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1 Introduction

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and a function $u_0 \in W^1_1(\Omega;\mathbb{R}^N)$ we consider the variational problem

$$J[w] = \int_\Omega f(\nabla w)\, \mathrm{d}x \to \min \quad \text{in } u_0 + \overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)\ , \tag{1}$$

where $f\colon \mathbb{R}^{nN} \to [0,\infty)$ is a strictly convex integrand of linear growth, i.e.

$$a|Z| - b \leq f(Z) \leq A|Z| + B \quad \text{for all } Z \in \mathbb{R}^{nN} \tag{2}$$

holds with suitable constants a, $A > 0$, b, $B \in \mathbb{R}$. Clearly (1) fails to have solutions in general, therefore we introduce the set

$$\begin{aligned} \mathcal{M} \;=\; \Big\{ & u \in \mathrm{BV}(\Omega;\mathbb{R}^N) : u \text{ is the } L^1\text{-limit of some} \\ & J\text{-minimizing sequence } \{u_k\} \subset u_0 + \overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N) \Big\} \end{aligned}$$

of generalized minimizers of problem (1). It is well known that the elements u of $\mathcal{M}$ naturally occur as minimizers of suitable relaxed versions of problem (1), precisely (see [11] or [7]) we have: let

$$\begin{aligned} K[w] \;=\; & \int_\Omega f(\nabla^a w)\, \mathrm{d}x + \int_\Omega f_\infty\Big(\frac{\nabla^s w}{|\nabla^s w|}\Big)\, \mathrm{d}|\nabla^s w| \\ & + \int_{\partial\Omega} f_\infty\big((u_0 - u)\otimes \nu\big)\, \mathrm{d}\mathcal{H}^{n-1}\ , \quad w \in \mathrm{BV}(\Omega;\mathbb{R}^N)\ , \end{aligned}$$

where ν is the outward unit normal to $\partial\Omega$, f_∞ is the recession function of f, and $\nabla^a w$ and $\nabla^s w$ denote the regular and singular part of ∇w w.r.t. the Lebesgue measure, respectively. Then

- $K[w] \to \min$ in $BV(\Omega;\mathbb{R}^N)$ has at least one solution;
- $\displaystyle\inf_{u_0 + \overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)} J = \inf_{\mathrm{BV}(\Omega;\mathbb{R}^N)} K$;
- u is K-minimizing $\Leftrightarrow u \in \mathcal{M}$.

It should be noted that there exists a formally different approach to relaxation based on the notion of a suitable Lagrangian (see [20] and [17]): let

$$\begin{aligned} & L(w,\kappa) = \int_\Omega \mathrm{div}\, \kappa \cdot (u_0 - w)\, \mathrm{d}x - \int_\Omega f^*(\kappa)\, \mathrm{d}x + \int_\Omega \kappa : \nabla u_0\, \mathrm{d}x\ , \\ & w \in \mathrm{BV}(\Omega;\mathbb{R}^N),\ \ \kappa \in \mathcal{U} := \big\{\sigma \in L^\infty(\Omega;\mathbb{R}^{nN}) :\ \mathrm{div}\, \sigma \in L^n(\Omega;\mathbb{R}^N)\big\}\ , \end{aligned}$$

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where f^* is the conjugate function of f. We introduce

$$\tilde{J}[w] = \sup_{\sigma \in \mathcal{U}} L(w, \sigma)$$

as relaxation of J to the space $BV(\Omega; \mathbb{R}^N)$, but in [7] we showed that K and $\tilde{J}$ coincide on $\mathrm{BV}(\Omega; \mathbb{R}^N)$.
In our note we like to investigate the regularity properties of generalized minimizers $u \in \mathcal{M}$. To this purpose we assume in addition to (2) that f is of class $C^2(\mathbb{R}^{nN})$. Then, according to [2], we know $\nabla u \in C^{0,\alpha}(\Omega_0; \mathbb{R}^{nN})$ for any $0 < \alpha < 1$, where Ω_0 is an open subset of Ω with full Lebesgue measure, provided we have $D^2 f(Z)(Y, Y) > 0$ for any Z, $Y \in \mathbb{R}^{nN}$, $Y \neq 0$. Another regularity result concerning the scalar case $N = 1$ is established in [10]: generalized minimizers $u \in \mathcal{M}$ are smooth in the interior of Ω if f satisfies a minimal surface type ellipticity condition. In order to get rid of this quite restrictive assumption we introduce the class of μ-elliptic, linear growth integrands.

Definition 1 *Let $f \in C^2(\mathbb{R}^{nN})$ satisfy (2) together with*

$$|\nabla f(Z)| \leq M < \infty \quad \text{for any } Z \in \mathbb{R}^{nN} . \tag{3}$$

Then, we say that f is μ-elliptic for some number $\mu > 1$ if and only if

$$\lambda(1 + |Z|^2)^{-\frac{\mu}{2}} |Y|^2 \leq D^2 f(Z)(Y, Y) \leq \Lambda(1 + |Z|^2)^{-\frac{1}{2}} |Y|^2 \tag{4}$$

holds with positive constants λ, Λ and for all Y, $Z \in \mathbb{R}^{nN}$.

