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Large-time behavior of discrete kinetic equations with non-symmetric interactions

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Abstract

We consider the initial-boundary value problem for general linear discrete velocity models appearing in kinetic theory. With time independent inflow boundary data we prove the existence of a unique steady state and the exponential convergence in time towards the steady state. The proof is based on the construction of suitable multiplyers used in a weighted L^2-norm.

1 Introduction

This note is devoted to show the exponential decay towards equilibrium for general discrete velocity models (DVM) obtained from some kinetic boundary value problems. The general DVM system we deal with reads

$$\left\{\begin{array}{rcl} \frac{\partial \mathbf{f}}{\partial t} + V\mathbf{f}_x + A(x)\mathbf{f} &=& 0, \qquad 0 < x < L,\ t > 0, \\ \mathbf{f}(x,0) &=& \mathbf{f}_0(x), \\ \mathbf{f}^+(0,t) &=& \mathbf{g}^+, \\ \mathbf{f}^-(L,t) &=& \mathbf{g}^-, \end{array}\right. \tag{1.1}$$

where $\mathbf{f}(x,t) = (f_j(x,t), j \in J)$ is the kinetic phase-space distribution function. Here and in the sequel we denote vectors in $I\!R^{|J|}$ by bold face letters. We assume that the discrete velocities $v_j \in I\!R$ are strictly ordered $v_j < v_{j+1}$ with the finite index set $J \subset Z\!\!Z$; moreover, $v_j > 0$ for $j > 0$ (i.e. $j \in J^+ := J \cap I\!N$) and $v_j < 0$ for $j < 0$ (i.e. $j \in J^- := J \cap (-I\!N)$). $\mathbf{f}^\pm$ denote the restrictions of $\mathbf{f}$ onto the index sets $J^\pm$. $\mathbf{f}_0$ is the given initial condition and $\mathbf{g}^\pm$ are the prescribed time-independent inflow boundary conditions.

The matrix $V = diag(v_j, j \in J)$, represents the discrete velocities matrix of the free-streeming term and $A(x)$, $0 < x < L$ is the interaction matrix (real valued, square matrix of size $|J|$). From now on we assume the following hypotheses on $A(x)$:

$$\left\{ \begin{array}{rcl} A & \in & L^\infty((0,L), I\!R^{|J|\times|J|}), \text{ piecewise continuous} \\ A^s & := & \frac{1}{2}(A + A^\top), \quad A^{as} := \frac{1}{2}(A - A^\top) \\ A^s(x) & \geq & 0 \quad \forall x \in (0,L). \end{array} \right. \tag{H1}$$

In this paper we study the exponential decay in time of the solution $\mathbf{f}(x,t)$ of (1.1) to its steady state $\mathbf{f}_\infty$, which is uniquely determined (as we shall prove) by the boundary data. For $A^s(x) \geq \lambda I$ with $\lambda > 0$ this decay would of course be trivial. Here we are interested in exponential decay that is induced by the outflux through the boundary under the weaker assumption $A^s(x) \geq 0$.

If the DVM includes the zero velocity we shall make another hypothesis on $A(x)$:

$$\text{if } 0 \in J \quad \text{let } a_{00}(x) > 0, \quad \forall x \in (0,L). \tag{H2}$$

This is motivated by the following consideration: Assume that $a_{00}(x_0) = 0$ for some x_0, first for a symmetric interaction matrix, i.e. $A(x) = A^s(x)$. $A^s(x_0) \geq 0$ would then imply $a^s_{0j}(x_0) = a^s_{j0}(x_0) = 0 \ \ \forall j \in J$. Hence, the equation for f_0 would then be decoupled from the remaining system. And since f_0 is not convected ($v_0 = 0$) and without decay by itself ($a_{00}(x_0) = 0$ by assumption) no exponential decay would be possible. In fact, (1.1) would then not even have a unique steady state. For a general interaction matrix with $a_{00}(x_0) = 0$ the stationary equivalent of (1.1) would be a differential-algebraic equation (DAE) of index 2 and the stationary boundary value problem with fixed $\mathbf{g}^\pm$ would be overdetermined (cf. [8], §4 of [1] for details).

We remark that (H2) does not permit to include $v_0 = 0$ as a discrete velocity for a purely anti-symmetric interaction matrix.

