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Abstract

We study some variational problems involving energy densities (functions that have to be minimized) experiencing an infinite number of wells. Such densities are encountered in the study of microstructure of some materials as crystals. We consider the energy minimization problem with a fixed Dirichlet boundary data related by a convex relation to some number N of wells. We give a necessary and sufficient condition for nonexistence of minimizers. In the absence of minimizers, we prove that the minimizing sequences converge to the boundary data and choose their gradients around each of the N wells with a probability which tends to be constant. Moreover, they generate a unique Young measure that represents the microstructure. Our analysis shows that the deformation gradient of such materials is only governed by the N wells even if the energy density vanishes at an infinite number of wells. Our results agree with the assumption made in most of analytical and computational investigations that the deformation gradient can be modeled by a limited number of wells.

AMS Subject Classification: 49, 74

Key words: microstructure, minimizers, variational problems, Young measures, wells.

1. Introduction

Let Ω be a bounded domain of $\mathbf{R}^N$ $(N \geq 2)$ with Lipschitz boundary Γ. Let $\Phi : \mathbf{R}^N \longrightarrow \mathbf{R}$ be a continuous function. We denote by $\mathcal{W}(\Phi)$ the set of wells of Φ i.e.

$$\mathcal{W}(\Phi) = \{ \ w \in \mathbf{R}^N \ / \ \Phi(w) = 0 \ \}. \tag{1.1}$$

We assume that Φ verifies

$$\Phi(w) > 0 \ \ \forall w \notin \mathcal{W}(\Phi). \tag{1.2}$$

If H denotes the following hyperplane of $\mathbf{R}^N$ characterized by $\mu \in \mathbf{R}^N$ and $\alpha \in \mathbf{R}$

$$H = \{ \ w \in \mathbf{R}^N \ / \ w \cdot \mu = \alpha \ \} \tag{1.3}$$

($w \cdot \mu$ denotes the scalar product of w and μ), we assume that there exist $w_1, \ldots, w_N$ N elements of $\mathcal{W}(\Phi)$ such that

$$w_i \ \in H \ \forall \ i = 1, \ldots, N \tag{1.4}$$

$$ri_H(\mathrm{Co}(w_i)) \neq \emptyset \tag{1.5}$$

where $\mathrm{Co}(w_i)$ is the convex hull of the w_i's, $ri_H(\mathrm{Co}(w_i))$ denotes its interior relatively to the topology of H. If $a \in ri_H(\mathrm{Co}(w_i))$, we denote by $W_a^{1,\infty}(\Omega)$ the set

$$W_a^{1,\infty}(\Omega) = \{ \ v \in W^{1,\infty}(\Omega) \ / \ v(x) = a \cdot x \text{ on } \Gamma \}.$$

Then, we would like to consider the problem

$$\inf_{v \in W_a^{1,\infty}(\Omega)} \int_\Omega \Phi(\nabla v(x)) \ dx. \tag{1.6}$$

It is well known that

$$\inf_{v \in W_a^{1,\infty}(\Omega)} \int_\Omega \Phi(\nabla v(x)) \ dx = |\Omega| \ \Phi^{**}(a),$$

($|\Omega|$ is the Lebesgue measure of Ω and Φ^{**} is the convex envelope of Φ (see [D.])).

When we write "inf" in (1.6) we have three different questions in mind : first, what is the value of the infimum? Is there a minimizer and if there is no minimizer, then what are the properties of the minimizing sequences. The answer to the first question is 0. Indeed, since $a \in ri_H(\mathrm{Co}(w_i))$, one has using the convexity of Φ^{**}

$$\Phi^{**}(a) = 0. \tag{1.7}$$

First, remark that there is no loss of generality in assuming

$$a = 0 \text{ and } \alpha = 0. \tag{1.8}$$

Indeed, the problem (1.6) is identical to the following one

$$\inf_{v \in W_0^{1,\infty}(\Omega)} \int_\Omega \tilde{\Phi}(\nabla v(x)) \ dx \tag{1.9}$$

where $\tilde{\Phi}(w) = \Phi(w + a)$. So, we end up to deal with the same problem with

$$\mathcal{W}(\tilde{\Phi}) = \mathcal{W}(\Phi) - a$$

and the wells $w_i - a \in \mathcal{W}(\tilde{\Phi})$ such that

$$0 \in ri_H(\mathrm{Co}(w_i - a)), \ w_i - a \in H - a = \{ \ w \in \mathbf{R}^N \ / \ w \cdot \mu = 0 \ \}. \tag{1.10}$$

So, in what follows we will always consider $a = 0$ and $\alpha = 0$. Remark that due to (1.4), (1.10) one has

$$H = \ \mathrm{Span}(w_i)_{i=1,\ldots,N} = \ \mathrm{Span}(w_i - w_1)_{i=2,\ldots,N} \tag{1.11}$$

where $\mathrm{Span}(a_i)_i$ denotes the vector space spanned by the vectors $a_i \in \mathbf{R}^N$. Let us prove the first equality in (1.11), the second one is due to the first part of (1.10). Since

$$0 \in ri_H(\mathrm{Co}(w_i)) \tag{1.12}$$

there exists r positive such that

$$B(0,r) \subset \mathrm{Co}(w_i) \subset \ \mathrm{Span}(w_i)_{i=1,\ldots,N} \tag{1.13}$$

where $B(0,r) = \{h' \in H \ / \ |h'| < r\}$ ($|\cdot|$ denotes the Euclidean norm). Let h' be a nonzero element of H, then the vector

$$v = \frac{r}{2}\frac{h'}{|h'|} \in B(0,r). \tag{1.14}$$

By (1.13) one deduces that $h' \in \ \mathrm{Span}(w_i)$. Combining this with (1.4) one gets

$$H = \ \mathrm{Span}(w_i)_{i=1,\ldots,N}. \tag{1.15}$$

By (1.11) and (1.12), there exist $\alpha_i \in \ (0,1),\ i = 1,\ldots,N$ unique such that

$$0 = \sum_{i=1}^{N} \ \alpha_i w_i \ , \ \sum_{i=1}^{N} \ \alpha_i = 1. \tag{1.16}$$

