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the function field modular group  $GL(2, \mathbb{F}_q[T])$**

Enrico Varela Roldán

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**Enrico Varela Roldán**

Saarland University  
Department of Mathematics  
P.O. Box 15 11 50  
66041 Saarbrücken  
Germany  
`varela@math.uni-sb.de`

Edited by  
FR 6.1 – Mathematik  
Universität des Saarlandes  
Postfach 15 11 50  
66041 Saarbrücken  
Germany

Fax: + 49 681 302 4443  
e-Mail: [preprint@math.uni-sb.de](mailto:preprint@math.uni-sb.de)  
WWW: <http://www.math.uni-sb.de/>

# Arithmetic of Eisenstein series of level $T$ for the function field modular group $\mathrm{GL}(2, \mathbb{F}_q[T])$

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## Abstract

In this paper we study the structure of the algebra of Drinfeld modular forms for the principal congruence subgroup  $\Gamma(T)$  of the full modular group  $\mathrm{GL}(2, \mathbb{F}_q[T])$ .

To this end and based on work of Cornelissen, we introduce a new class of Eisenstein series and describe their fundamental properties. These so-called modified Eisenstein series allow for a simplification of prior results and are an indispensable tool for future work on Drinfeld modular forms of level  $T$ .

**Keywords:** Drinfeld modular forms, Eisenstein series, basis problem  
**MSC 2010:** 11F52

## 0 Introduction

Drinfeld modular forms are the function field analogue to classical elliptic modular forms. The theory has its origins in Drinfeld's work [Dri74] and has subsequently been further developed by many authors, for example by Deligne and Husemöller [DH87], Goss [Gos80a, Gos80b], and Gekeler [Gek86].

In the special case of Drinfeld modular forms of level  $T$  the structure of the algebra of modular forms has already been described by Cornelissen [Cor97b]. In particular, the algebra is known to be generated by the Eisenstein series of weight 1. These special modular forms are defined as lattice sums, analogously to their counterparts in the classical number field situation.

The aim of this paper is to introduce a new class of modular forms in the Drinfeld setting for level  $T$  based on Cornelissen's work, which we call *modified Eisenstein series*. They are better suited to the description of the structure of the algebra of modular forms than the ordinary Eisenstein series, since they satisfy relations better adapted to our considerations.

Besides their use as generators of the algebra and its subspaces, the modified Eisenstein series also play a central role for representation theoretical questions that arise naturally in the Drinfeld situation. We disregard this aspect for the moment; it will be studied in a future publication.

Our principal results Theorem 3.6, which describes relations between modified Eisenstein series of different weights, and Theorem 4.12, in which a basis for the space of cusp forms is given that is compatible with the cusp filtration. The present paper is a concise summary of the first three chapters of the author's dissertation [Var15]. The latter contains all results and proofs in more detail.

In the first section of this paper I am going to fix some general notation and give a brief overview of prior results.

The second section introduces the modified Eisenstein series and some fundamental properties.

In the third section I present results concerning relations between modified Eisenstein series of different weights.

In the fourth section I will use mixed products of Eisenstein series to construct a basis of the space of cusp forms that is compatible with the filtration induced by the order of cusp forms.

The fifth and final section contains rules for congruences of cusp forms of a certain shape.

## 1 Preliminaries

This section provides a brief overview of the well-established Drinfeld setting without any claim to completeness. While there are many possible references one could cite here, our notation follows most closely the one used in [Cor97b] and [GR96], respectively. A more in-depth description of the theory can be found for example in [Gos96] or [Gek86].

The following notation shall be fixed throughout this paper.

**1.1 Notation.** Let  $q = p^r$  be a prime power. Further, let  $A = \mathbb{F}_q[T]$  be the ring of polynomials over the field with  $q$  elements and  $K = \mathbb{F}_q(T)$  its field of quotients. On  $K$  we fix the normalized absolute value “ $|\cdot|$ ” induced by the degree valuation on  $A$ . The completion of  $K$  with respect to this absolute value is  $K_\infty = \mathbb{F}_q[[T^{-1}]]$ , the field of formal Laurent series in  $T^{-1}$ . The algebraic closure of  $K_\infty$  is not itself complete with respect to the unique continuation of “ $|\cdot|$ ”; yet its completion

$$\mathcal{C}_\infty = \widehat{K}_\infty$$

is again algebraically closed. We call

$$\Omega = \mathcal{C}_\infty \setminus K_\infty$$

the *Drinfeld upper half-plane*.

*Remark.* One should think of  $A$ ,  $K$ ,  $K_\infty$ ,  $\mathcal{C}_\infty$ , and  $\Omega$  as function field analogues to  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$ ,  $\mathbb{C}$ , and the complex upper half-plane  $\mathbb{H}$ , respectively.

The Drinfeld upper half-plane  $\Omega$  can be equipped with a rigid analytic structure. This allows us to define function theoretical objects such as modular forms analogously to classical elliptic modular forms. Since our focus in this paper does not lie on analytic problems, we will accept this well-established analysis of the Drinfeld upper half-plane as given.

For an introduction to rigid analysis in general see, for example [FvdP04].

**1.2 Definition.** 1. The group  $\Gamma(1) = \mathrm{GL}(2, A)$  is called the *full modular group*.

2. Let  $N \in A$ . A subgroup of shape

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \mid a \equiv d \equiv 1 \pmod{N}, b \equiv c \equiv 0 \pmod{N} \right\}$$

is called the *principal congruence subgroup of level  $N$* .

Throughout this paper we deal exclusively with the special case of modular forms for  $\Gamma(T)$ . This group has  $q + 1$  cusps, which can be canonically parametrized by  $\mathbb{P}^1(\mathbb{F}_q)$ . The corresponding modular curve has genus 0.

Even when not stated explicitly, concepts like “cusps” or “modular forms” always refer to the level- $T$ -situation.

We are going to study modular forms (labelled “of type 0” in [GR96, (2.8.2)]) that are defined as follows:

**1.3 Definition** ([GR96, (2.8.2)]). A (*Drinfeld*) *modular form* of weight  $k \in \mathbb{N}_0$  for the group  $\Gamma(T)$  is a rigid analytic function  $f : \Omega \rightarrow \mathcal{C}_\infty$  that satisfies the following three conditions:

1. For  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(T)$  we have

$$f|_{[\gamma]_k}(z) := (cz + d)^{-k} f(\gamma z) = f(z),$$

where  $\Gamma(T)$  acts on  $\Omega$  by Möbius transformation.

2. The function  $f$  is holomorphic on  $\Omega$ .

3. The function  $f$  is holomorphic at the cusps of  $\Gamma(T)$ .

*Remark.* In simplified terms, condition 3 means that there exists a uniformizer  $\tau$  (analogous to  $q = e^{2\pi iz}$  in the classical setting) such that the modular form  $f$  admits an expansion

$$f(s, z) = \sum_{i \geq 0} a_i \tau^i$$

at any cusp  $s$ . The expansion of  $f$  at a single cusp is sufficient to determine the function  $f$ .

