Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint

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Preprint No. 46
Saarbrücken 2001

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submitted: December 19, 2001

Preprint No. 46
Saarbrücken 2001

Edited by
FR 6.1 – Mathematik
Im Stadtwald
D–66041 Saarbrücken
Germany

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AMS Subject Classification: 35J25, 46E35, 35B45, 35D05, 35D10
Key words: elliptic equations, Dirichlet problem, weight spaces

Abstract

The solvability of the Dirichlet problem for quasilinear elliptic second-order equations of nondivergence form are studied in the weight spaces.

§0. Introduction

In the present paper, we study the solvability of the Dirichlet problem for quasilinear elliptic second-order equations of nondivergence form. In doing so we assume that the right-hand side of the equation belongs to the weight space $\mathbb{L}_p$ such that the weight is equal to some power of the distance from a point to the boundary of a domain.
The structure of the paper is as follows. In Sec. 1, we formulate the statement of the problem and the main result. A priori estimates are derived in Sec. 2. Finally, in Sec. 3, the existence theorems for solutions of the linear and quasilinear problems are proved.

Throughout the paper, we use the following notation:
$x = (x_1, x') = (x_1, x_2, \dots, x_n)$ is a vector in $\mathbb{R}^n$;
$|x|$, $|x'|$ are the Euclidean norms in the corresponding spaces;
$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_1 > 0\}$;
$\Omega \subset \{x \in \mathbb{R}^n : 0 < x_1 < R\}$ is a domain in $\mathbb{R}^n$ with boundary $\partial\Omega$;
$|\Omega|$ denotes the Lebesgue measure of Ω;
$d(x)$ is the distance between a point $x \in \Omega$ and $\partial\Omega$;
$B^n_\rho(x^0)$ is the open ball in $\mathbb{R}^n$ with center x^0 and radius ρ;
$\Pi_\rho = \{x \in \mathbb{R}^n : |x_1| < \rho, |x'| < \rho\}$;
Γ_ρ is the part of $\partial\Pi_\rho$ lying on the hyperplane $x_1 = 0$.
We adopt the convention that the indices i, and j run from 1 to n. We also adopt the convention regarding summation with respect to repeated indices.
D_i denotes the operator of differentiation with respect to the variable x_i;
$Du = (D_i u)$ is the gradient of u;
$D^2u = D(Du)$ is the Hessian matrix of u;
We introduce the following spaces:
$C(\overline{\Omega})$ is the space of continuous functions with the norm $\|\cdot\|_\Omega$;
$C^2(\overline{\Omega})$ is the space of functions continuous in Ω together with their derivatives up to the second order;

$W^2_p(\Omega)$ $(1 \leqslant p \leqslant \infty)$ is the Sobolev space with the norm

$$\|u\|_{W^2_p(\Omega)} = \|D(Du)\|_{p,\Omega} + \|u\|_{p,\Omega},$$

where $\|\cdot\|_{p,\Omega}$ denotes the standard norm in $L_p(\Omega)$;
$L_{p,(\alpha)}(\Omega)$ is the weight space with the norm $\|\cdot\|_{p,(\alpha),\Omega}$, where

$$\|u\|_{p,(\alpha),\Omega} = \|\,(x_1)^\alpha\, u\|_{p,\Omega};$$

$\mathbb{L}_{p,(\alpha)}(\Omega)$ is the weight space with the norm $|||\cdot|||_{p,(\alpha),\Omega}$, where

$$|||u|||_{p,(\alpha),\Omega} = \|(d(x))^\alpha\, u\|_{p,\Omega}\,;$$

$\widetilde{V}^2_{p,(\alpha)}(\Omega)$ is a set of functions with the finite semi-norm $\|D^2u\|_{p,(\alpha),\Omega}$;
$\widetilde{\mathbb{V}}^2_{p,(\alpha)}(\Omega)$ is a set of functions with the finite semi-norm $|||D^2u|||_{p,(\alpha),\Omega}$.
It should be noted that the semi-norms $\|D^2u\|_{p,(\alpha),\Omega}$ and $|||D^2u|||_{p,(\alpha),\Omega}$ become the norms if $u|_{\partial\Omega} = 0$.
We set

$$f_+ = \max\ \{f, 0\}\,, \qquad f_- = \max\ \{-f, 0\}\,, \qquad \operatorname*{osc}_\Omega\, f = \sup_\Omega\, f - \inf_\Omega\, f,$$

and assume that $q > n$ and $\widehat{\alpha}(q) = 1 - \frac{n}{q}$.
We use letters M, N, C (with or without indices) to denote various constants. To indicate that, say, N depends on some parameters, we list them in the parentheses: $N(\dots)$.

§1. Statement of the problem

We consider the equation

$$-a^{ij}(x,u,Du)D_iD_ju + a(x,u,Du) = 0 \tag{1}$$

in the domain Ω and the Dirichlet boundary condition

$$u = 0 \quad \text{on} \quad \partial\Omega. \tag{2}$$

We assume that the matrix (a^{ij}) is symmetric, and the functions a^{ij} have the first-order generalized derivatives with respect to all the arguments. We assume also that the functions a^{ij} and a in Eq. (1) satisfy the following structure conditions:

$$\nu|\xi|^2 \leqslant a^{ij}(x,z,p) \leqslant \nu^{-1}|\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \tag{A0}$$
$$|a(x,z,p)| \leqslant \mu|p|^2 + b(x)|p| + \Phi(x), \tag{A1}$$
$$|p|\left|\frac{\partial a^{ij}(x,z,p)}{\partial p_k}\right| \leqslant \mu \quad \text{for} \quad |p| \geqslant 1, \tag{A2}$$
$$\left|\frac{\partial a^{ij}(x,z,p)}{\partial z}p_k + \frac{\partial a^{ij}(x,z,p)}{\partial x_k}\right| \leqslant \mu|p| + \Phi(x), \tag{A3}$$
$$b, \Phi \in \mathbb{L}_{q,(\alpha)}(\Omega) \tag{A4}$$

for any $x \in \Omega$, $z \in \mathbb{R}^1$, and $p \in \mathbb{R}^n$, where ν and μ are some positive constants.