Then, we adopt the following notations

$$H_+ = \{ \ w \in \mathbf{R}^N \ / \ w \cdot \mu > 0 \ \}, \ \overline{H}_+ = \{ \ w \in \mathbf{R}^N \ / \ w \cdot \mu \geq 0 \ \} \tag{1.17}$$

$$H_- = \{ \ w \in \mathbf{R}^N \ / \ w \cdot \mu < 0 \ \}, \ \overline{H}_- = \{ \ w \in \mathbf{R}^N \ / \ w \cdot \mu \leq 0 \ \} \tag{1.18}$$

$$\mathcal{W}(\Phi)^* = \mathcal{W}(\Phi) \backslash \{w_i, \ i = 1,\ldots,N \ \}. \tag{1.19}$$

The case when $\mathcal{W}(\Phi)$ spans a proper subspace of $\mathbf{R}^N$ was studied by M. Chipot and C. Collins (see for example [C.$_2$], [C.C.]). Mainly they prove, when the number of wells is less than N, that the minimizing sequences have a common behaviour in the sense that they define the same homogeneous Young measure supported by this number of wells. In this paper, we are concerned by the case where $\mathcal{W}(\Phi)$ spans the whole $\mathbf{R}^N$ so that one can select N wells and a hyperplane H such that (1.4), (1.5) hold. The fact that the energy density is allowed to have an infinite number of wells makes difficult to predict a priori the behaviour of the minimizing sequences since the energy density has more freedom to choose some strategy to lower the global energy by making use of its available huge number of wells. Our analysis shows that, in the absence of minimizers, the behaviour of the minimizing

sequences is only governed by the wells related to the boundary data by a convex relation. The most interesting examples of applications of our results occur in the study of deformation of structured materials as crystals (see $[B.J._1]$, $[B.J._1]$, $[E.]$, $[F._1]$, $[F._2]$, $[J.K.]$, $[K.]$, $[Ko.]$). In this case the potential wells are matrices. Moreover, due to the frame indifference, these wells are in infinite number. Note that we are considering here the so called scalar case i.e. the wells are vectors and not matrices. We are out of important phenomena such as incompatibility between phases observed in three-dimensinal models. Our scalar case studied here is nevertheless of some interest as it arises when restricting a three-dimensinal stored energy to antiplane shear deformations (see [F.]).
We organize the rest of this paper as follows. In section 2 we give a sufficient condition for existence of minimizers which is the case when the set $\mathcal{W}(\Phi)$ contains some wells situated in the two disjoint half spaces separated by the hyperplane H. In section 3 we prove that this condition is also necessary. Then we show, in the absence of minimizers, that the corresponding bounded minimizing sequences converge towards the boundary data. We also demonstrate that when $H \cap \mathcal{W}(\Phi) = \{w_i,\ i = 1, \ldots, N\}$ the sequences made of gradient of bounded minimizing sequences generate a unique homogeneous Young measure. In section 4 we prove that the minimizing sequences are choosing their gradient around each of the N wells w_i with a probability which tends to be constant. Finally, in section 5 we give via Lagrange finite elements a bounded minimizing sequence as well as error estimate for the corresponding energy. This estimate is the same as in $[C_2.]$ since the energy density only makes use of the wells $w_i,\ i = 1, \ldots, N$ to lower the global energy. The other wells seem to have no influence.
In his interesting paper (see [F.]), Friesecke gives a necessary and sufficient condition for existence of minimizers (theorem 2.12) for more general densities Φ. In particular, Φ is allowed to have wells of nonequal depths. The proofs of the equivalent condition for our special case are relatively simple using also the construction done by Chipot (see [C.$_2$]) and lemma 1. in section 3 below. This condition can be easily checked : it does not involve the computation of the convex envelope of Φ which is not in general an easy task (compare with theorem 2.12 in [F.]). Let us illustrate our results by the following example

Example 1: The notations are as above, we assume for convenience that μ is a normed vector. Let $\varphi : \mathbf{R}^N \to \mathbf{R}_+$ be a continuous function such that

$$\mathcal{W}(\varphi) = \{w_1, w_2, \ldots, w_N\}. \tag{1.20}$$

Let us consider the function Φ defined as follows

$$\Phi(w) = \varphi(w)((w \cdot \mu - \alpha)^- + (|w| - r)^2), \tag{1.21}$$

where $(\alpha, r) \in \mathbf{R} \times (0, +\infty)$, and f^- denotes the negative part of the function f. Then we have

$$\mathcal{W}(\Phi) = \mathcal{W}(\varphi) \cup (H_\alpha \cap \partial B(0, r)). \tag{1.22}$$

where $H_\alpha = \{w \in \mathbf{R}^N | w \cdot \mu \geq \alpha\}$ and $\partial B(0, r)$ is the boundary of the ball of center 0 and radius r. First remark that if $\alpha > r$ then

$$\mathcal{W}(\Phi) = \mathcal{W}(\varphi) \tag{1.23}$$

so that this case coincides with the case studied in $[C_2.]$. If $\alpha < 0$ the problem (1.6) admits minimizers. If $\alpha > 0$ the infimum in (1.6) is not attained and the minimizing sequences generate a unique Young measure. Now if $\alpha = 0$ the problem (1.6) does not admit minimizers but the minimizing sequences may generate several Young measures.