Equations of type (1.1) appear in applications as velocity discretizations of various kinetic models. The mathematical analysis of such discrete velocity models has a long standing tradition (cf. [11, 17], e.g.).

Examples of kinetic models leading to DVM systems of type (1.1) are:

1. *Vlasov-Fokker-Planck equation* [6, 7, 9, 10, 20] for the distribution function $f(x,v,t) \geq 0$, $x \in (0,L)$, $v \in I\!R$:

   $$f_t + v f_x + \phi_x f_v = \beta(vf)_v + \sigma f_{vv}, \tag{1.2}$$

   where $\phi = \phi(x)$ is a given potential. The kinetic equation (1.2) represents the evolution of the distribution of particles in phase-space subject to an external force field ϕ_x and colliding with a surrounding "heat bath". Here, $\beta \geq 0$ is the friction with the particles of the "bath" and $\sigma > 0$ is related to the temperature of the thermal bath.

   Typical velocity discretizations of the diffusion term in (1.2) lead to the symmetric part of the interaction matrix $A^s(x)$, while the force and friction terms yield its anti-symmetric part $A^{as}(x)$.

2. *Wigner equation* [22, 18, 1] for a given potential ϕ:

   $$w_t + v w_x - \Theta[\phi] w = 0, \tag{1.3a}$$

   with the pseudo-differential operator

   $$\Theta[\phi] = i\left[\phi(x + \frac{\partial_v}{2i}) - \phi(x - \frac{\partial_v}{2i})\right]. \tag{1.3b}$$

The quasi-transport equation (1.3) for the (real-valued, but not necessarily positive) *Wigner function* $w(x,v,t)$ is a kinetic formulation of quantum mechanics, equivalent to the Schrödinger equation.

Alternatively to (1.3b), $\Theta[\phi]$ can (formally) be expressed as a convolution operator in v:

$$(\Theta[\phi]w)(x,v) = a(x,v) *_v w(x,v), \tag{1.4}$$

with

$$a(x,v) = \sqrt{\frac{8}{\pi}} Im[e^{2ivx}(\mathcal{F}_{x\to v}\,\phi)(2v)],$$

where $\mathcal{F}$ denotes the Fourier transform. These two definitions of $\Theta[\phi]$ coincide under some regularity and decay assumptions on ϕ (cf. [4]). One easily verifies that $\Theta[\phi]$ is skew-symmetric in $L^2(I\!R_v)$.

Due to the convolution form (1.4), reasonable velocity discretizations of $\Theta[\phi]w$ are of the form

$$(A(x)\mathbf{w})_j = \sum_{k\in J} w_k a_{j-k}(x), \quad j \in J.$$

Hence, the interaction matrix $A(x)$, $0 < x < L$ is here skew-symmetric and Toeplitz (cf. [1]).

3. The *Wigner-Fokker-Planck equation* [2, 3] describes quantum diffusion effects and reads in the simplest case:

$$w_t + v w_x - \Theta[\phi]w = \frac{1}{\tau} w_{vv}.$$

For these models both the symmetric and anti-symmetric parts of the interraction matrix do not vanish.

This paper is organized as follows: In §2 we prove unique solvability of the stationary boundary value problem, including the singular case when 0 is a discrete velocity. Using a multiplyer technique the exponential decay of the transient solution towards the steady state is then proven in §3 in the case $0 \notin J$. However, we include a numerical test indicating that such an exponential decay should also hold in the case $0 \in J$.

2 Unique steady states

In this section we shall discuss the well-posedness of (1.1) and its corresponding steady state. By a standard perturbation lemma of semigroup theory [19] (1.1) has a unique, global-in-time solution $f \in C([0,\infty), L^2((0,L), I\!R^{|J|}))$, when assuming $\mathbf{f}_0 \in L^2((0,L), I\!R^{|J|})$. To keep the notation simple we shall restrict ourselves here to this space, but the results carry over to other L^p-spaces.