Theorem 1. *Let the following conditions hold:*
(i) $n < q < \infty$, $0 < \alpha < \widehat{\alpha}(q)$, $\partial\Omega \in \widetilde{\mathbb{V}}^2_{q,(\alpha)}$;
(ii) *every solution* $u^{[\tau]}(\cdot) \in \widetilde{\mathbb{V}}^2_{q,(\alpha)}(\Omega)$ *to the problem*

$$\begin{aligned} &\tau\left(-a^{ij}(x,u,Du)D_iD_ju + a(x,u,Du)\right) - (1-\tau)\Delta u = 0 \quad \textit{in} \quad \Omega, \\ &u|_{\partial\Omega} = 0, \end{aligned} \tag{3}$$

where $\tau \in [0,1]$, *satisfies the estimate*

$$\|u^{[\tau]}(\cdot)\|_\Omega \leqslant M_0;$$

(iii) *if* $|z| \leqslant M_0$, *then conditions* (A0), (A1), (A2), (A3), *and* (A4) *hold*;
(iv) *the function* $a(\cdot,z,p)$ *regarded as an element of the space* $\mathbb{L}_{q,(\alpha)}(\Omega)$ *is continuous with respect to* (z,p).
Then the problem (3) *has at least one solution* $\widehat{u}^{[\tau]}(\cdot) \in \widetilde{\mathbb{V}}^2_{q,(\alpha)}(\Omega)$ *for each* $\tau \in [0,1]$. *In particular,* $\widehat{u}^{[1]}(\cdot)$ *is a solution to the problem* (1), (2).

§2. A priori estimates

Suppose that $\mathcal{L}$ is a uniformly elliptic linear operator in Ω,

$$\mathcal{L}u \equiv -a^{ij}(x)D_iD_ju + b^i(x)D_iu,$$

and ν is an ellipticity constant. We introduce the notation $\mathbf{b}(x) = (b^i(x))$.

<u>Step 1. MAXIMUM ESTIMATES</u>

We start with the case where b_i are **bounded** functions.

Lemma 2. *Suppose that a nonnegative function $B \in C^2(\overline{\Omega})$ satisfies*

$$\mathcal{L}B \geqslant |b^1| \left(1 + \ln \frac{R}{x_1}\right) \quad a.e. \ in \quad \Omega.$$

Then any function $u \in \widetilde{V}^2_{\infty,(1)}(\Omega)$ such that $u\big|_{\partial\Omega} \leqslant 0$ satisfies the estimate

$$u \leqslant \frac{\|B\|_\Omega + e^{-1}R}{\nu} \|(\mathcal{L}u)_+\|_{\infty,(1),\Omega^u}, \tag{4}$$

where $\Omega^u = \{x \in \Omega : u(x) > 0\}$.

Proof. We introduce the function

$$v = \frac{\|(\mathcal{L}u)_+\|_{\infty,(1),\Omega^u}}{\nu} \left(x_1 \ln \frac{R}{x_1} + B\right).$$

An elementary computation gives

$$\mathcal{L}v \geqslant \frac{\|(\mathcal{L}u)_+\|_{\infty,(1),\Omega}}{x_1} \geqslant \mathcal{L}u \qquad \text{in} \quad \Omega^u.$$

Then applying the Aleksandrov maximum principle [A] to the difference $u - v$ in Ω^u, we obtain $u \leqslant v$ which implies (4). $\square$

Lemma 3. *Suppose that a nonnegative function $B \in C^2(\overline{\Omega})$ satisfies*

$$\mathcal{L}B \geqslant |\mathbf{b}| \qquad a.e. \ in \quad \Omega.$$

Then any function $u \in W^2_n(\Omega)$ such that $u\big|_{\partial\Omega} \leqslant 0$ satisfies the estimate

$$u \leqslant N_1(n) \frac{\|B\|_\Omega + N_2(n,\nu)R}{\nu} \|(\mathcal{L}u)_+\|_{n,\Omega^u}.$$

Proof. This statement is proved by verbatim repetition of the proof of Lemma 1.1 [N1]. $\square$

Theorem 4. *Let $n < q \leqslant \infty$. Suppose that a nonnegative function $B \in C^2(\overline{\Omega})$ satisfies*

$$\mathcal{L}B \geqslant |\mathbf{b}| \left(1 + \ln \frac{R}{x_1}\right) \qquad a.e. \ in \quad \Omega. \tag{5}$$

Then any function $u \in \widetilde{V}^2_{q,(\widehat{\alpha}(q))}(\Omega)$ *such that* $u\big|_{\partial\Omega} \leqslant 0$ *satisfies the estimate*

$$u \leqslant N_3(n)\frac{\|B\|_{\Omega^u} + N_4(n,\nu)R}{\nu}\|(\mathcal{L}u)_+\|_{q,(\widehat{\alpha}(q)),\Omega^u}. \tag{6}$$

Proof. The proof of this assertion is similar to that of Theorem 2.1 [N1]. The only difference is that we must use Lemmas 2 and 3 instead of Lemmas 1.3 and 1.1 from [N1]. □

We now turn to the case of **unbounded** functions b^i.

Theorem 5. *Let* $n < q_0, q_1 \leqslant \infty$, $\delta = \widehat{\alpha}(q_1) - \alpha_1 > 0$, *and let* $R \leqslant 1$. *Then there exists a constant* $\sigma = \sigma(n, \nu, \delta)$ *such that the inequality*

$$\|\mathbf{b}\|_{q_1,(\alpha_1),\Omega} \leqslant \sigma$$

implies the estimate

$$u \leqslant C_0(n,\nu)R\|(\mathcal{L}u)_+\|_{q_0,(\widehat{\alpha}(q_0)),\Omega^u} \tag{7}$$

for any function $u \in \widetilde{V}^2_{q_0,(\widehat{\alpha}(q_0))}(\Omega)$ *such that* $u\big|_{\partial\Omega} \leqslant 0$.

Proof. It sufficies to obtain (7) only for smooth $\Omega \subset\subset \{0 < x_1 < R\}$. Then the general case follows in the same way as Item 3 in the proof of Theorem 2.1 [N1].

1. Let $q_1 = \infty$. Then the assumptions of Theorem 5 mean that

$$|\mathbf{b}(x)|(x_1)^{1-\delta} \leqslant \sigma \qquad \text{a.e. in} \quad \Omega.$$

Thus, for $B = x_1 \ln \frac{R}{x_1}$ we have

$$\mathcal{L}B \geqslant \frac{\nu}{x^1} - |\mathbf{b}|\left(1 + \ln\frac{R}{x_1}\right) \geqslant \frac{\nu}{x_1}\left(1 - \frac{\sigma}{\nu}x_1^\delta\left(1 + \ln\frac{R}{x_1}\right)\right) \qquad \text{a.e. in} \quad \Omega.$$

We note that the inequalities

$$x_1^\delta\left(1 + \ln\frac{R}{x_1}\right) \leqslant C(\delta)R^\delta \leqslant C(\delta)$$

are true on the segment $[0, R]$. Hence for $\sigma = \frac{\nu}{2C(\delta)}$ we obtain

$$\mathcal{L}B \geqslant |\mathbf{b}|\left(1 + \ln\frac{R}{x_1}\right) \qquad \text{a.e. in} \quad \Omega.$$

Then (7) follows from (6).

