Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint Nr. 356

$C^{1,\alpha}$-interior regularity for minimizers of a class of variational problems with linear growth related to image inpainting in higher dimensions

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Saarbrücken 2015

Fachrichtung 6.1 – Mathematik
Universität des Saarlandes

Preprint No. 356
submitted: February 5, 2015

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AMS Subject Classification: 49 N 60, 49 Q 20
Keywords: image inpainting, variational method, TV-regularization

Abstract

We investigate a modification of the total variation image inpainting method in higher dimensions, i.e. we assume $n \geq 3$ and discuss existence as well as smoothness of corresponding solutions to the underlying variational problem. Precisely, we are going to establish that our type of a linear growth regularization admits a unique solution in the Sobolev space $W^{1,1}(\Omega)$, i.e. we do not need to consider a suitable relaxed variant of our original variational problem in the space of functions having bounded variation. Furthermore, we will prove $C^{1,\alpha}$ interior differentiability of our unique solution using De Giorgi-type arguments.

1 Introduction

Let us consider a black and white image that is described by a function $u : \Omega \to [0,1]$, where Ω is supposed to be a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 3$ (e.g. a cuboid in the case $n = 3$). In the context of image processing, $u(x)$ can be interpreted as a measure of the intensity of the grey level. Further, we suppose that we are given a subset D of Ω which is assumed to be $\mathcal{L}^n$-measurable ($\mathcal{L}^n$ denoting Lebesgue's measure on $\mathbb{R}^n$) and satisfies

$$0 < \mathcal{L}^n(D) < \mathcal{L}^n(\Omega). \tag{1.1}$$

Precisely, the region D that is also called "inpainting domain" (see [14]) represents a certain part of the observed image for which image data are missing or inaccessible, i.e. the intensity of the grey level is only known for points $x \in \Omega - D$. Further, the partial observation is represented through a measurable function $f : \Omega - D \to [0,1]$.

Our goal now is to recover the original image in terms of a function u: $\Omega \to \mathbb{R}$ on the entire domain Ω based on the partial observation $f : \Omega - D \to [0,1]$ which is usually corrupted by noise stemming from transmission or measuring errors. In the image processing community, this kind of image interpolation is called

"inpainting" or "image inpainting", respectively (compare [14, 22, 23]).
On account of [22], there are essentially four different methods to handle the inpainting problem, depending on being variational or non-variational and local or non-local where we like to refer to [2, 5, 15, 17, 16, 18, 22] and the references quoted therein for more details.

In our note we concentrate on a TV-like variational approach being of non-local type that has already been proposed in [12] for instance. Precisely we seek minimizers of the following functional

$$J[w] := \int_\Omega \psi(|\nabla w|)\,\mathrm{d}x + \frac{\lambda}{2}\int_{\Omega-D}(w-f)^2\,\mathrm{d}x\ . \tag{1.2}$$

In this setting, λ is a positive regularization parameter and ψ is supposed to be a convex and increasing function with non-negative values. From the point of view of image processing, the second term on the right-hand side of (1.2) represents a measure for the quality of data fitting, i.e. the deviation of the original image w from the known data on $\Omega - D$ while the first term produces a kind of mollification and allows to incorporate some kind of a priori information of the generated image on the entire domain Ω into the minimization process.

In this context, a common choice of ψ is $\psi(|\nabla w|) := |\nabla w|$ leading to the total variation inpainting model (compare, e.g., [4, 22]). In order to investigate this variational problem, one has to work with functions $\Omega \to \mathbb{R}$ being of bounded variation, i.e. in the space $\mathrm{BV}(\Omega)$. This space covers all L^1-functions whose distributional gradient ∇w is represented by a vector valued Radon measure on Ω with finite total variation $\int_\Omega |\nabla w|$ (see, e.g., [19]).

Due to the lack of ellipticity, we cannot expect regular solutions of variational problems involving the total variation, in general. For that reason, as already carried out in the papers [8, 9, 12, 13] and also in the related work [7], the basic idea was to replace the TV-density $\psi(|\nabla w|) = |\nabla w|$ through a family of strictly convex densities $F(\nabla w)$ being still of linear growth w.r.t. to the modulus of the gradient but satisfying better ellipticity properties. However, the study of smoothness properties of corresponding solutions remains a difficult problem since the required linear growth of F admits only weak and anisotropic ellipticity conditions. Thus, it stands to reason that regularity highly depends on the modulus of ellipticity.

Let us now fix the basic setup of our note: As in [8],[9] and [13] we introduce the energy

$$I[w] := \int_\Omega F(\nabla w)\,\mathrm{d}x + \frac{\lambda}{2}\int_{\Omega-D}(w-f)^2\,\mathrm{d}x \tag{1.3}$$

for functions w from the Sobolev space $W^{1,1}(\Omega)$ being intersected with the Lebesgue space $L^2(\Omega - D)$ (for details concerning these spaces, we refer to [1]), i.e. the functional I is well defined. Further, F: $\mathbb{R}^n \to [0, \infty)$ represents a density of class C^2 satisfying $F(0) = 0$ and $DF(0) = 0$. More precisely, we impose the following conditions on F:
there exist positive constants ν_1, ν_2, ν_3 and a real number $\mu > 1$ such that for any $Y, Z \in \mathbb{R}^n$ we have

$$|DF(Z)| \leq \nu_1 \tag{1.4}$$

and

$$\nu_2 \frac{1}{(1+|Z|)^\mu} |Y|^2 \leq D^2F(Z)(Y,Y) \leq \nu_3 \frac{1}{1+|Z|} |Y|^2 \, . \tag{1.5}$$

Remark 1.1
An integrand $F : \mathbb{R}^n \to [0, \infty)$ being of class C^2 and satisfying (1.4) *as well as* (1.5) *is called μ-elliptic.*

Based on the above hypotheses of μ-elliptic densities we can state some useful conclusions that have been established on p. 97/98 in [6] for instance.

Lemma 1.2
Suppose that F satisfies (1.4) *and* (1.5) *for some number $\mu > 1$. Then F is strictly convex on $\mathbb{R}^n$ and it holds:*

(i) there are real constants $\nu_1 > 0$, $\nu_2 \in \mathbb{R}$ such that for all $Z \in \mathbb{R}^n$ we have

$$DF(Z) \cdot Z \geq \nu_1 |Z| - \nu_2 \, ;$$

(ii) F is of linear growth in the sense that for real numbers ν_3, $\nu_4 > 0$, ν_5, $\nu_6 \in \mathbb{R}$ and for all $Z \in \mathbb{R}^n$ it holds

$$\nu_3 |Z| - \nu_5 \leq F(Z) \leq \nu_4 |Z| + \nu_6 \, ; \tag{1.6}$$

(iii) the integrand satisfies a balancing condition: there exists a real constant $\nu_7 > 0$ such that

$$|D^2F(Z)||Z|^2 \leq \nu_7 (1 + F(Z))$$

for all $Z \in \mathbb{R}^n$.

Following the lines of [11] there are prominent examples of integrands being μ-elliptic where without any doubt, the most prominent one is the minimal surface

integrand given by $F(Z) := \sqrt{1+|Z|^2}$, $Z \in \mathbb{R}^n$, fulfilling (1.5) for the choice $\mu = 3$. Since we cannot expect any smoothness results if the ellipticity exponent satisfies $\mu > 3$ (see [6], Section 4.4 for a counterexample), the limit $\mu = 3$ serves as an optimal choice.
Consulting [7], another example of a μ-elliptic density seems to be more serious in the context of TV-regularization: Let us fix $\mu > 1$ and set

$$\varphi_\mu(r) := \int_0^r \int_0^s (1+t^2)^{-\frac{\mu}{2}} dt ds, \quad r \in \mathbb{R}_0^+.$$

Since, in the TV-case, the density just depends on the modulus of the gradient, it does make sense to consider a function Φ_μ with

$$\Phi_\mu(Z) := \varphi_\mu(|Z|), \quad Z \in \mathbb{R}^n \tag{1.7}$$

Here, $\Phi_\mu : \mathbb{R}^n \to [0, \infty)$ is of class C^2 satisfying (1.4) and (1.5) with the prescribed elliptic parameter μ.
Moreover, for $\mu \neq 2$, we get a clearer representation of $\varphi_\mu(r)$, precisely

$$\varphi_\mu(r) = \frac{r}{\mu-1} + \frac{1}{\mu-1}\frac{1}{\mu-2}(r+1)^{-\mu+2} - \frac{1}{\mu-1}\frac{1}{\mu-2}, \tag{1.8}$$

whereas for $\mu = 2$ it holds

$$\varphi_2(r) = r - \log(1+r).$$

Observing next that we obtain

$$(\mu-1)\Phi_\mu(Z) \to |Z| \quad \text{as } \mu \to \infty \tag{1.9}$$

for all $Z \in \mathbb{R}^n$, it becomes evident that the density $\Phi_\mu(\nabla u)$ serves as a very good candidate w.r.t. approximating $|\nabla u|$ by integrands of linear growth satisfying better ellipticity properties.

In this note, we will prove two results: Assuming from now on that it holds $\mu \in (1, 2)$ for our elliptic parameter we primarily consider the variational problem $I \to \min$ with I from (1.3) in the space $W^{1,1}(\Omega) \cap L^2(\Omega - D)$ and show that we are able to produce a unique minimizer u belonging to the class $W^{1,1}(\Omega) \cap L^2(\Omega - D)$ with the additional property $0 \leq u \leq 1$ a.e. on Ω by benefiting from fine properties of a suitable regularization sequence $(u_\delta)_{\delta \in (0,1)}$. Subsequently, we prove full interior $C^{1,\alpha}$-regularity of our minimizer u for any $\alpha \in (0, 1)$ on the whole domain Ω by performing a De Giorgi-type iteration and quoting standard arguments from elliptic regularity theory. At this point, we highly emphasize that the condition $\mu < 2$ is of fundamental meaning in the course of both proofs.

To be more precise, we formulate our existence and regularity results for the case of involving μ-elliptic densities.

Theorem 1.3
Let (1.1) *hold where we assume* $n \geq 3$ *and suppose that* F *satisfies* (1.4) *and* (1.5) *for some* $\mu \in (1,2)$. *Then the problem* $I \to min$ *admits a unique solution* $u \in W^{1,1}(\Omega) \cap L^2(\Omega - D)$ *satisfying* $0 \leq u(x) \leq 1$ *almost everywhere on the entire domain* Ω.