Remark 1 *It is easy to show that (3) and (4) imply (2).*

In Sect. 2 we will give a list of examples satisfying (4) for any given number $\mu > 1$.
From now on we restrict ourselves to the scalar case $N = 1$, the reader will find further comments concerning vectorial problems in [3].
Let us first recall some recent results on the regularity of generalized minimizers.

Theorem 1 *([6]) Consider an integrand f as in Definition 1 and assume $\mu < 1 + 2/n$. Then we have:*

i) *$\mathcal{M} \subset C^{1,\alpha}(\Omega)$ for any $0 < \alpha < 1$.*

ii) *If u, $v \in \mathcal{M}$, then there is a real number c such that $u = v + c$.*

So the strong assumption $\mu < 1 + 2/n$ implies regularity together with uniqueness up to a constant. Unfortunately the condition on μ becomes more and more restrictive if $n \to \infty$. But we can compensate this effect by considering good boundary data.

Theorem 2 *([4]) Suppose that $u_0 \in L^\infty(\Omega)$ and let the assumptions of Theorem 1 hold for some $\mu < 3$. Then i) and ii) of Theorem 1 continue to hold.*

The next result concerns the limit case $\mu = 3$:

Theorem 3 *([4]) Suppose that we are in the situation of Theorem 2 with $\mu = 3$. Then there is a generalized minimizer $u^* \in \mathcal{M}$ such that*

i) $\nabla^s u^* = 0$.

ii) *For any $\Omega' \Subset \Omega$ we have*

$$\int_{\Omega'} |\nabla u^*| \ln^2(1 + |\nabla u^*|^2) \, \mathrm{d}x < \infty \, .$$

iii) *u^* is (up to a constant) the unique solution of the problem*

$$\int_{\Omega} f(\nabla w) \, \mathrm{d}x + \int_{\partial\Omega} f_\infty\big((u_0 - w) \otimes \nu\big) \, \mathrm{d}\mathcal{H}^{n-1} \to \min \quad \textit{in} \;\; W_1^1(\Omega) \, .$$

Note that from iii) of Theorem 3 we deduce that the relaxed problem

$$K[w] \to \min \;\; \text{in} \;\; \mathrm{BV}(\Omega)$$

admits a solution in $W_1^1(\Omega)$.
The main concern of our paper is to emphasize the role of the limit exponent $\mu = 3$. To this purpose we first extend Theorem 3 to integrands with an additional smooth x-dependence (Sect. 3), in a second step we construct an example which shows that we can not go beyond the limit case $\mu = 3$ in Theorem 4 below. This strongly indicates that Theorem 3 is sharp in the sense that also in the x-independent situation studied in Theorem 3 there exist generalized minimizers which are not of class W_1^1 if we replace the exponent $\mu = 3$ by some slightly larger number.

2 Examples of μ-elliptic integrands

i) Let us start with the most prominent example: the minimal surface integrand $f(Z) = \sqrt{1 + |Z|^2}$ satisfies Definition 1 with the limit exponent $\mu = 3$. However, there is much better information on account of the geometric structure of this example, in particular we have

$$\begin{aligned} \frac{c_1}{\sqrt{1+|Z|^2}}\left[|Y|^2 - \frac{(Y\cdot Z)^2}{1+|Z|^2}\right] &\leq D^2 f(Z)(Y,Y) \\ &\leq \frac{c_2}{\sqrt{1+|Z|^2}}\left[|Y|^2 - \frac{(Y\cdot Z)^2}{1+|Z|^2}\right] \end{aligned} \tag{5}$$

for all Z, $Y \in \mathbb{R}^n$ with some real numbers c_1, c_2.
Given an integrand satisfying this condition, Ladyzhenskaya/Ural'tseva ([13]) and Giaquinta/Modica/Souček ([10]) then use Sobolev's inequality for functions defined on minimal hypersurfaces (compare [15] and [9]) as an essential tool for proving their regularity results.