2. Let $q_0, q_1 < \infty$. In this case, it sufficies to establish (7) for smooth coefficients and smooth functions u, and then use the passage to the limit. We define B as a solution of the boundary value problem

$$\mathcal{L}B = |\mathbf{b}|\left(1 + \ln\frac{R}{x_1}\right) \equiv \widetilde{b} \quad \text{in } \Omega, \qquad B\Big|_{\partial\Omega} = 0.$$

The solution B satisfies the inequality (5) from Theorem 4. Consequently, the estimate (6) for $q = q_1$ and $u = B$ gives

$$\begin{aligned} B &\leqslant N_3 \frac{\|B\|_\Omega + N_4 R}{\nu} \|\widetilde{b}\|_{q_1,(\widehat{\alpha}(q_1)),\Omega} \\ &\leqslant N_3 \frac{\|B\|_\Omega + N_4 R}{\nu} C(\delta) R^\delta \|\mathbf{b}\|_{q_1,(\alpha_1),\Omega} \\ &\leqslant \frac{N_3 \sigma C(\delta)}{\nu} (\|B\|_\Omega + N_4 R). \end{aligned} \tag{8}$$

Taking $\sigma = \frac{\nu}{2N_3C(\delta)}$, from (8) we get

$$\|B\|_\Omega \leq N_4 R.$$

Then the estimate (7) again follows from (6)

3. Finally, let $q_0 = \infty, q_1 < \infty$.
For sufficiently large finite q from Item 2 it follows that

$$u \leqslant C_0 R \|(\mathcal{L}u)_+\|_{q,(\widehat{\alpha}(q)),\Omega^u} \leqslant C_0 R \|(\mathcal{L}u)_+\|_{\infty,(1),\Omega^u} (\inf_{\Omega} |x'|)^{-\frac{n}{q}} |\Omega|^{\frac{1}{q}}.$$

Now passing to the limit as $q \to \infty$ we obtain (7) for $q_0 = \infty, q_1 < \infty$. $\square$

Step 2. HÖLDER ESTIMATES FOR SOLUTIONS

Hereinafter, we suppose that

$$n < q < \infty, \qquad \delta = \widehat{\alpha}(q) - \alpha > 0.$$

Lemma 6. *Let $\rho \leqslant 1$. Then for any $\beta_1 \in]0,1[$ there exist positive constants $\sigma_1 = \sigma_1(n,\nu,\beta_1,\delta)$ and $\varepsilon = \varepsilon(n,\nu,\beta_1)$ such that for a function $u \in \widetilde{V}^2_{q,(\widehat{\alpha}(q))}(\Pi_\rho) \cap C(\overline{\Pi}_\rho)$ the conditions*

$$u \geqslant 0 \quad in \quad \overline{\Pi}_\rho, \qquad u \geqslant k > 0 \quad on \quad \Gamma_\rho,$$
$$\|\mathbf{b}\|_{q,(\alpha),\Pi_\rho} \leqslant \sigma_1$$

imply

$$u \geqslant (1-\beta_1)k - N_5(n,\nu)\rho\|(\mathcal{L}u)_-\|_{q,(\widehat{\alpha}(q)),\Pi_\rho} \quad \text{in} \quad \Pi_{\varepsilon\rho}. \tag{9}$$

Proof. Without loss of generality, we assume $k=1$. By change of the scale we transform Π_ρ into Π_1. In the new coordinates we have

$$\widetilde{\mathcal{L}}\widetilde{u} \equiv -\widetilde{a}^{ij}D_iD_j\widetilde{u} + \rho\widetilde{b}^iD_i\widetilde{u} = \rho^2(\mathcal{L}u) \qquad \text{in} \quad \Pi_1, \tag{10}$$

where $\widetilde{u}(x) = u(\rho x)$, $\widetilde{a}^{ij}(x) = a^{ij}(\rho x)$, etc.
Define a barrier function ψ as

$$\psi(x) = 1 - |x'|^2 + \left(\frac{n-1}{\nu}\right)^2 x_1^2 - \frac{4(n-1)}{\nu}x_1,$$

and set $\Pi = \Pi_1 \cap \{x_1 < \frac{\nu}{n-1}\}$.
From the assumptions of Lemma 6 it follows that

$$\psi - \widetilde{u} \leqslant 0 \quad \text{on} \quad \partial\Pi, \qquad\qquad \widetilde{\mathcal{L}}\psi \leqslant N_6(n,\nu)\rho|\widetilde{\mathbf{b}}| \quad \text{in} \quad \Pi,$$

and

$$\|\rho\widetilde{\mathbf{b}}\|_{q,(\alpha),\Pi} \leqslant \|\rho\widetilde{\mathbf{b}}\|_{q,(\alpha),\Pi_1} = \rho^\delta\|\mathbf{b}\|_{q,(\alpha),\Pi_\rho} \leqslant \sigma_1. \tag{11}$$

Suppose that $\sigma_1 \leqslant \sigma$, where $\sigma = \sigma(n,\nu,\delta)$ is the constant from Theorem 5 (with $q_0 = q_1 = q$). Next, applying Theorem 5 to the function $\psi - \widetilde{u}$ in Π and using (11) and the evident relation

$$\|(\widetilde{\mathcal{L}}\widetilde{u})_-\|_{q,(\widehat{\alpha}(q)),\Pi_1} = \rho\|(\mathcal{L}u)_-\|_{q,(\widehat{\alpha}(q)),\Pi_\rho},$$

which followed from (10), we obtain

$$\widetilde{u}(x) \geqslant \psi(x) - N_7(n,\nu)\rho\|(\mathcal{L}u)_-\|_{q,(\widehat{\alpha}(q)),\Pi_\rho} - N_8(n,\nu)\sigma_1 \quad \text{in} \quad \Pi. \tag{12}$$

Since $\psi(0) = 1$, we may choose $\varepsilon = \varepsilon(n,\nu,\beta_1)$ such that $\Pi_\varepsilon \subset \Pi$ and $\psi \geqslant 1-\beta_1/2$ in Π_ε. Then, taking σ_1 sufficiently small and returning to the original variables, we get (9) from (12). $\square$

Theorem 7. *Let $\rho_0 \leqslant 1$. Suppose that $u \in \widetilde{V}^2_{q,(\widehat{\alpha}(q))}(\Pi_{\rho_0}) \cap C(\overline{\Pi}_{\rho_0})$ is a function such that*

$$|u| \leqslant M_0 \quad \textit{in} \quad \Pi_{\rho_0}, \qquad\qquad u = 0 \quad \textit{on} \quad \Gamma_{\rho_0}$$

and

$$|a^{ij}(x,u,Du)D_iD_ju| \leqslant \mu|Du|^2 + b(x)|Du| + \Phi(x) \qquad \textit{in} \quad \Pi_{\rho_0},$$

where

$$b \in L_{q,(\alpha)}(\Pi_{\rho_0}), \qquad \Phi \in L_{q,(\widehat{\alpha}(q))}(\Pi_{\rho_0}),$$