By abuse of notation we write  $f(s)$  for such an expansion of  $f$  at a cusp  $s$  although the coefficients depend on choices of representatives. The vanishing order of  $f$  at the cusps is defined accordingly and is independent of these choices. If every term in the expansion is of degree at least  $n$  in  $\tau$ , we write  $f(s) = o(\tau^n)$ .

See [GR96, loc. cit.] for more details.

**1.4 Definition.** The modular forms of weight  $k \in \mathbb{N}_0$  for  $\Gamma(T)$  form a  $\mathcal{C}_\infty$ -vector space  $M_k = M_k(\Gamma(T))$ . The graded  $\mathcal{C}_\infty$ -algebra

$$M = M(\Gamma(T)) = \bigoplus_{k \in \mathbb{N}_0} M_k$$

of all modular forms of level  $T$  is in fact a direct sum in the space of all functions from  $\Omega$  to  $\mathcal{C}_\infty$ .

For weight  $k \in \mathbb{N}_0$  those modular forms in  $M_k$  that vanish of order at least  $n \in \mathbb{N}_0$  at all cusps of  $\Gamma(T)$  form a subspace

$$M_k^n = M_k^n(\Gamma(T)) \subseteq M_k.$$

Modular forms in  $M_k^1$  are called *cuspidal forms*. Modular forms in  $M_k^n$ ,  $n \in \mathbb{N}$ , are called  *$n$ -fold cuspidal forms* or *cuspidal forms of order  $n$* .

The resulting filtration of finite length

$$M_k = M_k^0 \supseteq M_k^1 \supseteq M_k^2 \supseteq \dots$$

is called the *cuspidal filtration* of  $M_k$ .

As in the classical setting, Drinfeld modular forms can be viewed as sections of line bundles. Since the line bundle of modular forms of weight 1 has degree  $q$  (see [Gek86, VII, (6.1)]), the following two results are immediate:

**1.5 Proposition.** For  $k \in \mathbb{N}_0$

$$\dim M_k = kq + 1.$$



**1.6 Proposition.** *Let  $k \in \mathbb{N}_0$ . For  $0 \neq f \in M_k$  the sum of the zeroes of  $f$  at the cusps and in  $\Gamma \backslash \Omega$  is  $kq$  (counting multiplicities).*

An important example of modular forms for  $\Gamma(T)$  are the Eisenstein series:

**1.7 Definition** ([Cor97b, I, (6.2)], [Gos80b]). Let  $k \in \mathbb{N}$  and  $\nu = (\nu_1, \nu_2) \in \mathbb{F}_q^2 \setminus \{(0, 0)\}$ . The (ordinary) Eisenstein series  $E_\nu^{(k)}$  of weight  $k$  and level  $T$  is given by

$$E_\nu^{(k)}(z) := \frac{1}{T} \sum_{\substack{(a,b) \in A^2 \\ (a,b) \equiv \nu \pmod{T}}} \left( \frac{1}{az + b} \right)^k.$$

This way, we obtain  $q^2 - 1$  modular forms of weight  $k$  for  $\Gamma(T)$ . We write  $E_\nu$  for  $E_\nu^{(1)}$ .

*Remark.* In the present paper, we do not concern ourselves with questions of convergence. However, let us point out that, unlike their elliptic counterparts, the Eisenstein series do in fact converge for all  $k \in \mathbb{N}$ . See for example [Gos80b] for further details.

In his dissertation [Cor97b], Cornelissen described the central part Eisenstein series play for the structure of the algebra  $M$ . We cite some fundamental results:

**1.8 Theorem** (Cornelissen [Cor97b, IV, Proposition 1.1]). *Let  $k \in \mathbb{N}$ . The Eisenstein series*

$$\begin{aligned} E_u^{(k)} &:= E_{(1,u)}^{(k)}, & u \in \mathbb{F}_q, \\ E_\infty^{(k)} &:= E_{(0,1)}^{(k)}. \end{aligned}$$

*form a maximal linearly independent subset of the space of all Eisenstein series of weight  $k$ . We call*

$$\text{Eis}_k := \left\langle E_\infty^{(k)}, E_u^{(k)} \mid u \in \mathbb{F}_q \right\rangle$$

*the space of Eisenstein series of weight  $k$ . Then*

$$M_k = \text{Eis}_k \oplus M_k^1$$

*is a direct sum of  $\mathcal{C}_\infty$ -vector spaces. In particular,*

$$M_1 = \text{Eis}_1.$$

*Remark.* In fact, Cornelissen's proposition holds for arbitrary principal congruence subgroups  $\Gamma(N)$  with non-constant  $N \in A$ . In this general setting, the basis of  $\text{Eis}_k$  is indexed accordingly by the cusps of  $\Gamma(N)$ .

When we speak of *the* ordinary Eisenstein series of weight  $k$  we always refer to the subset distinguished in the above theorem.

The importance of the Eisenstein series of weight 1 is illustrated by the following theorem:

**1.9 Theorem** (Cornelissen [Cor97b, III, Theorem 3.4]). *The algebra  $M$  of modular forms for  $\Gamma(T)$  is generated by the Eisenstein series of weight 1. More precisely,  $M$  admits a presentation*

$$M = \mathcal{C}_\infty [E_\nu \mid \nu \in \mathbb{F}_q \cup \{\infty\}] / I,$$

where the ideal  $I$  of relations is generated by the expressions

$$f_{i,j} := \sum_{\alpha, \beta \in \mathbb{F}_q} \alpha^{i-1} \beta^j (\alpha - \beta) E_\alpha E_\beta - \delta_{j,q-1} \cdot \sum_{\alpha \in \mathbb{F}_q} \alpha^{i-1} E_\alpha E_\infty$$

with  $1 \leq i \leq j \leq q-1$ . Here,  $\delta_{\cdot, \cdot}$  is the Kronecker delta and  $0^0 = 1$  by convention.

## 2 Modified Eisenstein series

In order to make the relations stated in Theorem 1.9 more suitable for the following calculations we now introduce certain linear combinations of the ordinary Eisenstein series.

**2.1 Definition.** Let  $k \in \mathbb{N}$ . The *modified Eisenstein series of weight  $k$  (and level  $T$ )* are given by

$$\begin{aligned} \mathcal{E}_i^{(k)} &:= \sum_{u \in \mathbb{F}_q} u^i E_u^{(k)}, & 0 \leq i \leq q-1, \\ \mathcal{E}_\infty^{(k)} &:= \sum_{u \in \mathbb{F}_q} u^k E_u^{(k)} + E_\infty^{(k)}, \end{aligned}$$

with the convention  $0^0 = 1$ . In the case  $k = 1$  we also write  $\mathcal{E}_i$  for  $\mathcal{E}_i^{(1)}$  and

$$\mathcal{E}_q := \mathcal{E}_\infty^{(1)}.$$

*Remark.* The modular forms  $\mathcal{E}_i = \mathcal{E}_i^{(1)}$  already appear in Cornelissen's work. Up to a normalizing factor  $\bar{\pi} \in \mathcal{C}_\infty$  (a generator of the lattice of the Carlitz

module) they agree with the modular forms  $Z_j$ ,  $0 \leq j \leq q$ , which are used in the proof of Theorem 3.4 in [Cor97b, chapter III].

