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Abstract

We construct a new counterexample confirming the sharpness of the Dini-type condition for the boundary of Ω. In particular, we show that for convex domains the Dini-type assumption is the necessary and sufficient condition which guarantees the Hopf-Oleinik type estimates.

1 Introduction

The influence of the properties of a domain to the behavior of a solution is one of the most important topic in the qualitative analysis of partial differential equations.

The significant result in this field is the Hopf-Oleinik lemma, known also as the "Boundary Point Principle". This celebrated lemma states:

Let u be a nonconstant solution to a second-order uniformly elliptic nondivergence equation with bounded measurable coefficients, and let u attend its extremum at a point x^0 located on the boundary of a domain $\Omega \subset \mathbb{R}^n$. Then $\frac{\partial u}{\partial n}(x^0)$ is necessarily nonzero provided that $\partial\Omega$ satisfies the proper assumptions at x^0.

This result was established in a pioneering paper of S. Zaremba [Zar10] for the Laplace equation in a 3-dimensional domain Ω having interior touching ball at x^0 and generalized by G. Giraud [Gir32]-[Gir33] to equations with Hölder continuous leading coefficients and continuous lower order coefficients in domains Ω belonging to the class $C^{1,\alpha}$ with $\alpha \in (0,1)$.

Notice that a related assertion about the negativity on $\partial\Omega$ of the normal derivative of the Green's function corresponding to the Dirichlet problem for the Laplace operator was proved much earlier for 2-dimensional smooth domains by C. Neumann in [Neu88] (see also [Kor01]). The result of [Neu88] was extended for operators with the lower order coefficients by L. Lichtenstein [Lic24]. The same version of the Boundary Point Principle for the Laplacian and 3-dimensional domains satisfying a more flexible interior paraboloid condition was obtained by M.V. Keldysch and M.A. Lavrentiev in [KL37].

A crucial step in studying the Boundary Point Principle was made by E. Hopf [Hop52] and O.A. Oleinik [Ole52], who simultaneously and independently proved the statement for the general elliptic equations with bounded coefficients and domains satisfying an interior ball condition at x^0.

Later the efforts of many mathematicians were focused on generalization of the Boundary Point Principle in several directions (for the details we refer the reader to [ABM$^+$11] and [Alv11] and references therein). Among these directions are the extension of the class of operators and the class of solutions, as well as the weakening of assumptions on the boundary.

The widening of the class of operators to singular/degenerate ones was made in the papers [KH75], [KH77] and [ABM$^+$11], while the uniform elliptic operators with unbounded lower order coefficients were studied in [Saf10] and [Naz12] (see also [NU09]). We mention also the publications [Tol83] and [MS15] where the Boundary Point Principle was established for a class of degenerate quasilinear operators including the p-Laplacian.
We note that before 2010 all the results were formulated for classical solutions, i.e. $u \in C^2(\Omega)$. The class of solutions was expanded in [Saf10] to strong generalized solutions with Sobolev's second order derivatives. The latter requirement seems to be natural in studying of nondivergence elliptic equations.
The reduction of the assumptions on the boundary of Ω up to $C^{1,\mathrm{Dini}}$-regularity was realized for various elliptic operators in the papers [Wid67], [Him70] and [Lie85] (see also [Saf08]). A weakened form of the Hopf-Oleinik lemma (the existence of a boundary point x^1 in any neighborhood of x^0 and a direction ℓ such that $\frac{\partial u}{\partial \ell}(x^1) \neq 0$) was proved in [Nad83] for a much wider class of domains including all Lipschitz ones.
The sharpness of some requirements was confirmed by corresponding counterexamples constructed in [Wid67], [Him70], [KH75], [Saf08], [ABM$^+$11] and [Naz12]. In particular, the counterexamples from [Wid67], [Him70] and [Saf08] show that the Hopf-Oleinik result fails for domains lying entirely in non-Dini paraboloids.
The main result of our paper is a new counterexample showing the sharpness of the Dini-type condition for the boundary of Ω. The simplest version of this counterexample can be formulated as follows:
Let Ω be a convex domain in $\mathbb{R}^n$, let $\partial\Omega$ in a neighborhood of the origin be described by the equation $x_n = F(x')$ with $F \geqslant 0$ and $F(0) = 0$, and let $u \in W^2_{n,loc}(\Omega) \cap C(\overline{\Omega})$ be a solution of the uniformly elliptic equation

$$-a^{ij}(x) D_i D_j u = 0 \quad in \quad \Omega.$$

Suppose also that $u\big|_{\partial\Omega}$ vanishes at a neighborhood of the origin. If, in addition, the function $\delta(r) = \sup\limits_{|x'|\leqslant r} \frac{F(x')}{|x'|}$ is not Dini continuous at zero, then $\frac{\partial u}{\partial \mathbf{n}}(0) = 0$.
It turns out that for convex domains the Dini-type assumption is necessary and sufficient for the validity of the Boundary Point Principle. We emphasize that in our counterexample the Dini-type condition fails for supremum of $F(x')/|x'|$, while in all the previous results of this kind it fails for infimum of $F(x')/|x'|$. In other words, we show that the violating of the Dini-condition just in one direction causes the lack of the Hopf-Oleinik lemma.

1.1 Notation and Conventions

Throughout the paper we use the following notation:
$x = (x', x_n) = (x_1, \dots, x_{n-1}, x_n)$ is a point in $\mathbb{R}^n$;
$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$;
$|x|, |x'|$ are the Euclidean norms in the corresponding spaces;
$x \cdot y$ is the inner product in $\mathbb{R}^n$;
Ω is a bounded convex domain in $\mathbb{R}^n$ with boundary $\partial\Omega$;
$\partial^*\Omega$ is the set of points of $\partial\Omega$ at which the normal to $\partial\Omega$ exists;
$\mathbf{n}(x^0)$ is the unit vector of the inner normal to $\partial\Omega$ at the point x^0.
$\mathcal{P}_r(\overline{x}) = \{x \in \mathbb{R}^n : |x' - \overline{x}'| < r, 0 < x_n - \overline{x}_n < r\}$; $\quad \mathcal{P}_r = \mathcal{P}_r(0)$;
$B_r(x^0)$ is the open ball in $\mathbb{R}^n$ with center x^0 and radius r; $\quad B_r = B_r(0)$;

For $r_1 < r_2$ we define the annulus $\mathcal{B}(x^0, r_1, r_2) = B_{r_2}(x^0) \setminus \overline{B_{r_1}(x^0)}$.
$v_+ = \max\{v, 0\}$, $\quad v_- = \max\{-v, 0\}$.
$\|\cdot\|_{\infty,\Omega}$ denotes the norm in $L_\infty(\Omega)$.

We adopt the convention that the indices i and j run from 1 to n. We also adopt the convention regarding summation with respect to repeated indices.

D_i denotes the operator of differentiation with respect to the variable x_i;

$\mathcal{L}$ is a linear uniformly elliptic operator with measurable coefficients:

$$\mathcal{L}u \equiv -a^{ij}(x)D_iD_ju + b^i(x)D_iu, \qquad \nu\mathcal{I}_n \leq (a^{ij}(x)) \leq \nu^{-1}\mathcal{I}_n, \tag{1}$$

where $\mathcal{I}_n$ is identity $(n \times n)$-matrix. We denote $\mathbf{b}(x) = (b^1(x), \dots, b^n(x))$.

We use letters C and N (with or without indices) to denote various constants. To indicate that, say, C depends on some parameters, we list them in the parenthesis: $C(\dots)$.