$(a^{ij}(x,z,p))$ *is a symmetric matrix satisfying the inequalities* (A0) *for any* $x \in \overline{\Pi}_{\rho_0}$, $z \in \mathbb{R}^1$, $p \in \mathbb{R}^n$.
If, in addition,

$$\|b\|_{q,(\alpha),\Pi_{\rho_0}} \leqslant \sigma_1,$$

where $\sigma_1 = \sigma_1(n,\nu,1/2,\delta)$ *is the constant from Lemma* 6, *then the estimate*

$$\operatorname*{osc}_{\Pi_\rho} u \leqslant C_1 \left(\frac{\rho}{\rho_0}\right)^{\gamma_1} \qquad \forall \rho \leqslant \rho_0$$

is valid with some $\gamma_1 = \gamma_1(n,\nu) \in]0,1[$ *and* $C_1 > 0$ *depending on the same arguments as* γ_1 *as well as on* μ, M_0, *and* $\|\Phi\|_{q,(\widehat{\alpha}(q)),\Pi_{\rho_0}}$.

Proof. This statement is deduced from Lemma 6 in the same way as Lemma 2.2 [AN1] is deduced from Lemma 2.1 [AN1]. □

Step 3. BOUNDARY ESTIMATES FOR THE GRADIENT

Before passing to estimates of the gradient on the boundary, we formulate several auxiliary statements. The first statement is concerned to the interior estimate for the gradients of solutions of elliptic equations. The second statement deals with the estimates for functions satisfying a linear elliptic inequality with specific type singularities. Finally, the third statement is devoted to a special decomposition of functions belonging to the weight space $L_{q,(\alpha)}$.
From now on we make the additional assumption: $0 < \alpha < \widehat{\alpha}(q)$.

Lemma 8. *Let* $\rho_0 \leqslant 1$. *Suppose that* $u \in \widetilde{V}^2_{q,(\alpha)}(\Pi_{\rho_0}) \cap C(\overline{\Pi}_{\rho_0})$ *is a function such that*

$$|u| \leqslant M_0 \qquad in \quad \Pi_{\rho_0},$$

and u *satisfies Eq.* (1) *in* Π_{ρ_0}.
Suppose also that for any $x \in \overline{\Pi}_{\rho_0}$, $z \in \mathbb{R}^1$, $p \in \mathbb{R}^n$ *the functions* $a^{ij}(x,z,p)$ *have the first-oder generalized derivatives with respect to all their arguments, and conditions* (A0), (A1), (A2), *and* (A3) *hold with* b, $\Phi \in L_{q,(\alpha)}(\Pi_{\rho_0})$.
Then there exists $\lambda = \lambda(n,\nu,q,\delta,\mu) > 0$ *with the following property: if the ball* $B^n_{x_1^*/2}(x^*)$ *does not intersect* $\partial\Pi_{\rho_0}$, *and*

$$\operatorname*{osc}_{B^n_{x_1^*/2}(x^*)} u \leqslant \omega_0,$$

then

$$|Du(x^*)| \leqslant N_9 \,(x^*)^{-1} \left[\operatorname*{osc}_{B^n_{x_1^*/2}(x^*)} u \right]^{\lambda}. \tag{13}$$

Here N_9 and ω_0 are positive constants depending on n, ν, q, δ, μ, $\|b\|_{q,(\alpha),\Pi_{\rho_0}}$, and $\|\Phi\|_{q,(\alpha),\Pi_{\rho_0}}$.

Proof. Consider $\varepsilon_1 \leqslant \frac{nq\delta}{(n+q\alpha)}$, and set $\delta_1 = \frac{nq\delta-(n+q\alpha)\varepsilon_1}{(n+\varepsilon_1)q}$. It should be noted also that according to definition $n+\varepsilon_1 < q$. By the Hölder inequality, for any function $f \in L_{q,(\alpha)}(\Pi_{\rho_0})$ and $\rho^* \leqslant \frac{x_1^*}{2}$ we have

$$\begin{aligned} \|f\|_{n+\varepsilon_1,B^n_{\rho^*}(x^*)} &\leqslant \|f\|_{q,(\alpha),B^n_{\rho^*}(x^*)} \left(\int_{B^n_{\rho^*}(x^*)} x_1^{\frac{-\alpha q(n+\varepsilon_1)}{q-n-\varepsilon_1}} \, dx \right)^{\frac{q-n-\varepsilon_1}{q(n+\varepsilon_1)}} \\ &\leqslant \|f\|_{q,(\alpha),\Pi_{\rho_0}} \,(\rho^*)^{-\alpha} \left| B^n_{\rho^*}(x^*) \right|^{\frac{q-n-\varepsilon_1}{q(n+\varepsilon_1)}} \\ &\leqslant (\rho^*)^{\delta_1} N_{10}(n) \|f\|_{q,(\alpha),\Pi_{\rho_0}}. \end{aligned} \tag{14}$$

This means that in $B^n_{\rho^*}(x^*)$ the statement of Theorem 2.1 [LU1] is valid and hence (13) is actually proved (for $b=0$; the case $b \in L_{n+\varepsilon_1}$ can be treated similarly); it sufficies to write out explicitly the dependence of the constant C occuring in the theorem mentioned on the parameter ρ^*. □

Lemma 9. *Let $r>n$, let $u \in W^2_r(\Pi_1) \cap C(\overline{\Pi}_1)$, and let $u|_{\Gamma_1} = 0$. In the cylinder Π_1 we define an operator*

$$\mathbb{L} \equiv -a^{ij}(x) D_i D_j + \left[b_1^i(x) + b_2^i(x) \right] D_i,$$

whose coefficients $a^{ij} = a^{ji}$, b_1^i, b_2^i are measurable functions,

$$\nu|\eta|^2 \leqslant a^{ij}(x)\eta_i\eta_j \leqslant \nu^{-1}|\eta|^2 \qquad \forall x \in \overline{\Pi}_1, \quad \eta \in \mathbb{R}^n,$$

$$\mathbf{b}_1 \in L_r(\Pi_1), \quad |\mathbf{b}_2(x)| \leqslant (x_1)^{\beta-1} F \quad in \quad \Pi_1, \quad F>0, \quad \beta \in]0,1[.$$

Suppose also that

$$|(\mathbb{L}u)(x)| \leqslant \Phi(x) + (x_1)^{\beta-1} H \qquad in \quad \Pi_1, \quad \Phi \in L_r(\Pi_1), \quad H>0.$$

Then there exists a positive number $\mathcal{R} = \mathcal{R}(\nu,\beta,F)$ such that the following estimates are valid

$$\frac{u(x)}{x_1} \leqslant N_{12} \qquad in \quad \Pi_{\mathcal{R}/2}, \tag{15}$$

$$\operatorname*{osc}_{\Pi_\rho} \frac{u(x)}{x_1} \leqslant N_{13}\rho^{\gamma_2} \quad \forall \rho \leqslant \mathcal{R}/4. \tag{16}$$

Here $\gamma_2 = \gamma_2(n, \nu, \beta, \widehat{\alpha}(r)) \in]0,1[$, *while the positive constants* N_{12} *and* N_{13} *depends on the same arguments as* γ_2 *as well as on* F, H, $\mathcal{R}$, $\sup_{\Pi_1} u$, $\|\mathbf{b}_1\|_{r,\Pi_1}$, *and* $\|\Phi\|_{r,\Pi_1}$.