Theorem 1.4
Under the assumptions (in particular, we assume $\mu \in (1,2)$*) and with the notation of Theorem 1.3 it holds* $u \in C^{1,\alpha}(\Omega)$ *for any* $0 < \alpha < 1$ *where* u *is the solution from Theorem 1.3.*

Remark 1.5
Since the unique I*-minimizer* u *automatically satisfies the maximum priniciple* $0 \leq u(x) \leq 1$ *a.e. on* Ω*, it can be interpreted as a measure for the intensity of the grey level.*

Remark 1.6
Considering the case $n = 2$ *we like to remark the following:*

- *The statement of Theorem 1.3 has already been proven in [8] (see Theorem 1.3 therein). In this situation we do not have to force well definity of our functional* I *in such a way as we did above since an application of Sobolev's embedding theorem ensures well definity of* I *for functions from the space* $W^{1,1}(\Omega)$. *In fact, following the arguments of the proof of Theorem 1.3 in [8] it turns out that the unique* I*-minimizer* $u \in W^{1,1}(\Omega)$ *belongs to the Sobolev space* $W^{2,s}_{loc}(\Omega)$ *for any exponent* $s \in (1,2)$ *and a new application of Sobolev's embedding theorem directly implies local* p*-integrability of* ∇u *for any finite exponent* p.
- *Full interior* $C^{1,\alpha}$*-regularity of the unique* I*-minimizer* $u \in W^{1,1}(\Omega)$ *for any* $\alpha \in (0,1)$ *on the whole domain* Ω *has been showed in [13], Theorem 2. This result was a substantial improvement of Theorem 1.4 in [8] where one could show partial* $C^{1,\beta}$*-regularity of* u *on* Ω *for any* $\beta < 1$*, i.e. there exists an open subset* Ω_0 *of* Ω *satisfying* $dim_{\mathcal{H}}(\Omega - \Omega_0) = 0$. *By definition,* $dim_{\mathcal{H}}(\Omega - \Omega_0) = 0$ *means that* $\mathcal{H}^{\varepsilon}(\Omega - \Omega_0) = 0$ *((*$\mathcal{H}^{\varepsilon}$ *denoting the Hausdorff-measure of dimension* ε*) for any* $\varepsilon > 0$*, i.e. the set of singular points is in some sense very small.*

Remark 1.7
The occurrence of the inpainting quantity $\int_{\Omega - D}(f - w)^2 \, \mathrm{d}x$ *causes some severe problems, if one likes to prove interior* $C^{1,\alpha}$*-regularity of minimizers by using De Giorgi-type arguments. Precisely, we cannot immediately refer to e.g. [6] adding some obvious modifications. Besides, as done in [13], in the course of the proof*

of Theorem 1.4 we also like to investigate in detail (see Lemma 3.6) what starting integrability of ∇u is actually needed to obtain its local boundedness.

Remark 1.8
In fact, we cannot expect solvability of the problem $I \to min$ in $W^{1,1}(\Omega) \cap L^2(\Omega - D)$ for arbitrary values of μ. Nevertheless, we can consider suitable relaxed variants of the original problem, e.g. an appropriate BV-variant by using the notion of a convex measure function or passing to the dual variational problem, for large values of μ. In fact, we can establish existence of generalized solutions to these relaxed variants of our original problem under much weaker assumptions on our density F (for details we refer to [9] and [Preprint mit Jan Müller]).

Remark 1.9
- *It is easy to check that Theorem 1.3 and Theorem 1.4 also extend to the case $D = \emptyset$ ("pure denoising of f").*
- *Applying minor adjustments, the results of Theorem 1.3 as well as Theorem 1.4 are also valid in the case that an additional boundary condition as $u = u_0$ on $\partial\Omega$ is involved where u_0 denotes a sufficently regular function fulfilling $0 \leq u_0 \leq 1$ (see [7] for more details).*

2 Proof of Theorem 1.3

Let the inpainting region D satisfy (1.1) where we assume $n \geq 3$ and consider a density F fulfilling (1.4) and (1.5) for some $\mu \in (1, 2)$. As in [9] (compare Lemma 2.1 in this reference) and [13] (see Lemma 2 therein) we introduce a suitable regularization of our original problem, i.e. we regularize the energy density F with F_δ in such a way that we let for fixed $\delta \in (0, 1]$

$$I_\delta[w] := \int_\Omega F_\delta(\nabla w)dx + \frac{\lambda}{2} \int_{\Omega - D} (w - f)^2 dx$$

where

$$F_\delta(P) := \frac{\delta}{2}|P|^2 + F(P), P \in \mathbb{R}^n. \tag{2.1}$$

Then we can give the following lemma:

Lemma 2.1
Suppose that (1.1) *holds for the damaged region D and let us assume the validity of* (1.4) *as well as* (1.5) *for some $\mu > 1$. Then the problem $I_\delta \to min$ in $W^{1,2}(\Omega)$ admits a unique solution u_δ satisfying*

(a) $0 \le u_\delta \le 1$ *a.e. on* Ω.

(b) $u_\delta \in W^{2,2}_{loc}(\Omega) \cap C^{1,\alpha}(\Omega)$ *for all* $0 < \alpha < 1$.

Proof of Lemma 2.1. A proof of this lemma under weaker assumptions on F can be found in [9], Lemma 3.1. □

In order to prove Theorem 1.3 we proceed as in the proof of Theorem 1.3 in [8]. Precisely, the first step is to establish that $u_\delta \in W^{1,2}_{\mathrm{loc}}(\Omega)$ uniformly in δ which serves as an appropriate auxiliary result for showing Theorem 1.3 subsequently.

Lemma 2.2
It holds $u_\delta \in W^{1,2}_{loc}(\Omega)$ *uniformly in* δ.

Proof of Lemma 2.2. Having Lemma 2.1 at hand, u_δ solves the euler equation

$$\int_\Omega DF_\delta(\nabla u_\delta)\cdot \nabla\varphi dx + \lambda \int_{\Omega - D} (u_\delta - f)\varphi dx = 0 \tag{2.2}$$

for all $\varphi \in C^\infty_0(\Omega)$. Recalling Lemma 2.1, (b), u_δ is of class $W^{2,2}_{\mathrm{loc}}(\Omega)$. Furthermore, since $|D^2 F_\delta|$ is bounded, $DF_\delta(\nabla u_\delta)$ is of class $W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ having partial derivatives

$$\partial_\gamma(DF_\delta(\nabla u_\delta)) = D^2F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \cdot) \quad \text{a.e. on } \Omega$$

where $\gamma \in \{1, \dots, n\}$. Observing that $\partial_\gamma \varphi$ represents an admissible choice in the euler equation (2.2) we arrive at the differentiated version of equation (2.2) by using an integration by parts. More precisely, we get

$$\int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma\nabla u_\delta, \nabla\varphi)dx = \lambda \int_{\Omega - D} (u_\delta - f)\partial_\gamma\varphi dx$$

and by approximation, the above equality remains valid for functions $\varphi \in W^{1,2}(\Omega)$ having compact support in Ω.

Next, we fix a point $x_0 \in \Omega$, a radius $R > 0$ such that $B_{2R}(x_0) \Subset \Omega$ and let $\eta \in C^\infty_0(B_{2R}(x_0))$ with $\eta \equiv 1$ on $B_R(x_0)$, $0 \le \eta \le 1$ and $|\nabla \eta| \le \frac{c}{R}$.

Noticing that $\varphi := \eta^2 \partial_\gamma u_\delta$ is admissible we may follow the calculations in [8], Proof of Theorem 1.2 with F_δ in place of H_δ and the condition of μ-ellipticity

(1.5) yields (compare (3.5) in [8] and taking the sum w.r.t. γ)

$$\begin{aligned} & c \int\limits_{B_{2R}(x_0)} \eta^2 \frac{|\nabla^2 u_\delta|^2}{(1+|\nabla u_\delta|^2)^{\mu/2}} dx \\ & \leq c(R) + \lambda \int\limits_{B_{2R}(x_0)} \mathbb{1}_{\Omega - D}(u_\delta - f)\partial_\gamma(\eta^2 \partial_\gamma u_\delta) dx \end{aligned} \tag{2.3}$$

where c is a positive constant being independent of δ and $B_R(x_0)$.
From (2.3) we may conclude

$$\int\limits_{B_R(x_0)} \frac{|\nabla^2 u_\delta|^2}{(1+|\nabla u_\delta|^2)^{\mu/2}} dx + \int\limits_{B_R(x_0)} |\nabla u_\delta|^2 dx \leq c(R) \tag{2.4}$$

exactly as in [8] by starting with (3.5) in this reference. Here, $c(R)$ represents a local constant being independent of δ. This proves the claim of Lemma 2.2 after using a covering argument. □

Remark 2.3
Setting $\varphi_\delta := (1+|\nabla u_\delta|^2)^{1-\frac{\mu}{4}}$ *it follows*

$$\int\limits_{B_R(x_0)} |\nabla \varphi_\delta|^2 dx \leq c(R)$$

by taking (2.4) *into account. This implies* $\nabla\varphi_\delta \in W^{1,2}_{loc}(\Omega)$ *uniformly w.r.t.* δ *after exploiting a covering argument. Considering the case* $n = 2$*, an application of Sobolev's embedding theorem immediately implies local uniform (in* δ*) higher integrability of* ∇u_δ*. Underlying the assumption* $n \geq 3$*, Sobolev's embedding theorem merely gives* $\nabla u_\delta \in L^q_{loc}(\Omega)$ *uniformly w.r.t.* δ *where* $1 \leq q < \frac{2n}{n-2}$.

After proving Lemma 2.2, we now give the proof of Theorem 1.3:
Initially we notice that from $I_\delta[u_\delta] \leq I[0]$ and from the linear growth of F w.r.t. the modulus of the gradient we get existence of a positive constant c such that

$$\sup_\delta \int\limits_\Omega |\nabla u_\delta| dx \leq c < \infty. \tag{2.5}$$

Quoting Lemma 2.1 and (2.5), an application of the BV-compactness theorem leads to existence of a function $\overline{u}$ belonging to the space $BV(\Omega)$ with $u_\delta \to \overline{u}$ in $L^1(\Omega)$ (and a.e. on Ω) by passing to appropriate subsequences. Consequently, Lemma 2.2 gives $\overline{u} \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and by remarking that $BV(\Omega) \cap W^{1,2}_{\mathrm{loc}}(\Omega)$ is a subspace of $W^{1,1}(\Omega)$, it follows $\overline{u} \in W^{1,1}(\Omega)$. Further we can arrange $u_\delta \rightharpoondown: \widetilde{u}$ in

$L^2(\Omega - D)$ by passing to an appropriate subsequence (recall $I_\delta[u_\delta] \le I[0]$) and an application of Egorov's theorem gives $\widetilde{u} = \overline{u}$ a.e. on $\Omega - D$. As a consequence we get $\overline{u} \in W^{1,1}(\Omega) \cap L^2(\Omega - D)$, i.e. $I[\overline{u}]$ is well defined.