However, this particular proof was the only instance in which this class of modular forms is studied in [Cor97b]. In addition, no generalization of this concept for weights  $\geq 2$  was considered.

Based on the definition we immediately see:

**2.2 Proposition.** *Let  $k \in \mathbb{N}$ . The modified Eisenstein series*

$$\begin{aligned} \mathcal{E}_i^{(k)}, \quad 0 \leq i \leq q-1, \\ \mathcal{E}_\infty^{(k)}, \end{aligned}$$

*are linearly independent. In particular, they form another basis of  $\text{Eis}_k$ .*

We can now restate Theorem 1.9 with much simpler relations:

**2.3 Theorem** (Cornelissen [Cor97b, III, Theorem 3.4], [Cor97a, (3.1.3), Theorem]). *The algebra  $M$  of modular forms for  $\Gamma(T)$  admits a presentation*

$$M = \mathcal{C}_\infty[\mathcal{E}_i \mid 0 \leq i \leq q]/I,$$

*where the ideal  $I$  of relations is generated by the expressions*

$$\mathcal{E}_i \mathcal{E}_j - \mathcal{E}_{i-1} \mathcal{E}_{j+1} \quad \text{for } 1 \leq i \leq j \leq q-1. \quad (1)$$

*Proof.* The new relations can easily be derived from the relations  $f_{i,j}$  by rearranging the terms in the sums according to the definition of the modified Eisenstein series.  $\square$

Since different products of Eisenstein series of weight 1 may represent the same modular form, it is advisable to establish the concept of a standard form of such expressions. We do this by repeatedly applying the identity

$$\mathcal{E}_i \mathcal{E}_j = \mathcal{E}_{i-1} \mathcal{E}_{j+1} \quad \text{for } 1 \leq i \leq j \leq q-1 \quad (2)$$

to eliminate successively all but (at most) one occurrence of an index different from 0 or  $q$ .

**2.4 Definition.** We call a product of modified Eisenstein series of weight 1 a *standard monomial* if it contains at most one factor  $\mathcal{E}_b$  with  $1 \leq b \leq q-1$ ; that is, if it is of shape

$$\mathcal{E}_0^m \mathcal{E}_b \mathcal{E}_q^n \quad \text{or} \quad \mathcal{E}_0^m \mathcal{E}_q^n$$

with  $m, n \in \mathbb{N}_0$ .

**2.5 Proposition.** *For every product*

$$\mathcal{E}_{b_1}^{m_1} \dots \mathcal{E}_{b_s}^{m_s}$$

*of modified Eisenstein series of weight 1 with  $0 \leq b_i \leq q$  and  $m_i \in \mathbb{N}$  there exists precisely one standard monomial which describes the same modular form.*

*Write*

$$m := m_1 + \dots + m_s$$

*and*

$$\mathfrak{b} := m_1 b_1 + \dots + m_s b_s$$

*with the unique decomposition*

$$\mathfrak{b} = cq + b, \quad 0 \leq b \leq q - 1.$$

*The corresponding standard monomial is then given by*

$$\mathcal{E}_{b_1}^{m_1} \dots \mathcal{E}_{b_s}^{m_s} = \begin{cases} \mathcal{E}_0^{m-1-c} \mathcal{E}_b \mathcal{E}_q^c & b > 0 \\ \mathcal{E}_0^{m-c} \mathcal{E}_q^c & b = 0. \end{cases}$$

*Proof.* Let us assume that the given product is not already a standard monomial (up to sorting).

Hence it contains two factors  $\mathcal{E}_{b'}$  and  $\mathcal{E}_{b''}$  with  $1 \leq b', b'' \leq q - 1$ . We can apply relation (2) to increase simultaneously the larger of these indices and decrease the smaller one. After a finite number of steps, at least one of the indices reaches  $q$  or  $0$ . This means

$$\mathcal{E}_{b'} \mathcal{E}_{b''} = \begin{cases} \mathcal{E}_0 \mathcal{E}_{b'+b''} & b' + b'' \leq q, \\ \mathcal{E}_{b'+b''-q} \mathcal{E}_q & b' + b'' > q. \end{cases}$$

By repeating this process, the initial product of modified Eisenstein series can be written as a product, in which at most one factor remains with an index in  $\{1, \dots, q - 1\}$  but which still represents the same modular form. We have found the desired standard monomial. Since the value of the sum of indices is fixed in each step, the resulting standard monomial has to be of the stated shape. The uniqueness is therefore obvious.  $\square$

By a simple dimension argument we immediately obtain:

**2.6 Corollary.** *The standard monomials of weight  $k$  form a basis of  $M_k$  for  $k \in \mathbb{N}$ .*

Certain mixed products of Eisenstein series can be transformed according to the following relation.

**2.7 Lemma.** *Let  $0 \leq k \leq i \leq q$ . Then*

$$\mathcal{E}_i E_0^k = (-1)^k \mathcal{E}_{i-k} E_\infty^k.$$

*Proof.* By definition we have  $E_0 = \mathcal{E}_0 - \mathcal{E}_{q-1}$  and  $\mathcal{E}_q = \mathcal{E}_1 + E_\infty$ . Thus for  $k = 1$  we can use relation (2) to get

$$\begin{aligned} \mathcal{E}_i E_0 &= \mathcal{E}_i(\mathcal{E}_0 - \mathcal{E}_{q-1}) = \mathcal{E}_0 \mathcal{E}_i - \mathcal{E}_{i-1} \mathcal{E}_q \\ &= \mathcal{E}_0 \mathcal{E}_i - \mathcal{E}_{i-1}(\mathcal{E}_1 + E_\infty) = \mathcal{E}_0 \mathcal{E}_i - \mathcal{E}_0 \mathcal{E}_i - \mathcal{E}_{i-1} E_\infty \\ &= -\mathcal{E}_{i-1} E_\infty. \end{aligned}$$

The relation follows inductively for  $2 \leq k \leq i$ . For  $k = 0$  the statement is tautological.  $\square$

To study the zeroes of modified Eisenstein series let us first recall the following result by Cornelissen concerning the ordinary Eisenstein series of weight 1:

**2.8 Proposition** (Cornelissen [Cor97b, III, Proposition 2.2]). *Let  $\tau$  be the uniformizer mentioned in the remark to Definition 1.3 and let  $\bar{\pi}$  be chosen as in the remark to Theorem 1.9.*