Definition 1. We say that a function $\sigma : [0,1] \to \mathbb{R}_+$ belongs to the class $\mathcal{D}_1$ if

- $\sigma(0) = 0, \quad \sigma(1) = 1$;
- σ is increasing and concave;
- $\sigma(t)/t$ is summable.

Remark 1.1. *We say that a function ζ satisfies the Dini condition at zero if*

$$|\zeta(r)| \leqslant C\sigma(r),$$

and σ belongs to the class $\mathcal{D}_1$.

Definition 2. Let a function σ belong to the class $\mathcal{D}_1$. We define the function $\mathcal{J}_\sigma$ as follows

$$\mathcal{J}_\sigma(s) := \int_0^s \frac{\sigma(\tau)}{\tau} d\tau. \tag{2}$$

Remark 1.2. *Due to concavity of σ the function $\sigma(t)/t$ decreases and, consequently,*

$$\sigma(t) \leqslant \mathcal{J}_\sigma(t) \qquad \forall t \in [0,1]. \tag{3}$$

In addition, for $t \leqslant t_0 \leqslant 1$ we have

$$\sigma\left(t/t_0\right) = \frac{\sigma\left(t/t_0\right)}{t/t_0} \cdot t/t_0 \leqslant \frac{\sigma(t)}{t} \cdot t/t_0 = \frac{\sigma(t)}{t_0}, \tag{4}$$

and, similarly,

$$\mathcal{J}_\sigma\left(t/t_0\right) \leqslant \frac{\mathcal{J}_\sigma(t)}{t_0}. \tag{5}$$

2 Preliminaries

2.1 Properties of Ω

Let Ω be a bounded convex domain in $\mathbb{R}^n$. The convexity implies the existence of $\mathcal{R}_0 > 0$ such that for any $x^0 \in \partial\Omega$ the set $\partial\Omega \cap \mathcal{P}_{\mathcal{R}_0}(x^0)$ in local cartesian coordinate system is the graph of a nonnegative function satisfying the Lipschitz condition. There is no restriction in supposing that $\mathcal{R}_0 \leqslant 1$. Without loss of generality we may also assume that the origin belongs to $\partial\Omega$ and

$$\mathcal{P}_{\mathcal{R}_0} \cap \Omega = \left\{(x', x_n) \in \mathbb{R}^n : |x'| \leqslant \mathcal{R}_0, 0 \leqslant F(x') < x_n < \mathcal{R}_0\right\}.$$

For $r \in (0, \mathcal{R}_0)$ we define the functions $\delta = \delta(r)$ and $\delta_1 = \delta_1(r)$ by the formulas

$$\delta(r) := \max_{|x'| \leqslant r} \frac{F(x')}{|x'|}, \qquad \delta_1(r) := \max_{|x'| \leqslant r} |\nabla F(x')|. \tag{6}$$

Lemma 2.1. *The following statements hold:*

(a) $\delta_1(r) \to 0$ *as* $r \to 0$ *iff* $\delta(r) \to 0$ *as* $r \to 0$.

(b) $\delta_1(r)$ *satisfies the Dini-condition at zero iff* $\delta(r)$ *satisfies the Dini-condition at zero.*

Proof. By convexity of F, we have for any x' and z' the estimate

$$F(z') \geqslant F(x') + \nabla F(x') \cdot (z' - x'). \tag{7}$$

Therefore,

$$|\nabla F(x')| \geqslant \nabla F(x') \cdot \frac{x'}{|x'|} \geqslant \frac{F(x')}{|x'|},$$

and, consequently,

$$\delta_1(r) \geqslant \delta(r). \tag{8}$$

On the other hand, for any $r < \mathcal{R}_0$ we can find a point x'_* such that

$$|\nabla F(x'_*)| = \delta_1(r).$$

Chosing $z' = x'_* + r\dfrac{\nabla F(x'_*)}{|\nabla F(x'_*)|}$, we easily deduce from (7) the inequalities

$$|z'| \leqslant 2r \quad \text{and} \quad F(z') \geqslant r\delta_1(r),$$

which provide

$$\delta(2r) \geqslant \delta(|z'|) \geqslant \frac{\delta_1(r)}{2}. \tag{9}$$

Combining (8) and (9) we conclude that statement *(a)* is obvious and the integrals

$$\int_0^{\mathcal{R}_0} \frac{\delta(r)}{r} dr \quad \text{and} \quad \int_0^{\mathcal{R}_0} \frac{\delta_1(r)}{r} dr$$

converge simultaneously. □

If $\delta(r)$ does not converge to zero as $r \to 0$, we can easily see that the domain Ω is contained in a dihedral wedge with the angle less than π and the edge going through the origin. For this case the statement of **Main Theorem** is proved already in [AN00, Theorem 4.3]. By this reason we will assume throughout this paper that

$$\delta(r) \to 0 \quad \text{as} \quad r \to 0. \tag{10}$$

In view of (10), it is evident that δ and δ_1 are moduli of continuity at the origin of the functions $F(x')/|x'|$ and $|\nabla F(x')|$, respectively.

If $x^0 \in \partial\Omega$ is not the origin, we will denote the coordinates in the above-mentioned local cartesian system by $y_1, \dots, y_n$. The unit vector directed along the y_n-axes will be denoted by $\mathbf{n}(x^0)$. Observe that $\mathbf{n}(x^0)$ is the inward normal vector to $\partial\Omega$ if x^0 is a smooth point of $\partial\Omega$.

2.2 Properties of $\mathcal{X}\,(\Omega)$

Let $\mathcal{X}(\Omega)$ be a function space with the norm $\|\cdot\|_{\mathcal{X},\Omega}$.

We suppose that $\mathcal{X}(\Omega)$ has the following properties:

(i) *For arbitrary measurable function g defined in Ω and any function $f \in \mathcal{X}\,(\Omega)$ the inequality $|g(x)| \leqslant |f(x)|$ implies $g \in \mathcal{X}\,(\Omega)$ and $\|g\|_{\mathcal{X},\Omega} \leqslant \|f\|_{\mathcal{X},\Omega}$;*

(ii) *For $f_k \in \mathcal{X}\,(\Omega)$ the convergence $f_k \searrow 0$ a.e. in Ω implies $\|f_k\|_{\mathcal{X},\Omega} \to 0$.*

Using the terminology of classic monograph of Kantorovich and Akilov [KA82] we may say that $\mathcal{X}\,(\Omega)$ is the ideal functional space with order continuous monotone norm (see [KA82, §3, Chapter IV, Part I] for more details).