Moreover, the functional I is weakly lower semicontinuous in $W^{1,2}_{\mathrm{loc}}(\Omega)$ which follows exactly as in the proof of Theorem 1.3 in [8], i.e. we have

$$\int\limits_{\omega} F(\nabla \overline{u}) dx + \int\limits_{\omega \cap (\Omega - D)} (\overline{u} - f)^2 dx \le \liminf_{\delta \to 0} I[u_\delta]$$

for compact subregions ω of Ω. Considering a compact exhaustion of Ω and using $\mathbb{1}_\omega \to \mathbb{1}_\Omega$ a.e. on Ω it follows by using Lebesgue's theorem on dominated convergence

$$I[\overline{u}] \le \liminf_{\delta \to 0} I[u_\delta] \tag{2.6}$$

which corresponds to (3.3) in [8].
Thanks to the I_δ-minimality of u_δ in $W^{1,2}(\Omega)$ we obtain

$$I[\overline{u}] \le \liminf_{\delta \to 0} I[u_\delta] \le \liminf_{\delta \to 0} I_\delta[u_\delta] \le \liminf_{\delta \to 0} I_\delta[v] = I[v]$$

where $v \in W^{1,2}(\Omega)$.
By approximation we can find a sequence $(v_k) \subset W^{1,2}(\Omega)$ satisfying $v_k \to w$ in $W^{1,1}(\Omega) \cap L^2(\Omega - D)$ (see [Preprint mit Jan Müller]) and as a consequence we finally get $I[\overline{u}] \le I[w]$ for any $w \in W^{1,1}(\Omega) \cap L^2(\Omega - D)$. Thus, $\overline{u}$ represents a I-minimizer belonging to the class $W^{1,1}(\Omega) \cap L^2(\Omega - D)$.

In accordance with Lemma 2.1 we additionally get $0 \le \overline{u} \le 1$ a.e. on Ω as well as $\overline{u} \in W^{1,2}_{\mathrm{loc}}(\Omega)$ by construction. In order to face the uniqueness problem, let $\overline{w}$ denote a second function from the space $W^{1,1}(\Omega) \cap L^2(\Omega - D)$ being I-minimizing. By strict convexity it holds $\nabla \overline{u} = \nabla \overline{v}$ a.e. on Ω together with $\overline{u} = \overline{v}$ a.e. on $\Omega - D$. Thus, it follows $\overline{u} = \overline{v} + c$ a.e. on Ω for a suitable constant c and thanks to (1.1) we may infer $c = 0$ which completes the proof of the theorem.

Remark 2.4
We can state that it holds

$$\begin{aligned} I_\delta[u_\delta] &\to I[\overline{u}], \\ \delta \int\limits_{\Omega} |\nabla u_\delta|^2 dx &\to 0, \\ u_\delta \to \overline{u} &\quad \textit{in } L^1(\Omega), \\ u_\delta \rightharpoonup \overline{u} &\quad \textit{in } W^{1,2}_{loc}(\Omega) \end{aligned}$$

as $\delta \to 0$ not only for a subsequence.

3 Proof of Theorem 1.4

Suppose that the inpainting region D satisfies (1.1) and that our density F is μ-elliptic with some $\mu \in (1,2)$. Moreover, we will assume $n \geq 3$ in the following that leads to severe problems since we need existence of local uniform (in δ) L^p-estimates of ∇u_δ (or local uniform L^p-estimates up to a fixed exponent, respectively) in order to perform a De Giorgi type iteration which gives local uniform (in δ) a priori gradient bounds of ∇u_δ after applying Stampacchia's lemma (compare, e.g., [24], Lemma 5.1, p.219). Recalling Remark 2.3, Sobolev's embedding theorem merely implies $\nabla u_\delta \in L^q_{\mathrm{loc}}(\Omega)$ uniformly w.r.t. δ where $1 \leq q < \frac{2n}{n-2}$ and it turns out that this starting integrability of ∇u_δ is not good enough for performing a De Giorgi-type iteration (compare Remark 3.8).
The proof of Theorem 1.4 is organized in four steps: Regularization and local uniform L^p-estimates of ∇u_δ for any finite exponent $1 < p < \infty$, Caccioppoli-type inequality, De Giorgi-type iteration and the Conclusions.

Step 1. Regularization and local uniform L^p-estimates of ∇u_δ

As in the proof of Theorem 1.3, we are going to regularize the energy density F with F_δ from (2.1), i.e. we let

$$I_\delta[w] := \int_\Omega F_\delta(\nabla w)dx + \frac{\lambda}{2}\int_\Omega (w-f)^2 dx.$$

Recalling the statements of Lemma 2.1 and Remark 2.4 we are able to deduce higher local uniform (in δ) p-integrability of ∇u_δ for any finite exponent $1 < p < \infty$. This result will serve as an important tool for carrying out a De Giorgi-type iteration in step 3 of the present proof.

Lemma 3.1
Suppose that we have (1.1) *for D and that F satisfies* (1.4) *and* (1.5) *for some $\mu \in (1,2)$. Then for any $1 < p < \infty$ and for any $\omega \Subset \Omega$ there is a constant $c(p,\omega)$, which in particular does not depend on δ, such that*

$$\|\nabla u_\delta\|_{L^p(\omega,\mathbb{R}^n)} \leq c(p,\omega) < \infty. \tag{3.1}$$

Proof of Lemma 3.1. Primarily we are going to establish a variant of Caccioppoli's inequality which only holds for $\mu \in (1,2)$.

Lemma 3.2
Let (1.1) *hold and suppose that F satisfies* (1.4) *and* (1.5) *for some $\mu \in (1,2)$. Then there is a real number $c > 0$ such that for any $s_0 \geq 0$, for all $\eta \in C_0^\infty(\Omega)$*

satisfying $0 \le \eta \le 1$ *and for any* $\delta \in (0,1)$ *it holds*

$$
\begin{aligned}
&\int_\Omega |\nabla^2 u_\delta|^2 \Gamma_\delta^{s_0-\frac{\mu}{2}} \eta^2 dx + \delta \int_\Omega |\nabla^2 u_\delta|^2 \Gamma_\delta^{s_0} \eta^2 dx \\
&\le \int_\Omega D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta) \eta^2 \Gamma_\delta^{s_0} dx \\
&\le c \int_\Omega D^2 F_\delta(\nabla u_\delta)(\partial_\gamma u_\delta \nabla \eta, \partial_\gamma u_\delta \nabla \eta) \Gamma_\delta^{s_0} dx + c \int_\Omega |\nabla \eta|^2 \Gamma_\delta^{s_0} dx \\
&+ c \int_\Omega \eta^2 \Gamma_\delta^{s_0} dx
\end{aligned}
\tag{3.2}
$$

where we have set $\Gamma_\delta := 1 + |\nabla u_\delta|^2$ *and* c*, in particular, is independent of* δ*.*

Proof of Lemma 3.2. We start by noting that the first inequality follows from (1.5). Next, we fix $s_0 > 0$ (For the case $s_0 = 0$ we refer to Lemma 2.2). Since u_δ is I_δ-minimal we get

$$
\int_\Omega DF_\delta(\nabla u_\delta) \cdot \nabla \varphi dx + \lambda \int_{\Omega - D} (u_\delta - f) \varphi dx = 0 \tag{3.3}
$$

for all $\varphi \in C_0^\infty(\Omega)$. Recalling the statements of Lemma 2.1 and as already discussed in the proof of Lemma 2.2 the differentiated version of euler's equation (3.3) reads as

$$
\int_\Omega D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \nabla \psi) dx = \lambda \int_{\Omega - D} (u_\delta - f) \partial_\gamma \psi dx \tag{3.4}
$$

for all $\psi \in C_0^\infty(\Omega)$ and by standard approximation arguments for all $\psi \in W^{1,2}(\Omega)$ with compact support in Ω.

Next we state that $\psi = \eta^2 \partial_\gamma u_\delta \Gamma_\delta^{s_0}$ is admissible in (3.4) (recall Lemma 2.1 (b) as well as the product and the chain rule for Sobolev functions) and as a consequence we get (from now on summation w.r.t. $\gamma \in \{1, \dots, n\}$)

$$
\begin{aligned}
&\int_\Omega D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\eta^2 \Gamma_\delta^{s_0} dx \\
&+ s_0 \int_\Omega D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \nabla \Gamma_\delta)\Gamma_\delta^{s_0-1}\eta^2 dx \\
&= -2 \int_\Omega D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \nabla \eta)\eta \Gamma_\delta^{s_0} dx \\
&+ \lambda \int_{\Omega - D} (u_\delta - f)\partial_\gamma(\eta^2 \partial_\gamma u_\delta \Gamma_\delta^{s_0}) dx.
\end{aligned}
\tag{3.5}
$$

Studying the last integral on the r.h.s. of (3.5), we obtain

$$
\begin{aligned}
&\int_{\Omega - D} (u_\delta - f)\partial_\gamma(\eta^2 \partial_\gamma u_\delta \Gamma_\delta^{s_0}) dx \\
&= \int_\Omega (u_\delta - f)\partial_\gamma(\eta^2 \partial_\gamma u_\delta \Gamma_\delta^{s_0}) dx - \int_{\Omega \cap D} (u_\delta - f)\partial_\gamma(\eta^2 \partial_\gamma u_\delta \Gamma_\delta^{s_0}) dx \\
&= - \int_\Omega |\nabla u_\delta|^2 \eta^2 \Gamma_\delta^{s_0} dx - \int_\Omega f \partial_\gamma(\eta^2 \partial_\gamma u_\delta \Gamma_\delta^{s_0}) dx \\
&- \int_{\Omega \cap D} (u_\delta - f)\partial_\gamma(\eta^2 \partial_\gamma u_\delta \Gamma_\delta^{s_0}) dx
\end{aligned}
\tag{3.6}
$$

where the last equality follows by performing an integration by parts.

