Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint Nr. 377

Some remarks on the (non-) attainment of the boundary data for variational problems in the space BV

Michael Bildhauer and Martin Fuchs

Saarbrücken 2016

Fachrichtung 6.1 – Mathematik
Universität des Saarlandes

Preprint No. 377
submitted: March 10, 2016

Some remarks on the (non-) attainment of the boundary data for variational problems in the space BV

Michael Bildhauer

Saarland University
Department of Mathematics
P.O. Box 15 11 50
66041 Saarbrücken
Germany
bibi@math.uni-sb.de

Martin Fuchs

Saarland University
Department of Mathematics
P.O. Box 15 11 50
66041 Saarbrücken
Germany
fuchs@math.uni-sb.de

Edited by
FR 6.1 – Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-Mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

AMS Subject-Classification: 49 J 40, 49 J 45, 49 Q 05.

Keywords: variational problems of linear growth, boundary behaviour, minimal surfaces.

Abstract

We discuss the standard relaxed version of a minimization problem for variational integrals of linear growth together with prescribed Dirichlet boundary data u_0 and give estimates for the size of the set $\{x \in \partial\Omega : u(x) \neq u_0(x)\}$ for BV-minimizers u which imply $\mathcal{H}^{n-1}(\{x \in \partial\Omega : u(x) < u_0(x)\}) = \mathcal{H}^{n-1}(\{x \in \partial\Omega : u(x) > u_0(x)\})$ in the case of minimal surfaces u not attaining the boundary values u_0 on a subset of $\partial\Omega$ with positive measure.

To explain our results we first look at the classical variational problem for minimal surfaces

$$J[u] := \int_\Omega \sqrt{1+|\nabla u|^2}\,\mathrm{d}x \to \min \text{ in } u_0 + \overset{\circ}{W}{}^1_1(\Omega)\,, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded Lipschitz domain and u_0 denotes a given function from the Sobolev space $W^1_1(\Omega)$ (see, e.g., [Ad] for a definition). As it is outlined for example in [GMS], the variational problem (1) admits a natural extension to the class $\mathrm{BV}(\Omega)$ consisting of all functions $u \in L^1(\Omega)$ having finite total variation (compare [Gi] or [AFP] for details) i.e. one studies the problem

$$K[u] := \int_\Omega \sqrt{1+|\nabla u|^2} + \int_{\partial\Omega} |u-u_0|\,\mathrm{d}\mathcal{H}^{n-1} \to \min \text{ in } \mathrm{BV}(\Omega)\,. \tag{2}$$

Note that the expression $\int_\Omega \sqrt{1+|\nabla u|^2}$ has to be understood as a convex function of a measure (see [DT]) which equals the quantity $J[u]$ from formula (1) in case that $u \in W^1_1(\Omega)$. Moreover, we recall that in $\int_{\partial\Omega} |u-u_0|\,\mathrm{d}\mathcal{H}^{n-1}$ the trace of $u \in \mathrm{BV}(\Omega)$ on $\partial\Omega$ is considered.

We also note that due to famous examples (see, e.g., Santis example [Sa], compare also [Gi]), in general the uniqueness of solutions and the attainment of the boundary data cannot be expected. We wish to mention that a discussion of the (non-) attainment of the boundary data can also be found in [BS], Theorem 1.4 and 1.5.

Here we can state and prove by an elementary reasoning

Theorem 1. *Let $u \in BV(\Omega)$ denote a solution of the variational problem (2). We define the sets (up to subsets of $\partial\Omega$ with vanishing $\mathcal{H}^{n-1}$-measure)*

$$\begin{aligned} \partial_+\Omega &:= \{x \in \partial\Omega : u(x) > u_0(x)\}\,, \\ \partial_-\Omega &:= \{x \in \partial\Omega : u(x) < u_0(x)\}\,, \\ \partial_0\Omega &:= \{x \in \partial\Omega : u(x) = u_0(x)\}\,. \end{aligned}$$

Then it holds

$$\left|\mathcal{H}^{n-1}(\partial_+\Omega) - \mathcal{H}^{n-1}(\partial_-\Omega)\right| \leq \mathcal{H}^{n-1}(\partial_0\Omega)\,. \tag{3}$$

Formula (3) admits the following nice interpretation:

Corollary 1. *If the generalized minimal surface u ignores the boundary data u_0 completely in the sense that $\mathcal{H}^{n-1}(\partial\Omega_0) = 0$, then – at least in measure – the sets $\partial_\pm\Omega$ are equally distributed.*

The proof of Theorem 1 will be obtained as a by-product of the following slightly more general investigations whose framework is collected in Appendix A.1 of [Bi], where the interested reader will also find a list of further references.