The steady state problem of (1.1) for $\mathbf{f} = \mathbf{f}(x)$ reads

$$\left\{ \begin{array}{rcll} V\mathbf{f}_x + A(x)\mathbf{f} &=& 0, & 0 < x < L, \\ \mathbf{f}^+(0) &=& \mathbf{g}^+, & \\ \mathbf{f}^-(L) &=& \mathbf{g}^-. & \end{array} \right. \tag{2.1}$$

Theorem 2.1 *Let A satisfy (H1) and (H2). Then (2.1) with given boundary data $\mathbf{g}^\pm$ has a unique solution $\mathbf{f}_\infty \in L^2((0,L), I\!R^{|J|})$.*

PROOF: The inhomogeneous boundary value problem (2.1) is equivalent to

$$\begin{cases} V\mathbf{f}_x + A(x)\mathbf{f} = \mathbf{h}(x), & 0 < x < L, \\ \mathbf{f}^+(0) = 0, & \mathbf{f}^-(L) = 0. \end{cases} \tag{2.2}$$

Here $\mathbf{h} = -A(x)\tilde{\mathbf{g}} - V\tilde{\mathbf{g}}_x$ and $\tilde{\mathbf{g}}$ is a smooth continuation of the boundary data $(\mathbf{g}^+, \mathbf{g}^-)$ to the phase-space $(0, L) \times J$. If $0 \in J$ we shall assume $h_0 = 0$, which can be obtained by an appropriate choice of $\tilde{g}_0$.

Case (a) - no zero velocity:

For $0 \notin J$ the operator $T = V\partial_x$, defined with homogeneous inflow boundary data is clearly invertible and T^{-1} is compact. Thus, by the Fredholm alternative, the existence and uniqueness of a solution to (2.2) is equivalent to proving that the homogeneous version of (2.2) only admits the trivial solution. Multiplying (2.2) (with $\mathbf{h} = 0$) by $\mathbf{f}^\top$ and integrating in x gives:

$$\mathbf{f}^\top(L)V\mathbf{f}(L) - \mathbf{f}^\top(0)V\mathbf{f}(0) + \int_0^L \mathbf{f}^\top(x)A(x)\mathbf{f}(x)\,dx = 0. \tag{2.3}$$

Note that $\mathbf{f}^-(L) = 0, \mathbf{f}^+(0) = 0$ and $A^s(x) \geq 0$, which gives $\mathbf{f}(x) = 0\ \forall x$. This is a slight generalization of Theorem 3.6 in [1] and rather standard in transport theory [14, 13]. Hence (1.1) has a unique steady state $\mathbf{f}_\infty \in L^2((0, L), I\!R^{|J|})$.

Case (b) - zero is a discrete velocity:

Now we shall reduce the second case ($0 \in J$) to the previous one. For $0 \in J$ Equation (2.2) is a DAE, and due to (H2) its index is 1. By using the (homogeneous – since $h_0 = 0$) algebraic constraint (line $j = 0$ of (2.2)) and (H2) we can eliminate the variable f_0 from (2.2):

We define the regular elimination matrix $M(x) \in I\!R^{|J|\times|J|}$ with the elements

$$m_{jk}(x) = \begin{cases} \delta_{jk}\,; & j \in J,\, k \in \tilde{J}, \\ 1\,; & j = k = 0, \\ -\frac{a_{j0}(x)}{a_{00}(x)}\,; & j \in \tilde{J},\, k = 0, \end{cases} \tag{2.4}$$

with the index set $\tilde{J} := J \setminus \{0\}$, and $\tilde{N} := |\tilde{J}|$. Multiplying (2.2) from the left by $M(x)$ yields (note that $M\,V = V$, $M\,\mathbf{h} = \mathbf{h}$):

$$\begin{cases} T\mathbf{f} + \tilde{A}(x)\mathbf{f} = \mathbf{h}(x), & 0 < x < L, \\ \mathbf{f}^+(0) = 0, & \mathbf{f}^-(L) = 0, \end{cases} \tag{2.5}$$

where $\tilde{A}(x) = M(x)\,A(x)$. Because of (2.4) we have

$$\tilde{a}_{j0}(x) = 0 \quad \forall j \in \tilde{J}, \tag{2.6}$$

and hence the $\tilde{N}$ differential equations in (2.5) for the variables f_j, $j \in \tilde{J}$ are decoupled from the one algebraic constraint.