Moreover we have

$$
\begin{aligned}
&s_0 \int_\Omega D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \nabla \Gamma_\delta)\Gamma_\delta^{s_0-1}\eta^2 dx \\
&= \frac{s_0}{2} \int_\Omega D^2 F_\delta(\nabla u_\delta)(\nabla \Gamma_\delta, \nabla \Gamma_\delta)\Gamma_\delta^{s_0-1}\eta^2 dx.
\end{aligned}
\tag{3.7}
$$

Incorporating (3.6) and (3.7) in (3.5), it follows

$$
\begin{aligned}
&\int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma\nabla u_\delta,\partial_\gamma\nabla u_\delta)\eta^2\Gamma_\delta^{s_0}dx\\
&+\frac{s_0}{2}\int_\Omega D^2F_\delta(\nabla u_\delta)(\nabla\Gamma_\delta,\nabla\Gamma_\delta)\Gamma_\delta^{s_0-1}\eta^2dx+\lambda\int_\Omega|\nabla u_\delta|^2\eta^2\Gamma_\delta^{s_0}dx\\
&=-2\int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma\nabla u_\delta,\partial_\gamma u_\delta\nabla\eta)\eta\Gamma_\delta^{s_0}dx\\
&-\lambda\int_\Omega f\partial_\gamma(\eta^2\partial_\gamma u_\delta\Gamma_\delta^{s_0})dx-\lambda\int_{\Omega\cap D}(u_\delta-f)\partial_\gamma(\eta^2\partial_\gamma u_\delta\Gamma_\delta^{s_0})dx.
\end{aligned}
\tag{3.8}
$$

An application of the inequality of Cauchy-Schwarz to the bilinear form $D^2F_\delta(\nabla u_\delta)$ and using Young's inequality ($\varepsilon>0$) subsequently, it holds

$$
\begin{aligned}
&\left|\int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma\nabla u_\delta,\partial_\gamma u_\delta\nabla\eta)\eta\Gamma_\delta^{s_0}dx\right|\\
&\leq\int_\Omega(D^2F_\delta(\nabla u_\delta)(\partial_\gamma\nabla u_\delta,\partial_\gamma\nabla u_\delta))^{\frac{1}{2}}(D^2F_\delta(\nabla u_\delta)(\partial_\gamma u_\delta\nabla\eta,\partial_\gamma u_\delta\nabla\eta))^{\frac{1}{2}}\eta\Gamma_\delta^{s_0}dx\\
&\leq\varepsilon\int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma\nabla u_\delta,\partial_\gamma\nabla u_\delta)\eta^2\Gamma_\delta^{s_0}dx\\
&+\varepsilon^{-1}\int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma u_\delta\nabla\eta,\partial_\gamma u_\delta\nabla\eta)\Gamma_\delta^{s_0}dx.
\end{aligned}
$$

Recalling $0 \le u_\delta, f \le 1$ a.e. and absorbing terms by choosing $\varepsilon = \frac{1}{4}$ it follows

$$
\begin{aligned}
&\int_\Omega D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\eta^2 \Gamma_\delta^{s_0} dx \\
&+ s_0 \int_\Omega D^2 F_\delta(\nabla u_\delta)(\nabla \Gamma_\delta, \nabla \Gamma_\delta)\Gamma_\delta^{s_0-1}\eta^2 dx \\
&+ 2\lambda \int_\Omega |\nabla u_\delta|^2 \eta^2 \Gamma_\delta^{s_0} dx \\
&\le c \int_\Omega D^2 F_\delta(\nabla u_\delta)(\partial_\gamma u_\delta \nabla \eta, \partial_\gamma u_\delta \nabla \eta)\Gamma_\delta^{s_0} dx + c \int_\Omega \eta |\nabla \eta| |\nabla u_\delta| \Gamma_\delta^{s_0} dx \\
&+ c \int_\Omega \eta^2 |\nabla^2 u_\delta| \Gamma_\delta^{s_0} dx + c s_0 \int_\Omega \eta^2 |\nabla u_\delta| |\nabla \Gamma_\delta| \Gamma_\delta^{s_0-1} dx \\
&=: c \int_\Omega D^2 F_\delta(\nabla u_\delta)(\partial_\gamma u_\delta \nabla \eta, \partial_\gamma u_\delta \nabla \eta)\Gamma_\delta^{s_0} dx + \sum_{j=1}^{3} I_j.
\end{aligned}
\tag{3.9}
$$

Starting with I_1 we use Young's inequality ($\varepsilon > 0$) and get

$$
I_1 \le \lambda \int_\Omega \eta^2 |\nabla u_\delta|^2 \Gamma_\delta^{s_0} dx + c \int_\Omega |\nabla \eta|^2 \Gamma_\delta^{s_0} dx. \tag{3.10}
$$

Studying I_3 we obtain by noting $|\nabla \Gamma_\delta| \le c |\nabla u_\delta| |\nabla^2 u_\delta|$

$$
I_3 \le c s_0 \int_\Omega \eta^2 |\nabla^2 u_\delta| \Gamma_\delta^{s_0} dx. \tag{3.11}
$$

As a consequence of (3.11) we may put I_2 and I_3 together and an application of Young's inequality ($\varepsilon > 0$) to this new integral leads to

$$
\begin{aligned}
&c(s_0) \int_\Omega \eta^2 |\nabla^2 u_\delta| \Gamma_\delta^{s_0} dx \\
&\le c(s_0) \int_\Omega \left[\varepsilon \eta^2 \Gamma_\delta^{-\frac{\mu}{2}} |\nabla^2 u_\delta|^2 \Gamma_\delta^{s_0} + \varepsilon^{-1} \eta^2 \Gamma_\delta^{s_0 + \frac{\mu}{2}} \right] dx \\
&\le c(s_0) \int_\Omega \left[\varepsilon D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta) \eta^2 \Gamma_\delta^{s_0} + \varepsilon^{-1} \eta^2 \Gamma_\delta^{s_0 + \frac{\mu}{2}} \right] dx
\end{aligned}
\tag{3.12}
$$

where we used (1.5) in the last inequality.

Incorporating (3.10) and (3.12) in (3.9) it follows by absorbing terms (we choose $\varepsilon > 0$ sufficiently small)

$$\begin{aligned}
&\int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\eta^2\Gamma_\delta^{s_0} dx \\
&+ s_0 \int_\Omega D^2F_\delta(\nabla u_\delta)(\nabla\Gamma_\delta, \nabla\Gamma_\delta)\Gamma_\delta^{s_0-1}\eta^2 dx \\
&+ \lambda \int_\Omega |\nabla u_\delta|^2\eta^2\Gamma_\delta^{s_0} dx \\
&\leq c \int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma u_\delta \nabla\eta, \partial_\gamma u_\delta \nabla\eta)\Gamma_\delta^{s_0} dx + c \int_\Omega |\nabla\eta|^2\Gamma_\delta^{s_0} dx \\
&+ c \int_\Omega \eta^2\Gamma_\delta^{s_0+\frac{\mu}{2}} dx.
\end{aligned} \tag{3.13}$$

Now, we investigate the last integral on the right-hand side of (3.13). Recalling $\mu < 2$ at this point and setting $p := \frac{2}{\mu} > 1$ as well as $q := \frac{2}{2-\mu} > 1$ we get by using Young's inequality one more time (observe that it holds $\frac{1}{p} + \frac{1}{q} = 1$)

$$\begin{aligned}
c \int_\Omega \eta^2\Gamma_\delta^{s_0+\frac{\mu}{2}} dx &\leq \varepsilon \int_\Omega \eta^2\Gamma_\delta^{s_0+1} dx + c\varepsilon^{-1} \int_\Omega \eta^2\Gamma_\delta^{s_0} dx \\
&= \varepsilon \int_\Omega \eta^2\Gamma_\delta^{s_0}|\nabla u_\delta|^2 dx + c(\varepsilon) \int_\Omega \eta^2\Gamma_\delta^{s_0} dx.
\end{aligned}$$

Hence, absorbing terms, (3.13) yields

$$\begin{aligned}
&\int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\eta^2\Gamma_\delta^{s_0} dx \\
&+ s_0 \int_\Omega D^2F_\delta(\nabla u_\delta)(\nabla\Gamma_\delta, \nabla\Gamma_\delta)\Gamma_\delta^{s_0-1}\eta^2 dx \\
&+ \int_\Omega |\nabla u_\delta|^2\eta^2\Gamma_\delta^{s_0} dx \\
&\leq c \int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma u_\delta \nabla\eta, \partial_\gamma u_\delta \nabla\eta)\Gamma_\delta^{s_0} dx + c \int_\Omega |\nabla\eta|^2\Gamma_\delta^{s_0} dx \\
&+ c \int_\Omega \eta^2\Gamma_\delta^{s_0} dx
\end{aligned} \tag{3.14}$$

where c, in particular, does not depend on δ. Neglecting the non-negative second and the non-negative third integral on the l.h.s. of (3.14) we immediately get the desired variant of Caccioppoli's inequality (3.2). $\square$

Next, we are going to establish the local uniform p-integrability of ∇u_δ for any finite exponent p where the variant of Caccioppoli's inequality which was deduced in Lemma 3.2 will serve as an important tool. In order to prove local uniform L^p-estimates of ∇u_δ we adopt techniques as already applied on pp.116 in [6].

Initially we fix a ball $B_{2R_0}(x_0) \Subset \Omega$ where $R_0 > 0$ denotes a real number being sufficently small. Next, we assume that there is a real number $\alpha_0 \geq 0$ such that

$$\int\limits_{B_{2R_0}(x_0)} \Gamma_\delta^{\alpha_0+\frac{1}{2}} dx + \delta \int\limits_{B_{2R_0}(x_0)} \Gamma_\delta^{\alpha_0+1} dx \leq c := c(R_0, \alpha_0) \tag{3.15}$$

where c, in particular, is independent of δ and note that (3.15) is valid for $\alpha_0 = 0$ since we have

$$\begin{aligned} &\int\limits_{B_{2R_0}(x_0)} \Gamma_\delta^{\frac{1}{2}} dx + \delta \int\limits_{B_{2R_0}(x_0)} \Gamma_\delta dx \\ &\leq \int\limits_{B_{2R_0}(x_0)} (1 + |\nabla u_\delta|) dx + \delta \int\limits_{B_{2R_0}(x_0)} \Gamma_\delta dx \\ &\leq c \int\limits_{B_{2R_0}(x_0)} [1 + F_\delta(\nabla u_\delta)] dx \\ &\leq c(R_0) \end{aligned} \tag{3.16}$$

where the last inequality holds true by recalling (1.6) and that $\int\limits_\Omega F_\delta(\nabla u_\delta) dx$ is uniformly bounded in δ.