*There is a constant  $\zeta \in \mathcal{C}_\infty^\times$  such that the ordinary Eisenstein series of weight 1 admit expansions at the cusps as follows:*

$$\begin{aligned} \bar{\pi}^{-1} E_\infty(\infty) &= \zeta + o(\tau^2) \\ \bar{\pi}^{-1} E_u(\infty) &= \tau + o(\tau^2), \quad u \in \mathbb{F}_q \\ \bar{\pi}^{-1} E_\infty((\alpha : 1)) &= \tau + o(\tau^2), \quad \alpha \in \mathbb{F}_q \\ \bar{\pi}^{-1} E_u((\alpha : 1)) &= (\alpha + u)^{-1} \tau + o(\tau^2), \quad \alpha, u \in \mathbb{F}_q, \alpha \neq -u \\ \bar{\pi}^{-1} E_u((\alpha : 1)) &= \zeta + o(\tau^2), \quad \alpha = -u \in \mathbb{F}_q. \end{aligned}$$

For the modified Eisenstein series of weight 1 we obtain the following result:

**2.9 Proposition.** *Let  $0 \leq i \leq q$ . The modified Eisenstein series  $\mathcal{E}_i$  has a zero*

*of order  $q - i$  at the cusp  $\infty$ ,*

*of order  $i$  at the cusp  $(0 : 1)$ ,*

*and no further zeroes at the cusps  $(\alpha : 1)$  with  $\alpha \in \mathbb{F}_q^\times$  or in  $\Gamma(T) \backslash \Omega$ . With the notation from Proposition 2.8 we observe*

$$\bar{\pi}^{-1} \mathcal{E}_i((\alpha : 1)) = (-\alpha)^i \zeta + o(\tau), \quad \alpha \in \mathbb{F}_q^\times.$$

*Proof.* First we determine the vanishing order at the cusp  $\infty$ . Cornelissen shows in the proof of [Cor97b, III, Theorem 3.4] that the modular form called  $Z_q$  in his notation does not vanish at  $\infty$ . According to the remark following Theorem 1.9 the same is therefore true for  $\mathcal{E}_q$ . By Lemma 2.7 we have

$$\mathcal{E}_q E_0 = -\mathcal{E}_{q-1} E_\infty.$$

Since  $E_0$  has a simple zero at  $\infty$  and  $E_\infty$  does not vanish at  $\infty$ , this equation shows that  $\mathcal{E}_{q-1}$  has vanishing order exactly 1 at  $\infty$ . A second application of Lemma 2.7, this time for  $k = 1$  and  $i = q - 1$ , results in the equation

$$\mathcal{E}_{q-1} E_0 = -\mathcal{E}_{q-2} E_\infty.$$

Thus  $\mathcal{E}_{q-2}$  vanishes at  $\infty$  of order exactly 2. In this way we obtain successively the vanishing order of all  $\mathcal{E}_i$  at  $\infty$ .

The stated vanishing order at  $(0 : 1)$  can be shown to be correct by a similar argument, starting from the equation

$$\mathcal{E}_1 E_0 = -\mathcal{E}_0 E_\infty,$$

and using the fact that the modular form  $\mathcal{E}_0$  does not vanish at  $(0 : 1)$  according to [Cor97b, loc. cit.].

Since the vanishing orders of all  $\mathcal{E}_i$  at the cusps  $\infty$  and  $(0 : 1)$  add up to  $q$  there can be no further zeroes, see Proposition 1.6.

The stated shape of the series expansion follows directly from the definition of the modified Eisenstein series by using Proposition 2.8 and has already been described for Cornelissen's  $Z_j$  in [Cor97b, loc. cit.]  $\square$

The corresponding question for modified Eisenstein series of weight  $k \geq 2$  is more difficult to answer. We are going to give results for a special case in the next section. However, we can immediately describe the standard monomials at the cusps.

**2.10 Corollary.** *Let  $k \in \mathbb{N}$ . The modular form  $\mathcal{E}_0^{k-1-l} \mathcal{E}_b \mathcal{E}_q^l$  with  $0 \leq b \leq q-1$  and  $0 \leq l \leq k-1$  has a zero*

*of order  $(k-l)q - b$  at the cusp  $\infty$ ,*

*of order  $lq + b$  at the cusp  $(0 : 1)$ ,*

*and no further zeroes.*

*The modular form  $\mathcal{E}_q^k$  vanishes at  $(0 : 1)$  of order  $kq$  and has no further zeroes.*

*Proof.* The stated vanishing orders at the cusps are a direct consequence of Proposition 2.9. Since their sum is  $kq$ , there can be no further zeroes according to Proposition 1.6.  $\square$

### 3 Relations for weight $k \leq q$

How can modified Eisenstein series of higher weight be expressed as polynomials in terms of modified Eisenstein series of weight 1 (which is possible according to Theorem 2.3)? In this section we give a partial answer to this question. Some further results can be found in the author's dissertation. The full generalization still remains an open question.

The answer to the corresponding question for ordinary Eisenstein series involves the theory of Goss polynomials. In the simplest case one finds:

**3.1 Proposition** ([Gek12, Corollary 2.8]). *Let  $k$  be of shape  $k = k'p^n$  for some  $n \in \mathbb{N}_0$  and  $1 \leq k' \leq q$ . Then*

$$E_\nu^{(k)} = (E_\nu^{(1)})^k = E_\nu^k$$

*holds for all  $\nu \in \mathbb{F}_q \cup \{\infty\}$ .*

For weight  $k$  as in the proposition, the modified Eisenstein series can therefore be written as linear combinations of powers of ordinary Eisenstein series of weight 1.

**3.2 Lemma.** *Let  $k = k'p^n$  for  $n \in \mathbb{N}_0$  and  $1 \leq k' \leq q$ . We have*

$$\begin{aligned} \mathcal{E}_i^{(k)} &= \sum_{u \in \mathbb{F}_q} u^i E_u^{(k)} = \sum_{u \in \mathbb{F}_q} u^i E_u^k, \quad 0 \leq i \leq q-1, \\ \mathcal{E}_\infty^{(k)} &= \sum_{u \in \mathbb{F}_q} u^k E_u^{(k)} + E_\infty^{(k)} = \sum_{u \in \mathbb{F}_q} u^k E_u^k + E_\infty^k. \end{aligned}$$

At this point, the naive approach would be to express each ordinary Eisenstein series of weight 1 on the right hand side as a linear combination of modified Eisenstein series and to expand the resulting expression. In practice this method fails due to the technical complexity as it requires to expand powers of sums with up to  $q$  terms.

Instead, we choose an inductive approach. In each step we simplify the arising expressions by means of the following variant of the relations between products of Eisenstein series, which is due to Cornelissen and Zagier:

**3.3 Lemma** ([Cor97a, Addendum 6]). *The relations (1) from Theorem 2.3 are equivalent to*

$$(u - v)E_u E_v + (E_u - E_v)E_\infty = 0 \quad \text{for all } u, v \in \mathbb{F}_q.$$

First we generalize this for higher weights.

**3.4 Lemma.** *Let  $k \in \mathbb{N}$  and  $u \neq v \in \mathbb{F}_q$ . Then*

$$E_u^k E_v = - \sum_{j=1}^k \frac{1}{(u-v)^j} E_u^{k+1-j} E_\infty^j + \frac{1}{(u-v)^k} E_v E_\infty^k.$$

*Proof.* The statement is proven by a straightforward induction with respect to  $k$ .  $\square$

The following lemma will be used to show the cancellation of certain terms.