ii) We fix $\mu > 1$, let

$$\varphi(r) = \int_0^r \int_0^s (1+t^2)^{-\frac{\mu}{2}} \, \mathrm{d}t \, \mathrm{d}s \, , \quad r \in \mathbb{R}_0^+ \, ,$$

and consider $\Phi_\mu(Z) = \varphi(|Z|)$. By direct calculations it is easy to see that Φ_μ is an μ-elliptic integrand of linear growth in the sense of Definition 1, in particular (4) is satisfied with the optimal exponent μ.
If we choose $\mu = 1$, then we obtain an integrand of nearly linear growth which, at least for large $|Z|$, behaves like $|Z| \ln(1+|Z|)$.
If $\mu = 2$, then the explicit representation reads as

$$\Phi_2(Z) = |Z| \arctan |Z| - \frac{1}{2} \ln(1+|Z|^2) \, .$$

In the limit case $\mu = 3$, it is easy to perform the integrations and the minimal surface example is exactly recovered.
iii) On account of the above observation we need to give some examples with limit ellipticity $\mu = 3$, with linear growth and which are not of minimal surface structure in the sense of (5).
A somehow technical example can be constructed by considering the function Φ_μ for some fixed $1 < \mu < 3$, where we now "destroy" ellipticity by inserting "linear pieces" (compare the construction of integrands with (s, μ, q)-growth given in [8]). Then, on one hand, the inequality on the right-hand side of (5) is no longer valid. On the other hand, we have degenerate ellipticity and by adding some minimal surface part we get an "$\mu = 3$"-elliptic energy density which is not of minimal surface type.
We did this piecewise construction for the following reason: the Ansatz $f(Z) = g(|Z|)$ automatically leads (more or less) to the minimal surface structure. In particular, if $Z \perp Y \in \mathbb{R}^n$ and if f is of linear growth, hence $g'(t) \to c$ as $t \to +\infty$, then

$$D^2 f(Z)(Y,Y) = \frac{g'(|Z|)}{|Z|} |Y|^2 \approx (1+|Z|^2)^{-\frac{1}{2}} |Y|^2$$

if $|Z|$ is sufficiently large.
Nevertheless, there exists a different natural class of examples where the above structure is lost: the idea is to replace $|Z|$ by the distance to a convex set by the way obtaining a variety of interesting energy densities. Let us just sketch a very easy example in the case $n = 2$, $Z = (z_1, z_2)$. Denote by C the upper unit half disc, i.e. $C = \{Z : \, |Z| < 1, \, z_2 > 0\}$ (for the sake of simplicity we neglect a smoothing procedure at the edges). Note that the distance function $\rho(Z) := \mathrm{dist}(Z, C)$ coincides (up to the constant 1) in the upper half plane (for $|Z| > 1$) with $|Z|$. Now let

$$f(Z) = \sqrt{1 + \rho^2(Z)} \, .$$

We are mainly interested in the points $Z = (0, z_2)$, $z_2 < 0$, $|z_2| \gg 1$: it is immediately verified that in this case

$$\begin{aligned} D^2 f\big((0,z_2)\big)(e_1, e_1) &= 0 \, , \\ D^2 f\big((0,z_2)\big)(e_2, e_2) &= \big(1 + |\rho^2\big((0,z_2)\big)|\big)^{-\frac{3}{2}} = (1+|Z|^2)^{-\frac{3}{2}} \, , \end{aligned}$$

where e_i, $i = 1, 2$, denotes the i^{th} unit coordinate vector. In particular we observe that the minimal surface structure is completely destroyed on account of the degeneracy of C. This of course induces degeneracy of f as well. The first way of obtaining μ-elliptic integrands with linear growth evidently is to change the geometry in a suitable way. We prefer a simple and more anisotropic idea: let (for $|Z| > 1$)

$$\begin{aligned} \tau\big((z_1, z_2)\big) &= \frac{1}{2} z_1 + \rho\big((z_1, z_2)\big) \, , \\ \tilde{f}\big((z_1, z_2)\big) &= \sqrt{1 + \tau^2\big(z_1, z_2)\big)} \, . \end{aligned}$$

Then there is a positive constant c such that

$$c^{-1}|Z| \leq \tau\big((z_1, z_2)\big) \leq c|Z|$$

for all $|Z|$ sufficiently large, and if $z_2 < 0$, $|z_2| \gg 1$, then we obtain in both coordinate directions e_i, $i = 1, 2$, with suitable constants c_i

$$D^2 \tilde{f}\big((0, z_2)\big)(e_i, e_i) = c_i (1 + |\tau\big((0, z_2)\big)|^2)^{-\frac{3}{2}} \, .$$

Summarizing the properties of $\tilde{f}$ we see that this function is of linear growth and satisfies the μ-ellipticity condition with limit exponent $\mu = 3$. Moreover, $\tilde{f}$ does not satisfy the minimal surface ellipticity condition (5), and there is no chance to get something analogous: given the points $(0, z_2)$ as above, both eigenvalues of $D^2 \tilde{f}\big((0, z_2)\big)$ grow like $(1 + |Z|^2)^{-3/2}$.