Let $N \geq 1$ and consider a strictly convex integrand F: $\mathbb{R}^{nN} \to [0,\infty)$ with (w.l.o.g.) $F(0) = 0$ being of linear growth in the sense that

$$a|Z| - b \leq F(Z) \leq A|Z| + B \quad \text{for all } Z \in \mathbb{R}^{nN} \tag{4}$$

holds with constants a, $A > 0$, b, $B \geq 0$. Letting

$$F_\infty(Z) := \lim_{t\to\infty} \frac{1}{t} F(tZ)$$

we consider for $u \in \mathrm{BV}(\Omega, \mathbb{R}^N)$ the functional

$$L[u] := \int_\Omega F(\nabla^a u)\,\mathrm{d}x + \int_\Omega F_\infty\left(\frac{\nabla^s u}{|\nabla^s u|}\right) \mathrm{d}\,|\nabla^s u| + \int_{\partial\Omega} F_\infty\left((u_0 - u) \otimes \nu\right) \mathrm{d}\mathcal{H}^{n-1}\,, \tag{5}$$

where now u_0 is a given function from the space $W_1^1(\Omega, \mathbb{R}^N)$, $\otimes$ is the tensor product of vectors and ν stands for the exterior normal of $\partial\Omega$. Moreover, we use the symbols $\nabla^a u$ ($\nabla^s u$) to denote the regular (singular) part of the tensor-valued measure ∇u w.r.t. Lebesgue's measure.

We have

Proposition 1. *Let F satisfy (4) and define L according to (5).*

i) The minimization problem

$$L[w] \to \min \quad \textit{in } \mathrm{BV}(\Omega, \mathbb{R}^N) \tag{6}$$

admits at least one solution.

ii) It holds

$$\inf_{u_0 + \overset{\circ}{W}{}_1^1(\Omega,\mathbb{R}^N)} \int_\Omega F(\nabla w)\,\mathrm{d}x = \inf_{BV(\Omega,\mathbb{R}^N)} L[w]$$

iii) The solutions of problem (6) are in one-to-one correspondence with the L^1–limits of minimizing sequences for problem (6) restricted to the class of functions from $u_0 + \overset{\circ}{W}{}_1^1(\Omega, \mathbb{R}^N)$.

With a given function $f\colon\ [0,\infty) \to [0,\infty)$, $f(0) = 0$, being strictly increasing and strictly convex we additionally impose the structure condition

$$F(Z) = f\left(|Z|\right), \quad Z \in \mathbb{R}^{nN}, \tag{7}$$

so that

$$F_\infty(Z) = f_\infty(|Z|), \quad f_\infty := \lim_{t\to\infty} \frac{1}{t} f(t).$$

Then it holds

Theorem 2. *Let F satisfy (4) and (7) and consider a solution $u \in \mathrm{BV}(\Omega, \mathbb{R}^N)$ of problem (6) with L defined in formula (5).*

Then we have the estimate (as above $\partial_0\Omega := \{x \in \partial\Omega : u(x) = u_0(x)\}$)

$$\left| \int_{\partial\Omega\cap[u\neq u_0]} \frac{u-u_0}{|u-u_0|}\, \mathrm{d}\mathcal{H}^{n-1} \right| \le \mathcal{H}^{n-1}(\partial_0\Omega). \tag{8}$$

Clearly (3) is a consequence of (8): if we are in the scalar case, then it holds

$$\int_{\partial\Omega\cap[u\neq u_0]} \frac{u-u_0}{|u-u_0|}\, \mathrm{d}\mathcal{H}^{n-1} = \mathcal{H}^{n-1}(\partial_+\Omega) - \mathcal{H}^{n-1}(\partial_-\Omega)$$

and we obtain the following

Corollary 2. *Let the assumptions of Theorem 2 hold together with $N = 1$. Then inequality (3) is true.*

Proof of Theorem 2. From $L[u] \le L[u+\xi]$ for any $\xi \in \mathbb{R}^N$ we find that

$$\xi \mapsto \int_{\partial\Omega} F_\infty\left((u_0 - (u+\xi)) \otimes \nu\right)\, \mathrm{d}\mathcal{H}^{n-1}$$

attains its minimum at $\xi = 0$, which by the structure condition (7) means that we just have to discuss the convex function

$$g : \mathbb{R}^N \to [0,\infty), \quad g(\xi) := \int_{\partial\Omega} |u - u_0 + \xi|\, \mathrm{d}\mathcal{H}^{n-1}.$$