Proof. Consider $\mathcal{R} \leqslant \min\left\{1, \left(\frac{\beta(\beta+1)\nu}{14F}\right)^{1/\beta}\right\}$. To prove (15) and (16) it sufficies to apply successively an elliptic versions of Lemma 3.4 [AN3] and Lemma 4.5 [AN3] to the function u. The first application gives us (15), while the second one gives (16). $\square$

Lemma 10. *Let* $\rho_0 \leqslant 1$, *let* ε_1 *be the same constant as in the proof of Lemma* 8, *and let* $\delta_2 = \frac{\delta q - \varepsilon_1}{q - n - \varepsilon_1} \in]0,1[$.
Then for any function $g \in L_{q,(\alpha)}(\Pi_{\rho_0})$ *there exist* $g_1 \in L_{n+\varepsilon_1}(\Pi_{\rho_0})$ *and* $g_2 \in L_{\infty,(1-\delta_2)}(\Pi_{\rho_0})$ *such that*

$$g(x) = g_1(x) + g_2(x),$$

and

$$\|g_1\|_{n+\varepsilon_1,\Pi_{\rho_0}} + \|g_2\|_{\infty,(1-\delta_2),\Pi_{\rho_0}} \leqslant 2\|g\|_{q,(\alpha),\Pi_{\rho_0}}. \tag{17}$$

Proof. Without loss of generality, we assume $g \geqslant 0$ in Π_{ρ_0}. We set

$$g_1(x) = \left(g(x) - x_1^{\delta_2 - 1}\varkappa\right)_+, \qquad g_2(x) = g(x) - g_1(x),$$

where $\varkappa$ is a positive parameter to be specified later.
It is obvious that

$$\|g_2\|_{\infty,(1-\delta_2),\Pi_{\rho_0}} \leqslant \varkappa. \tag{18}$$

Moreover, due to the Hölder inequality it follows that

$$\begin{aligned} \|g_1\|_{n+\varepsilon_1,\Pi_{\rho_0}} &= \left(\int_{\mathcal{G}} \left(g(x) - x_1^{\delta_2-1}\varkappa\right)^{n+\varepsilon_1} dx\right)^{\frac{1}{n+\varepsilon_1}} \\ \leqslant \left(\int_{\mathcal{G}} x_1^{\alpha q}\left(g(x) - x_1^{\delta_2-1}\varkappa\right)^q dx\right)^{\frac{1}{q}} &\left(\int_{\mathcal{G}} x_1^{-(n+\varepsilon_1)(1-\delta_2)} dx\right)^{\frac{\alpha}{(n+\varepsilon_1)(1-\delta_2)}} \\ &\leqslant \|g\|_{q,(\alpha),\Pi_{\rho_0}} \left(\int_{\mathcal{G}} x_1^{-(n+\varepsilon_1)(1-\delta_2)} dx\right)^{\frac{\alpha}{(n+\varepsilon_1)(1-\delta_2)}}, \end{aligned} \tag{19}$$

where $\mathcal{G} = \left\{x : g(x) > x_1^{\delta_2-1}\varkappa\right\}$. On the other hand, we have

$$\|g\|_{q,(\alpha),\Pi_{\rho_0}} \geqslant \left(\int_{\mathcal{G}} x_1^{\alpha q}\,(g(x))^q\,dx\right)^{1/q}$$
$$\geqslant \varkappa\left(\int_{\mathcal{G}} x_1^{q(\alpha+\delta_2-1)}dx\right)^{1/q} = \varkappa\left(\int_{\mathcal{G}} x_1^{-(n+\varepsilon_1)(1-\delta_2)}dx\right)^{1/q}. \quad (20)$$

Combining (19) and (20) we obtain

$$\|g_1\|_{n+\varepsilon_1,\Pi_{\rho_0}} \leqslant \varkappa^{\frac{-\alpha q}{(n+\varepsilon_1)(1-\delta_2)}}\left(\|g\|_{q,(\alpha),\Pi_{\rho_0}}^{1+\frac{\alpha q}{(n+\varepsilon_1)(1-\delta_2)}}\right). \quad (21)$$

Now choosing $\varkappa = \|g\|_{q,(\alpha),\Pi_{\rho_0}}$, and adding (21) and (18) we immediately get the desired estimate (17). □

Theorem 11. *Let the assumptions of Lemma* 8 *be valid. Suppose also that* $u|_{\Gamma_{\rho_0}} = 0$, *and*

$$\|b\|_{q,(\alpha),\Pi_{\rho_0}} \leqslant \sigma_1, \quad (22)$$

where $\sigma_1 = \sigma_1(n,\nu,1/2,\delta)$ *is the constant from Lemma* 6.
Then the following estimates are valid:

$$\sup_{\Gamma_{\rho_0/4}} |Du| \leqslant C_2, \quad (23)$$

$$\operatorname*{osc}_{\Gamma_\rho} Du \leqslant C_3\left(\frac{\rho}{\rho_0}\right)^{\gamma_3} \qquad \forall\rho \leqslant \rho_0/8. \quad (24)$$

Here $\gamma_3 = \gamma_3(n,\nu,q,\delta,\mu) \in]0,1[$, *while the positive constants* C_2-C_3 *depend on the same parameters as* γ_3 *as well as on* ρ_0, M_0, $\|b\|_{q,(\alpha),\Pi_{\rho_0}}$, *and* $\|\Phi\|_{q,(\alpha),\Pi_{\rho_0}}$.