3 Smooth x-dependence

Now we are going to prove that Theorem 3 remains valid if we admit an additional smooth x-dependence of the energy density f. This will enable us in the next section to discuss the sharpness of our results. To be precise, let us suppose that we now have

Assumption 1 *There are constants $c_1, \dots, c_7$ such that for all $x \in \overline{\Omega}$, for all P, U, $V \in \mathbb{R}^n$ and for any $\gamma = 1, \dots, n$*

i) *the variational integrand $f = f(x, P)$ is of linear growth in P, uniformly w.r.t. x, i.e.*

$$a|P| - b \leq f(x, P) \leq A|P| + B$$

holds with constants which are not depending on x;

ii) *$f(x, P)$ is of class $C^2\big(\overline{\Omega} \times \mathbb{R}^n\big)$ and any of the derivatives occurring below exist;*

iii) $\big|\nabla_P f(x, P)\big| \leq c_1$;

iv) $c_2(1+|P|^2)^{-\frac{3}{2}}|U|^2 \le D_P^2 f(x,P)(U,U) \le c_3(1+|P|^2)^{-\frac{1}{2}}|U|^2$;

v) $\big|\partial_\gamma \nabla_P f(x,P)\big| \le c_4$;

vi) $\big|\partial_\gamma\partial_\gamma \nabla_P f(x,P)\big| \le c_5$;

vii) $\big|\partial_\gamma D_P^2 f(x,P)(U,V)\big| \le c_6\big|D_P^2 f(x,P)(U,V)\big| + \dfrac{c_7}{1+|P|^2}|U||V|$.

Remark 2 *Maybe, assumption vii) needs some brief comment: if we want to include integrands of the type $f(x,P)=g(\alpha(x)P)$ with some scalar function α in our considerations, then we can not expect that $\partial_\gamma D_P^2 f$ and $D_P^2 f$ define equivalent bilinear forms on $\mathbb{R}^n$. However, the admissible pertubation on the right-hand side of vii) in particular is weak enough to be verified for the counterexample of the next section.*

The x-dependent variant of Theorem 3 then reads as

Theorem 4 *Theorem 3 remains valid also for energy densities satisfying Assumption 1.*

Proof. We follow the lines of [4] (see also [3]) and start by letting

$$J_\delta[w] := \frac{\delta}{2}\int_\Omega |\nabla w|^2 \,\mathrm{d}x + \int_\Omega f(x,\nabla w)\,\mathrm{d}x\ , \quad w \in u_0 + \overset{\circ}{W}{}_2^1(\Omega)\ , \quad \delta \in (0,1)\ .$$

Here and in the following we may assume in addition that $u_0 \in L^\infty \cap W_2^1(\Omega)$, the well known approximation procedure needed to handle the case $u_0 \in L^\infty \cap W_1^1(\Omega)$ is outlined, for instance, in [5]. Next, let u_δ denote the unique solution of the variational problem

$$J_\delta[w] \to \min \quad \text{ in } u_0 + \overset{\circ}{W}{}_2^1(\Omega)$$

and abbreviate $f_\delta(x,P) = \frac{\delta}{2}|P|^2 + f(x,P)$. Then the main properties of the regularization $\{u_\delta\}$ are summarized in

Lemma 1 **i)** *The regularizing sequence $\{u_\delta\}$ is a J-minimizing sequence from $u_0 + \overset{\circ}{W}{}_1^1(\Omega)$;*

ii) *there is a real number c, independent of δ, such that*

$$\delta \int_\Omega |\nabla u_\delta|^2 \,\mathrm{d}x \le c\ , \qquad \int_\Omega |\nabla u_\delta|\,\mathrm{d}x \le c\ ;$$

iii) *u_δ is of class $W^2_{2,loc} \cap W^1_{\infty,loc}(\Omega)$;*

iv)

$$\int_\Omega \nabla_P f_\delta(x, \nabla u_\delta) \cdot \nabla \varphi \, \mathrm{d}x = 0 \quad \textit{for all } \varphi \in C_0^\infty(\Omega) \, ;$$

v) *for all* $\varphi \in C_0^\infty(\Omega)$, $\gamma = 1, \dots, n$, *we also have*

$$\int_\Omega D_P^2 f_\delta(x, \nabla u_\delta)(\partial_\gamma \nabla u_\delta, \nabla \varphi) \, \mathrm{d}x + \int_\Omega (\partial_\gamma \nabla_P f_\delta)(x, \nabla u_\delta) \cdot \nabla \varphi \, \mathrm{d}x = 0 \, ,$$

where iii)–v) are valid for any $\delta \in (0,1)$.