**3.5 Lemma.** *Let  $1 \leq k \leq q-1$  and  $0 \leq b \leq q-1$ . For  $0 \leq i \leq k$  we have*

$$\sum_{\substack{u, v \in \mathbb{F}_q \\ u \neq v}} u^i v^b E_u^k E_v = \sum_{j=1}^k \binom{b}{j} (-1)^j \sum_{u \in \mathbb{F}_q} u^{i+b-j} E_u^{k+1-j} E_\infty^j - \delta_{i,k} \sum_{v \in \mathbb{F}_q} v^b E_v E_\infty^k.$$

*As before, we use the convention  $0^0 = 1$  and  $\delta_{\cdot, \cdot}$  is the Kronecker delta. In the special case  $b = 0$  we get*

$$\sum_{\substack{u, v \in \mathbb{F}_q \\ u \neq v}} u^i E_u^k E_v = -\delta_{i,k} \sum_{v \in \mathbb{F}_q} E_v E_\infty^k.$$

*Proof.* Using Lemma 3.4, we obtain for the expression on the left hand side:

$$\begin{aligned} \sum_{\substack{u, v \in \mathbb{F}_q \\ u \neq v}} u^i v^b E_u^k E_v &= \sum_{\substack{u, v \in \mathbb{F}_q \\ u \neq v}} u^i v^b \left( - \sum_{j=1}^k \frac{1}{(u-v)^j} E_u^{k+1-j} E_\infty^j + \frac{1}{(u-v)^k} E_v E_\infty^k \right) \\ &= - \sum_{j=1}^k \sum_{u \in \mathbb{F}_q} u^i \underbrace{\left( \sum_{\substack{v \in \mathbb{F}_q \\ v \neq u}} \frac{v^b}{(u-v)^j} \right)}_{=:\lambda_1(b,j)} E_u^{k+1-j} E_\infty^j \\ &\quad + \sum_{v \in \mathbb{F}_q} v^b \underbrace{\left( \sum_{\substack{u \in \mathbb{F}_q \\ u \neq v}} \frac{u^i}{(u-v)^k} \right)}_{=:\lambda_2(i,k)} E_v E_\infty^k. \end{aligned}$$

The indicated sums can be simplified as follows:

1. First we compute

$$\lambda_1(b, j) = \sum_{m=0}^b \binom{b}{m} u^{b-m} (-1)^m \sum_{w \in \mathbb{F}_q^\times} w^{m-j}.$$



The inner sum vanishes except for  $q-1 \mid m-j$ . Taking into account the possible values for  $m$  and  $j$ , we observe

$$-(q-1) \leq -k \leq m-j \leq b-1 \leq q-2.$$

Thus  $m-j$  is only divisible by  $q-1$  if  $m-j = 0$  or  $m-j = -k = -(q-1)$  holds. Consequently, we have

$$\lambda_1(b, j) = -\binom{b}{j} u^{b-j} (-1)^j - \delta_{k, q-1} \delta_{j, q-1} u^b.$$

2. Analogously, the second sum can be written

$$\lambda_2(i, k) = \sum_{m=0}^i \binom{i}{m} v^{i-m} \sum_{w \in \mathbb{F}_q^\times} w^{m-k}.$$

Again we see that the inner sum vanishes except if  $m = k$  (which implies  $i = k$ ) or  $m = 0, k = q-1$  (for arbitrary  $i$ ). Thus

$$\lambda_2(i, k) = -\delta_{i, k} - \delta_{k, q-1} v^i.$$

By combining these results we get

$$\begin{aligned} & \sum_{\substack{u, v \in \mathbb{F}_q \\ u \neq v}} u^i v^b E_u^k E_v \\ &= \sum_{j=1}^k \sum_{u \in \mathbb{F}_q} u^i \binom{b}{j} u^{b-j} (-1)^j E_u^{k+1-j} E_\infty^j + \delta_{k, q-1} \sum_{u \in \mathbb{F}_q} u^i u^b E_u E_\infty^{q-1} \\ & \quad - \delta_{i, k} \sum_{v \in \mathbb{F}_q} v^b E_v E_\infty^k - \delta_{k, q-1} \sum_{v \in \mathbb{F}_q} v^b v^i E_v E_\infty^{q-1} \\ &= \sum_{j=1}^k \binom{b}{j} (-1)^j \sum_{u \in \mathbb{F}_q} u^{i+b-j} E_u^{k+1-j} E_\infty^j - \delta_{i, k} \sum_{v \in \mathbb{F}_q} v^b E_v E_\infty^k. \end{aligned}$$

The claim for the special case  $b = 0$  follows from  $\binom{0}{j} = 0$  for  $1 \leq j \leq k$ .  $\square$

We can now establish the central result of this section:

**3.6 Theorem.** *Let  $1 \leq k \leq q$ . Then*

$$\mathcal{E}_0^{k-i} \mathcal{E}_q^i = \begin{cases} \mathcal{E}_i^{(k)} & 0 \leq i \leq k-1 \\ \mathcal{E}_\infty^{(k)} & i = k. \end{cases}$$

*Proof.* Obviously, the theorem is a tautology in the case  $k = 1$ .

We complete the proof by showing that the statement is true for  $k + 1$  if it holds for  $k$ .

Assume the statement is true for  $k \leq q - 1$ . For  $0 \leq i \leq k$  we compute

$$\begin{aligned}
\mathcal{E}_0^{k+1-i} \mathcal{E}_q^i &= \mathcal{E}_0(\mathcal{E}_0^{k-i} \mathcal{E}_q^i) = \left( \sum_{v \in \mathbb{F}_q} E_v \right) \left( \sum_{u \in \mathbb{F}_q} u^i E_u^k + \delta_{i,k} E_\infty^k \right) \\
&= \sum_{u,v \in \mathbb{F}_q} u^i E_u^k E_v + \delta_{i,k} \sum_{v \in \mathbb{F}_q} E_v E_\infty^k \\
&= \sum_{v \in \mathbb{F}_q} v^i E_v^{k+1} + \sum_{\substack{u,v \in \mathbb{F}_q \\ u \neq v}} u^i E_u^k E_v + \delta_{i,k} \sum_{v \in \mathbb{F}_q} E_v E_\infty^k \\
&= \sum_{v \in \mathbb{F}_q} v^i E_v^{k+1} = \mathcal{E}_i^{(k+1)}.
\end{aligned}$$

Besides the inductive hypothesis we have used the fact that we may write the modified Eisenstein series of weight  $k \leq q$  as sums of powers of the ordinary Eisenstein series as in Lemma 3.2.

The additional sums in the second to last line cancel out according to the case  $b = 0$  of Lemma 3.5.