Let $\tilde{B}(x) \in I\!R^{\tilde{N}\times\tilde{N}}$ be the system matrix of this decoupled ODE-system, obtained by canceling row $j = 0$ and column $j = 0$ of the matrix $\tilde{A}(x)$. We shall now show that $\tilde{B}^s(x) \geq 0$ for any fixed $x \in (0, L)$: For $\tilde{\mathbf{l}} \in I\!R^{\tilde{N}}$ we define

$$\mathbf{l}^\top := (\tilde{\mathbf{l}}^\top\Big|_{J^+}, 0, \tilde{\mathbf{l}}^\top\Big|_{J^-}) \cdot M(x) = (\tilde{\mathbf{l}}\Big|_{J^+}, l_0, \tilde{\mathbf{l}}\Big|_{J^-}).$$

Hence we get:

$$\mathbf{l}^\top \cdot A(x) \cdot \mathbf{l} = (\tilde{\mathbf{l}}^\top\Big|_{J^+}, 0, \tilde{\mathbf{l}}^\top\Big|_{J^-}) \cdot M(x) \cdot M^{-1}(x) \cdot \tilde{A}(x) \cdot \begin{pmatrix} \tilde{\mathbf{l}}\big|_{J^+} \\ l_0 \\ \tilde{\mathbf{l}}\big|_{J^-} \end{pmatrix} = \tilde{\mathbf{l}}^\top \cdot \tilde{B}(x) \cdot \tilde{\mathbf{l}}, \tag{2.7}$$

where we have used (2.6).

Therefore we also obtain

$$\tilde{\mathbf{l}}^\top \cdot \tilde{B}^s(x) \cdot \tilde{\mathbf{l}} = \mathbf{l}^\top \cdot A^s(x) \cdot \mathbf{l} \geq 0 \quad \forall \tilde{\mathbf{l}} \in I\!R^{\tilde{N}},$$

and the decoupled ODE-subsystem of (2.5) satisfies (H1). According to case (a) this subsystem admits a unique stationary solution $\tilde{\mathbf{f}}_\infty(x) = (\tilde{f}_j(x),\, j \in \tilde{J})$ and the remaining component $f_0(x)$ is obtained from the algebraic constraint. ■

3 Exponential decay in time

After having established the existence of a unique steady state $\mathbf{f}_\infty(x)$ we now turn to our main result - the exponential decay of $\mathbf{f}(t)$ in case 0 is not a discrete velocity.

We first point out that the particular case $J = J^+$ ($v_j > 0$ for all j) is trivial since the solution with homogeneous inflow data becomes identically zero in finite time $T = L/v_1$. This is based on the fact that the solution is transported out of the domain with at least velocity v_1. The analogous result holds in the case $J = J^-$ ($v_j < 0$ for all j).

Theorem 3.1 *Let $\mathbf{f}_0 \in L^2((0,L), I\!R^{|J|})$ and assume (H1) and $0 \notin J$. Then*

$$\|\mathbf{f}(t) - \mathbf{f}_\infty\|_2 = O(e^{-\lambda t})$$

for some $\lambda > 0$.

The key point of this theorem is that we only assumed $A^s(x) \geq 0$ in (H1). So the exponential decay is only due to mass transport out of the considered interval $(0, L)$. Under the stronger assumption $A^s(x) \geq \lambda I$ exponential decay with the rate λ would of course follow from a trivial energy estimate.

The idea of the proof consists in introducing a weighted space $L^2_\varphi(0, L)$ with the positive weights $\varphi(x) = (\varphi_j(x), j \in J)$ (in $W^{1,\infty}(0, L)$) to be constructed. This method was first used for a 2-velocity model by M. Tidriri in [21]. We subtract from (1.1) equation (2.1) for the unique steady state f_∞. Hence, we shall consider in the sequel only homogeneous inflow data, i.e. $\mathbf{g}^+ = 0$, $\mathbf{g}^- = 0$ and hence $\mathbf{f}_\infty = 0$.

Proof: With the diagonal matrix $\Phi(x) = diag(\varphi_j(x), j \in J)$ we define the norm in L^2_φ as

$$\|\mathbf{f}\|^2_\varphi = <\Phi\mathbf{f}, \mathbf{f}>,$$

where

$$<\mathbf{f}^1, \mathbf{f}^2> = \int_0^L \sum_{j\in J} f_j^1 f_j^2 dx$$

is the standard L^2-inner product.