Proof. It sufficies to establish (23) and (24) for smooth functions u, and then use the passage to the limit.
By Lemma 10 we can decompose b and Φ into the sums $b_1 + b_2$ and $\Phi_1 + \Phi_2$ such that (17) is fulfilled for these decompositions.
In accordance with Theorem 7 and Lemma 8, we have the estimate

$$|Du(x)| \leqslant N_{14}\,(x_1)^{\lambda_1 - 1} \qquad \text{in} \quad \Pi_{\rho_0/2},$$

where $\lambda_1 = \lambda\gamma_1$, and the constant N_{14} is determined by the known parameters listed in the statement of the theorem. There is no loss of generality in assuming that $\lambda_1 \leqslant \delta_2$, where δ_2 is that constant from Lemma 10. Hence in $\Pi_{\rho_0/2}$ the inequality (A1) can be rewritten as follows:

$$|a(x,u,Du)| \leqslant \left(b_1(x) + b_2(x) + \widehat{b}(x)\right)|Du| + \Phi_1(x) + \Phi_2(x), \quad (25)$$

where $\widehat{b}(x) = \mu N_{14}\,(x_1)^{\lambda_1 - 1}$.
Arguing in the same way as in Theorem 3.1 [AN2], from (25) we find

$$|\widetilde{\mathcal{L}}u| \leqslant \Phi_1(x) + \Phi_2(x) \qquad \text{in} \quad \Pi_{\rho_0/2}, \tag{26}$$

where

$$\widetilde{\mathcal{L}} \equiv -\widetilde{a}^{ij}_{(u)}(x) D_i D_j + \left[\widetilde{b}^i_1(x) + \widetilde{b}^i_2(x)\right] D_i u,$$
$$\widetilde{a}^{ij}_{(u)}(x) = a^{ij}(x, u, Du), \qquad |\widetilde{\mathbf{b}}_1(x)| \leqslant b_1(x), \qquad |\widetilde{\mathbf{b}}_2(x)| \leqslant b_2(x) + \widehat{b}(x).$$

Obviously,

$$b^i_1,\ \Phi_1 \in L_{n+\varepsilon_1}(\Pi_{\rho_0}), \tag{27}$$

and the inequality (17) leads to the estimates

$$|\widetilde{\mathbf{b}}_2(x)| \leqslant (x_1)^{\lambda_1 - 1}\left[\|b_2\|_{\infty,(1-\delta_2),\Pi_{\rho_0}} + \mu N_{13}\right], \tag{28}$$
$$|\Phi_2(x)| \leqslant (x_1)^{\lambda_1 - 1}\,\|\Phi_2\|_{\infty,(1-\delta_2),\Pi_{\rho_0}}. \tag{29}$$

Relations (26)-(29) and the condition $u|_{\Gamma_{\rho_0}} = 0$ show that Lemma 9 with

$$F = \left[\|b_2\|_{\infty,(1-\delta_2),\Pi_{\rho_0}} + \mu N_{14}\right], \qquad H = \|\Phi_2\|_{\infty,(1-\delta_2),\Pi_{\rho_0}}$$

can be applied to the function u. This application gives us (23), and (24) for any $\rho \leqslant \rho_1 = \min\{\rho_0/8, \mathcal{R}_0/4\}$, where $\mathcal{R}_0 := \mathcal{R}(\nu, \lambda_1, F)$ is the constant from Lemma 9. In the case $\rho_1 < \rho \leqslant \rho_0/8$ we estimate the oscillation of Du on Γ_ρ as follows

$$\operatorname*{osc}_{\Gamma_\rho} Du \leqslant 2C_2 \leqslant 2C_2\left(\frac{\rho}{\rho_1}\right)^{\gamma_3} \leqslant C_3\left(\frac{\rho}{\rho_0}\right)^{\gamma_3}$$

with $C_3 = 2C_2\left(\frac{\rho_0}{\rho_1}\right)^{\gamma_3}$. □

Step 4. ESTIMATES FOR THE GRADIENT NEAR THE BOUNDARY

Theorem 12. *Let the assumptions of Lemma* 8 *and the condition* (22) *be valid, let* $\mathcal{R}_0$ *be the same constant as in the proof of Theorem* 11, *and let* $\mathcal{R}_1 = \min\{\rho_0, \mathcal{R}_0\}$.
If, moreover, $u|_{\Gamma_{\rho_0}} = 0$, *then for* $x^* \in \Pi_{\mathcal{R}_1/3}$ *such that the ball* $B^n_{x^*_1/2}(x^*)$ *does not intersect* $\partial\Pi_{\mathcal{R}_1/3}$ *we have the following estimate*

$$|Du(x^*)| \leqslant C_4 \tag{30}$$

Here C_4 is the positive constant depending on n, ν, q, δ, μ, $\|b\|_{q,(\alpha),\Pi_{\rho_0}}$, and $\|\Phi\|_{q,(\alpha),\Pi_{\rho_0}}$.

Proof. Consider $\rho = \frac{x_1^*}{2}$ and define in B_1^n the function $v(x) = \frac{u(\rho x + x^*)}{\rho}$. We shall have established (30) if we obtain for v the estimate for the gradient which does not depend on ρ.
Arguing in the same way as in the proof of Theorem 11 we find that Lemma 9 is applicable to u. Observe also that $B_\rho^n(x^*) \subset \Pi_{\mathcal{R}_0/2}$. Hence the inequality (15) is valid and, consequently,

$$|v| \leqslant N_{15} \qquad \text{in} \quad B_1^n,$$

where $N_{15} = N_{15}(n, \nu, \mu, \delta, M_0, \|\Phi\|_{q,(\alpha),\Pi_{\rho_0}})$.
Observe also that the function v satisfies the equation

$$-\widetilde{a}^{ij}(x, v, Dv) D_i D_j v + \widetilde{a}(x, v, Dv) = 0 \qquad \text{in} \quad B_1^n,$$

where $\widetilde{a}^{ij}(x, v, Dv) = a^{ij}(\rho x + x^*, \rho v, Dv)$, $\widetilde{a}(x, v, Dv) = \rho a(\rho x + x^*, \rho v, Dv)$.
Moreover, for any $x \in \overline{B}_1$, $z \in \mathbb{R}^1$, $p \in \mathbb{R}^n$ conditions (A0), (A1), (A2) and (A3) are transformed respectively as follows:

$$\nu|\zeta|^2 \leqslant \widetilde{a}^{ij}(x, z, p)\zeta_i\zeta_j \leqslant \nu^{-1}|\zeta|^2 \qquad \forall \zeta \in \mathbb{R}^n, \tag{A0'}$$

$$|\widetilde{a}(x, z, p)| \leqslant \widetilde{\mu}|p|^2 + \widetilde{b}(x)|p| + \widetilde{\Phi}(x), \tag{A1'}$$

$$|p| \left| \frac{\partial \widetilde{a}^{ij}(x, z, p)}{\partial p_k} \right| \leqslant \mu \quad \text{for} \quad |p| \geqslant 1, \tag{A2'}$$

$$\left| \frac{\partial \widetilde{a}^{ij}(x, z, p)}{\partial z} p_k + \frac{\partial \widetilde{a}^{ij}(x, z, p)}{\partial x_k} \right| \leqslant \widetilde{\mu}|p| + \widetilde{\Phi}(x), \tag{A3'}$$