We will also assume that

(iii) *$\mathcal{X}_{loc}\,(\Omega)$ contains the Orlicz space $L_{\Phi,loc}\,(\Omega)$ with $\Phi(\varsigma) = e^{\varsigma} - \varsigma - 1$.*

Finally, the basic assumption about $\mathcal{X}\,(\Omega)$ is the Aleksandrov-type maximum principle. It means that if $D\,(Du) \in \mathcal{X}_{loc}\,(\Omega)$, $u|_{\partial\Omega} \leq 0$, and $|\mathbf{b}| \in \mathcal{X}\,(\Omega)$ then

$$u \leqslant N_0(n, \nu, \|\mathbf{b}\|_{\mathcal{X},\Omega}) \cdot \operatorname{diam}(\Omega) \cdot \|(\mathcal{L}u)_+\|_{\mathcal{X},\{u>0\}}. \tag{11}$$

Remark 2.2. *It is well known from [Ale60], [Bak61] and [Ale63] (see also survey [Naz05] for further references) that $L_n(\Omega)$ has property (11). It is also evident that properties (i)-(iii) are satisfied in $L_n\,(\Omega)$. Therefore, $L_n(\Omega)$ can be treated as a "basic" example of $\mathcal{X}(\Omega)$. As other examples of the space $\mathcal{X}(\Omega)$ we mention some Lebesgue weighted spaces with power weights (see [Naz01]).*

Remark 2.3. *Unlike the natural properties (i)-(ii), assumption (iii) is rather "technical" one. Without (iii), our arguments from the proof of Step 3 in Theorem 4.1 are not applicable to the approximating operator $\mathcal{L}_\varepsilon$. So, we can not withdraw (iii) in abstract setting. However, in all known examples of $\mathcal{X}\,(\Omega)$ the property (iii) is satisfied.*

Remark 2.4. *Some of the statements, that will be referred to in the sequel, were proved earlier just for the case $\mathcal{X}(\Omega) = L_n(\Omega)$. However, if all the arguments are based only on the Aleksandrov-type maximum principle, these statements remain valid for an arbitrary considered space $\mathcal{X}(\Omega)$. In such cases, we will refer without any further explanation.*

We also need the following convergence lemmas.

Lemma 2.5. *Let $\{f_j\}$ be a sequence of measurable functions on Ω, and let $f \in \mathcal{X}(\Omega)$. Suppose also that $f_j \to 0$ in measure on Ω, and $|f_j(x)| \leqslant |f(x)|$. Then*

$$\|f_j\|_{\mathcal{X},\Omega} \to 0 \quad as \quad j \to \infty. \tag{12}$$

Proof. We argue by a contradiction. Suppose (12) fails. Then there exists a subsequence $\{f_{j_k}\}$ satisfying

$$\|f_{j_k}\|_{\mathcal{X},\Omega} \geqslant \varepsilon > 0, \qquad \forall k \in \mathbb{N}. \tag{13}$$

Due to the Riesz theorem, there exists also a sub-subsequence $\left\{f_{j_{k_l}}\right\}$ such that

$$f_{j_{k_l}} \to 0 \quad \text{a.e. in} \quad \Omega.$$

For simplicity of notation we renumber the latter subsequence $\left\{f_{j_{k_l}}\right\}$ and denote its elements again by f_j.

Setting $\tilde{f}_k := \sup\limits_{j \geqslant k} |f_j|$ we can easily see that $\tilde{f}_k \searrow 0$ a.e. in Ω. Now, taking into account properties (i) and (ii) of the space $\mathcal{X}(\Omega)$ we immediately get a contradiction with inequalities (13). The proof is complete. $\square$

Lemma 2.6. *Let $f \in \mathcal{X}(\Omega)$, and let $\mu(\rho) := \sup\limits_{x \in \Omega} \|f\|_{\mathcal{X},B_\rho(x)\cap\Omega}$.*
Then

$$\mu(\rho) \to 0 \quad as \ \rho \to 0.$$

Proof. For every $\rho > 0$ there exists a point $x^* = x^*(\rho) \in \Omega$ such that

$$\|f\|_{\mathcal{X},B_\rho(x^*)\cap\Omega} \geqslant \frac{1}{2}\mu(\rho).$$

Next, we denote by $\chi_{B_\rho(x^*)}$ the the characteristic function of the set $B_\rho(x^*)$, and set

$$f_\rho := f \cdot \chi_{B_\rho(x^*)}.$$

It is evident that $|f_\rho| \to 0$ in measure on Ω. Application of Lemma 2.5 finishes the proof. $\square$

Remark 2.7. *We call $\mu(\rho) := \sup\limits_{x \in \Omega} \|f\|_{\mathcal{X},B_\rho(x)\cap\Omega}$ the modulus of continuity of function f in $\mathcal{X}(\Omega)$.*

Lemma 2.8. *Let $D(Du) \in \mathcal{X}(\Omega)$, let $\mathcal{L}$ be defined by (1), and let $\mathcal{L}u \in \mathcal{X}(\Omega)$. There exist the family of operators*

$$\mathcal{L}_\varepsilon = -a_\varepsilon^{ij}(x) D_i D_j + b_\varepsilon^i(x) D_i$$

with smooth coefficients a_ε^{ij} and the bounded coefficients b_ε^i satisfying

$$\nu \mathcal{I}_n \leq (a_\varepsilon^{ij}(x)) \leq \nu^{-1} \mathcal{I}_n, \qquad x \in \Omega, \tag{14}$$

$$|b_\varepsilon^i(x)| \leqslant |b^i(x)|, \qquad x \in \Omega, \tag{15}$$

$$\| (\mathcal{L} - \mathcal{L}_\varepsilon) u \|_{\mathcal{X},\Omega} \to 0 \quad as \quad \varepsilon \to 0, \tag{16}$$

respectively.

Proof. We start with extension of a^{ij} on the whole $\mathbb{R}^n$ by the identity matrix and denote by a_ε^{ij} the standard mollification of extended functions a^{ij}. By construction, the coefficients a_ε^{ij} are smooth functions converging as $\varepsilon \to 0$ to a^{ij} a.e. in Ω. Moreover, it is clear that inequalities (14) are true.
Further, we set

$$\widetilde{b}_\varepsilon^i(x) := \min\left\{|b^i(x)|, \varepsilon^{-1}\right\} \cdot \operatorname{sign} b^i(x). \tag{17}$$

In view of (17), it is evident that $\widetilde{b}_\varepsilon^i D_i u$ converges as $\varepsilon \to 0$ to $b^i D_i u$ almost everywhere in Ω. We claim that it is possible to change $\widetilde{b}_\varepsilon^i$ such that the "corrected coefficients" b_ε^i satisfy

$$|b_\varepsilon^i D_i u| \leqslant |b^i D_i u| \quad \text{in} \quad \Omega. \tag{18}$$

Indeed, if $|\widetilde{b}_\varepsilon^i D_i u| \leqslant |b^i D_i u|$ in Ω then (18) holds with $b_\varepsilon^i \equiv \widetilde{b}_\varepsilon^i$. Otherwise, consider a point $x^0 \in \Omega$ where $|\widetilde{b}_\varepsilon^i(x^0) D_i u(x^0)| > |b^i(x^0) D_i u(x^0)|$.

- $\boxed{a)}$ Let $\widetilde{b}_\varepsilon^i(x^0) D_i u(x^0) > b^i(x^0) D_i u(x^0) \geqslant 0$. In this case we decrease all the coefficients $\widetilde{b}_\varepsilon^i(x^0)$ corresponding to the positive summands such that the both sums $b_\varepsilon^i D_i u$ and $b^i D_i u$ becomes equal.

- $\boxed{b)}$ Let $\widetilde{b}_\varepsilon^i(x^0) D_i u(x^0) < b^i(x^0) D_i u(x^0) \leqslant 0$. In this case we decrease all the coefficients $\widetilde{b}_\varepsilon^i(x^0)$ corresponding to the negative summands such that the both sums $b_\varepsilon^i D_i u$ and $b^i D_i u$ becomes equal.

- $\boxed{c)}$ Finally, let $\widetilde{b}_\varepsilon^i(x^0) D_i u(x^0)$ and $b^i(x^0) D_i u(x^0)$ have different signs. In this case we apply to $-b_\varepsilon^i(x^0)$ the arguments from case a) or from case b), respectively.