Now, we set $\alpha := \alpha_0 + 2 - \mu$ and choose $\varphi = \eta^2 \Gamma_\delta^\alpha u_\delta$ where $\eta \in C_0^\infty(B_{2R_0}(x_0))$ satisfying $0 \leq \eta \leq 1, \eta \equiv 1$ on $B_{R_0}(x_0)$ and $|\nabla \eta| \leq \frac{c}{R_0}$. Quoting Lemma 2.1, u_δ is of class $W^{2,2}_{\mathrm{loc}}(\Omega) \cap C^{1,\alpha}(\Omega) \subset W^{2,2}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$, thus φ is admissible in the euler equation (3.3) (recall the product and the chain rule for Sobolev functions) and we obtain

$$
\begin{aligned}
0 &= \int\limits_{B_{2R_0}(x_0)} DF_\delta(\nabla u_\delta)\cdot\nabla(\eta^2\Gamma_\delta^\alpha u_\delta)dx + \lambda \int\limits_{B_{2R_0}(x_0)-D} (u_\delta - f)\eta^2\Gamma_\delta^\alpha u_\delta dx \\
&= \int\limits_{B_{2R_0}(x_0)} DF(\nabla u_\delta)\cdot\nabla(\eta^2\Gamma_\delta^\alpha u_\delta)dx + \delta \int\limits_{B_{2R_0}(x_0)} \nabla u_\delta\cdot\nabla(\eta^2\Gamma_\delta^\alpha u_\delta)dx \\
&+ \lambda \int\limits_{B_{2R_0}(x_0)-D} (u_\delta - f)\eta^2\Gamma_\delta^\alpha u_\delta dx \\
&= \int\limits_{B_{2R_0}(x_0)} DF(\nabla u_\delta)\cdot\nabla u_\delta\,\eta^2\Gamma_\delta^\alpha dx + 2\int\limits_{B_{2R_0}(x_0)} DF(\nabla u_\delta)\cdot\nabla\eta\,\Gamma_\delta^\alpha\eta u_\delta dx \\
&+ \alpha \int\limits_{B_{2R_0}(x_0)} DF(\nabla u_\delta)\cdot\nabla\Gamma_\delta\,\eta^2 u_\delta\Gamma_\delta^{\alpha-1}dx + \delta\int\limits_{B_{2R_0}(x_0)} |\nabla u_\delta|^2\eta^2\Gamma_\delta^\alpha dx \\
&+ 2\delta\int\limits_{B_{2R_0}(x_0)} \nabla u_\delta\cdot\nabla\eta\,\Gamma_\delta^\alpha u_\delta\eta dx + \delta\alpha\int\limits_{B_{2R_0}(x_0)} \nabla u_\delta\cdot\nabla\Gamma_\delta\,\eta^2\Gamma_\delta^{\alpha-1}u_\delta dx \\
&+ \lambda\int\limits_{B_{2R_0}(x_0)-D} (u_\delta - f)\eta^2\Gamma_\delta^\alpha u_\delta dx.
\end{aligned}
\tag{3.17}
$$

Recalling $0 \leq u_\delta, f \leq 1$ and the boundness of DF (see (1.4)) we get from (3.17) by exploiting $|\nabla u_\delta| \leq \Gamma_\delta^{\frac{1}{2}}$ as well as $|\nabla\Gamma_\delta| \leq c|\nabla u_\delta||\nabla^2 u_\delta|$ in addition

$$
\begin{aligned}
&\int\limits_{B_{2R_0}(x_0)} DF(\nabla u_\delta)\cdot\nabla u_\delta\eta^2\Gamma_\delta^\alpha dx + \delta\int\limits_{B_{2R_0}(x_0)} |\nabla u_\delta|^2\eta^2\Gamma_\delta^\alpha dx \\
&\leq c\int\limits_{B_{2R_0}(x_0)} |\nabla\eta|\eta\Gamma_\delta^\alpha dx + c(\alpha)\int\limits_{B_{2R_0}(x_0)} |\nabla^2 u_\delta|\eta^2\Gamma_\delta^{\alpha-\frac{1}{2}}dx \\
&+ c\delta\int\limits_{B_{2R_0}(x_0)} |\nabla\eta|\Gamma_\delta^{\alpha+\frac{1}{2}}\eta dx + c(\alpha)\delta\int\limits_{B_{2R_0}(x_0)} |\nabla^2 u_\delta|\eta^2\Gamma_\delta^\alpha dx \\
&+ c(\lambda)\int\limits_{B_{2R_0}(x_0)} \eta^2\Gamma_\delta^\alpha dx.
\end{aligned}
\tag{3.18}
$$

By means of Lemma 1.2 (i) we may estimate the l.h.s. of (3.18) as follows

$$
\int\limits_{B_{2R_0}(x_0)} DF(\nabla u_\delta)\cdot\nabla u_\delta\eta^2\Gamma_\delta^\alpha dx + \delta\int\limits_{B_{2R_0}(x_0)} |\nabla u_\delta|^2\eta^2\Gamma_\delta^\alpha dx
$$

$$\geq \nu_1 \int\limits_{B_{2R_0}(x_0)} \eta^2 \Gamma_\delta^{\alpha+\frac{1}{2}} dx - \nu_2 \int\limits_{B_{2R_0}(x_0)} \eta^2 \Gamma_\delta^{\alpha} dx$$
$$+ \delta \int\limits_{B_{2R_0}(x_0)} \eta^2 \Gamma_\delta^{\alpha+1} dx - \delta \int\limits_{B_{2R_0}(x_0)} \eta^2 \Gamma_\delta^{\alpha} dx.$$

Fixing $\varepsilon > 0$ and using Young's inequality, the r.h.s. of (3.18) turns into

$$\begin{aligned}
\text{r.h.s.} \leq{}& c\varepsilon \int\limits_{B_{2R_0}(x_0)} \eta^2 \Gamma_\delta^{\alpha+\frac{1}{2}} dx + c\varepsilon^{-1} \int\limits_{B_{2R_0}(x_0)} |\nabla \eta|^2 \Gamma_\delta^{\alpha-\frac{1}{2}} dx \\
&+ c\varepsilon \int\limits_{B_{2R_0}(x_0)} \eta^2 \Gamma_\delta^{\alpha+\frac{1}{2}} dx + c\varepsilon^{-1} \int\limits_{B_{2R_0}(x_0)} |\nabla^2 u_\delta|^2 \eta^2 \Gamma_\delta^{\alpha-\frac{3}{2}} dx \\
&+ c\delta\varepsilon \int\limits_{B_{2R_0}(x_0)} \eta^2 \Gamma_\delta^{\alpha+1} dx + c\delta\varepsilon^{-1} \int\limits_{B_{2R_0}(x_0)} |\nabla \eta|^2 \Gamma_\delta^{\alpha} dx \\
&+ c\delta\varepsilon \int\limits_{B_{2R_0}(x_0)} \eta^2 \Gamma_\delta^{\alpha+1} dx + c\delta\varepsilon^{-1} \int\limits_{B_{2R_0}(x_0)} |\nabla^2 u_\delta|^2 \eta^2 \Gamma_\delta^{\alpha-1} dx \\
&+ c \int\limits_{B_{2R_0}(x_0)} \eta^2 \Gamma_\delta^{\alpha} dx.
\end{aligned}$$

Hence, by absorbing terms (choose $\varepsilon > 0$ sufficiently small), (3.18) turns into

$$\begin{aligned}
&\int\limits_{B_{R_0}(x_0)} \Gamma_\delta^{\alpha+\frac{1}{2}} dx + \delta \int\limits_{B_{R_0}(x_0)} \Gamma_\delta^{\alpha+1} dx \\
&\leq c\Big[\int\limits_{B_{2R_0}(x_0)} |\nabla \eta|^2 \Gamma_\delta^{\alpha-\frac{1}{2}} dx + \int\limits_{B_{2R_0}(x_0)} |\nabla^2 u_\delta|^2 \eta^2 \Gamma_\delta^{\alpha-\frac{3}{2}} dx \\
&+ \int\limits_{B_{2R_0}(x_0)} \eta^2 \Gamma_\delta^{\alpha} dx \Big] \\
&+ c\delta\Big[\int\limits_{B_{2R_0}(x_0)} |\nabla \eta|^2 \Gamma_\delta^{\alpha} dx + \int\limits_{B_{2R_0}(x_0)} |\nabla^2 u_\delta|^2 \eta^2 \Gamma_\delta^{\alpha-1} dx \\
&+ \int\limits_{B_{2R_0}(x_0)} \eta^2 \Gamma_\delta^{\alpha} dx \Big] \\
&=: c \sum_{j=1}^{3} I_j + c\delta \sum_{j=4}^{6} I_j.
\end{aligned} \tag{3.19}$$

Starting with I_1, we recall that by definition of α it holds $\alpha - \frac{1}{2} = \alpha_0 + \frac{3}{2} - \mu \leq \alpha_0 + \frac{1}{2}$. As a consequence it follows

$$\int\limits_{B_{2R_0}(x_0)} |\nabla\eta|^2 \Gamma_\delta^{\alpha-\frac{1}{2}} dx \leq c(R_0) \int\limits_{B_{2R_0}(x_0)} \Gamma_\delta^{\alpha_0+\frac{1}{2}} dx \leq c$$

on account of (3.15) where c does not depend on δ.

Since we may assume w.l.o.g. that $\mu \geq \frac{3}{2}$ it holds $\alpha \leq \alpha_0 + \frac{1}{2}$. Hence, an upper bound for $I_3, \delta I_4$ and δI_6, which is not depending on δ, can easily be found by using (3.15).