where $\widetilde{\mu} = \mu\rho \leqslant \mu$, $\widetilde{b}(x) = \rho b(\rho x + x^*)$, $\widetilde{\Phi}(x) = \rho \Phi(\rho x + x^*)$.
In addition, the simplest computation, in view of (14), yields

$$\begin{aligned} \|\widetilde{\Phi}\|_{n+\varepsilon_1, B_1^n} &= \rho^{\frac{\varepsilon_1}{n+\varepsilon_1}} \|\Phi\|_{n+\varepsilon_1, B_\rho^n(x^*)} \\ &\leqslant \rho^{\delta_1 + \frac{\varepsilon_1}{n+\varepsilon_1}} N_{10} \|\Phi\|_{q,(\alpha),\Pi_{\rho_0}} \leqslant N_{10} \|\Phi\|_{q,(\alpha),\Pi_{\rho_0}}, \end{aligned}$$

where ε_1 and δ_1 are the same constants as in the proof of Lemma 8.
Analogously, we have

$$\|\widetilde{b}\|_{n+\varepsilon_1, B_1^n} \leqslant N_{10} \|b\|_{q,(\alpha),\Pi_{\rho_0}}.$$

Therefore all hypothesis of Theorem 2.1 [LU1] are valid for the function v in B_1^n. Application of this theorem gives us

$$\|Dv\|_{B_{1/2}^n} \leqslant C_4,$$

which immediately leads to (30). $\square$

Theorem 13. *Assume $\rho_0 \leqslant \mathcal{R}_0$, where $\mathcal{R}_0$ is the same constant as in the proof of Theorem* 11. *Let the assumptions of Lemma* 8 *and the condition* (22) *be valid. If, moreover, $u|_{\Gamma_{\rho_0}} = 0$, and*

$$|Du| \leqslant M_1 \quad in \quad \Pi_{\rho_0},$$

then for $x, y \in \Pi_{\rho_0/8}$ we have the following estimate

$$|Du(x) - Du(y)| \leqslant C_5|x-y|^{\gamma_4}, \tag{31}$$

where $\gamma_4 \in]0,1[$ and $C_5 > 0$ depend on n, ν, q, δ, ρ_0, M_0, M_1, $\|b\|_{q,(\alpha),\Pi_{\rho_0}}$, and $\|\Phi\|_{q,(\alpha),\Pi_{\rho_0}}$.

Proof. It is obvious that for any $x \in \overline{\Pi}_{\rho_0}$, $|z| \leqslant M_0$, $|p| \leqslant M_1$ conditions (A1), (A2), and (A3) take the following form:

$$|a(x,z,p)| \leqslant \mu M_1^2 + b(x)M_1 + \Phi(x) \equiv \widehat{\Phi}_{\{1\}}(x), \tag{A1"}$$

$$\left|\frac{\partial a^{ij}(x,z,p)}{\partial p_k}\right| \leqslant \mu, \tag{A2"}$$

$$\left|\frac{\partial a^{ij}(x,z,p)}{\partial z}\right|, \quad \left|\frac{\partial a^{ij}(x,z,p)}{\partial x_k}\right| \leqslant \mu M_1 + \Phi(x) \equiv \widehat{\Phi}_{\{2\}}(x). \tag{A3"}$$

We consider the function $w(x) = u(x) - x_1 D_1 u(0)$. It is evident that w satisfies the equation

$$-\widehat{a}^{ij}(x,w,Dw)D_iD_jw + \widehat{a}(x,w,Dw) = 0 \quad \text{in} \quad \Pi_{\rho_0},$$

where

$$\begin{aligned}
\widehat{a}^{ij}(x,w,Dw) &= \widehat{a}^{ij}(x,w,D_1w,D'w) \\
&= a^{ij}(x, w + x_1D_1u(0), D_1w + D_1u(0), D'w), \\
\widehat{a}(x,w,Dw) &= \widehat{a}(x,w,D_1w,D'w) \\
&= a(x, w + x_1D_1u(0), D_1w + D_1u(0), D'w).
\end{aligned}$$

Decomposing $\widehat{\Phi}_{\{1\}}$ and $\widehat{\Phi}_{\{2\}}$ in the same way as in the proof of Lemma 10 we see that w satisfies the assumptions of Lemma 9 (with $b_1^i(x) = b_2^i(x) = 0$). Hence the inequality (16) with u replaced by w is true. Therefore, taking into account that

$$\lim_{x\to 0} \frac{w(x)}{x_1} = D_1w(0) = 0,$$

we get

$$|w(x)| \leqslant (x_1)^{1+\gamma_2} N_{16} \quad \text{in} \quad \Pi_{\rho_0/4}, \tag{32}$$

where the constant $N_{16} > 0$ is determined by the known parameters listed in the statement of the theorem. There is no loss in generality in assuming that $\gamma_2 \leqslant \frac{\varepsilon_1}{n+\varepsilon_1}$, where ε_1 is the same constant as in the proof of Lemma 8.
Now we consider $x^* \in \Pi_{\rho_0/8}$ and define in B_1^n for $\rho = \frac{x_1^*}{2}$ the function $v(x) = \frac{w(\rho x + x^*)}{\rho^{1+\gamma_2}}$. First of all we obtain for v the estimates for the gradient which do not depend on ρ.
From (32) it follows that $|v| \leqslant N_{16}$ in B_1^n. Observe also that the function v satisfies the equation

$$-\widetilde{a}^{ij}(x) D_i D_j v + \widetilde{a}(x) = 0 \quad \text{in} \quad B_1^n,$$

where

$$\begin{aligned} \widetilde{a}^{ij}(x) &= \widehat{a}^{ij}(\rho x + x^*, \rho^{1+\gamma_2} v, \rho^{\gamma_2} Dv), \\ \widetilde{a}(x) &= \rho^{1-\gamma_2} \widehat{a}(\rho x + x^*, \rho^{1+\gamma_2} v, \rho^{\gamma_2} Dv). \end{aligned} \tag{33}$$