2. A sufficient condition for existence of minimizers

We investigate in this section the case when the set $\mathcal{W}(\Phi)$ contains some wells situated in the two half spaces defined by the hyperplane H. More precisely we have

Theorem 1. *If $\mathcal{W}(\Phi) \cap H_+$ and $\mathcal{W}(\Phi) \cap H_-$ are not empty, then the infimum in (1.6) is attained.*

Proof : Let $w_{-1} \in \mathcal{W}(\Phi) \cap H_+$ and $w_0 \in \mathcal{W}(\Phi) \cap H_-$. Since

$$0 \in ri_H(\mathrm{Co}(w_i))_{i=1,\ldots,N} \tag{2.1}$$

there exist k wells ($k \geq N - 1$) among the w_i's that we can assume to be $w_1, \ldots, w_k$ such that

$$\mathrm{Span}(w_i)_{i=-1,0,\ldots,k} = \mathbf{R}^N \tag{2.2}$$

$$0 \in \mathrm{Int}(\mathrm{Co}(w_i)_{i=-1,0,\ldots,k}) \tag{2.3}$$

where $\mathrm{Int}(A)$ denotes the interior of A. Therefore, there exist $\beta_{-1}, \beta_0, \ldots, \beta_k \in (0, 1)$ such that

$$0 = \sum_{i=-1}^{k} \beta_i w_i, \quad \sum_{i=-1}^{k} \beta_i = 1. \tag{2.4}$$

Then, we consider the function

$$u(x) = \bigwedge_{i=-1}^{k} w_i \cdot x + 1, \ x \in \mathbf{R}^N \tag{2.5}$$

where $\bigwedge$ denotes the infimum of functions. It is clear that

$$\nabla u(x) = w_i \ \text{ a.e. in } \mathbf{R}^N \quad (i = -1, 0, \ldots, k). \tag{2.6}$$

Due to (2.4) one has

$$\bigwedge_{i=-1}^{k} w_i \cdot x \leq 0, \ \forall \ x \in \mathbf{R}^N. \tag{2.7}$$

We denote by S the following open subset of $\mathbf{R}^N$

$$S = \{x \in \mathbf{R}^N \ / \ u(x) = \bigwedge_{i=-1}^{k} w_i \cdot x + 1 > 0\}. \tag{2.8}$$

We claim that S is bounded. Indeed, one has

$$\forall \ x \in S, \quad \bigwedge_{i=-1}^{k} w_i \cdot x + 1 > 0. \tag{2.9}$$

Then

$$\forall \ x \in S, \quad \bigvee_{i=-1}^{k} w_i \cdot (-x) < 1 \tag{2.10}$$

where $\bigvee$ denotes the supremum of numbers. Thus

$$\forall \ x \in S, \quad |x| \bigvee_{i=-1}^{k} w_i \cdot \frac{(-x)}{|x|} < 1. \tag{2.11}$$

Hence

$$\forall \ x \in S, \quad |x| \inf_{y \in S^{N-1}} \bigvee_{i=-1}^{k} w_i \cdot y < 1 \tag{2.12}$$

where $S^{N-1} = \{y \in \mathbf{R}^N \ / \ |y| = 1\}$. Since S^{N-1} is compact, there exists $y^* \in S^{N-1}$ such that

$$\inf_{y \in S^{N-1}} \bigvee_{i=-1}^{k} w_i \cdot y = \bigvee_{i=-1}^{k} w_i \cdot y^*. \tag{2.13}$$

¿From (2.4) one has

$$\bigvee_{i=-1}^{k} w_i \cdot y^* \geq 0. \tag{2.14}$$

We claim that

$$\bigvee_{i=-1}^{k} w_i \cdot y^* > 0. \tag{2.15}$$

Indeed, assume that

$$\bigvee_{i=-1}^{k} w_i \cdot y^* = 0. \tag{2.16}$$

Then

$$\forall\ i = -1, 0, \ldots, k \quad w_i \cdot y^* \leq 0, \tag{2.17}$$

but from (2.4) one has

$$\sum_{i=-1}^{k} \beta_i w_i \cdot y^* = 0, \ \ \beta_i \in (0,1) \tag{2.18}$$

then

$$w_i \cdot y^* = 0 \ \ \forall i = 1, \ldots, k \tag{2.19}$$

It results from (2.2) that $y^* = 0$ which contradicts the fact that $y^* \in S^{N-1}$. Thus (2.15) holds and S is then a bounded open set. Taking $\Omega = S$, one has

$$u \in W_0^{1,\infty}(\Omega), \tag{2.20}$$

since the w_i's belong to $\mathcal{W}(\Phi)$ and u verifies (2.6) we deduce that u is a minimizer of problem (1.6) with $\Omega = S$. Then, one can construct a minimizer on Ω by covering it with scaled copies of S using Vitali's covering lemma.