Studying I_2 we state that by definition of α we have $\alpha + \frac{\mu}{2} - \frac{3}{2} \leq \alpha_0$ and since $\alpha_0 \geq 0$, Lemma 3.2 and (1.5) give

$$\begin{aligned}
I_2 &\leq \int\limits_{B_{2R_0}(x_0)} |\nabla^2 u_\delta|^2 \eta^2 \Gamma_\delta^{\alpha_0-\frac{\mu}{2}} dx \\
&\leq c \int\limits_{B_{2R_0}(x_0)} D^2F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta) \eta^2 \Gamma_\delta^{\alpha_0} dx \\
&\leq c \int\limits_{B_{2R_0}(x_0)} D^2F_\delta(\nabla u_\delta)(\partial_\gamma u_\delta \nabla\eta, \partial_\gamma u_\delta \nabla\eta) \Gamma_\delta^{\alpha_0} dx \\
&+ c \int\limits_{B_{2R_0}(x_0)} |\nabla\eta|^2 \Gamma_\delta^{\alpha_0} dx + c \int\limits_{B_{2R_0}(x_0)} \eta^2 \Gamma_\delta^{\alpha_0} dx \\
&\leq c(R_0) \int\limits_{B_{2R_0}(x_0)} \left[\Gamma_\delta^{-\frac{1}{2}} + \delta\right] \Gamma_\delta^{1+\alpha_0} dx + c(R_0) \int\limits_{B_{2R_0}(x_0)} \Gamma_\delta^{\alpha_0+\frac{1}{2}} dx \\
&+ c \int\limits_{B_{2R_0}(x_0)} \Gamma_\delta^{\alpha_0+\frac{1}{2}} dx \\
&\leq c(R_0, \alpha_0)
\end{aligned}$$

where the last inequality holds in accordance with (3.15), thus we have found an upper bound of I_2 not depending on δ.

Proceeding with δI_5 we distinguish between two cases whereby we first assume that $\alpha \leq 1$. It follows on account of (1.5) and Lemma 3.2

$$\delta I_5 \leq \delta \int\limits_{B_{2R_0}(x_0)} |\nabla^2 u_\delta|^2 \eta^2 dx$$

$$
\begin{aligned}
&\leq c \int\limits_{B_{2R_0}(x_0)} D^2F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\eta^2 dx \\
&\leq c \int\limits_{B_{2R_0}(x_0)} D^2F_\delta(\nabla u_\delta)(\partial_\gamma u_\delta \nabla\eta, \partial_\gamma u_\delta \nabla \eta) dx + c(R_0) \\
&\leq c(R_0) \int\limits_{B_{2R_0}(x_0)} \left[\Gamma_\delta^{-\frac{1}{2}} + \delta\right]\Gamma_\delta dx + c(R_0) \\
&\leq c(R_0)
\end{aligned}
$$

where the last inequality is valid by taking (3.15) and (3.16), respectively, into account. In particular, the upper bound of I_5 does not depend on δ in this case.

Considering the second case (i.e. $\alpha > 1$), (1.5) and Lemma 3.2 yield

$$
\begin{aligned}
\delta I_5 &\leq c \int\limits_{B_{2R_0}(x_0)} D^2F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\eta^2 \Gamma_\delta^{\alpha-1} dx \\
&\leq c \int\limits_{B_{2R_0}(x_0)} D^2F_\delta(\nabla u_\delta)(\partial_\gamma u_\delta \nabla\eta, \partial_\gamma u_\delta \nabla\eta)\Gamma_\delta^{\alpha-1} dx + c \int\limits_{B_{2R_0}(x_0)} |\nabla\eta|^2 \Gamma_\delta^{\alpha-1} dx \\
&+ c \int\limits_{B_{2R_0}(x_0)} \eta^2 \Gamma_\delta^{\alpha-1} dx \\
&\leq c(R_0) \int\limits_{B_{2R_0}(x_0)} \left[\Gamma_\delta^{-\frac{1}{2}} + \delta\right]\Gamma_\delta^{\alpha} dx + c(R_0) \int\limits_{B_{2R_0}(x_0)} \Gamma_\delta^{\alpha-1} dx \\
&+ c \int\limits_{B_{2R_0}(x_0)} \Gamma_\delta^{\alpha-1} dx \\
&\leq c(R_0, \alpha_0).
\end{aligned}
$$

where we took (3.15) into account once more by noting that $\alpha - \frac{1}{2} \leq \alpha_0 + \frac{1}{2}$. Summarizing, we have proved that δI_5 is bounded from above by a constant being independent of δ.

Altogether, by means of (3.19) we were able to show the following statement: Suppose that (3.15) holds for some given $R_0 > 0$ and $\alpha_0 \geq 0$. Then there is a constant which is not depending on δ with

$$
\int\limits_{B_{R_0}(x_0)} \Gamma_\delta^{\alpha_0+2-\mu+\frac{1}{2}} dx + \delta \int\limits_{B_{R_0}(x_0)} \Gamma_\delta^{\alpha_0+2-\mu+1} dx \leq c. \tag{3.20}
$$

We now prove by induction that for any $m \in \mathbb{N}$ there is a constant $c(m) > 0$, independent of δ, such that for all $\delta \in (0,1)$

$$\int\limits_{B_{R_0/2^{m-1}}(x_0)} \Gamma_\delta^{m(2-\mu)+\frac{1}{2}} dx + \delta \int\limits_{B_{R_0/2^{m-1}}(x_0)} \Gamma_\delta^{m(2-\mu)+1} dx \le c \tag{3.21}$$

Since (3.15) holds for $\alpha_0 = 0$, it follows that $\alpha_0 = 0$ is also an admissible choice in (3.20). Thus, (3.21) extends to $m = 1$.
Next we assume by induction that (3.21) is true for some $m \in \mathbb{N}$. As a consequence, $\alpha_0 = m(2-\mu)$ serves as an admissible choice in (3.15) and (3.20) leads to

$$\begin{aligned} &\int\limits_{B_{R_0/2^m}(x_0)} \Gamma_\delta^{(m+1)(2-\mu)+\frac{1}{2}} dx + \delta \int\limits_{B_{R_0/2^m}(x_0)} \Gamma_\delta^{(m+1)(2-\mu)+1} dx \\ &\le \int\limits_{B_{R_0}(x_0)} \Gamma_\delta^{\alpha_0+2-\mu+\frac{1}{2}} dx + \delta \int\limits_{B_{R_0}(x_0)} \Gamma_\delta^{\alpha_0+2-\mu+1} dx \\ &\le c \end{aligned}$$

where we like to stress that c does not depend on δ. Hence, (3.21) remains valid for any choice of $m \in \mathbb{N}$.

Next, we let $\omega \Subset \Omega$ and let $p \in (1,\infty)$ denote some number. Then there exists another number $m = m(p) \in \mathbb{N}$ with $p \le 1 + 2m(2-\mu)$ and a finite number of balls $B_{R_i}(x_i) \Subset \Omega$ $(i = 1, \dots, M)$ such that

$$\omega \Subset \bigcup_{i=1}^{M} B_{\rho_i}(x_i) \subset \Omega \quad \text{where } \rho_i := \frac{R_i}{2^{m-1}}.$$

Making use of (3.21) we infer

$$\begin{aligned} \|\nabla u_\delta\|_{p,\omega}^p \le \sum_{i=1}^{M} \|\nabla u_\delta\|_{p,B_{\rho_i}(x_i)}^p &\le \sum_{i=1}^{M} \int\limits_{B_{\rho_i}(x_i)} \Gamma_\delta^{\frac{p}{2}} dx \\ &\le \sum_{i=1}^{M} \int\limits_{B_{\rho_i}(x_i)} \Gamma_\delta^{m(2-\mu)+\frac{1}{2}} dx \\ &\le c(p,\omega) \end{aligned}$$

where the local constant $c(p,\omega)$ in particular is independent of δ. This proves the local uniform p-integrability of ∇u_δ w.r.t. δ for any finite exponent p and therewith Lemma 3.1. □

Remark 3.3
Considering the case $n = 2$, we directly obtain Lemma 3.1 by quoting Remark 2.3.

Step 2. Caccioppoli-type inequality

As the second step we are going to establish a Caccioppoli-type inequality which in particular is valid for any $\mu > 1$ and will play an important role when performing a De Giorgi-type iteration in step 3.
Initially, we introduce some notation: We fix a point $x_0 \in \Omega$ and consider radii $0 < r < R < R_0$ with $B_{R_0}(x_0) \Subset \Omega$. Moreover, we let $A_\delta(k, R) := \{x \in B_R(x_0) : \Gamma_\delta > k\}$ where $k > 0$ and Γ_δ denotes the function from Lemma 3.2. Further we consider $\eta \in C_0^\infty(B_R(x_0))$ with $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_r(x_0)$ and $|\nabla \eta| \le \frac{c}{R-r}$. Finally, for functions $v : \Omega \to \mathbb{R}$ we denote $\max\{v, 0\}$ by v^+. Then, the following variant of Caccioppoli's inequality can be derived.

Lemma 3.4
Suppose that the inpainting region D satisfies (1.1) *where* $n \ge 3$ *and that F is μ-elliptic with the prescribed parameter $\mu > 1$. Then we have the following variant of Caccioppoli's inequality*

$$\begin{aligned}\int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{-\mu}{2}} |\nabla \Gamma_\delta|^2 \eta^2 dx \le c \int\limits_{A_\delta(k,R)} |D^2 F_\delta(\nabla u_\delta)||\nabla \eta|^2 (\Gamma_\delta - k)^2 dx \\ + c \int\limits_{A_\delta(k,R)} \eta^2 |\nabla u_\delta|^{2+\mu} dx + \int\limits_{A_\delta(k,R)} \eta |\nabla \eta| |\nabla u_\delta|^3 dx \\ \le \frac{c}{(R-r)^2} \int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\nu}{2}} dx\end{aligned} \tag{3.22}$$

where $\nu := \max\{4, 2+\mu\}$ *and for a suitable positive constant c independent of δ, r and R.*

Proof of Lemma 3.4. A proof can be found in [13] (see Lemma 3 in this reference). □

Remark 3.5
As already outlined in [13], Remark 3, the choice of the parameter ν in Lemma 3.4 is not optimal. In fact, considering the regularization

$$F_{\delta,q}(Z) := \frac{\delta}{q}|Z|^q + F(Z)$$

for our density F where $Z \in \mathbb{R}^n$ and $q > 1$ denotes a number being sufficiently close to 1, we may choose any $\nu > max\{3, 2+\mu\} = 2+\mu$.