Proof. For i) we observe that the additional x-dependence does not affect the corresponding arguments of [5] (compare [18] for the case of integrands depending on the modulus of the gradient). Alternatively, we can follow the reasoning outlined in [10] which is based on Reshetnyak's lower semicontinuity theorem (see [16]). Claim ii) is immediate by $J_\delta[u_\delta] \le J_\delta[u_0] \le J_1[u_0]$ and the linear growth of f, iv) is the Euler equation for u_δ which implies iii) by Theorem 5.2, Chapter 4, of [12]. With the higher integrability and differentiability given in iii), we finally may differentiate iv) to obtain v). ■

Next we state the main

Lemma 2 *Suppose that the hypotheses of Theorem 4 are valid and let* $\{u_\delta\}$ *be given as above. Then, for any domain* $\Omega' \Subset \Omega$, *there is a real number* $c(\Omega')$ *– independent of* δ *– such that*

$$\int_{\Omega'} |\nabla u_\delta| \ln^2(1 + |\nabla u_\delta|^2) \, \mathrm{d}x \le c(\Omega') < \infty \, .$$

Proof. Let us abbreviate $\Gamma_\delta = 1 + |\nabla u_\delta|^2$, $\omega_\delta = \ln(\Gamma_\delta)$ and fix some ball $B_{2r}(x_0) \Subset \Omega$. Given $\eta \in C_0^\infty(B_{2r}(x_0))$, $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_r(x_0)$, we conclude from Lemma 1, iii), and from standard density arguments that $\varphi = u_\delta \omega_\delta^2 \eta^2$ is an admissible choice in Lemma 1, iv), hence

$$\begin{aligned}
&\int_{B_{2r}(x_0)} \nabla_P f(x, \nabla u_\delta) \cdot \nabla u_\delta \omega_\delta^2 \eta^2 \, \mathrm{d}x + \delta \int_{B_{2r}(x_0)} |\nabla u_\delta|^2 \omega_\delta^2 \eta^2 \, \mathrm{d}x \\
&= \; - \int_{B_{2r}(x_0)} \nabla_P f(x, \nabla u_\delta) \cdot u_\delta \big[\nabla \omega_\delta^2 \eta^2 + \nabla \eta^2 \omega_\delta^2 \big] \, \mathrm{d}x \\
&\quad\; - \delta \int_{B_{2r}(x_0)} \nabla u_\delta \cdot u_\delta \big[\nabla \omega_\delta^2 \eta^2 + \nabla \eta^2 \omega_\delta^2 \big] \, \mathrm{d}x \, . \qquad (6)
\end{aligned}$$

Note that the equations iv) and v) of Lemma 1 remain unchanged if we replace $f(x,P)$ by $f_{x_0} := f(x,P) - \nabla_P f(x_0, 0) \cdot P$. Moreover, without changing the constants c_2–c_7, the inequalities iv)–vii) of Assumption 1 are valid with f replaced by f_{x_0}. Finally, we find a positive number c_1 such that iii) of Assumption 1 holds uniformly w.r.t. x_0 for any f_{x_0} as above. As a consequence, we may assume w.l.o.g. that $\nabla_P f(x_0, 0) = 0$. This implies by Assumption 1, iv),

$$\nabla_P f(x_0, P) \cdot P = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta} \nabla_P f(x_0, \theta P) \cdot P \, \mathrm{d}\theta \ge c_2 |P| \int_0^{|P|} (1 + \rho^2)^{-\frac{3}{2}} \, \mathrm{d}\rho \, ,$$

thus, recalling Assumption 1, v), we may choose r sufficiently small such that for all $x \in B_{2r}(x_0)$

$$\nabla_P f(x,P) \cdot P \geq c'(1+|P|^2)^{\frac{1}{2}} - c'' \tag{7}$$

holds for some real numbers c', c'' which are not depending on x. Inequality (7) implies that the left-hand side of (6) is greater than or equal to

$$\int_{B_{2r}(x_0)} \Big[c' \Gamma_\delta^{\frac{1}{2}} \omega_\delta^2 \eta^2 - c'' \omega_\delta^2 \eta^2\Big] \, \mathrm{d}x + \delta \int_{B_{2r}(x_0)} |\nabla u_\delta|^2 \omega_\delta^2 \eta^2 \, \mathrm{d}x \, . \tag{8}$$

Since $|\nabla f|$ and $|u_\delta|$ are bounded, we find the following estimate for the right-hand side of (6) (using Young's inequality with $\varepsilon > 0$ fixed)

$$\begin{aligned} \text{r.h.s} \;\; \leq \;\; & c \int_{B_{2r}(x_0)} \eta^2 \Big[\varepsilon \Gamma_\delta^{\frac{1}{2}} \omega_\delta^2 + \varepsilon^{-1} \Gamma_\delta^{-\frac{1}{2}} |\nabla \omega_\delta|^2\Big] \, \mathrm{d}x \\ & + c(r) \int_{B_{2r}(x_0)} \omega_\delta^2 \, \mathrm{d}x \\ & + c\delta \int_{B_{2r}(x_0)} \eta^2 \Big[\varepsilon |\nabla u_\delta|^2 \omega_\delta^2 + \varepsilon^{-1} |\nabla \omega_\delta|^2\Big] \, \mathrm{d}x \\ & + c(r)\delta \int_{B_{2r}(x_0)} |\nabla u_\delta| \omega_\delta^2 \, \mathrm{d}x \, . \end{aligned} \tag{9}$$