Let now  $i = k + 1$ . In this case we have

$$\begin{aligned}
\mathcal{E}_q^{k+1} &= \mathcal{E}_q \mathcal{E}_q^k = \left( \sum_{v \in \mathbb{F}_q} v E_v + E_\infty \right) \left( \sum_{u \in \mathbb{F}_q} u^k E_u^k + E_\infty^k \right) \\
&= \sum_{v \in \mathbb{F}_q} v^{k+1} E_v^{k+1} + E_\infty^{k+1} \\
&\quad + \sum_{\substack{u,v \in \mathbb{F}_q \\ u \neq v}} u^k v E_u^k E_v + \sum_{u \in \mathbb{F}_q} u^k E_u^k E_\infty + \sum_{v \in \mathbb{F}_q} v E_v E_\infty^k \\
&= \mathcal{E}_\infty^{(k+1)},
\end{aligned}$$

since

$$\sum_{\substack{u,v \in \mathbb{F}_q \\ u \neq v}} u^k v E_u^k E_v = - \sum_{u \in \mathbb{F}_q} u^k E_u^k E_\infty - \sum_{v \in \mathbb{F}_q} v E_v E_\infty^k$$

according to Lemma 3.5 for  $i = k$  and  $b = 1$ . □

*Remark.* The above theorem has an interesting interpretation from a representation theoretic point of view: A subspace of  $\text{Eis}_k$  can be expressed as the  $k$ -th symmetric power of the space generated by  $\mathcal{E}_0$  and  $\mathcal{E}_q$ . Since this leaves the scope of this paper we will deal with the details in a future publication.

Theorem 3.6 can also be used to describe the behavior of those particular modified Eisenstein series of weight  $k \leq q$  at the cusps.

**3.7 Proposition.** *Let  $1 \leq k \leq q$  and  $0 \leq i \leq k-1$ . The modified Eisenstein series  $\mathcal{E}_i^{(k)}$  has*

*vanishing order  $(k-i)q$  at the cusp  $\infty$ ,*

*vanishing order  $iq$  at the cusp  $(0:1)$ ,*

*and no further zeroes.*

*The modified Eisenstein series  $\mathcal{E}_\infty^{(k)}$  vanishes at  $(0:1)$  of order  $kq$  and has no further zeroes.*

*Proof.* This is a direct consequence of the presentations given in Theorem 3.6 and the behavior of  $\mathcal{E}_0$  and  $\mathcal{E}_q$  at the cusps as determined in Proposition 2.9.  $\square$

*Remark.* For those modified Eisenstein series of weight  $k \leq q$  not covered by Theorem 3.6 the expression as a linear combination of standard monomials can be found in [Var15, Satz 2.32]. The resulting formulae are more involved than the case considered above.

## 4 The cusp filtration

Let us now study the cusp filtration introduced in Definition 1.4. Our goal is to construct a basis of  $M_k^1$  that is compatible with this filtration in the following sense: The intersection of the basis with a subspace of the filtration is in turn a basis of the subspace.

As we have seen in Theorem 1.8 there are no non-trivial cusp forms of weight 1 (in fact, we know this to be true for any arithmetic subgroup of  $\Gamma(1)$  by a result of Gekeler and Teitelbaum [Gek90]). Therefore we may assume  $k \geq 2$  in this section. Nevertheless, some notation will still include the trivial case  $k = 1$ .

Throughout this section we will make use of the following notation:

**4.1 Notation.** For weight  $k \in \mathbb{N}$  we fix the decomposition

$$k = \mathfrak{k} + \widehat{\mathfrak{k}}(q+1)$$

with uniquely determined non-negative integers  $1 \leq \mathfrak{k} \leq q+1$  and  $\widehat{\mathfrak{k}} \in \mathbb{N}_0$ . In addition, we define

$$\mathbf{m}(k) := \left\lfloor \frac{kq}{q+1} \right\rfloor = \mathfrak{k} - 1 + \widehat{\mathfrak{k}}q.$$

It will turn out later that  $\mathbf{m}(k)$  is the length of the cusp filtration of  $M_k$ .

Consider the following construction of sets of modular forms:

**4.2 Notation.** Let  $k \geq 2$ . For each  $1 \leq i \leq \mathbf{m}(k) - 1$  we define a set  $\mathcal{B}_k^i$  consisting of the modular forms

$$\begin{aligned}\mathcal{F}_b^{(i,k)} &:= \mathcal{E}_0^{k-i-1} \mathcal{E}_b E_\infty^i, & 0 \leq b \leq q-1, \\ \mathcal{F}_\infty^{(i,k)} &:= (-1)^i \mathcal{E}_q^{k-i} E_0^i.\end{aligned}$$

Let further  $\mathcal{B}_k^{\mathbf{m}(k)}$  be the set with elements

$$\mathcal{F}_b^{(\mathbf{m}(k),k)} := \mathcal{E}_0^{\hat{\mathbf{k}}} \mathcal{E}_b E_\infty^{\mathbf{m}(k)}, \quad 0 \leq b \leq q+1-\hat{\mathbf{k}}.$$

Finally, for  $1 \leq i \leq \mathbf{m}(k)$  we define

$$\mathcal{B}_k^{i,+} := \bigcup_{i \leq j \leq \mathbf{m}(k)} \mathcal{B}_k^j \quad (\text{disjoint union}).$$

All of the monomials defined above describe modular forms of weight  $k$  since

$$k = \mathbf{m}(k) + \hat{\mathbf{k}} + 1.$$

In the remainder of this section we prove that the construction provides a system of bases of the cusp filtration. We choose an approach that focuses heavily on the arithmetic of Eisenstein series. An alternative approach using the theorem of Riemann-Roch is also viable.

In order to show that the constructed modular forms are indeed pairwise distinct, we study their behavior at the cusps.

**4.3 Lemma.** *Let  $k \geq 2$ .*

1. *If  $1 \leq i \leq \mathbf{m}(k) - 1$  then we have for  $0 \leq b \leq q - 1$ :*

$$\begin{aligned}\text{The vanishing order of } \mathcal{F}_b^{(i,k)} & \text{ at } \infty \text{ is} && (k-i)q - b, \\ & \text{ at } (0 : 1) \text{ is} && b + i, \\ & \text{ at } (\alpha : 1), \alpha \in \mathbb{F}_q^\times, \text{ is} && i.\end{aligned}$$

$$\begin{aligned}\text{The vanishing order of } \mathcal{F}_\infty^{(i,k)} & \text{ at } \infty \text{ is} && i, \\ & \text{ at } (0 : 1) \text{ is} && (k-i)q, \\ & \text{ at } (\alpha : 1), \alpha \in \mathbb{F}_q^\times, \text{ is} && i.\end{aligned}$$

2. *For  $0 \leq b \leq q + 1 - \hat{\mathbf{k}}$  we have:*

$$\begin{aligned}\text{The vanishing order of } \mathcal{F}_b^{(\mathbf{m}(k),k)} & \text{ at } \infty \text{ is} && q - b + \hat{\mathbf{k}}q, \\ & \text{ at } (0 : 1) \text{ is} && b + \mathbf{m}(k), \\ & \text{ at } (\alpha : 1), \alpha \in \mathbb{F}_q^\times, \text{ is} && \mathbf{m}(k).\end{aligned}$$

*Proof.* In each case the vanishing order can be read off directly using Proposition 2.8 and Proposition 2.9.  $\square$

**4.4 Lemma.** *For every integer  $1 \leq l \leq (k-1)q$  there exists precisely one modular form in  $\mathcal{B}_k^{1,+}$  which has vanishing order  $l$  at the cusp  $\infty$ .*

*Proof.* We only have to show the existence of modular forms with the required vanishing orders. Uniqueness follows from the cardinality of  $\mathcal{B}_k^{1,+}$ .