Part (a) - decay estimate:
We first take the inner product of equation (1.1) with $\Phi(x)\mathbf{f}$ and obtain after an integration by parts:

$$\frac{1}{2}\frac{d}{dt} <\mathbf{f}, \Phi\mathbf{f}> - \frac{1}{2} <\mathbf{f}, V(\partial_x\Phi)\mathbf{f}> + <A\mathbf{f}, \Phi\mathbf{f}> \;\leq\; 0, \tag{3.1}$$

where we have used $\mathbf{f}^+(0) = 0, \mathbf{f}^-(L) = 0$, and $\varphi_j(x) \geq 0$. The key part of the proof will be to construct the weights Φ and a diagonal matrix $B(\Phi, x)$ with the properties:

$$0 < c_1 \leq \varphi_j(x) \leq c_2, \quad \forall x, \forall j, \tag{3.2}$$

$$-\frac{1}{2} < \mathbf{f}, V(\partial_x \Phi)\mathbf{f} > + < A(x)\mathbf{f}, \Phi \mathbf{f} > \ \geq \ < B(\Phi, x)\mathbf{f}, \mathbf{f} > \qquad \forall \mathbf{f} \in I\!R^{|J|}, \quad (3.3)$$

$$B(\Phi, x) \geq \lambda \Phi \qquad \text{for some } \lambda > 0. \quad (3.4)$$

We remark that c_1, c_2, λ will be independent of $j, x, \mathbf{f}$. Once we have proved (3.3) and (3.4), one trivially obtains from (3.1):

$$\frac{1}{2}\frac{d}{dt} < \mathbf{f}, \Phi \mathbf{f} > +\lambda < \mathbf{f}, \Phi \mathbf{f} > \ \leq \ 0,$$

and hence

$$\|\mathbf{f}(t)\|_\varphi \leq e^{-\lambda t}\|\mathbf{f}_0\|_\varphi, \quad t > 0.$$

Finally, using (3.2) proves the assertion:

$$\|\mathbf{f}(t)\|_2 \leq e^{-\lambda t}\sqrt{\frac{c_2}{c_1}}\|\mathbf{f}_0\|_2, \quad t > 0.$$

Part (b) - construction of $B(\Phi)$:
We split the matrix $A(x)$ into its symmetric and skew-symmetric part: $A = A^s + A^{as}$. First we shall derive an estimate of type (3.3) pertaining to A^s. We have

$$< A^s\mathbf{f}, \Phi\mathbf{f} > = \frac{1}{2} < (A^s\Phi + \Phi A^s)\mathbf{f}, \mathbf{f} > \ \geq \ \mu(\Phi) < \mathbf{f}, \mathbf{f} >,$$

where $\mu(\Phi, x) \in I\!R$ is the smallest eigenvalue of $\frac{1}{2}(A^s(x)\Phi + \Phi A^s(x))$. Due to hypothesis (H1) we have $\mu(I, x) \geq 0$, where I is the identity matrix. Let us define the diagonal matrix $D^s(\Phi, x) := (\mu(\Phi, x) - \mu(I, x))I$. Then we trivially have

$$< A^s\mathbf{f}, \Phi\mathbf{f} > \geq < D^s(\Phi)\mathbf{f}, \mathbf{f} > \qquad \forall \mathbf{f}. \quad (3.5)$$

Moreover, $D^s(I, x) = 0$ and $D^s(\Phi, x)$ is globally Lipschitz in Φ, independent of x. The last assertion is a consequence of Weyl's theorem [16] (page 198) that gives

$$|\mu(\Phi_1) - \mu(\Phi_2)| \leq \rho(\frac{1}{2}(A^s(\Phi_1 - \Phi_2) + (\Phi_1 - \Phi_2)A^s)) \ \leq \ \|A^s\|\|\Phi_1 - \Phi_2\|$$

in any matrix norm (ρ denotes the spectral radius).

Next we shall derive a similar estimate for $A^{as} = (c_{ij})$:

$$\begin{aligned} < A^{as}\mathbf{f}, \Phi\mathbf{f} > &= \int_0^L \sum_{i<j} c_{ij}(x) f_i f_j [\varphi_i(x) - \varphi_j(x)]\, dx \\ &\geq -\frac{1}{2}\int_0^L \sum_{i<j} |c_{ij}(x)|\, |\varphi_i(x) - \varphi_j(x)| (f_i^2 + f_j^2)\, dx \\ &= -< D^{as}(\Phi)\mathbf{f}, \mathbf{f} > \qquad \forall \mathbf{f}, \end{aligned} \quad (3.6)$$

with

$$D^{as}(\Phi, x) := diag\left(\frac{1}{2}\sum_{j\in J} |c_{ij}(x)|\, |\varphi_i - \varphi_j|\right).$$

As before we have $D^{as}(I, x) = 0$ and $D^{as}(\Phi, x)$ is globally Lipschitz in Φ.