Clearly $\int_{B_{2r}(x_0)} \omega_\delta^2 \, \mathrm{d}x$ and $\delta \int_{B_{2r}(x_0)} |\nabla u_\delta| \omega_\delta^2 \, \mathrm{d}x$ are uniformly bounded with respect to δ (compare Lemma 1, ii)). Hence (6), (8) and (9) imply after absorbing terms (for ε sufficiently small)

$$\begin{aligned} \int_{B_r(x_0)} \Gamma_\delta^{\frac{1}{2}} \omega_\delta^2 \, \mathrm{d}x \;\; \leq \;\; & c \Big[1 + \int_{B_{2r}(x_0)} \Gamma_\delta^{-\frac{1}{2}} |\nabla \omega_\delta|^2 \eta^2 \, \mathrm{d}x \\ & + \delta \int_{B_{2r}(x_0)} |\nabla \omega_\delta|^2 \eta^2 \, \mathrm{d}x\Big] \, . \end{aligned} \tag{10}$$

Given (10), we observe that a.e.

$$|\nabla \omega_\delta|^2 \leq c \frac{1}{1+|\nabla u_\delta|^2} |\nabla^2 u_\delta|^2 \, ,$$

thus we may use Assumption 1, *iv*), with the result

$$\begin{aligned} \int_{B_r(x_0)} \Gamma_\delta^{\frac{1}{2}} \omega_\delta^2 \, \mathrm{d}x \;\; \leq \;\; & c \Big[1 + c \int_{B_{2r}(x_0)} \Big(\Gamma_\delta^{-\frac{1}{2}} + \delta\Big) \Gamma_\delta^{-1} |\nabla^2 u_\delta|^2 \eta^2 \, \mathrm{d}x\Big] \\ \leq \;\; & c \Big[1 + c \int_{B_{2r}(x_0)} D^2 f_\delta(\nabla u_\delta) \big(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta\big) \eta^2 \, \mathrm{d}x\Big] \, . \end{aligned} \tag{11}$$

Here and in the following we always take the sum w.r.t. repeated Greek indices $\gamma = 1, \dots, n$. Now, Lemma 2 is proved once we have found a uniform bound for the right-hand side of (11). To this purpose we observe that the starting integrability of u_δ is good enough (recall Lemma 1, iii)) to take $\varphi = \eta^2 \partial_\gamma u_\delta$, η as above, as an admissible test function in the differentiated Euler equation v) of Lemma 1 (of course we again need some standard density argument). As a result we obtain

$$\begin{aligned}
&\int_{B_{2r}(x_0)} D_P^2 f_\delta(x, \nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\eta^2 \,\mathrm{d}x \\
= \;& -2\int_{B_{2r}(x_0)} D_P^2 f_\delta(x, \nabla u_\delta)(\partial_\gamma \nabla u_\delta, \nabla \eta)\eta \partial_\gamma u_\delta \,\mathrm{d}x \\
& -2\int_{B_{2r}(x_0)} (\partial_\gamma \nabla_P f_\delta)(x, \nabla u_\delta) \cdot \partial_\gamma u_\delta \nabla \eta \eta \,\mathrm{d}x \\
& -\int_{B_{2r}(x_0)} (\partial_\gamma \nabla_P f_\delta)(x, \nabla u_\delta) \cdot \partial_\gamma \nabla u_\delta \eta^2 \,\mathrm{d}x \\
=: \;& I + II + III \, .
\end{aligned} \tag{12}$$

By Assumption 1, v), we have

$$|II| \leq c(\eta) \int_{B_{2r}(x_0)} |\nabla u_\delta| \,\mathrm{d}x \leq c \, ,$$

whereas the first integral on the right-hand side of (12) is handled with Young's inequality for $\varepsilon > 0$ sufficiently small

$$\begin{aligned}
|I| \;\leq\; & \varepsilon \int_{B_{2r}(x_0)} D_P^2 f_\delta(x, \nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\eta^2 \,\mathrm{d}x \\
& + c\varepsilon^{-1} \int_{B_{2r}(x_0)} |D_P^2 f_\delta(x, \nabla u_\delta)||\nabla \eta|^2 |\nabla u_\delta|^2 \,\mathrm{d}x \, .
\end{aligned}$$

Here the second integral on the right-hand side is uniformly bounded, the first one can be absorbed on the left-hand side of (12), hence it remains to find an upper bound for III. We perform a partial integration to obtain

$$\begin{aligned}
III \;=\; & \int_{B_{2r}(x_0)} (\partial_\gamma \partial_\gamma \nabla_P f_\delta)(x, \nabla u_\delta) \cdot \nabla u_\delta \eta^2 \,\mathrm{d}x \\
& + \int_{B_{2r}(x_0)} (\partial_\gamma D_P^2 f_\delta)(x, \nabla u_\delta)(\partial_\gamma \nabla u_\delta, \nabla u_\delta)\eta^2 \,\mathrm{d}x \\
& + \int_{B_{2r}(x_0)} (\partial_\gamma \nabla_P f_\delta)(x, \nabla u_\delta) \cdot \nabla u_\delta \partial_\gamma \eta^2 \,\mathrm{d}x \\
=: \;& III_1 + III_2 + III_3 \, .
\end{aligned}$$