■

3. Nonexistence of minimizer-Young measure.

In this section we show that there is no minimizer when we are out of the situation of section 2. This proves the fact that the sufficient condition in theorem 1 is also necessary. So, we assume that

$$\mathcal{W}(\Phi) \cap H_+ = \emptyset \ \text{ or } \ \mathcal{W}(\Phi) \cap H_- = \emptyset \tag{3.1}$$

i.e.

$$\mathcal{W}(\Phi) \subset \overline{H}_+ \ \text{ or } \ \mathcal{W}(\Phi) \subset \overline{H}_-. \tag{3.2}$$

In the sequel, we will treat the case where

$$\mathcal{W}(\Phi) \subset \overline{H}_+ \tag{3.3}$$

the negative case is handeled similarly. We will frequently use the following lemma

Lemma 1. *Let Ω be a bounded domain of $\mathbf{R}^N$ and $u \in W_0^{1,1}(\Omega)$ such that*

$$\frac{\partial u}{\partial \nu}(x) = \nabla u(x) \cdot \nu \geq 0 \quad a.e. \ in \ \ \Omega \tag{3.4}$$

where ν is a nonzero vector of $\mathbf{R}^N$. Then one has

$$u = 0 \quad a.e. \ in \ \ \Omega. \tag{3.5}$$

Proof : Since $u \in W_0^{1,1}(\Omega)$ one has the following variant of Poincaré's inequality

$$\int_\Omega |u(x)|dx \leq C \int_\Omega |\frac{\partial u}{\partial \nu}(x)|dx. \tag{3.6}$$

Using (3.4) one has

$$\int_\Omega |\frac{\partial u}{\partial \nu}(x)|dx = \int_\Omega \frac{\partial u}{\partial \nu}(x)dx = \int_\Omega \nabla u(x)dx \cdot \nu. \tag{3.7}$$

Integrating by part one gets

$$\int_\Omega \nabla u(x)dx = 0. \tag{3.8}$$

Combining (3.6)-(3.8) one deduces that

$$u(x) = 0 \text{ a.e. in } \Omega. \tag{3.9}$$

Remark 1: Let $u \in W_0^{1,1}(\Omega)$ such that

$$\frac{\partial u}{\partial \nu} \leq 0 \quad \text{a.e. in } \Omega$$

then $u = 0$ a.e. in Ω. Indeed, it suffices to apply Lemma 1 to $-u$.

Then one can prove:

Theorem 2. *Assume that $0 \notin \mathcal{W}(\Phi)$, and $\mathcal{W}(\Phi) \subset \overline{H}_+$, then the infimum in (1.6) is not achieved.*

Proof : If u is a minimizer one has

$$\int_\Omega \Phi(\nabla u(x))dx = 0 \tag{3.10}$$

hence by (1.1), (1.2) one gets

$$\nabla u(x) \in \mathcal{W}(\Phi) \subset \overline{H}_+ \quad \text{a.e. in } \Omega. \tag{3.11}$$

Therefore

$$\nabla u(x) \cdot \mu \geq 0 \quad \text{a.e. in } \Omega. \tag{3.12}$$

Using Lemma 1 one gets

$$u = 0 \quad \text{a.e. in } \Omega. \tag{3.13}$$

Since $0 \notin \mathcal{W}(\Phi)$ one has by (1.2)

$$\Phi(0) > 0 \tag{3.14}$$

so that (3.10) is impossible.

■

In the absence of minimizers we turn to the study of minimizing sequences. We have

Theorem 3. *Assume that*

$$\mathcal{W}(\Phi) \subset \overline{H}_+. \tag{3.15}$$

Let $(u_n)_n$ be a minimizing sequence of (1.6) such that

$$||u_n(x)||,\ ||\nabla u_n(x)|| \leq C \quad a.e.\ in\ \Omega \tag{3.16}$$

for some constant C independent of n ($||\cdot||$ denotes a norm in $\mathbf{R}$ or $\mathbf{R}^N$), then one has

$$u_n \longrightarrow 0 \text{ uniformly in } \Omega. \tag{3.17}$$

Proof : From (3.16), by a compactness argument there exist $u \in W_0^{1,\infty}(\Omega)$ and a subsequence u_{n_k} such that

$$u_{n_k} \to u \text{ uniformly in } \Omega \tag{3.18}$$

$$\nabla u_{n_k} \rightharpoonup \nabla u \text{ in } L^\infty(\Omega)^N - * \text{ weak} \tag{3.19}$$

when $k \to \infty$. Now, the bounded sequence of gradients generates a Young measure on $\mathbf{R}^N$ (see [P.]) in the sense that there is a probability measure

ν_x on $\mathbf{R}^N$ and a subsequence of ∇u_{n_k} -still labeled ∇u_{n_k}- such that for any continuous function F on $\mathbf{R}^N$ one has

$$F(\nabla u_{n_k}) \rightharpoonup \int_{\mathbf{R}^N} F(\lambda) d\nu_x(\lambda) \text{ in } L^\infty(\Omega) - * \text{ weak.} \tag{3.20}$$

Considering first (3.20) with $F = \Phi$ and since u_n is a minimizing sequence one gets

$$\int_\Omega 1 \cdot \Phi(\nabla u_{n_k}) \longrightarrow 0 = \int_\Omega \int_{\mathbf{R}^N} \Phi(\lambda)\ d\nu_x(\lambda) dx. \tag{3.21}$$

It follows that

$$\int_{\mathbf{R}^N} \Phi(\lambda)\ d\nu_x(\lambda) = 0 \text{ a.e. in } \Omega. \tag{3.22}$$