Due to construction, the "corrected sum" $b_\varepsilon^i D_i u$ also converges as $\varepsilon \to 0$ to $b^i D_i u$ a.e. in Ω, and pointwise inequalities (15) hold true.
Finally, taking into account (18) and applying Lemma 2.5 we get (16). $\square$

3 Gradient estimates near the boundary

Lemma 3.1. *Let $\mathcal{N} \subset \mathbb{R}^n_+$ be an open set, let $\gamma = \frac{\nu}{\sqrt{n-1}}$, let $\rho > 0$, and let*

$$\Pi_\rho = \{y \in \mathbb{R}^n : |y_i| < \rho \quad \textit{for} \quad i = 1, \dots, n-1; \quad 0 < y_n < \gamma\rho\}.$$

We assume that $|\mathbf{b}| \in \mathcal{X}(\mathcal{N})$ and a function v satisfies the conditions

$$D(Dv) \in \mathcal{X}_{loc}(\mathcal{N}), \quad v \geqslant 0 \quad \textit{in} \quad \Pi_\rho, \quad v \geqslant k = const > 0 \quad \textit{on} \quad \partial\mathcal{N} \cap \overline{\Pi}_\rho.$$

Then

$$v \geqslant C_1 k - C_2 k \|\mathbf{b}\|_{\mathcal{X}, \mathcal{N}\cap\Pi_\rho} - C_3 \rho \|(\mathcal{L}v)_-\|_{\mathcal{X}, \mathcal{N}\cap\Pi_\rho} \quad \textit{in} \quad \mathcal{N} \cap B_{\frac{\gamma\rho}{4}}(z),$$

where $z = (0, \dots, 0, \frac{1}{2}\gamma\rho)$, while $C_1 = \frac{1}{16}(1-\gamma^2)$, $C_2 = C_2(n,\nu)$, and $C_3 = C_3(n,\nu)$.

Proof. The proof is similar in spirit to [AU95, Lemma 1].
Consider the barrier function

$$\psi(y) = k\left[1 - \frac{|y'|^2}{\rho^2} + \frac{y_n^2}{\gamma^2\rho^2} - 2\frac{y_n}{\gamma\rho}\right].$$

An elementary computation gives

$$\mathcal{L}\psi \leqslant k\left(\frac{2(n-1)}{\rho^2}\nu^{-1} - \frac{2}{\gamma^2\rho^2}\nu\right) + |\mathbf{b}||D\psi| \leqslant N_1(n,\nu)|\mathbf{b}|\frac{k}{\rho} \quad \text{in} \quad \Pi_\rho.$$

Moreover, setting

$$\mathcal{S}_1 = \{y \in \partial(\mathcal{N} \cap \Pi_\rho) : |y_i| = \rho \quad \text{for some} \quad i = 1, \dots, n-1\},$$
$$\mathcal{S}_2 = \{y \in \partial(\mathcal{N} \cap \Pi_\rho) : y_n = \gamma\rho\}$$

we have

$$\psi\big|_{\mathcal{S}_1 \cup \mathcal{S}_2} \leqslant 0 \leqslant v,$$
$$\psi\big|_{\partial\mathcal{N}\cap\overline{\Pi}_\rho} \leqslant k \leqslant v\big|_{\partial\mathcal{N}\cap\overline{\Pi}_\rho}.$$

Applying inequality (11) in $\mathcal{N} \cap \Pi_\rho$ to the difference $\psi - v$ we obtain

$$\psi - v \leqslant N_0 \cdot \operatorname{diam}(\Pi_\rho) \cdot \|(\mathcal{L}\psi - \mathcal{L}v)_+\|_{\mathcal{X}, \mathcal{N}\cap\Pi_\rho} \quad \text{in} \quad \mathcal{N} \cap \Pi_\rho,$$

and, consequently,

$$v \geqslant k\left[1 - \frac{\gamma^2\rho^2}{16\rho^2} + \frac{9\gamma^2\rho^2}{16\gamma^2\rho^2} - 2\frac{3\gamma\rho}{4\gamma\rho}\right] - C_2 k \|\mathbf{b}\|_{\mathcal{X}, \mathcal{N}\cap\Pi_\rho} - C_3\rho\|(\mathcal{L}v)_-\|_{\mathcal{X}, \mathcal{N}\cap\Pi_\rho}$$
$$= \frac{(1-\gamma^2)}{16}k - C_2 k \|\mathbf{b}\|_{\mathcal{X}, \mathcal{N}\cap\Pi_\rho} - C_3\rho\|(\mathcal{L}v)_-\|_{\mathcal{X}, \mathcal{N}\cap\Pi_\rho} \quad \text{in} \quad \mathcal{N} \cap B_{\theta\rho}(z).$$

□

Our next statement is a version of Theorem 2.3 [Naz12].

Lemma 3.2. *Let $D(Dv) \in \mathcal{X}_{loc}(\Omega)$, let $v\big|_{\partial\Omega} = 0$, let $|\mathbf{b}| \in \mathcal{X}(\Omega)$, and let $0 \in \partial\Omega$. Suppose also that for all $\rho \leqslant \rho_*$ the inequalities*

$$\|b^n\|_{\mathcal{X},\mathcal{P}_\rho\cap\Omega} \leqslant \mathfrak{B}\sigma(\rho/\rho_*), \quad \|(\mathcal{L}v)_+\|_{\mathcal{X},\mathcal{P}_\rho\cap\Omega} \leqslant \mathfrak{F}\sigma(\rho/\rho_*)$$

hold true. Here $\mathfrak{B}$ and $\mathfrak{F}$ are some positive constants, while a function σ belongs to $\mathcal{D}_1$.
Then

$$\sup_{0<x_n<\rho} \frac{v(0,x_n)}{x_n} \leqslant C_4\left(\rho^{-1}\sup_{\mathcal{P}_\rho\cap\Omega} v + \mathfrak{F}\mathcal{J}_\sigma(\rho/\rho_*)\right), \quad \forall\rho \leqslant \rho_*. \tag{19}$$

Here the constant C_4 depends on n, ν, $\mathfrak{B}$, σ, and on the moduli of continuity of $|\mathbf{b}|$ in $\mathcal{X}(\mathcal{P}_{\rho_*}\cap\Omega)$, whereas $\mathcal{J}_\sigma$ is a function defined by formula (2).

Proof. Carefully repeating in $\mathcal{P}_\rho \cap \Omega$ all the arguments necessary for proving Theorem 2.3 from [Naz12] and taking into account Remark 2.4 from the present paper we arrive at the inequality

$$\sup_{0<x_n<\rho/2} \frac{v(0,x_n)}{x_n} \leqslant N\left(\rho^{-1}\sup_{\mathcal{P}_{\rho/2}\cap\Omega} v + \mathfrak{F}\mathcal{J}_\sigma(\rho/\rho_*)\right), \tag{20}$$

where the constant N depends only on n, ν, $\mathfrak{B}$, σ and the moduli of continuity of $|\mathbf{b}|$ in $\mathcal{X}(\mathcal{P}_{\rho_*}\cap\Omega)$.
Further, it is easy to find a majorant for $\dfrac{v(0,x_n)}{x_n}$ for any $x_n \in [\rho/2,\rho)$ since

$$\sup_{\rho/2\leqslant x_n<\rho} \frac{v(0,x_n)}{x_n} \leqslant 2\rho^{-1}\sup_{\rho/2\leqslant x_n<\rho} v(0,x_n) \leqslant 2\rho^{-1}\sup_{\mathcal{P}_\rho\cap\Omega} v. \tag{21}$$