Finally we choose the diagonal matrix B as

$$B(\Phi, x) := -\frac{1}{2}V(\partial_x\Phi) + D^s(\Phi, x) - D^{as}(\Phi, x),$$

which satisfies (3.3) and $B(I, x) = 0$.

Part (c) - construction of Φ:
Let the diagonal matrix $\Phi(x)$ be obtained as the solution of the non-linear ODE-system for the $\varphi_j(x)$, $j \in J$:

$$\begin{cases} B(\Phi, x) - \lambda\Phi = 0, & 0 < x < L, \\ \Phi(\frac{L}{2}) = I \end{cases} \tag{3.7}$$

for some (small enough) $\lambda > 0$. We remark that (3.7) has a unique (global-in-x) solution $\Phi_\lambda(x)$ since $D^s(\Phi, x), D^{as}(\Phi, x)$ are globally Lipschitz in Φ and $A(x)$ is piecewise continuous. Hence, Φ is Lipschitz continuous in x and piecewise C^1. Also Φ_λ depends continuously on λ. Since $\Phi_0 \equiv I, \Phi_\lambda(x)$ will be strictly positive and satisfy (3.2) for some $\lambda > 0$ small enough. This finishes the construction of Φ verifying (3.2)–(3.4) and the proof of the theorem. ∎

Remark 3.2 *The case in which 0 is a discrete velocity cannot be concluded with the same procedure as in the previous theorem. Let us illustrate this for the case in which A is symmetric, positive semidefinite and singular. Then, the algebraic constraint due to the zero velocity in eq. (3.7) reads simply $\mu(\Phi) = \lambda\varphi_0$ with $\lambda > 0$ and $\varphi_0(x) > 0$. However, under our previous hypotheses on A, the* general inertia theorem *[15, Theorem 2.4.10] asserts that $\mu(\Phi) \leq 0$ for any diagonal matrix Φ. Therefore, the algebraic constraint is never satisfied. Hence, the above method does not extend to cover the case $0 \in J$.*

To finish our discussion we consider numerically a specific example including 0 as a discrete velocity. As we shall see below there is numerical evidence of exponential decay towards the unique steady state in this situation.

Example 3.3 *We consider the 3×3 system (1.1) with velocities $V = diag(-1, 0, 1)$ and interaction matrix $A = (a_{ij})$ with $a_{ij} = 1$ for any $i, j \in \{-1, 0, 1\}$ and homogeneous boundary conditions, i.e., $f_1(0) = 0$, $f_{-1}(L) = 0$. We use upwind 2-point finite differences in the x-direction and explicit Euler in time. Finally, in Figure 1 we compare the L^2-norm of the computed solution with a fitted exponential function: $g(t) = 0.4 * \exp(-0.915\,t)$. For large times ($t > 50$) $g(t)$ and $\|f(t)\|_2$ coincide almost exactly.*

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References

[1] A. Arnold, H. Lange, P.F. Zweifel, *A discrete-velocity, stationary Wigner equation*, J. Math. Phys. **41**-11, 7167–7180, 2000.

[2] A. Arnold, J.L. Lopez, P.A. Markowich, J. Soler, *An Analysis of Quantum Fokker-Planck Models: A Wigner Function Approach*, submitted to Rev. Mat. Iberoam. 2001.