Assumption 1, vi), shows that $|III_1|$ is bounded independent of δ, the uniform estimate for $|III_3|$ again follows from v) of Assumption 1. Finally, for the consideration of $|III_2|$ we make use of Assumption 1, vii), which, together with Young's inequality, gives for $\varepsilon > 0$

(note that the γ-derivative of the δ-part vanishes)

$$\begin{aligned}
|III_2| &\le c\int_{B_{2r}(x_0)} |D_P^2 f(x,\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \nabla u_\delta)|\eta^2 \,\mathrm{d}x \\
&\quad + c\int_{B_{2r}(x_0)} (1+|\nabla u_\delta|^2)^{-1}|\nabla^2 u_\delta||\nabla u_\delta|\eta^2 \,\mathrm{d}x \\
&\le c\varepsilon \int_{B_{2r}(x_0)} D_P^2 f(x,\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\eta^2 \,\mathrm{d}x \\
&\quad + c\varepsilon^{-1}\int_{B_{2r}(x_0)} D_P^2 f(x,\nabla u_\delta)(\nabla u_\delta, \nabla u_\delta)\eta^2 \,\mathrm{d}x \\
&\quad + c\varepsilon \int_{B_{2r}(x_0)} (1+|\nabla u_\delta|^2)^{-\frac{3}{2}}|\nabla^2 u_\delta|^2\eta^2 \,\mathrm{d}x \\
&\quad + c\varepsilon^{-1}\int_{B_{2r}(x_0)} (1+|\nabla u_\delta|^2)^{-\frac{1}{2}}|\nabla u_\delta|^2\eta^2 \,\mathrm{d}x \,. \qquad (13)
\end{aligned}$$

Note that, on account of the ellipticity Assumption 1, *iv*), the third integral on the right-hand side is estimated by the first one which in turn is absorbed on the left-hand side of (12). The remaining two integrals on the right-hand side of (13) are handled with the linear growth of f and Lemma 2 is proved. ■

Since the regularization $\{u_\delta\}$ is a J-minimizing sequence (recall Lemma 1, i)), each L^1-cluster point u^* is a generalized minimizer which satisfies on account of Lemma 2 the higher integrability claimed in Theorem 4. Once higher integrability of the gradient is shown, the last assertion (concerning uniqueness) can be taken from [7] and the proof of Theorem 4 is complete. ■

4 A counterexample to Theorem 4 in case $\mu > 3$

We proceed with an example on the sharpness of Theorem 4. The idea originates from [10], Example 3.2, where the authors restrict themselves to the one-dimensional situation. We follow the proposal of Giaquinta, Modica and Souček and give a rigorous proof that the arguments extend to higher-dimensional annuli Ω. What is more, the example given in [10] is degenerated which is not the case in the modification outlined below. As a consequence, we precisely can verify the assumptions of Sect. 3 with the exception that we now have $\mu > 3$.

The general setting is the following: let $n = 2$ and $|x| = \sqrt{x_1^2 + x_2^2} = r$. We fix some positive numbers $0 < \rho_1 < \rho_2$, $\rho := (\rho_2 - \rho_1)/2$ and choose

$$\Omega := \left\{x \in \mathbb{R}^2 : \rho_1 < r < \rho_2\right\} .$$

Moreover, α: $\Omega \to \mathbb{R}$ is defined by

$$\alpha(r) := 1 + \gamma|r-\rho|^2 ,$$

where the positive parameter γ is chosen later on (see (18)). If $k > 2$ is fixed, then the energy density under consideration reads as

$$f(x,P) = f(r,P) = \begin{cases} (1+\alpha(r)|P|^k)^{\frac{1}{k}} & \text{if } |P| > \varepsilon\,, \\ h(r,P) & \text{if } |P| \le \varepsilon\,. \end{cases}$$