One deduces that

$$\text{Supp}\nu_x \subset \mathcal{W}(\Phi) \subset \overline{H}_+, \quad \text{for a.e. } x \in \Omega \tag{3.23}$$

where $\text{Supp}\nu_x$ denotes the support of ν_x. Considering then for F in (3.20) the function $F(\lambda) = \lambda \cdot \mu$

$$\nabla u_{n_k}(x) \cdot \mu \rightharpoonup \int_{\mathbf{R}^N} \lambda \cdot \mu\ d\nu_x(\lambda) \text{ in } L^\infty(\Omega) - * \text{ weak.} \tag{3.24}$$

Combining this with (3.19) one deduces that

$$\nabla u(x) \cdot \mu = \int_{\mathbf{R}^N} \lambda \cdot \mu\ d\nu_x(\lambda) \text{ a.e. in } \Omega, \tag{3.25}$$

but, by (3.23) one has

$$\int_{\mathbf{R}^N} \lambda \cdot \mu\ d\nu_x(\lambda) \geq 0 \text{ a.e. in } \Omega. \tag{3.26}$$

Combining (3.25), (3.26) and Lemma 1, one deduces that $u \equiv 0$. Since the sequence has 0 as unique limit point the whole sequence converges towards 0. In other words one has obtained:

$$u_n \to 0 \text{ uniformly in } \Omega, \tag{3.27}$$

$$\nabla u_n \rightharpoonup 0 \text{ in } L^\infty(\Omega)^N - * \text{ weak,} \tag{3.28}$$

$$\int_{\mathbf{R}^N} \lambda \cdot \mu\ d\nu_x(\lambda) = 0 \text{ a.e. in } \Omega. \tag{3.29}$$

This completes the proof of theorem 3.

∎

Due to (3.29) one has

$$\mathrm{Supp}\nu_x \subset H \cap \mathcal{W}(\Phi). \tag{3.30}$$

so that when n is large the gradient $\nabla u_n(x)$ tends to take one of the wells situated in the Hyperplane H with a probability which tends to ν_x. Let us make this more precise by isolating the wells $w_i,\ i = 1, \dots, N$. Then we have

Theorem 4. *Assume that*

$$\mathcal{W}(\Phi)^* \subset H_+. \tag{3.31}$$

If (u_n) is a minimizing sequence of problem (1.6) satisfying (3.16) then

$$u_n \to 0 \ \textit{uniformly in} \ \ \Omega, \tag{3.32}$$

moreover, the sequence of gradients ∇u_n generates a unique homogeneous Young Measure on $\mathbf{R}^N$ given by

$$\nu_x = \sum_{i=1}^{N} \alpha_i \delta_{w_i} \tag{3.33}$$

where δ_{w_i} denotes the Dirac mass at the point w_i and α_i are the constants appearing in (1.16).

Proof : The convergence in (3.32) is a simple consequence of theorem 3. Let $(\nu_x)_{x\in\Omega}$ be the Young measure associated to ∇u_n verifying (3.20). Due to (3.30) and (3.31) one has

$$\mathrm{Supp}\nu_x \subset \{w_i, i = 1, \dots, N\} \tag{3.34}$$

and the proof follows as in $[C_2]$. ∎

Remark 2 : If $\mathcal{W}(\Phi)^* \cap H$ is not empty, then the bounded minimizing sequences of (1.6) converge toward 0 (cf Theorem 3) but they may not generate the same homogeneous Young measure. Indeed, when we have wells satisfying (1.16) one can construct a minimizing sequence of problem (1.6) satisfying (3.16) and whose sequence of gradients generates a Young measure given by (3.31) (see $[C._2]$). Note also that the function Φ defined in example 1. with $\alpha \geq 0$ (respectively $\alpha > 0$) satisfies the hypothesis of theorem 3 (respectively theorem 4.).

4. Probabilistic analysis

In this section we would like to analyse further the behaviour of the minimizing sequences of (1.6). As we will see, under some assumptions, they have a common pattern. In particular in order to minimize the energy they are choosing their gradients around each of the N wells w_i with a probability which tends to be constant. First let us make precise our assumptions. We will assume that

$$\begin{cases} \mathcal{W}(\Phi)^* \subset H_+, \\ \\ \mathcal{W}(\Phi)^* \ \text{ is compact.} \end{cases} \tag{4.1}$$

Moreover, we denote by π the projection on $\mathcal{W}(\Phi)$ defined by

$$\pi(\xi) = w \tag{4.2}$$

where w is the unique element of $K_N(\xi) \subset K_{N-1}(\xi) \subset \ldots \subset K_0(\xi) \subset \mathcal{W}(\Phi)$ where $K_i, i = 0, \ldots, N$ are defined as follows

$$K_0(\xi) = \{w \in \mathcal{W}(\Phi) \mid |\xi - w| = \min_{w' \in \mathcal{W}(\Phi)} |\xi - w'|\}$$

$$K_i(\xi) = \{w \in K_{i-1} \mid w \cdot e_i = \min_{w' \in K_{i-1}} w' \cdot e_i\}, \ i = 1, \ldots, N. \tag{4.3}$$

where $(e_1, \ldots, e_N)$ is the canonical basis of $\mathbf{R}^N$. By (4.1) and since the K_i's are compact such an operator is clearly defined. Then, we will assume that there exist $\lambda > 0, \ p \geq 1$ such that

$$\Phi(\xi) \geq \lambda |\xi - \pi(\xi)|^p \ \forall \xi \ \in \mathbf{R}^N. \tag{4.4}$$