Combination of (20) and (21) finishes the proof. □

4 Main results

Throughout this section we shall suppose that $\mathcal{L}$ is defined by (1), $|\mathbf{b}| \in \mathcal{X}(\Omega)$, and a function u satisfies the following assumptions:

$$D(Du) \in \mathcal{X}_{loc}(\Omega), \quad u \in C\left(\overline{\Omega}\right), \quad \mathcal{L}u = 0 \quad in \quad \Omega, \quad u\big|_{\partial\Omega\cap\mathcal{P}_{\mathcal{R}_0}} = 0. \tag{22}$$

Theorem 4.1. *Let $0 \in \partial\Omega$, and let the inequality*

$$\sup_{x\in\mathcal{P}_{\mathcal{R}_0/2}\cap\{x_n=0\}} \|b^n\|_{\mathcal{X},\mathcal{P}_\rho(x',0)\cap\Omega} \leqslant \mathfrak{B}\sigma\left(\rho/\mathcal{R}_0\right)$$

hold true for all $\rho \leqslant \mathcal{R}_0/2$. Here $\mathfrak{B}$ is a positive constant, and a function $\sigma \in \mathcal{D}_1$ satisfies

$$\mathcal{J}_\sigma(t) = o(\delta(t)) \quad \text{as} \quad t \to 0. \tag{23}$$

Then, there exists a sufficiently small positive number R_0 completely defined by n, ν, $\mathcal{R}_0$, $\mathfrak{B}$, by the functions σ, δ and δ_1, and by the moduli of continuity of $|\boldsymbol{b}'|$ in $\mathcal{X}(\Omega)$ such that for any $r \in (0, R_0/2)$ we have

$$\underset{\Omega\cap\mathcal{P}_{r/4}}{\operatorname{osc}} \frac{u(x)}{x_n} \leqslant (1 - \varkappa\delta(r)) \underset{\Omega\cap\mathcal{P}_{2r}}{\operatorname{osc}} \frac{u(x)}{x_n}. \tag{24}$$

Here the constant $\varkappa \in]0;1[$ is completely determined by n, ν.

Proof. The proof will be divided into 3 steps.

$\boxed{1.}$ Our arguments are adapted from [AU95, Lemma 2] and [Ura96, Lemma 3]. Let us denote

$$m^{\pm} = \sup_{\Omega\cap\mathcal{P}_{2r}} \pm\frac{u(x)}{x_n}, \qquad \omega = m^+ + m^- = \underset{\Omega\cap\mathcal{P}_{2r}}{\operatorname{osc}} \frac{u(x)}{x_n}.$$

Since $u\big|_{\partial\Omega} = 0$ we have $m^{\pm} \geqslant 0$. Therefore, at least one of the numbers $m^{\pm}$ is not less than $\frac{\omega}{2}$, and both of the numbers $m^{\pm}$ are less than ω.

Let $m^+ \geqslant \frac{\omega}{2}$ for definiteness. Then we consider the nonnegative function $v(x) = m^+x_n - u(x)$ in $\Omega \cap \mathcal{P}_{2r}$; (if $m^- > \frac{\omega}{2}$ then we consider the function $v(x) = m^-x_n + u(x)$).

Due to definition of δ, for any sufficiently small $r > 0$ we can find a point $x^* \in \partial\mathcal{P}_r \cap \partial\Omega$ such that $x_n^* = r\delta(r)$. Without loss of generality we may assume that $x_1^* = r$ and $x_\tau^* = 0$ for $\tau = 2, \dots, n-1$.
Next we assign to x^* a local coordinate system $y_1, \dots, y_n$ such that

(a) y_1- axis is directed along the projection of the vector $(x_1^*, \dots, x_{n-1}^*)$ onto tangential hyperplane to $\partial\Omega$ at x^*;

(b) y_2, …, y_{n-1}-axes are parallel to x_2, …, x_{n-1}-axes, respectively;

(c) y_n-axis is directed along $\mathbf{n}(x^*)$.

Setting $\gamma = \frac{\nu}{\sqrt{n-1}}$ we consider in y-coordinates the cylinder

$$\Pi := \left\{ y \in \mathbb{R}^n : \left| y_1 - \frac{r}{2} \right| < \frac{r}{2},\ |y_\tau| < \frac{r}{2},\ 0 < y_n < \frac{1}{2}\gamma r \right\},$$

and the ball $B_{\rho_0}(z^0)$ with $\rho_0 = \frac{1}{8}\gamma r$ and $z^0 = \left(\frac{r}{2}, 0, \ldots, 0, \frac{1}{4}\gamma r\right)$.
It should be emphasized that from now on, all considerations will be carried out in x-coordinates.

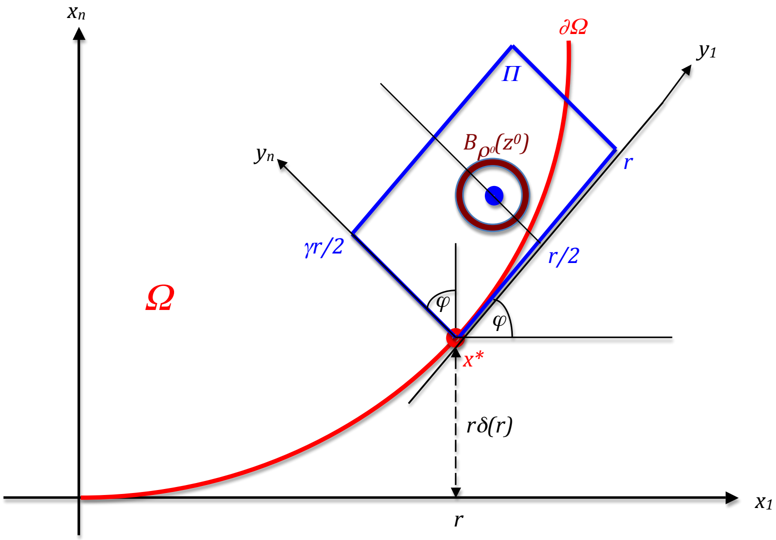

Figure 1: Schematic view of Π and $B_{\rho_0}(z^0)$.

We claim that

$$B_{\rho_0}(z^0) \subset \Omega. \tag{25}$$

Indeed, assume that (25) fails. Then there is a point $\hat{x} \in B_{\rho_0}(z^0)$ satisfying (in x-coordinates) the inequalities

$$F(\hat{x}') \geqslant \hat{x}_n \geqslant z_n^0 - \rho_0. \tag{26}$$

Since $\hat{x} \in B_{\rho_0}(z^0)$ it is clear that $|\hat{x}'| \leqslant 2r$ and

$$F(\hat{x}') \leqslant 2r\delta(2r).$$

On the other hand, denoting by φ the angle between x_n- and y_n-axis (see Fig. 1) we conclude that

$$z_n^0 - \rho_0 = r\delta(r) + \frac{r}{2}\sin\varphi + \frac{\gamma r}{4}\cos\varphi - \frac{\gamma r}{8} \geqslant \frac{\gamma r}{8}\left(2\cos\varphi - 1\right).$$

Thus (26) is transformed into

$$\gamma\,(2\cos\varphi - 1) \leqslant 16\delta(2r). \tag{27}$$

In view of (10) and Lemma 2.1, one can choose R_0 so small that $\delta_1(R_0) \leqslant 3/4$. It guarantees for all $r \leqslant R_0/2$ the inequalities

$$\cos\varphi = \frac{1}{\sqrt{1+\tan^2\varphi}} \geqslant \frac{1}{\sqrt{1+\delta_1^2(r)}} \geqslant \frac{1}{\sqrt{1+\delta_1^2(R_0)}} \geqslant \frac{4}{5}. \tag{28}$$

Now, combining (28) and (27) we get a contradiction with relation (10) provided $\delta(R_0)$ being small enough. The proof of (25) is complete.