Step 3. De Giorgi-type iteration

The third step is devoted to the derivation of local uniform a priori gradient bounds of u_δ. In this context the variant of Caccioppoli's inequality that has been deduced in the second step as well as the well-known Lemma of Stampacchia (see, e.g., [24], Lemma 5.1, p.219 or [6], Lemma 3.26, p. 66) act as essential tools when performing a De Giorgi-type iteration.
Actually we are going to prove a De Giorgi-type lemma that provides a sufficient condition in order to close the gap between local uniform $\overline{p}$-integrability of the gradients for a certain exponent $\overline{p}$ and local uniform a priori gradient bounds. Hence, concerning future problems or applications, respectively, it might be of interest to take note of this sufficient condition that is formulated in the following

Lemma 3.6
Suppose that $n \geq 3$, u_δ is a sequence of class $W^{2,2}_{loc}(\Omega)$ and that we are given real numbers $\overline{p}, \nu > 3$, $\mu > 1$ satisfying

$$\frac{\mu+\nu}{2} n < \overline{p}.$$

Moreover, suppose that we have a uniform constant $c > 0$ (with $\Gamma_\delta, A_\delta(k,R), r, R, R_0, \eta$ as above) such that it holds

$$\int\limits_{A_\delta(k,R)} \Gamma_\delta^{-\frac{\mu}{2}} |\nabla \Gamma_\delta|^2 \eta^2 dx \leq \frac{c}{(R-r)^2} \int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\nu}{2}} dx \tag{3.23}$$

and assume in addition that ∇u_δ is locally $\overline{p}$-integrable uniformly in δ, i.e.

$$\sup_\delta \int\limits_{\Omega'} |\nabla u_\delta|^{\overline{p}} dx = c(\overline{p}, \Omega') < \infty, \tag{3.24}$$

where $\Omega' \Subset \Omega$. Then it holds $\nabla u_\delta \in L^\infty_{loc}(\Omega, \mathbb{R}^n)$ uniformly in δ.

Having the inpainting model at hand, we may even use (3.1) for any finite p but as already mentioned in [13], a replacement of condition (3.24) from Lemma 3.6 by (3.1) does not simplify the following proof in an essential way. Precisely we can state an immediate conclusion of Lemma 3.6.

Proposition 3.7
Suppose that $n \geq 3$, $\mu \in (1,2)$ and that u_δ denotes the approximating sequence

from Lemma 2.1 to the inpainting model under consideration. Then we have local uniform (in δ) a priori gradient bounds for u_δ.

Remark 3.8
Considering the inpainting model with $n \geq 3$, assuming $\mu \in (1,2)$ and denoting by u_δ the approximating sequence from Lemma 2.1, we could show that a priori, we have local uniform (in δ) $L^{\overline{p}}$-estimates of ∇u_δ for all $1 \leq \overline{p} < \frac{2n}{n-2}$ (see Remark 2.3). Consulting Lemma 3.6 now, it turns out that this initial local uniform starting integrability of ∇u_δ is not sufficient in order to derive uniform local a priori gradient bounds by citing Lemma 3.6 since we have to require $\overline{p} > 2n$ at least (notice that Lemma 3.4 provides a variant of Caccioppoli's inequality being in the spirit of (3.23)). Consequently, we had to show higher local uniform (in δ) $\overline{p}$-integrability of ∇u_δ up to the fixed exponent $\frac{\mu+\nu}{2}n + \varepsilon$ with $\varepsilon > 0$ sufficently small before quoting Lemma 3.6 for getting uniform (in δ) local a priori gradient bounds for u_δ. In fact, in Lemma 3.1, we could even show local uniform (in δ) $\overline{p}$-integrability of ∇u_δ for any finite exponent $\overline{p}$.

Remark 3.9
Note that Lemma 3.6 has already been established in [13] (compare Lemma 4 in this reference) in the case $n = 2$.

Proof of Lemma 3.6. For proving Lemma 3.6 we adopt techniques as already applied in [13], Proof of Lemma 4 and in [6], pp.119.

As in [13], we primarily establish a technical proposition being of pure algebraic nature. Its proof is given in the Appendix.

Proposition 3.10
Consider real numbers $\overline{p}, \nu > 3, \mu > 1$ with

$$\frac{\mu+\nu}{2}n < \overline{p}. \tag{3.25}$$

Then, there exist real numbers $s_1, s_2, s_3 > 1$ such that

(i) $2\frac{s_1}{s_1-1} < \overline{p}$, *(ii)* $\frac{1}{s_1}\frac{n}{n-1} > 1$,

(iii) $\mu\frac{s_2}{s_2-1} < \overline{p}$, *(iv)* $\nu\frac{s_3}{s_3-1} < \overline{p}$,

(v) $\frac{1}{2}\frac{n}{n-1}\left(\frac{1}{s_3} + \frac{1}{s_2}\right) > 1$.

Now, we start proving Lemma 3.6. Recalling the previous notation and applying

Sobolev's inequality we get

$$\int_{A_\delta(k,r)} (\Gamma_\delta - k)^{\frac{n}{n-1}} dx \le \int_{B_R(x_0)} (\eta(\Gamma_\delta - k)^+)^{\frac{n}{n-1}} dx$$
$$\le c\Big(\int_{B_R(x_0)} |\nabla[\eta(\Gamma_\delta - k)^+]| dx\Big)^{\frac{n}{n-1}}.$$

Moreover it holds

$$c\Big(\int_{B_R(x_0)} |\nabla[\eta(\Gamma_\delta - k)^+]| dx\Big)^{\frac{n}{n-1}} = c\Big(\int_{A_\delta(k,R)} |\nabla[\eta(\Gamma_\delta - k)]| dx\Big)^{\frac{n}{n-1}}$$
$$\le c\Big(\int_{A_\delta(k,R)} |\nabla\eta|(\Gamma_\delta - k) dx\Big)^{\frac{n}{n-1}}$$
$$+ c\Big(\int_{A_\delta(k,R)} \eta|\nabla\Gamma_\delta| dx\Big)^{\frac{n}{n-1}}$$
$$=: c\Big[I_1^{\frac{n}{n-1}} + I_2^{\frac{n}{n-1}}\Big].$$

As a consequence we can state

$$\int_{A_\delta(k,r)} (\Gamma_\delta - k)^{\frac{n}{n-1}} dx \le c\Big[I_1^{\frac{n}{n-1}} + I_2^{\frac{n}{n-1}}\Big]. \tag{3.26}$$

At this point, we are going to use the algebraic Proposition 3.10 with the same parameters as given in Lemma 3.6. Since we may assume the validity of (3.25) we consequently get existence of real numbers $s_i > 1$, $i = 1,2,3$ fulfilling the claims (i)–(v) of Proposition 3.10.
Incorporating (3.24) we obtain $\Gamma_\delta - k \in L^{\frac{\overline{p}}{2}}(B_R(x_0))$ uniformly in δ. In accordance with Proposition 3.10, (i) and (3.24), we may therefore conclude $\Gamma_\delta - k \in L^{\frac{s_1}{s_1-1}}(B_R(x_0))$ uniformly in δ. By using Hölder's inequality it follows

$$I_1^{\frac{n}{n-1}} = \Big(\int_{A_\delta(k,R)} |\nabla\eta|(\Gamma_\delta - k) dx\Big)^{\frac{n}{n-1}}$$
$$\le \frac{c}{(R-r)^{\frac{n}{n-1}}} (\mathcal{L}^n(A_\delta(k,R)))^{\frac{n}{n-1}\frac{1}{s_1}} \Big(\int_{A_\delta(k,R)} (\Gamma_\delta - k)^{\frac{s_1}{s_1-1}} dx\Big)^{\frac{n}{n-1}\frac{s_1-1}{s_1}} \tag{3.27}$$
$$\le \frac{c}{(R-r)^{\frac{n}{n-1}}} (\mathcal{L}^n(A_\delta(k,R)))^{\frac{n}{n-1}\frac{1}{s_1}}$$

and since on account of Proposition 3.10, (ii), there exists a real number $\overline{\beta} := \frac{n}{n-1}\frac{1}{s_1} > 1$, (3.27) turns into

$$I_1^{\frac{n}{n-1}} \le \frac{c}{(R-r)^{\frac{n}{n-1}}}(\mathcal{L}^n(A_\delta(k,R)))^{\overline{\beta}}. \tag{3.28}$$

Next we discuss I_2: Applying Hölder's inequality and the special type of Caccioppoli's inequality of condition (3.23) we get

$$\begin{aligned} I_2^{\frac{n}{n-1}} &\le \Bigg[\int\limits_{A_\delta(k,R)} \eta^2|\nabla\Gamma_\delta|^2\Gamma_\delta^{\frac{-\mu}{2}}\,dx\Bigg]^{\frac{1}{2}\frac{n}{n-1}}\Bigg[\int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\mu}{2}}\,dx\Bigg]^{\frac{1}{2}\frac{n}{n-1}} \\ &\le \Bigg[\int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\mu}{2}}\,dx\Bigg]^{\frac{1}{2}\frac{n}{n-1}}\Bigg[\frac{c}{(R-r)^2}\int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\nu}{2}}\,dx\Bigg]^{\frac{1}{2}\frac{n}{n-1}}. \end{aligned} \tag{3.29}$$

Quoting (3.24) once again we have $\Gamma_\delta^{\frac{\mu}{2}} \in L^{\frac{\overline{p}}{\mu}}(B_R(x_0))$ uniformly in δ and by using Proposition 3.10, (iii) and (3.24) again, we get $\Gamma_\delta^{\frac{\mu}{2}} \in L^{\frac{s_2}{s_2-1}}(B_R(x_0))$ uniformly in δ. Thanks to Hölder's inequality we may estimate

$$\begin{aligned} &\Bigg[\int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\mu}{2}}\,dx\Bigg]^{\frac{1}{2}\frac{n}{n-1}} \\ &\le \Bigg(\int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\mu s_2}{2(s_2-1)}}\,dx\Bigg)^{\frac{1}{2}\frac{n}{n-1}\frac{s_2-1}{s_2}} \mathcal{L}^n(A_\delta(k,R))^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s_2}} \\ &\le c\mathcal{L}^n(A_\delta(k,R))^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s_2}}. \end{aligned} \tag{3.30}$$

By means of (3.24) it follows $\Gamma_\delta^{\frac{\nu}{2}} \in L^{\frac{\overline{p}}{\nu}}(B_R(x_0))$ uniformly in δ. Taking Proposition 3.10, (iv) and (3.24) into account it holds $\Gamma_\delta^{\frac{\nu}{2}} \in L^{\frac{s_3}{s_3-1}}(B_R(x_0))$ uniformly in δ and Hölder's inequality implies

$$\begin{aligned} \Bigg[\int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\nu}{2}}\,dx\Bigg]^{\frac{1}{2}\frac{n}{n-1}} &\le \Bigg[\int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\nu}{2}\frac{s_3}{s_3-1}}\,dx\Bigg]^{\frac{1}{2}\frac{n}{n-1}\frac{s_3-1}{s_3}} \mathcal{L}^n(A_\delta(k,R))^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s_3}} \\ &\le c\mathcal{L}^n(A_\delta(k,R))^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s_3}}. \end{aligned} \tag{3.31}$$