For  $1 \leq l \leq \mathfrak{m}(k) - 1$  we may choose  $\mathcal{F}_\infty^{(l,k)} \in \mathcal{B}_k^l$ .

For  $\mathfrak{m}(k) \leq l \leq (\widehat{\mathfrak{k}} + 1)q$  the modular form

$$\mathcal{F}_{(\widehat{\mathfrak{k}}+1)q-l}^{(\mathfrak{m}(k),k)} \in \mathcal{B}_k^{\mathfrak{m}(k)}$$

is well-defined and has vanishing order  $l$  at  $\infty$ .

For  $1 + (\widehat{\mathfrak{k}} + 1)q \leq l \leq (k-1)q$  there is  $1 \leq j \leq \mathfrak{m}(k) - 1$  such that

$$1 + (k-j-1)q \leq l \leq (k-j)q.$$

The modular form

$$\mathcal{F}_{(k-j)q-l}^{(j,k)} \in \mathcal{B}_k^j$$

is well-defined and has the desired vanishing order.  $\square$

The following properties of the sets constructed in Notation 4.2 are now obvious:

**4.5 Proposition.** *The set  $\mathcal{B}_k^{1,+}$  consists of  $(k-1)q$  linearly independent modular forms of weight  $k$ . In particular, all monomials constructed in Notation 4.2 represent pairwise distinct modular forms.*

*For  $1 \leq i \leq \mathfrak{m}(k)$  we have*

$$\#\mathcal{B}_k^{i,+} = kq + 1 - i(q+1).$$

**4.6 Lemma.** *Let  $k \geq 2$  and  $1 \leq i \leq \mathfrak{m}(k)$ . Then*

$$\mathcal{B}_k^i \subseteq M_k^i$$

*and no element of  $\mathcal{B}_k^i$  lies in one of the subspaces  $M_k^j$  with  $j > i$ .*

*In other words: The subspace  $M_k^i$  is the smallest subspace of the cusp filtration that contains elements of  $\mathcal{B}_k^i$ .*

*Proof.* In Lemma 4.3 we have already determined the vanishing orders at the cusps. By applying the estimates

$$(k-i)q - b > \mathfrak{k} - 1 + \widehat{\mathfrak{k}}q = \mathfrak{m}(k) > i \tag{3}$$

for  $1 \leq i \leq \mathbf{m}(k) - 1$  and  $0 \leq b \leq q - 1$ , and

$$q - b + \widehat{\mathfrak{k}}q \geq \mathfrak{k} - 1 + \widehat{\mathfrak{k}}q = \mathbf{m}(k) \quad (4)$$

for  $0 \leq b \leq q + 1 - \mathfrak{k}$  we see that each element of  $\mathcal{B}_k^i$  is indeed a cusp form of order  $i$  and that there is at least one cusp such that the vanishing order is exactly  $i$ .  $\square$

**4.7 Corollary.** *Let  $k \geq 2$  and  $1 \leq i \leq \mathbf{m}(k)$ . Then*

$$\mathcal{B}_k^{i,+} \subseteq M_k^i.$$

With a simple dimension argument we observe:

**4.8 Proposition.** *Let  $k \geq 2$ . The set  $\mathcal{B}_k^{1,+}$  forms a basis of  $M_k^1$ .*

The following lemma describes the interaction between the subsets of the basis and the cusp filtration in more detail.

**4.9 Lemma.** *Let  $k \geq 2$  and  $1 \leq i \leq \mathbf{m}(k)$ . There is no non-trivial linear combination of elements of  $\mathcal{B}_k^i$  which lies in  $M_k^{i+1}$ .*

*Proof.* Let us first consider the case where  $1 \leq i \leq \mathbf{m}(k) - 1$ . Assume there is a non-trivial linear combination

$$\mathcal{F} := \sum_{b=0}^{q-1} \lambda_b \mathcal{F}_b^{(i,k)} + \lambda_\infty \mathcal{F}_\infty^{(i,k)} \in M_k^{i+1}$$

with coefficients in  $\mathcal{C}_\infty$ . That is, we assume that  $\mathcal{F}$  has vanishing order at least  $i + 1$  at each cusp.

If we compare the vanishing orders at the cusps of the elements of  $\mathcal{B}_k^i$  (remembering the estimates given in (3) and (4)) we see that  $\mathcal{F}_0^{(i,k)}$  is the only modular form in  $\mathcal{B}_k^i$  which vanishes at  $(0 : 1)$  of order exactly  $i$ . Similarly,  $\mathcal{F}_\infty^{(i,k)}$  is the only modular form that has vanishing order exactly  $i$  at  $\infty$ .

In order for  $\mathcal{F}$  to be an  $i + 1$ -fold cusp form we must therefore have

$$\lambda_0 = \lambda_\infty = 0.$$

We can now write

$$\mathcal{F} = \sum_{b=1}^{q-1} \lambda_b \mathcal{F}_b^{(i,k)} = \sum_{b=1}^{q-1} \lambda_b \mathcal{E}_0^{k-i-1} \mathcal{E}_b E_\infty^i = \mathcal{E}_0^{k-i-1} E_\infty^i \sum_{b=1}^{q-1} \lambda_b \mathcal{E}_b.$$

Here, the factor  $\mathcal{E}_0^{k-i-1} E_\infty^i$  has vanishing order exactly  $i$  at the cusps  $(\alpha : 1)$  with  $\alpha \in \mathbb{F}_q^\times$ . According to our initial assumption this means that the modular form  $\sum_{b=1}^{q-1} \lambda_b \mathcal{E}_b$  vanishes at all cusps  $(\alpha : 1)$  with  $\alpha \in \mathbb{F}_q^\times$ . Since it also vanishes at  $(0 : 1)$  and  $\infty$ , we have a modular form of weight 1 with  $q + 1$  zeroes; a contradiction.

Analogously, we prove that the statement is true in the case  $i = \mathbf{m}(k)$ .  $\square$

**4.10 Corollary.** *Let  $k \geq 2$ . For  $i > \mathbf{m}(k)$  we have*

$$M_k^i = \{0\}.$$

**4.11 Lemma.** *Let  $k \geq 2$ . Let  $0 \neq \mathcal{F} \in M_k^1$  and let  $1 \leq i \leq \mathbf{m}(k)$  be such that the decomposition of  $\mathcal{F}$  with respect to the basis  $\mathcal{B}_k^{1,+}$  contains an element of  $\mathcal{B}_k^i$  with a non-zero coefficient. Then there is at least one cusp at which  $\mathcal{F}$  has vanishing order at most  $i$ .*

*If, in addition,  $i$  is minimal with these properties, then there exists at least one cusp such that the vanishing order of  $\mathcal{F}$  at this cusp is exactly  $i$ .*

*Proof.* Write  $\mathcal{F} = \mathcal{F}_1 + \dots + \mathcal{F}_{\mathbf{m}(k)}$  where for each  $1 \leq j \leq \mathbf{m}(k)$  the summand  $\mathcal{F}_j$  is a linear combination of basis elements in  $\mathcal{B}_k^j$ . This decomposition is unique.