$\boxed{2.}$ With (25) at hands, we observe that

$$\inf\{x_n : x \in \Omega \cap \Pi\} \geqslant r\delta(r).$$

On the other hand, the condition $u = 0$ for $x \in \partial\Omega \cap \Pi$ gives the estimate

$$v = m^+ x_n \geqslant \frac{\omega}{2} x_n \quad \text{on} \quad \partial\Omega \cap \Pi.$$

Hence,

$$v \geqslant \frac{\omega}{2} r\delta(r) =: k_0 \quad \text{on} \quad \partial\Omega \cap \Pi. \tag{29}$$

So, we can apply Lemma 3.1 to the function v in cylinder Π. This gives the estimate

$$\inf_{B_{\rho_0}(z^0)} v \geqslant \left(k_0\left[C_1 - C_2\|\mathbf{b}\|_{\mathcal{X},\Omega\cap\mathcal{P}_{2r}}\right] - C_3\omega r\|b^n\|_{\mathcal{X},\Omega\cap\mathcal{P}_{2r}}\right)_+,$$

where C_1, C_2 and C_3 are the constants from Lemma 3.1. Decreasing R_0, if necessary, we may assume that $\|\mathbf{b}\|_{\mathcal{X},\Omega\cap\mathcal{P}_{R_0}} \leqslant C_1/\left(2C_2\right)$. Thus, we arrive at

$$\inf_{B_{\rho_0}(z^0)} v \geqslant \left(k_0\frac{C_1}{2} - C_3\omega r\|b^n\|_{\mathcal{X},\Omega\cap\mathcal{P}_{2r}}\right)_+ =: k_1. \tag{30}$$

Consider now an arbitrary point $\widetilde{z} = (\widetilde{z}', r/4+\rho_0/8)$ such that $|\widetilde{z}'| \leqslant \dfrac{r}{4}$. Observe also that $B_{\rho_0}(\widetilde{z}) \subset \Omega$, otherwise we get a contradiction with definition of $\delta(r)$.
We claim that

$$\inf_{B_{\rho_0/8}(\widetilde{z})} v \geqslant \left(k_0\widetilde{C}_1 - \widetilde{C}_2\omega r\|b^n\|_{\mathcal{X},\Omega\cap\mathcal{P}_{2r}}\right)_+, \tag{31}$$

where $\widetilde{C}_1 = \widetilde{C}_1(n,\nu)$, whereas $\widetilde{C}_2$ is determined completely by n, ν, and $\|\mathbf{b}\|_{\mathcal{X},\Omega}$. Indeed, due to convexity of Ω, for l running from 1 to a finite number $\mathfrak{N} = \mathfrak{N}(n,\nu)$ chosen so that

$$\frac{4}{3\rho_0}|z^0 - \widetilde{z}| \leqslant \mathfrak{N} \leqslant \frac{2}{\rho_0}|z^0 - \widetilde{z}|, \tag{32}$$

and for points $z^{[l]} := z^0 - \frac{l}{\mathfrak{N}}(z^0 - \widetilde{z})$ we have $B_{\rho_0}(z^{[l]}) \subset \Omega$. It should be emphasized that the lower and the upper bounds in (32) do not depend on r. In view of (30) we can compare in $\mathcal{B}(z^{[1]}, \rho_0/8, \rho_0)$ the function v with the standard barrier function

$$w(x) = k_1 \frac{|x - z^{[1]}|^{-s} - \rho_0^{-s}}{(\rho_0/8)^{-s} - \rho_0^{-s}}.$$

If $s = n\nu^{-2}$ then elementary calculation garantees the estimates

$$\begin{aligned}
\mathcal{L}w \leqslant |\mathbf{b}||Dw| \leqslant c(n,\nu)k_1|\mathbf{b}|\rho_0^{-1} \quad &\text{in} \quad \mathcal{B}(z^{[1]}, \rho_0/8, \rho_0), \\
w(x) = k_1 \leqslant v(x) \quad \text{on the sphere} \quad &|x - z^{[1]}| = \frac{\rho_0}{8} \\
w(x) = 0 \ \leqslant v(x) \quad \text{on the sphere} \quad &|x - z^{[1]}| = \rho_0.
\end{aligned}$$

Application of the maximum principle (11) in $\mathcal{B}(z^{[1]}, \rho_0/8, \rho_0)$ to the difference $w - v$ gives us the inequality

$$v(x) \geqslant \big(k_1 \left[w(x) - 2cN_0\|\mathbf{b}\|_{\mathcal{X},\Omega\cap\mathcal{P}_{2r}}\right] - N_0\frac{\gamma r}{4}\omega\|b^n\|_{\mathcal{X},\Omega\cap\mathcal{P}_{2r}}\big)_+.$$

Since $B_{\rho_0/8}(z^{[2]}) \subset \mathcal{B}\left(z^{[1]}, \rho_0/8, 7\rho_0/8\right)$, the evident bound $w \geqslant \theta(n,\nu)$ holds true in $B_{\rho_0/8}(z^{[2]})$.

Decreasing R_0, if necessary, we ensure that $\|\mathbf{b}\|_{\mathcal{X},\Omega\cap\mathcal{P}_{R_0}} \leqslant (4cN_0)^{-1}\theta$. This implies

$$\inf_{B_{\rho_0/8}(z^{[2]})} v(x) \geqslant \left(\frac{k_1\theta}{2} - N_0\frac{\gamma r}{4}\omega\|b^n\|_{\mathcal{X},\Omega\cap\mathcal{P}_{2r}}\right)_+ =: k_2.$$

Repeating this procedure for $\mathcal{B}(z^{[l]}, \rho_0/8, \rho_0)$ and $l = 2, \ldots, \mathfrak{N}$ we arrive at (31) with $\widetilde{C}_1 = (\theta/2)^{\mathfrak{N}}$ and $\widetilde{C}_2 = N_0\,\dfrac{\gamma}{4} \cdot \dfrac{1-(\theta/2)^{\mathfrak{N}}}{1-(\theta/2)}$.

Furthermore, it is clear that

$$\left(k_0\widetilde{C}_1 - \widetilde{C}_2 r\omega\|b^n\|_{\mathcal{X},\Omega\cap\mathcal{P}_{2r}}\right)_+ \geqslant \omega r\left(\frac{1}{2}\widetilde{C}_1\delta(r) - \widetilde{C}_2\mathfrak{B}\sigma\left(r/\mathcal{R}_0\right)\right)_+,$$

while inequalities (3) and (4) guarantee that

$$\sigma\left(r/\mathcal{R}_0\right) \leqslant \frac{\mathcal{J}_\sigma(r)}{\mathcal{R}_0}.$$

Decreasing again R_0 and taking into account the assumption (23) and the above inequalities, we can transform (31) into the form

$$\inf_{B_{\rho_0/8}(\widetilde{z})} v \geqslant \frac{1}{4}\widetilde{C}_1 \omega r \delta(r) =: \widetilde{k}. \tag{33}$$