Now, let us denote by R some positive number such that

$$\begin{cases} B(w_i, \ R) \cap B(w_j, \ R) = \emptyset \ \ i \neq j \\ \\ B(w_i, \ R) \cap B(w', \ R) = \emptyset \ \ \forall w' \in \mathcal{W}(\Phi)^*. \end{cases} \tag{4.5}$$

Such a R exists since $\mathcal{W}(\Phi)^*$ is compact. If $v \in W^{1,\infty}(\Omega)$ we denote by $E(v)$ the quantity

$$E(v) = \int_\Omega \Phi(\nabla v(x)) dx \tag{4.6}$$

and if B is some Lipschitz subdomain of Ω we will denote by $B_i^R(v)$ the set

$$B_i^R(v) = \{x \in B / \nabla v(x) \in B(w_i, \ R)\}. \tag{4.7}$$

If $|\ |$ denotes the Lebesgue measure in $\mathbf{R}^N$, $\frac{|B_i^R(v)|}{|B|}$ represents the probability for v to have its gradient on B in $B(w_i, \ R)$. We introduce also the notation

$$B_{ex}^R(v) = B \setminus \bigcup_{i=1}^N B_i^R(v). \tag{4.8}$$

Then we have:

Lemma 2. *Under the above assumptions, if u_n is a minimizing sequence verifying (3.16) then one has for some constant C independent of n*

$$|\sum_{i=1}^{N} |B_i^R(u_n)|w_i| \leq CE(u_n)^{\frac{1}{2p}} \tag{4.9}$$

when n is choosen large enough.

Proof : Note first that

$$\begin{aligned}
\sum_{i=1}^{N} |B_i^R(u_n)|w_i &= \int_{\bigcup_i B_i^R} \pi(\nabla u_n(x))dx \\
&= \int_B \pi(\nabla u_n(x))dx - \int_{B_{ex}^R} \pi(\nabla u_n(x))dx \qquad (4.10)\\
&= \int_B \pi(\nabla u_n(x)) - \nabla u_n(x)dx + \int_B \nabla u_n(x)dx \\
&\quad - \int_{B_{ex}^R} \pi(\nabla u_n(x))dx \\
&=^{def} I_1 + I_2 + I_3.
\end{aligned}$$

To estimate I_1 note that by Hölder's inequality and (4.4)

$$\begin{aligned}
|I_1| &\leq \int_B |\pi(\nabla u_n(x)) - \nabla u_n(x)|dx \\
&\leq |B|^{1-\frac{1}{p}}(\int_B |\pi(\nabla u_n(x)) - \nabla u_n(x)|^p dx)^{\frac{1}{p}} \\
&\leq \lambda^{-\frac{1}{p}}|B|^{1-\frac{1}{p}}E(u_n)^{\frac{1}{p}}. \qquad (4.11)
\end{aligned}$$

To estimate I_2 one applies the divergence theorem to get

$$I_2 = \int_B \nabla u_n(x)dx = \int_{\partial B} u_n(x)n(x)d\sigma(x)$$

where $n(x)$ denotes the outward normal to ∂B the boundary of B. Hence

$$|I_2| \leq \int_{\partial B} |u_n(x)|d\sigma(x) \leq |\partial B|^{\frac{1}{2}}(\int_{\partial B} |u_n(x)|^2 d\sigma(x))^{\frac{1}{2}}. \tag{4.12}$$

Applying the continuity of the trace function to u_n^2 one gets for a certain constant C

$$\int_{\partial B} |u_n(x)|^2 d\sigma(x) \leq C \int_B |u_n(x)|^2 + |u_n(x)||\nabla u_n(x)|dx. \tag{4.13}$$

Since

$$\int_B |u_n|^2 + |u_n(x)||\nabla u_n(x)|dx = \int_B |u_n(x)|(|u_n(x)| + |\nabla u_n(x)|)dx$$

one deduces by (3.16)

$$\int_{\partial B} |u_n(x)|^2 d\sigma(x) \leq C \int_B |u_n(x)|dx. \tag{4.14}$$

Then one obtains by Poincaré's inequality

$$\int_{\partial B} |u_n(x)|^2 d\sigma(x) \leq C \int_\Omega |u_n(x)|dx \leq C \int_\Omega |\nabla u_n(x) \cdot \mu|dx \tag{4.15}$$

(Recall that μ is an orthogonal vector to the hyperplane H that we can assume of norm equal to 1). By the triangle inequality one has

$$\int_\Omega |\nabla u_n(x) \cdot \mu|dx \leq \int_\Omega |(\nabla u_n(x) - \pi(\nabla u_n(x))) \cdot \mu|dx + \int_\Omega |\pi(\nabla u_n(x)) \cdot \mu|dx \tag{4.16}$$

but by (4.1) one has

$$\pi(\nabla u_n(x)) \cdot \mu \geq 0 \tag{4.17}$$

then

$$\begin{aligned}
\int_\Omega |\pi(\nabla u_n(x)) \cdot \mu|dx &= \int_\Omega \pi(\nabla u_n(x)) \cdot \mu dx \\
&= \int_\Omega (\pi(\nabla u_n(x)) - \nabla u_n(x)) \cdot \mu dx + \int_\Omega \nabla u_n(x) \cdot \mu dx \\
&= \int_\Omega (\pi(\nabla u_n(x)) - \nabla u_n(x)) \cdot \mu dx \qquad (4.18)
\end{aligned}$$

since

$$\int_\Omega \nabla u_n(x) \cdot \mu dx = \int_\Omega \nabla u_n(x) dx \cdot \mu = 0.$$