According to Lemma 4.9 each of these  $\mathcal{F}_j$  is a cusp form of order exactly  $j$  if it is non-trivial. If  $i$  is minimal such that  $\mathcal{F}_i \neq 0$  there can be no cancellations and the vanishing order of  $\mathcal{F}$  is exactly  $i$  for at least one cusp.  $\square$

We are now able to prove the main result of this section.

**4.12 Theorem.** *Let  $k \geq 2$  and  $1 \leq i \leq \mathbf{m}(k)$ . The set  $\mathcal{B}_k^{i,+}$  is a basis of  $M_k^i$ .*

*Proof.* Obviously,  $\mathcal{B}_k^{i,+}$  spans a subspace of  $M_k^i$ .

If a cusp form  $\mathcal{F}$  is not an element of the space generated by  $\mathcal{B}_k^{i,+}$  there must be at least one cusp such that the vanishing order of  $\mathcal{F}$  at this cusp is strictly less than  $i$  by Lemma 4.11. Therefore  $\mathcal{F}$  is not in  $M_k^i$  and the proof is complete.  $\square$

*Remark.* For some applications it is helpful to extend the above concepts to the trivial case  $k = 1$ .

The statement

$$M_k^i = \{0\} \quad \text{for } i > \mathbf{m}(k)$$

is again true in this situation since  $\mathbf{m}(1) = 0$ .

In analogy to Notation 4.2, we define a set  $\mathcal{B}_1^0 = \mathcal{B}_1^{\mathbf{m}(1)}$  consisting of the modular forms

$$\mathcal{F}_b^{(0,1)} := \mathcal{E}_b, \quad 0 \leq b \leq q.$$

This set is in fact a basis of

$$M_1^{\mathbf{m}(1)} = M_1 = \text{Eis}_1.$$

## 5 Congruences of cusp forms

In this final section we study a certain type of cusp forms under reduction modulo cusp forms of higher order. The resulting congruence formula Proposition 5.4 has applications for descriptions of transformation properties of the basis of  $M_k^1$  constructed in the previous section (see [Var16]).

**5.1 Lemma.** *Let  $1 \leq i \leq q - 1$  and  $0 \leq m < q - i$ . For any  $l \in \mathbb{N}$  the congruence*

$$\mathcal{E}_q^l \mathcal{E}_m E_\infty^i \equiv \mathcal{E}_1^l \mathcal{E}_m E_\infty^i \pmod{M_{i+l+1}^{i+1}}$$

*holds.*

*Proof.* According to the definition of the modified Eisenstein series we have

$$\mathcal{E}_q^l \mathcal{E}_m E_\infty^i = (\mathcal{E}_1 + E_\infty)^l \mathcal{E}_m E_\infty^i = \mathcal{E}_1^l \mathcal{E}_m E_\infty^i + \sum_{j=1}^l \binom{l}{j} \mathcal{E}_1^{l-j} \mathcal{E}_m E_\infty^{i+j}.$$

We determine the vanishing order of the modular form  $\mathcal{E}_1^{l-j} \mathcal{E}_m E_\infty^{i+j}$  for  $1 \leq j \leq l$  at each cusp by using Proposition 2.8 and Proposition 2.9.

In this way, we see that this modular form vanishes at the cusps  $(\alpha : 1)$  with  $\alpha \in \mathbb{F}_q^\times$  of order

$$i + j \geq i + 1.$$

The vanishing order at  $(0 : 1)$  is

$$l - j + m + i + j = i + m + l \geq i + 1,$$

since we assume  $l \geq 1$ . Finally, according to the prerequisites the vanishing order at  $\infty$  satisfies the inequality

$$q - m + (l - j)(q - 1) \geq q - m > i.$$

Thus we see that

$$\mathcal{E}_1^{l-j} \mathcal{E}_m E_\infty^{i+j} \in M_{i+l+1}^{i+1}$$

for  $1 \leq j \leq l$  and the proof is complete.  $\square$

For the next step the idea is to transform the factor  $\mathcal{E}_1^l \mathcal{E}_m$  into a standard monomial and to apply the lemma again to eliminate all occurrences of  $\mathcal{E}_q$ . By repeated iteration we receive the following lemma whose proof is straightforward but technical and will be omitted for brevity's sake.

**5.2 Lemma.** *Let  $1 \leq i \leq q - 1$ . For  $l \in \mathbb{N}$  and  $0 \leq b \leq q - 1$  we have*

$$\mathcal{E}_1^l \mathcal{E}_b E_\infty^i \equiv \mathcal{E}_0^l \mathcal{E}_{[l+b]} E_\infty^i \pmod{M_{i+l+1}^{i+1}},$$

where “[ $\cdot$ ]” denotes the unique representative modulo  $q - 1$  in  $\{1, \dots, q - 1\}$ .



In combination with Lemma 5.1 this leads to the following, more refined variant:

**5.3 Lemma.** *Let  $1 \leq i \leq q - 1$  and  $0 \leq m < q - i$ . For  $l \in \mathbb{N}_0$  we have*

$$\mathcal{E}_q^l \mathcal{E}_m E_\infty^i \equiv \mathcal{E}_0^l \mathcal{E}_{[l+m]} E_\infty^i \pmod{M_{i+l+1}^{i+1}}.$$

The lemma also holds in the trivial case  $l = 0$ , if we replace “[ $\cdot$ ]” with the symbol “ $\langle \cdot \rangle$ ” defined as

$$\langle x \rangle = \begin{cases} 0 & x = 0 \\ [x] & \text{else.} \end{cases}$$

We can now state the following general formula for reduction modulo cusp forms of higher order.

**5.4 Proposition.** *Let  $1 \leq i \leq q - 1$  and  $0 \leq m < q - i$ . For  $l \in \mathbb{N}$  let arbitrary  $a_1, \dots, a_l \in \{0, \dots, q\}$  be given. Then*

$$\left( \prod_{j=1}^l \mathcal{E}_{a_j} \right) \mathcal{E}_m E_\infty^i \equiv \mathcal{E}_0^l \mathcal{E}_{\langle a+m \rangle} E_\infty^i \pmod{M_{i+1+l}^{i+1}}$$

with  $a := a_1 + \dots + a_l$ .

*Proof.* In case all  $a_j$  equal  $q$ , this is just a restatement of Lemma 5.3. In the non-trivial case the proposition is proven by transforming products of modified Eisenstein series into standard monomials and applying the reduction rules established previously in this section.  $\square$

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