})$  with the Weyl fractional derivative (cf. (1.17) for the case  $\alpha = 1/2$ ). The potential operator for  $\{X_t\}$  is the Weyl fractional integral,

$$W^{-\alpha} f(x) = \int_0^\infty T_t f(x) dt = \frac{1}{\Gamma(\alpha)} \int_x^\infty f(z) (z-x)^{\alpha-1} dz, \quad f \in C_c(\mathbb{R}).$$

We note in passing that  $W^{-\alpha} \partial^\alpha = -I$  on  $C_c^1(\mathbb{R})$ . Schrödinger perturbations of  $W^{-\alpha}$  were discussed in [4, Example 2 and 3]. The discussion was facilitated by the fact that 3G Theorem holds for its kernel density  $(y-x)_+^{\alpha-1}/\Gamma(\alpha)$ .

*Example.* Since the inverse Gaussian subordinator is obtained by the Esscher transform (tempering) of the 1/2-stable subordinator (cf. [6], Sec. 4.4.2), for  $f \in C_c^1(\mathbb{R})$  the generator of the inverse Gaussian subordinator is given by (1.20).

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