Let $\partial g(0)$ denote the subgradient of g at 0, i.e. the closed and convex subset ($\neq \emptyset$) of $\mathbb{R}^N$ consisting of those vectors $\eta \in \mathbb{R}^N$ such that

$$g(y) \ge g(0) + \eta \cdot y \quad \text{for all } y \in \mathbb{R}^N.$$

Since $g(y) \ge g(0)$ for any $y \in \mathbb{R}^N$, it holds

$$0 \in \partial g(0). \tag{9}$$

Let us write

$$\begin{aligned} g(\xi) &= g_0(\xi) + g_1(\xi)\,, \\ g_0(\xi) &:= |\xi| \mathcal{H}^{n-1}(\partial_0 \Omega)\,, \\ g_1(\xi) &:= \int_{\partial\Omega \cap [u \neq u_0]} |u + \xi - u_0| \, \mathrm{d}\mathcal{H}^{n-1}\,. \end{aligned}$$

From Proposition 2 below we get

$$\partial g(0) = \overline{B_\alpha(\xi)}\,, \tag{10}$$

where we have set

$$\alpha := \mathcal{H}^{n-1}(\partial_0 \Omega), \quad \xi := \nabla g_1(0) = \int_{\partial\Omega \cap [u \neq u_0]} \frac{u - u_0}{|u - u_0|} \, \mathrm{d}\mathcal{H}^{n-1}\,.$$

Note that the existence of the gradient of g_1 at the origin together with the formula follows by elementary calculations.

Combining (9) and (10) we find that $|\xi| \leq \alpha$ holds, hence (8) is established.

Let us finally discuss

Proposition 2. *Let $h : \mathbb{R}^N \to [0, \infty)$ denote a convex function for which $\xi := \nabla h(0)$ exists. For a number $\alpha \geq 0$ consider the function $G := \alpha |\cdot| + h$.*

Then it holds

$$\partial G(0) = \overline{B_\alpha(\xi)}\,. \tag{11}$$

Proof of Proposition 2. W.l.o.g. consider the case $h(0) = 0$ together with $\alpha = 1$. For any vector $v \in \mathbb{R}^N$, $|v| \leq 1$, it holds

$$G(y) = |y| + h(y) \geq v \cdot y + \xi \cdot y \quad \text{for all } y \in \mathbb{R}^N\,,$$

hence $v + \xi \in \partial G(0)$ and in conclusion $\overline{B_1(\xi)} \subset \partial G(0)$.

Fix some vector $w \in \partial G(0)$, i.e.

$$G(y) \geq w \cdot y \quad \text{for all } y \in \mathbb{R}^N\,.$$

For $t > 0$ we get from this inequality and the definition of G

$$|y| + \frac{1}{t} h(ty) \geq w \cdot y\,,$$

thus after passing to the limit $t \to 0$

$$|y| + \xi \cdot y \geq w \cdot y\,,$$

which means $|y| \geq (w - \xi) \cdot y$ and thereby $|w - \xi| \leq 1$ by the arbitrariness of y.

This means that $w \in \overline{B_1(\xi)}$ and (11) follows. This completes the proof of Proposition 2 and thereby the proof of Theorem 2. $\square$

References

[Ad] Adams, R.A., Sobolev Spaces. Academic Press, New York - London (1975).

[AFP] Ambrosio, L., Fusco, N., Pallara, D., Functions of bounded variation and free discontinuity problems. Oxford University Press,New York (2000).

[BS] Beck, L., Schmidt, T., On the Dirichlet problem for variational integrals in BV. J. Reine Angew. Math. 674, 113–194 (2013).

[Bi] Bildhauer, M., Convex variational problems: linear, nearly linear and anisotropic growth conditions. Lect. Notes Math. 1818. Springer, New York (2003).

[DT] Demengel, F., Temam, R., Convex functions of a measure and applications. Indiana Univ. Math. J. 33, 673–709 (1984).

[GMS] Giaquinta, M., Modica, G., Souček, J., Functionals with linear growth in the calculus of variations. Comm. Math. Univ. Carolinae 20, 143–172 (1979).

[Gi] Giusti, E., Minimal surfaces and functions of bounded variation. Birkhäuser, Boston– Basel–Stuttgart (1984).

[Sa] Santi, E., Sul problema al contorno per l'equazione della superfici di area minima su domini limitati qualunque, Ann. Univ. Ferrara 17 (1972), 13–26.