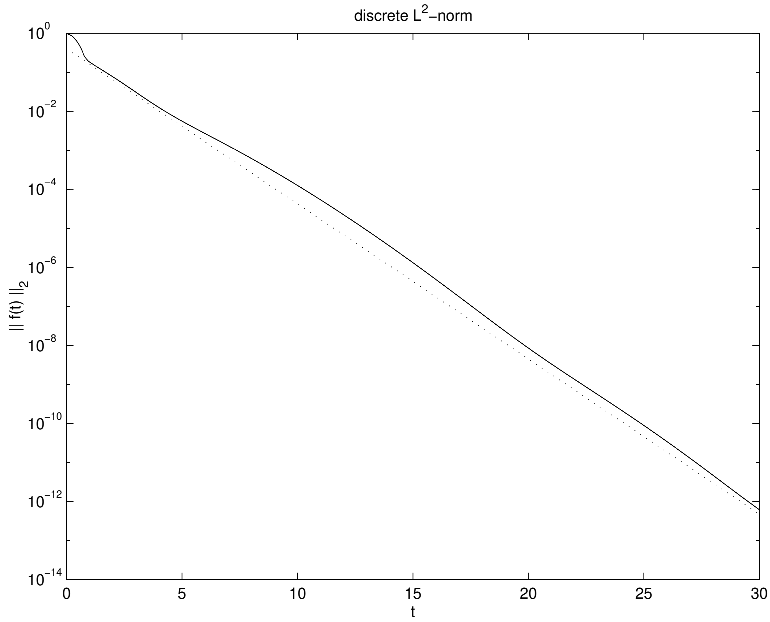

Figure 1: Time decay of the discrete L^2-norm of the solution for Example 3.3 (solid line) compared with its asymptotic decay function $g(t)$ (dotted line).

[3] A. Arnold, J.A. Carrillo, E. Dhamo, *On the periodic Wigner-Poisson-Fokker-Planck system: well-posedness and approximation by an operator splitting algorithm*, manuscript, 2001.

[4] A. Arnold, F. Nier, *Numerical Analysis of the Deterministic Particle Method Applied to the Wigner Equation*, Math. of Comp. **58**, 645–669, 1992.

[5] R. Beals, *An Abstract Treatment of Some Forward-Backward Problems of Transport and Scattering*, J. Funct. Anal. **34**, 1–20, 1979.

[6] L. L. Bonilla, J. A. Carrillo, J. Soler, *Asymptotic behavior of an initial-boundary value problem for the Vlasov-Poisson-Fokker-Planck system*, SIAM J. Appl. Math **57**, 1343-1372, 1997.

[7] F. Bouchut, J. Dolbeault, *On long asymptotics of the Vlasov-Fokker-Planck equation and of the Vlasov-Poisson-Fokker-Planck system with Coulombian and Newtonian potentials*, Diff. and Integ. Equations **8**, 487-514, 1995.

[8] K.E. Brenan, S.L. Campbell, L.R. Petzold, *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*, Classics in Applied Mathematics 14, SIAM, Philadelphia (1989)

[9] J. A. Carrillo, *Global weak solutions for the initial-boundary value problems to the Vlasov-Poisson-Fokker-Planck system*, Math. Methods Appl. Sci. **21**, 907–938, 1998.

[10] J. A. Carrillo, J. Soler, J. L. Vázquez, *Asymptotic behaviour and selfsimilarity for the three dimensional Vlasov-Poisson-Fokker-Planck system*, J. Funct. Anal. **141**, 99-132, 1996.

[11] C. Cercignani, R. Illner, M. Shinbrot, *A boundary value problem for discrete-velocity models*, Duke Math. J. **55**, 4, 889–900, 1987.

[12] M. Cessenat, *Théorèmes de trace pour des espaces de fonctions de la neutronique*, C. R. Acad. Sc. Paris, **300**, série I, n^o 3, 89–92, 1985.

[13] R. Dautray, J.L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, vol. 6, Springer Verlag, Berlin, 1993.

[14] W. Greenberg, C. van der Mee, V. Protopopescu, *Boundary Value Problems in Abstract Kinetic Theory*, Birkhäuser, Basel-Boston-Stuttgart, 1997.

[15] R.A. Horn, C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge-New York, 1991.

[16] R.A. Horn, C.R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge-New York, 1999.

[17] S. Kawashima, *Existence and Stability of Stationary Solutions to the Discrete Boltzmann Equation*, Japan J. Indust. Appl. Math. **8**, 389–429, 1991.

[18] P.A. Markowich, C. Ringhofer, C. Schmeiser, *Semiconductor Equations*, Springer, New York, 1990.

[19] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, 2nd edition, 1992.

[20] H. Risken, *The Fokker-Planck Equation*, Springer Berlin-Heidelberg, 1984.

[21] M.D. Tidriri, *Asymptotic analysis of a coupled system of kinetic equations*, C.R. Acad. Sci. Paris, **328**, Série I, 637–642, 1999.

[22] E. Wigner, *On the quantum correction for thermodynamic equilibrium*, Phys. Rev. **40**, 749–759, 1932.