$\boxed{3.}$ Now, we take a small $\eta > 0$, define the set

$$\mathcal{A}_\eta := \mathcal{B}(\widetilde{z}, \rho_0/8, \widetilde{z}_n) \cap \Omega \cap \{x \in \mathcal{P}_{\mathcal{R}_0} : F(x') + \eta < x_n < \mathcal{R}_0\}$$

and introduce in $\mathcal{A}_\eta$ the barrier function

$$W(x) = \mu \widetilde{k}\, \frac{|x - \widetilde{z}|^{-s} - (\widetilde{z}_n)^{-s}}{(\rho_0/8)^{-s} - (\widetilde{z}_n)^{-s}},$$

where $s = n\nu^{-2}$ and $0 < \mu \leqslant 1$.
Notice that $D\,(Du) \in \mathcal{X}\,(\mathcal{A}_\eta)$. Using Lemma 2.8 we construct the family of operators $\mathcal{L}_\varepsilon$ satisfying $\|\mathcal{L}_\varepsilon u\|_{\mathcal{X},\mathcal{A}_\eta} \to 0$ as $\varepsilon \to 0$.
Arguing in the spirit of the proof of Lemma 4.2 [LU88], we define $v_1(x)$ and $v_2(x)$ as solutions of the following problems:

$$\begin{cases} \mathcal{L}_\varepsilon v_1 = b^i_\varepsilon D_i W & \text{in } \mathcal{A}_\eta \\ \quad v_1 = v & \text{on } \partial\mathcal{A}_\eta \end{cases}, \qquad \begin{cases} \mathcal{L}_\varepsilon v_2 = b^i_\varepsilon D_i W - b^n_\varepsilon m^+ & \text{in } \mathcal{A}_\eta \\ \quad v_2 = 0 & \text{on } \partial\mathcal{A}_\eta \end{cases}.$$

It is well known (see, for instance, [Kry08, Chapter 6]) that $D(Dv_1)$ and $D(Dv_2)$ belong to the space $BMO_{loc}\,(\mathcal{A}_\eta)$. Moreover, the John-Nirenberg theorem [JN61] (see also [Duo01, §4, Chapter 6]) implies that $D(Dv_i)$, $i = 1, 2$, belong to the Orlicz space $L_{\Phi,loc}(\mathcal{A}_\eta)$ with $\Phi(\varsigma) = e^\varsigma - \varsigma - 1$. So, taking into account the property (iii) we may conclude that $D(Dv_i) \in \mathcal{X}_{loc}\,(\mathcal{A}_\eta)$, $i = 1, 2$.

Furthermore, in view of (33) and the direct calculation, we have the inequalities

$$\begin{aligned}
&\mathcal{L}_\varepsilon W \leqslant b^i_\varepsilon D_i W \quad \text{in} \quad \mathcal{A}_\eta,\\
&W(x) = \mu\widetilde{k} \leqslant v(x) = v_1(x) \quad \text{on the sphere} \quad |x - \widetilde{z}| = \frac{\rho_0}{8},\\
&W(x) = 0 \;\leqslant v(x) = v_1(x) \quad \text{on} \quad \partial\mathcal{A}_\eta \cap \{x \in \mathbb{R}^n : |x - \widetilde{z}| = \widetilde{z}_n\}.
\end{aligned}$$

On the rest of $\partial\mathcal{A}_\eta$ we have $x_n = F(x') + \eta$ and, consequently, $dist\,\{x, \partial\Omega\} \leqslant \eta$. Since $u \in C\left(\overline{\Omega}\right)$, the latter inequality implies the estimate $u \leqslant H(\eta)$ there, and therefore,

$$v_1(x) = v(x) = m^+ x_n - u \geqslant \frac{\omega}{2} x_n - H(\eta),$$

where H is a nonnegative function tending to zero as $\eta \to 0$.
In addition, it is easy to verify that

$$W(x) \leqslant \mu N_1(n,\nu)\widetilde{C}_1\omega\delta(r)x_n \quad \text{in } \overline{\mathcal{B}}(\widetilde{z},\rho_0/8,\widetilde{z}_n).$$

Choosing $\mu = \min\left\{1; \left(2N_1\widetilde{C}_1\right)^{-1}\right\}$, we get

$$v_1(x) \geqslant W(x) - H(\eta) \quad \text{on} \quad \partial\mathcal{A}_\eta.$$

The maximum principle (11) applied to the difference $W - H(\eta) - v_1$ in $\mathcal{A}_\eta$ provides the inequality

$$v_1(x) \geqslant W(x) - H(\eta) \geqslant \mu N_2(n,\nu)\widetilde{C}_1\omega\delta(r)\left(\widetilde{z}_n - |x-\widetilde{z}|\right) - H(\eta).$$

It follows from the last inequality with $x = (\widetilde{z}', x_n) \in \Omega$ and $0 < x_n \leqslant \widetilde{z}_n - \rho_0/8 = r/4$ that

$$v_1(\widetilde{z}', x_n) \geqslant N_3(n,\nu)\,\omega\,\delta(r)x_n - H(\eta). \tag{34}$$

Next, we look for a majorant for v_2. With this aim in view, we extend the coefficients a^{ij}_ε continuously and and the coefficients b^i_ε by zero to the whole annulus $\mathcal{B}(\widetilde{z},\rho_0/8,\widetilde{z}_n)$, and denote by $\widetilde{v}_2(x)$ the solution of the problem

$$\mathcal{L}_\varepsilon\widetilde{v}_2 = \begin{cases} (\mathcal{L}_\varepsilon v_2)_+ & \text{in} \quad \mathcal{A}_\eta, \\ \quad 0 & \text{in} \quad \mathcal{B}(\widetilde{z},\rho_0/8,\widetilde{z}_n)\setminus\mathcal{A}_\eta; \end{cases}$$
$$\widetilde{v}_2 = 0 \quad \text{on} \quad \partial\mathcal{B}(\widetilde{z},\rho_0/8,\widetilde{z}_n).$$

The maximum principle guarantees

$$v_2 \leqslant \widetilde{v}_2 \quad \text{in} \quad \mathcal{A}_\eta. \tag{35}$$

Direct computations show that for $\rho \leqslant r/4$ the barrier function W satisfies in the set $\mathcal{E}_\rho := \mathcal{P}_\rho(\widetilde{z}',0)\cap\mathcal{B}(\widetilde{z},\rho_0/8,\widetilde{z}_n)$ the following inequalities

$$|D_nW| \leqslant |DW| \leqslant N_4(n,\nu)\,\mu\,\frac{\widetilde{k}}{r} \leqslant N_4\,\omega\,\delta(r),$$
$$|D'W| \leqslant N_4\mu\frac{\widetilde{k}\rho}{r^2} \leqslant N_4\,\omega\,\frac{\delta(r)\rho}{r}.$$

So, in view of (15) and (10), we have for all $\rho \leqslant r/4$ the bounds

$$\|\,(\mathcal{L}_\varepsilon\widetilde{v}_2)_+\,\|_{\mathcal{X},\mathcal{E}_\rho} \leqslant \|b^n\|_{\mathcal{X},\mathcal{E}_\rho}\left(m^+ + \|D_nW\|_{\infty,\mathcal{E}_\rho}\right) + \|\mathbf{b}'\|_{\mathcal{X},\mathcal{E}_\rho}\|D'W\|_{\infty,\mathcal{E}_\rho}$$
$$\leqslant N_5(n,\nu)\,\omega\,\left[\mathfrak{B}\sigma\left(\rho/\mathcal{R}_0\right) + \frac{\delta(r)}{r}\rho\|\mathbf{b}'\|_{\mathcal{X},\mathcal{A}_\eta}\right].$$