Moreover, condition (A0) is transformed into (A0') with $\widetilde{a}^{ij}$ defined in (33), while conditions (A1"), (A2"), and (A3") take the following forms:

$$|\widetilde{a}(x,z,p)| \leqslant \widetilde{\Phi}_{\{1\}}(x),$$

$$\left| \frac{\partial \widetilde{a}^{ij}(x,z,p)}{\partial p_k} \right| \leqslant \mu,$$

$$\left| \frac{\partial \widetilde{a}^{ij}(x,z,p)}{\partial z} \right|, \quad \left| \frac{\partial \widetilde{a}^{ij}(x,z,p)}{\partial x_k} \right| \leqslant \widetilde{\Phi}_{\{2\}}(x),$$

where $\widetilde{\Phi}_{\{1\}}(x) = \rho^{1-\gamma_2} \widehat{\Phi}_{\{1\}}(\rho x + x^*)$ and $\widetilde{\Phi}_{\{2\}}(x) = \rho^1 \widehat{\Phi}_{\{2\}}(\rho x + x^*)$.
The simplest computations, in view of (14), yield

$$\begin{aligned} \|\widetilde{\Phi}_{\{1\}}\|_{n+\varepsilon_1, B_1^n} &= \rho^{\frac{\varepsilon_1}{n+\varepsilon_1} - \gamma_2} \|\widehat{\Phi}_{\{1\}}\|_{n+\varepsilon_1, B_\rho^n(x^*)} \\ &\leqslant N_{10} \rho^{\frac{\varepsilon_1}{n+\varepsilon_1} - \gamma_2 + \delta_1} \|\widehat{\Phi}_{\{1\}}\|_{q,(\alpha),\Pi_{\rho_0}} \leqslant N_{10} \|\widehat{\Phi}_{\{1\}}\|_{q,(\alpha),\Pi_{\rho_0}}, \\ \|\widetilde{\Phi}_{\{2\}}\|_{n+\varepsilon_1, B_1^n} &= \rho^{\frac{\varepsilon_1}{n+\varepsilon_1}} \|\widehat{\Phi}_{\{2\}}\|_{n+\varepsilon_1, B_\rho^n(x^*)} \\ &\leqslant N_{10} \rho^{\frac{\varepsilon_1}{n+\varepsilon_1} + \delta_1} \|\widehat{\Phi}_{\{2\}}\|_{q,(\alpha),\Pi_{\rho_0}} \leqslant N_{10} \|\widehat{\Phi}_{\{2\}}\|_{q,(\alpha),\Pi_{\rho_0}}. \end{aligned}$$

Hence we can apply Theorems 4.1 and 4.3 from [LU2] to the function v, which yields

$$|Dv(x)| \leqslant N_{17}, \qquad |Dv(x) - Dv(y)| \leqslant N_{18} |x-y|^{\widetilde{\gamma}}, \quad x, y \in B_{1/2}^n, \tag{34}$$

where the positive constants $\widetilde{\gamma}$, N_{17}, and N_{18} are determined by the known parameters listed in the statement of the theorem.

Furthermore, we note if $x, y \in \Pi_{\rho_0/8}$ and $|x-y| < \frac{1}{2}\max\{x_1, y_1\}$ then the second estimate in (34) leads to (31) with $\gamma_4 = \widetilde{\gamma}$.
Otherwise, $|x-y| \geqslant \frac{1}{2}\max\{x_1, y_1\}$. In view of the first estimate in (34), for $x \in \Pi_{\rho_0/8}$ we have

$$|Dw(x)| \leqslant N_{17}\,(x_1)^{\gamma_2}.$$

Consequently, we obtain

$$\begin{aligned}|Du(x) - Du(y)| &\leqslant |Dw(x) - Dw(y)| \\ &\leqslant 2N_{17}\,(\max\{x_1, y_1\})^{\gamma_2} \leqslant 2^{1+\gamma_2} N_{17}|x-y|^{\gamma_2}.\end{aligned}$$

So, in either case, we have (31) with $\gamma_4 = \min\{\widetilde{\gamma}, \gamma_2\}$. This completes the proof. □

§3. The solvability of linear and quasilinear Dirichlet problems

Theorem 14 *Let $1 < p < \infty$, and let $\alpha \in \left]-\frac{1}{p}, 2-\frac{1}{p}\right[$. Suppose also that the coefficients of the operator $\mathcal{L}$ satisfy the following conditions:*

$$a^{ij} \in C(\overline{\Omega}), \quad |\mathbf{b}| \in \mathbb{L}_{\overline{p},(\overline{\alpha})}(\Omega),$$

where

$$\overline{p} = \begin{cases} \max\{n, p\}, & \text{if} \quad p \neq n \\ n + \varepsilon, & \text{if} \quad p = n, \end{cases} \qquad \text{and} \quad \overline{\alpha} = \max\{\widehat{\alpha}(\overline{p}) - \varepsilon, 0\},$$

while ε is a small positive number.
If, in addition, $\partial\Omega \in \widetilde{\mathbb{V}}^2_{\overline{p},(\overline{\alpha})}(\Omega)$ then for any $f \in \mathbb{L}_{p,(\alpha)}(\Omega)$ the boundary value problem

$$\begin{aligned} \mathcal{L}u &= f \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} &= 0 \end{aligned} \tag{35}$$

has a unique solution $u \in \widetilde{\mathbb{V}}^2_{p,(\alpha)}(\Omega)$.

Proof. By the Hölder inequality and the embedding theorems (cf. Theorems 10.1 and 10.4 [BIN]) one can verify that the assumption $|\mathbf{b}| \in \mathbb{L}_{\overline{p},(\overline{\alpha})}(\Omega)$ provides the inclusion $b^i D_i u \in \mathbb{L}_{p,(\alpha)}(\Omega)$ for any $u \in \widetilde{\mathbb{V}}^2_{p,(\alpha)}(\Omega)$.
Moreover, the condition $\partial\Omega \in \widetilde{\mathbb{V}}^2_{\overline{p},(\overline{\alpha})}(\Omega)$ guarantees that the assumption on $|\mathbf{b}|$ is invariant under straightening the boundary $\partial\Omega$.

Using the coercive estimate for a model problem in the half-space, which follows from Theorem 7.6 [N2], and standard methods (partition of unity, straightening the boundary $\partial\Omega$, freezing coefficients, and reduction to the canonical form) we obtain the following inequality

$$\|D^2u\|_{p,(\alpha),\Omega} \leqslant N_{19}(n,p,\alpha)\left(\|\mathcal{L}u\|_{p,(\alpha),\Omega} + \|u\|_{p,(\alpha),\Omega}\right). \tag{36}$$

It is well known that the problem (35) with smooth coefficients b^i is uniquely solvable for any smooth function f. So, the inverse operator $\mathcal{L}^{-1}$ is defined on everywhere dense set in the weight space $\mathbb{L}_{p,(\alpha)}(\Omega)$. The estimate (36) shows that this inverse operator can be continuously extended on the whole space $\mathbb{L}_{p,(\alpha)}(\Omega)$.
Now approximation the non-smooth coefficients b^i by smooth functions finishes the proof. □

Proof of Theorem 1. The obtained a priori estimates (Theorems 5, 7, 11, 12, and 13) permit us to prove this assertion by a standard argument based on the Leray-Schauder fixed point principle (Theorem 10.1 [LU2]) and results on the solvability of the linear Dirichlet problem (Theorem 14). □
This work was supported by the Russian Foundation for Fundamental Research (grant No. 99-01-00684).

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