Thus

$$
\begin{aligned}
\int_\Omega |\nabla u_n(x)\cdot\mu|dx &\leq 2\int_\Omega |(\nabla u_n(x)-\pi(\nabla u_n(x)))\cdot\mu|dx \\
&\leq 2\int_\Omega |\nabla u_n(x)-\pi(\nabla u_n(x))|dx. \qquad (4.19)
\end{aligned}
$$

Applying Hölder's inequality one obtains

$$
\int_\Omega |\nabla u_n(x)\cdot\mu|dx \leq 2|\Omega|^{1-\frac{1}{p}}\int_\Omega (|\nabla u_n(x)-\pi(\nabla u_n(x))|^p dx)^{\frac{1}{p}}. \qquad (4.20)
$$

Combining (4.12), (4.15) and (4.20) one deduces by (4.4) that

$$
|I_2| \leq CE(u_n)^{\frac{1}{2p}}. \qquad (4.21)
$$

To estimate I_3, note that

$$
\begin{aligned}
|I_3| &\leq \int_{B^R_{ex}(u_n)} |\pi(\nabla u_n(x))|dx \\
&\leq (\max_{w\in\mathcal{W}(\Phi)}|w|)|B^R_{ex}(u_n)| \qquad (4.22)
\end{aligned}
$$

To estimate $|B^R_{ex}(u_n)|$ we denote by D and D' the sets

$$
D = \{x\in B^R_{ex}(u_n)\ /\ \pi(\nabla u_n(x))\in\{w_1,\dots,w_N\}\}
$$

$$
D' = B^R_{ex}(u_n)\backslash D = \{x\in B^R_{ex}(u_n)\ /\ \pi(\nabla u_n(x))\in\mathcal{W}(\Phi)^*\}
$$

then one has

$$
\begin{aligned}
R|D| &\leq \int_D |\pi(\nabla u_n(x))-\nabla u_n(x)|dx \\
&\leq \int_B |\pi(\nabla u_n(x))-\nabla u_n(x)|dx.
\end{aligned}
$$

Hence, using Hölder's inequality

$$
|D| \leq \frac{1}{R}|B|^{1-\frac{1}{p}}\lambda^{-\frac{1}{p}}E(u_n)^{\frac{1}{p}}. \qquad (4.23)
$$

On the other hand one has

$$
\begin{aligned}
|D'| \min_{w\in\mathcal{W}(\Phi)^*} w\cdot\mu &\le \int_{D'} \pi(\nabla u_n(x))\cdot\mu dx \quad (\min_{w\in\mathcal{W}(\Phi)^*} w\cdot\mu > 0 \text{ by } (4.1))\\
&\le \int_{\Omega} \pi(\nabla u_n(x))\cdot\mu dx \quad (\text{since } \pi(\nabla u_n(x))\cdot\mu \ge 0 \text{ for a.e } x\in\Omega)\\
&= \int_{\Omega} (\pi(\nabla u_n(x)) - \nabla u_n(x))\cdot\mu dx\\
&\le \int_{\Omega} |\pi(\nabla u_n(x)) - \nabla u_n(x)| dx
\end{aligned}
$$

by Hölder's inequality and (4.4) we have

$$
|D'| \le (\min_{w\in\mathcal{W}(\Phi)^*} w\cdot\mu)^{-1} \lambda^{-\frac{1}{p}} |\Omega|^{1-\frac{1}{p}} E(u_n)^{\frac{1}{p}}. \tag{4.24}
$$

Combining (4.23) and (4.24) we obtain

$$
|B^R_{ex}(u_n)| = |D| + |D'| \le \{\frac{1}{R}|B|^{1-\frac{1}{p}}\lambda^{-\frac{1}{p}} + (\min_{w\in\mathcal{W}(\Phi)^*} w\cdot\mu)^{-1}\lambda^{-\frac{1}{p}}|\Omega|^{1-\frac{1}{p}}\} E(u_n)^{\frac{1}{p}} \tag{4.25}
$$

so that

$$
|I_3| \le \max_{w\in\mathcal{W}(\Phi)} |w| \{\frac{1}{R}|B|^{1-\frac{1}{p}}\lambda^{-\frac{1}{p}} + (\min_{w\in\mathcal{W}(\Phi)^*} w\cdot\mu)^{-1}\lambda^{-\frac{1}{p}}|\Omega|^{1-\frac{1}{p}}\} E(u_n)^{\frac{1}{p}}. \tag{4.26}
$$

Combining (4.10), (4.11), (4.21), (4.26) we obtain

$$
|\sum_{i=1}^{n} |B^R_i(u_n)| w_i| \le C(E(u_n)^{\frac{1}{p}} + E(u_n)^{\frac{1}{2p}}). \tag{4.27}
$$

When n is so large we have

$$
E(u_n) < 1
$$

so that

$$
E(u_n)^{\frac{1}{p}} \le E(u_n)^{\frac{1}{2p}}
$$

then we have

$$
|\sum_{i=1}^{n} |B^R_i(u_n)| w_i| \le C E(u_n)^{\frac{1}{2p}}
$$

which is (4.9).