Since the function $\rho \mapsto \left[\mathfrak{B}\sigma\left(\rho/\mathcal{R}_0\right) + \frac{\delta(r)}{r}\rho\|\mathbf{b}'\|_{\mathcal{X},\mathcal{A}_\eta}\right]$ satisfies the Dini-condition at zero, there exist the uniquely defined function $\sigma_1 \in \mathcal{D}_1$ and a constant $\mathfrak{B}_1$ such that

$$\mathfrak{B}\sigma\left(\rho/\mathcal{R}_0\right) + \frac{\delta(r)}{r}\rho\|\mathbf{b}'\|_{\mathcal{X},\mathcal{A}_\eta} = \mathfrak{B}_1\sigma_1\left(4\rho/r\right).$$

Thus, we may apply Lemma 3.2 to the function $\widetilde{v}_2$. It gives for $\rho = r/4$ the estimate

$$\sup_{0<x_n<r/4} \frac{\widetilde{v}_2(\widetilde{z}', x_n)}{x_n} \leqslant C_4\left((r/4)^{-1}\sup_{\mathcal{E}_{r/4}}\widetilde{v}_2 + N_5\omega\mathfrak{B}_1\mathcal{J}_{\sigma_1}(1)\right). \tag{36}$$

It is easy to see that

$$\mathfrak{B}_1\mathcal{J}_{\sigma_1}(1) = \mathfrak{B}\mathcal{J}_\sigma\left(\frac{r}{4\mathcal{R}_0}\right) + \frac{\delta(r)}{4}\|\mathbf{b}'\|_{\mathcal{X},\mathcal{A}_\eta}.$$

Furthermore, applying (11) to $\widetilde{v}_2$ and to the operator $\mathcal{L}_\varepsilon$ in $\mathcal{B}(\widetilde{z}, \rho_0/8, \widetilde{z}_n)$, we obtain

$$\sup_{\mathcal{E}_{r/4}}\widetilde{v}_2 \leqslant \sup_{\mathcal{B}(\widetilde{z},\rho_0/8,\widetilde{z}_n)}\widetilde{v}_2 \leqslant N_6(n,\nu,\|\mathbf{b}\|_{\mathcal{X},\Omega})\,\omega r\left[\mathfrak{B}\sigma\left(\frac{r}{\mathcal{R}_0}\right) + \delta(r)\|\mathbf{b}'\|_{\mathcal{X},\mathcal{A}_\eta}\right].$$

Substitution of the above estimates in (36) and having regard to (3) provide

$$\sup_{0<x_n<r/4}\frac{\widetilde{v}_2(\widetilde{z}', x_n)}{x_n} \leqslant N_7\,\omega\left[\mathfrak{B}\mathcal{J}_\sigma\left(\frac{r}{\mathcal{R}_0}\right) + \delta(r)\|\mathbf{b}'\|_{\mathcal{X},\mathcal{A}_\eta}\right], \tag{37}$$

where the constant N_7 depends only on n, ν and $\|\mathbf{b}\|_{\mathcal{X},\Omega}$.
Taking into account the inequality (5), the assumption (23), and the evident relation $\|\mathbf{b}'\|_{\mathcal{X},\mathcal{A}} = o(1)$ as $r \to 0$, we decrease R_0 such that the property

$$\left[\mathfrak{B}\mathcal{J}_\sigma\left(\frac{r}{\mathcal{R}_0}\right) + \delta(r)\|\mathbf{b}'\|_{\mathcal{X},\mathcal{A}_\eta}\right] \leqslant \frac{N_3}{2N_7}\delta(r) \tag{38}$$

holds true for all $r \leqslant R_0$.
Finally, combining (34)-(35) with (37)-(38) we arrive at the estimate

$$v_1(\widetilde{z}', x_n) - v_2(\widetilde{z}', x_n) \geqslant \frac{N_3}{2}\omega\delta(r)x_n - H(\eta) \tag{39}$$

for $r \leqslant R_0$ and $x = (\widetilde{z}', x_n) \in \Omega$ with $x_n \in [F(\widetilde{z}') + \eta, r/4]$.
Considering in $\mathcal{A}_\eta$ the function $v_3(x) = v(x) - v_1(x) + v_2(x)$ one can easily see that

$$\mathcal{L}_\varepsilon v_3 = -\mathcal{L}_\varepsilon u \to 0 \quad \text{in} \quad \mathcal{X}\left(\mathcal{A}_\eta\right) \quad \text{as} \quad \varepsilon \to 0.$$

In addition, $v_3 = 0$ on $\partial\mathcal{A}_\eta$. Applying the maximum principle (11) to $\pm v_3$ and to the operator $\mathcal{L}_\varepsilon$ we obtain that the difference $v_1(x) - v_2(x)$ converges to $v(x)$ uniformly in $\mathcal{A}_\eta$. Therefore, passing in (39) first to the limit as $\varepsilon \to 0$ and then as $\eta \to 0$, we get

$$\frac{v(x)}{x_n} \geqslant \frac{N_3}{2}\omega\delta(r). \tag{40}$$

for $r \leqslant R_0$ and $x = (\widetilde{z}', x_n) \in \Omega$ with $x_n \in [F(\widetilde{z}'), r/4]$.
Since $\widetilde{z}'$ can be chosen arbitrarily with only $|\widetilde{z}'| \leqslant \dfrac{r}{4}$, the estimate (40) gives (24) with $\varkappa = N_3/2$. $\square$

Theorem 4.2 (**Main Theorem**). *Let the assumptions of Theorem 4.1 hold, and let $F(x')/|x'|$ do not satisfy the Dini-condition at the origin.*
Then for any function u satisfying (22) the equality

$$\frac{\partial u}{\partial \mathbf{n}}(0) = 0$$

holds true.

Proof. Consider the sequence $r_k = 8^{-k}R_0$, $k \geqslant 0$, where R_0 is the constant from Theorem 4.1.
Application of Theorem 4.1 to u guarantees for $k \geqslant 0$ the following inequalities

$$\operatorname*{osc}_{\Omega\cap\mathcal{P}_{r_{k+1}}} \frac{u(x)}{x_n} \leqslant (1 - \varkappa\delta(r_k/2)) \operatorname*{osc}_{\Omega\cap\mathcal{P}_{r_k}} \frac{u(x)}{x_n} \leqslant \operatorname*{osc}_{\Omega\cap\mathcal{P}_{R_0}} \frac{u(x)}{x_n} \cdot \prod_{j=0}^{k} (1 - \varkappa\delta(r_j/2)) \,.$$

Since

$$\sum_{j=0}^{\infty} \ln(1 - \varkappa\delta(r_j/2)) \asymp -\sum_{j=0}^{\infty} \delta(r_j/2) \asymp -\int_0^{r_0} \frac{\delta(r)}{r} dr = -\infty,$$

we have

$$\prod_{j=0}^{\ell} (1 - \varkappa\delta(r_j/2)) \to 0 \quad \text{as} \quad \ell \to \infty.$$

We recall also that Lemma 3.2 implies the finiteness of the quantity $\operatorname*{osc}\limits_{\Omega\cap\mathcal{P}_{R_0}} \dfrac{u(x)}{x_n}$.
Thus, taking into account that $u\big|_{\partial\Omega\cap\mathcal{P}_{\mathcal{R}_0}} = 0$ we get

$$\left|\frac{\partial u}{\partial \mathbf{n}}(0)\right| = \left|\lim_{x_n\to 0} \frac{u(0,x_n)}{x_n}\right| \leqslant \lim_{k\to\infty} \left|\operatorname*{osc}_{\Omega\cap\mathcal{P}_{r_k}} \frac{u(x)}{x_n}\right| = 0,$$

and complete the proof. $\square$

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