Putting (3.29) - (3.31) together and exploiting Proposition 3.10, (v) we may infer existence of a real number $\widetilde{\beta} := \frac{1}{2}\frac{n}{n-1}(\frac{1}{s_2}+\frac{1}{s_3}) > 1$ such that

$$I_2^{\frac{n}{n-1}} \le \frac{c}{(R-r)^{\frac{n}{n-1}}}\mathcal{L}^n(A_\delta(k,R))^{\widetilde{\beta}}. \tag{3.32}$$

Assuming w.l.o.g. that $\mathcal{L}^n(A_\delta(k,R)) < 1$, (3.26), (3.28) and (3.32) imply existence of a real number $\beta > 1$ with

$$\int\limits_{A_\delta(k,r)} (\Gamma_\delta - k)^{\frac{n}{n-1}} dx \leq \frac{c}{(R-r)^{\frac{n}{n-1}}} \mathcal{L}^n(A_\delta(k,R))^\beta. \tag{3.33}$$

At this point we define the following quantities for $k \geq 2$ and $r < R_0$:

$$\tau_\delta(k,r) := \int\limits_{A_\delta(k,r)} (\Gamma_\delta - k)^{\frac{n}{n-1}} dx, \quad a_\delta(k,r) := \mathcal{L}^n(A_\delta(k,r))$$

and observe that

$$\tau_\delta : (k_0,\infty) \times (0,R_0) \to [0,\infty), (h,\rho) \mapsto \tau_\delta(h,\rho) := \int\limits_{A_\delta(h,\rho)} (\Gamma_\delta - h)^{\frac{n}{n-1}} dx$$

is a non-negative real-valued function defined for $h > k_0$ and $\rho < R_0$. Moreover, τ_δ is for fixed ρ non-increasing in h and is non-decreasing in ρ if h is fixed. Now, suppose that there are given two real numbers h, k with $h > k > 2$, i.e. we have $\frac{\Gamma_\delta - k}{h-k} \geq 1$ on $A_\delta(h,R)$. Consequently it holds

$$a_\delta(h,R) \leq \int\limits_{A_\delta(h,R)} (\Gamma_\delta - k)^{\frac{n}{n-1}} (h-k)^{-\frac{n}{n-1}} dx,$$

thus

$$a_\delta(h,R) \leq \frac{1}{(h-k)^{\frac{n}{n-1}}} \tau_\delta(k,R). \tag{3.34}$$

From (3.33) and (3.34) it follows

$$\tau_\delta(h,r) \leq \frac{c}{(R-r)^\gamma (h-k)^\alpha} \tau_\delta(k,R)^\beta \tag{3.35}$$

where

$$\gamma := \frac{n}{n-1} > 0,\ \alpha := \frac{n}{n-1}\beta > 0,\ \beta > 1. \tag{3.36}$$

Having (3.35) and (3.36) at hand we may apply Stampacchia's well-known lemma (see, e.g., Lemma B.1, p. 63 in [21]) and obtain local uniform a priori gradient bounds of u_δ. To be more precise, an application of Stampacchia's lemma ensures existence of a positive quantity d_δ such that

$$\tau_\delta(d_\delta + k_0, R_0 - \sigma R_0) = 0$$

for all $\sigma \in (0,1)$ with

$$d_\delta^\alpha = \frac{2^{\frac{(\alpha+\beta)\beta}{\beta-1}} C}{\sigma^\gamma R_0^\gamma} [\tau_\delta(k_0, R_0)]^{\beta-1} \le d^\alpha.$$

where d is a constant not depending on δ since we may use (3.24) (recall $\overline{p} > 2n$). Choosing $k_0 = 2$ and $\sigma = \frac{1}{2}$ we infer

$$0 = \tau_\delta(d_\delta + 2, R_0/2) \ge \tau_\delta(d + 2, R_0/2) \ge 0,$$

i.e. it holds

$$\tau_\delta(d + 2, R_0/2) = 0. \tag{3.37}$$

Condition (3.37) finally leads to the uniform estimate

$$|\nabla u_\delta| \le c$$

a.e. on $B_{R_0/2}(x_0)$ for all $\delta \in (0,1)$ where c in particular is independent of δ since $\Gamma_\delta \le d$ a.e. on $B_{R_0/2}(x_0)$.

Using a covering argument, we finally get

$$||\nabla u_\delta||_{L^\infty(\omega, \mathbb{R}^n)} \le c(\omega)$$

for all $\omega \Subset \Omega$ and $\delta \in (0,1)$, i.e. u_δ is locally uniformly Lipschitz continuous with Lipschitz constant $c(\omega) > 0$. This completes the proof of Lemma 3.6. □

Step 4. Conclusions

Taking the assumption $\mu \in (1,2)$ into account, an application of Proposition 3.7 results in $\nabla u_\delta \in L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ uniformly in δ. Quoting Remark 2.4 we know $u_\delta \to u$ in $L^1_{\mathrm{loc}}(\Omega)$ and since u_δ is locally uniformly (in δ) Lipschitz continuous, the assumptions of Arzelà-Ascoli's theorem are satisfied. This theorem finally gives $u \in C^{0,1}(\Omega)$.

As the last step of the proof of Theorem 1.4 we are going to close the gap between local Lipschitz continuity of u and Hölder continuous first partial derivatives of u in Ω. For that reason we are going to show that u is weak solution of a partial differential equation having its prinicipal part in divergence form. Quoting standard results about elliptic partial differential equations of second order we finally get the desired result.

For proceeding with the proof of Theorem 1.4 we let $\omega \Subset \Omega$ be arbitrary and note that u is solution of the Euler equation

$$\int_{\Omega} DF(\nabla u)\nabla\varphi dx = -\int_{\Omega} g\varphi dx \tag{3.38}$$

for all $\varphi \in C_0^\infty(\Omega)$ where we have set $g := \lambda \mathbb{1}_{\Omega - D}(u - f)$.
Since u is Lipschitz continuous, we may argue with the standard difference quotient technique to get $u \in W^{2,2}_{\text{loc}}(\Omega)$. Moreover, we have $DF(\nabla u) \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^n)$ by using the chain rule for Sobolev functions. Thanks to these results, we obtain after performing an integration by parts

$$-\int_{\omega} D^2F(\nabla u)(\partial_\alpha \nabla u, \nabla\varphi)dx = -\int_{\omega} g\partial_\alpha\varphi dx.$$

for all $\varphi \in C_0^\infty(\Omega)$.
Setting $v := \partial_\alpha u$, we get

$$\int_{\omega} D^2F(\nabla u)(\nabla v, \nabla\psi)dx = \int_{\omega} g\partial_\alpha\psi dx.$$

where the coefficients $a_{\alpha\beta}(x) := \frac{\partial^2 F}{\partial p_\alpha \partial p_\beta}(\nabla u)$ are strictly elliptic and bounded on ω (This fact follows from (1.5) and from the local Lipschitz continuity of u). Finally, Theorem 8.22, p.200, of [20] ensures interior Hölder continuity of v and therefore of $\partial_\alpha u$ for all $\alpha \in \{1, \ldots, n\}$, i.e. u has locally Hölder continuous first partial derivatives in Ω. This completes the proof of Theorem 1.4.

4 Appendix

In the Appendix, we give the proof of the algebraic Lemma 3.10 that served as a technical tool during the proof of Lemma 3.6. Before proving this lemma we will repeat its statements below

Lemma 4.1
Consider real numbers $\overline{p}, \nu > 3, \mu > 1$ with

$$\frac{\mu + \nu}{2} n < \overline{p}. \tag{4.1}$$

Then, there exist real numbers $s_1, s_2, s_3 > 1$ such that

(i) $2\frac{s_1}{s_1 - 1} < \overline{p}$, *(ii)* $\frac{1}{s_1}\frac{n}{n-1} > 1$,

(iii) $\mu\frac{s_2}{s_2 - 1} < \overline{p}$, *(iv)* $\nu\frac{s_3}{s_3 - 1} < \overline{p}$,

(v) $\frac{1}{2}\frac{n}{n-1}\left(\frac{1}{s_3} + \frac{1}{s_2}\right) > 1$.

Proof of Lemma 4.1. Primarily we choose $\widetilde{p} < \overline{p}$ such that (4.1) still holds for $\widetilde{p}$ instead of $\overline{p}$.
Due to (4.1) it holds $\widetilde{p} > \nu\frac{n}{2}$. As a consequence, the statements (i) and (iv) are obvious by setting $s_1 := \frac{\widetilde{p}}{\widetilde{p}-2} > 1$ as well as $s_3 := \frac{\widetilde{p}}{\widetilde{p}-\nu} > 1$. Besides, combining $\mu > 1$ and (4.1) we may conclude the validity of the inequality $\widetilde{p} > 2n$. Recalling our choice of the parameter s_1 from above we immediately obtain (ii). For proving (v) we observe that we have

$$m := 2\frac{n-1}{n} - \frac{1}{s_3} = 2 - \frac{2}{n} - 1 + \frac{\nu}{\widetilde{p}} < 1 \tag{4.2}$$

since $\widetilde{p} > \nu\frac{n}{2}$.
Thanks to (4.2) we may choose $s_2 > 1$ in such a way that

$$m < \frac{1}{s_2} < 1 \tag{4.3}$$

and an application of (4.3) implies

$$\frac{1}{2}\frac{n}{n-1}\left(\frac{1}{s_3} + \frac{1}{s_2}\right) > \frac{1}{2}\frac{n}{n-1}\left(\frac{1}{s_3} + m\right) = 1$$

which shows (v).
In order to verify the last statement of Lemma 4.1 we claim that it holds

$$\frac{1}{s_2} < 1 - \frac{\mu}{\widetilde{p}} \tag{4.4}$$

and remark that the validity of (4.4) directly implies assertion (iii) of Lemma 4.1.
On account of (4.1) it follows

$$m - 1 + \frac{\mu}{\widetilde{p}} = \frac{\nu + \mu}{\widetilde{p}} - \frac{2}{n} < 0.$$

Thus, we get

$$m < 1 - \frac{\mu}{\widetilde{p}}$$

and consequently we may choose $s_2 > 1$ in addition to (4.3) in such a way that

$$m < \frac{1}{s_2} < 1 - \frac{\mu}{\widetilde{p}}.$$

This shows (4.4) and completes the proof of Lemma 4.1. $\square$

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