∎

As a consequence we have:

Theorem 5. *Let (u_n) be a minimizing sequence of problem (1.6) satisfying (3.16). Under the above assumptions one has for some constant C independent of n*

$$|\frac{|B_i^R(u_n)|}{|B|} - \alpha_i| \le CE(u_n)^{\frac{1}{2p}} \quad \forall\ i = 1, \dots, N \tag{4.28}$$

whenever n is choosen large enough.

Proof : We borrow the idea of the proof from [C.$_2$]. Let M denote the $(N+1) \times N$ matrix

$$M = \begin{pmatrix} w_1 & \dots & w_N \\ 1 & \dots & 1 \end{pmatrix}. \tag{4.29}$$

By (1.11) the vectors $w_i - w_1,\ i = 2, \dots, N$ are linearly independent so that this matrix has rank N. In particular, the system

$$My = b \tag{4.30}$$

has at most one solution which, when it exists, is given by

$$y = (M^T M)^{-1} M^T b$$

(M^T denotes the transpose of M) and one has

$$|y| \le ||(M^T M)^{-1} M^T|||b| \tag{4.31}$$

where $||\ \ ||$ denotes the matrix norm corresponding to the Euclidean norm. Since (see (1.16))

$$\sum_{i=1}^N \alpha_i w_i = 0, \qquad \sum_{i=1}^N \alpha_i = 1$$

one has

$$\begin{cases} \sum_{i=1}^N (\alpha_i|B| - |B_i^R(u_n)|) w_i = -\sum_{i=1}^N |B_i^R(u_n)| w_i \\ \sum_{i=1}^N (\alpha_i|B| - |B_i^R(u_n)|) = |B_{ex}^R|. \end{cases} \tag{4.32}$$

Hence, the vector with entries $\alpha_i|B| - |B_i^R(u_n)|$ satisfies (4.30) for b given by the right hand side of (4.32). The result follows then by combining (4.25), (4.9), (4.31) and the fact that

$$E(u_n)^{\frac{1}{p}} \le E(u_n)^{\frac{1}{2p}}$$

when n is large. As a consequence of this result one has

$$\frac{|B_i^R(u_n)|}{|B|} \to \alpha_i \quad \forall\, i = 1, \ldots, N$$

at a rate proportional to $E(u_n)^{\frac{1}{2p}}$.

Remark 3 : One can construct (see $[C._2]$) for each $h \in (0,1)$ a function $u_h \in W_0^{1,\infty}(\Omega)$ such that

$$|u_h(x)|,\ |\nabla u_h(x)| \leq C \text{ a.e. in } \Omega \tag{4.33}$$

where C is a constant independent of h. Moreover one has

$$E(u_h) \leq Ch \tag{4.34}$$

so that one has a minimizing sequence of problem (1.6) satisfying (3.16).

5. Numerical Analysis

In this section, we assume that Ω is a polyhedral domain for the simplicity of the numerical analysis. Let $(\mathcal{T}_h)_{h>0}$ be a family of regular triangulations of Ω (see [R.T.]), that is to say satisfying

$$\forall\, h > 0 \begin{cases} \forall\, K \in \mathcal{T}_h,\ K \text{ is a N-simplex,} \\ \\ \max\limits_{K \in \mathcal{T}_h} (h_K) = h, \\ \\ \exists \nu > 0 \text{ such that } \forall K \in \mathcal{T}_h \ \frac{h_K}{\rho_K} \leq \nu. \end{cases} \tag{5.1}$$

where h_K is the diameter of the N-simplex and ρ_K its roundness (i.e. the largest diameter of the balls that could fit in K). If $P_1(K)$ is the space of polynomials of degree 1 on K, set

$$V_h^0 = \{v\ :\ \Omega \longrightarrow \mathbf{R} \text{ continuous, } v/_K \in P_1(K),\ \forall\, K \in \mathcal{T}_h,\ v = 0 \text{ on } \Gamma\}. \tag{5.2}$$

The following theorem gives the existence of finite elements minimizing sequence of problem (1.6) satisfying (3.16) as well as an error estimate for the corresponding energy.

Theorem 6. *For each $h \in (0,1)$, there exists $u_h \in V_h^0$ such that*

$$|u_h(x)|,\ |\nabla u_h(x)| \leq C \ \textit{a.e. in } \Omega \tag{5.3}$$

for a certain constant C independent of h. Moreover

$$E(u_h) \leq Ch^{\frac{1}{2}}. \tag{5.4}$$

Proof : (see $[C._2]$)

∎

Corollary. *We assume that (4.1), (4.2), (4.4) hold. If B is a Lipschitz domain of Ω, there exists a constant C independent of h such that*

$$\int_\Omega |u_h(x)|dx \le Ch^{\frac{1}{2p}} \tag{5.5}$$

$$\int_\Omega |\nabla u_h(x)\cdot \mu|dx \le Ch^{\frac{1}{2p}} \tag{5.6}$$

$$\int_\Omega |\nabla u_h(x) - \pi(\nabla u_h(x))|dx \le Ch^{\frac{1}{2p}} \tag{5.7}$$

$$|\frac{|B_i^R(u_h)|}{|B|} - \alpha_i| \le Ch^{\frac{1}{4p}} \quad \forall\, i = 1,\ldots,N \tag{5.8}$$

Proof : The estimate in (5.8) is a consequence of theorem 5 and the estimate given by theorem 6. The estimate in (5.7) is infered from (4.11) and (5.4). The estimates in (5.5) and (5.6) are deduced by (5.4), (4.15) and (4.20).

∎

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