Here $h(r,P)$ is chosen such that $f(x,P)$ is strictly convex, non-degenerate in P and such that $f(x,P)$ of class $C^2(\overline{\Omega}\times\mathbb{R}^2)$. For an explicit construction we may consider

$$\tilde{f}(P) = \begin{cases} (1+|P|^k)^{\frac{1}{k}} & \text{if } |P| > \varepsilon\,, \\ \tilde{h}(P) & \text{if } |P| \le \varepsilon\,, \end{cases}$$

together with the Ansatz

$$\tilde{h}(P) = a + b(1+|P|^l)^{\frac{1}{l}} + c|P|^2\,,$$

where a, b and c are suitable constants and $l > k$. The requirement that $\tilde{f}$ is of class $C^2(\mathbb{R}^2)$ in particular implies

$$\begin{aligned} b &= (1+|\varepsilon|^k)^{\frac{1}{k}-2}(1+|\varepsilon|^l)^{2-\frac{1}{l}}|\varepsilon|^{k-l}\frac{k-2-|\varepsilon|^k}{l-2-|\varepsilon|^l} > 0\,, \\ c &= \frac{1}{2}|\varepsilon|^{k-2}(1+|\varepsilon|^k)^{\frac{1}{k}-2}\left[(1+|\varepsilon|^k) - (1+|\varepsilon|^l)\frac{k-2-|\varepsilon|^k}{k-2-|\varepsilon|^l}\right] > 0\,. \end{aligned}$$

We then let $h(r,P) := \tilde{h}([\alpha(r)]^{1/k}P)$. Finally, the choice of the second parameter $0 < \varepsilon < 1$ will be made in inequality (26) below.

Theorem 5 *With the above notation, the variational problem*

$$J[w] := \int_\Omega f(x,\nabla w)\,\mathrm{d}x \to \min \qquad \textit{in } u_0 + \overset{\circ}{W}{}^1_1(\Omega)$$

does not admit a generalized minimizer $v \in \mathcal{M}$ of class $W^1_1(\Omega)$ if u_0 is supposed to satisfy $u_0(\rho_1) = -a$ and $u_0(\rho_2) = a$ for a constant $a > 0$ sufficiently large (see (29)). Here and in the following – with a slight abuse of notation – we write $u(x_1,x_2) = u(r)$ whenever u is merely depending on $|x|$.

Remark 3 **i)** *Note that the ellipticity exponent of f is given by $\mu = k+1 > 3$, hence we really obtain an example on the sharpness of our results.*

ii) *Moreover, it should be emphasized that the boundary values u_0 may be chosen as a function of class $C^\infty(\overline{\Omega})$.*

Proof. Assume by contradiction that $v \in W^1_1(\Omega)$ is a generalized minimizer. Then the proof of Theorem 5 splits into three steps.

Step 1. First of all we note that by the symmetry of the problem and with the obvious meaning of notation (after introducing polar coordinates) we have

$$v(r,\varphi) = v(r)\ . \tag{14}$$

In fact, consider the regularization $\{u_\delta\}$ of Sect. 3 which clearly satisfies $u_\delta(r,\varphi) = u_\delta(r)$ since for any real number φ_0 the function $u_\delta(r,\varphi+\varphi_0)$ is J_δ-minimizing with respect to the boundary values u_0 as well. Hence, uniqueness of minimizers proves the claim for u_δ. Next, consider a L^1-cluster point $u^* \in \mathcal{M}$ of the sequence $\{u_\delta\}$, in particular we have $u^* = u^*(r)$. It was already mentioned above that there is an open set Ω_0 of full measure such that $u^* \in C^{1,\alpha}(\Omega_0)$. As an immediate consequence (see [5]) we obtain

$$\sigma = \nabla f(\nabla u^*) \quad \text{in } \Omega_0\ .$$

Moreover, following an idea of [19], a minimax inequality is proved in [7] (compare [3] for some additional details) which implies

$$\sigma = \nabla f(\nabla w) \quad \text{in } \Omega_0$$

for any generalized minimizer $w \in \mathcal{M}$, hence

$$\nabla u^* = \nabla^a u^* = \nabla w = \nabla^a w \quad \text{in } \Omega_0\ .$$

This, together with $u^* = u^*(r)$ and $v \in W^1_1(\Omega)$, immediately gives assertion (14).
Step 2. We next claim that v takes the boundary data u_0 in the sense that the trace of v on $\partial\Omega$ is just u_0, i.e.

$$v(\rho_1) = -a \quad \text{and} \quad v(\rho_2) = a\ . \tag{15}$$

In order to prove (15) we consider the comparison function

$$w(r) = \begin{cases} v(r) - v(\rho_1) - a & \rho_1 < r < \rho\ , \\ v(r) - v(\rho_2) + a & \rho \le r < \rho_2\ , \end{cases}$$

and assume by contradiction that (15) fails to be true. If we observe that

$$f_\infty(r,P) = \alpha^{\frac{1}{k}}(r)|P|\ , \quad \rho_1 < r < \rho_2\ , \quad P \in \mathbb{R}^2\ ,$$

then we obtain

$$K[w] = \int_\Omega f(r,\nabla^a w)\,\mathrm{d}x + \int_{\partial B_\rho(0)} d|\nabla^s w|\ . \tag{16}$$

Here we used the fact that $\nabla^s w$ is supported on $\partial B_\rho(0)$ and that w takes its boundary data u_0 on $\partial\Omega$. From [1], Theorem 3.77, p. 171, one gets

$$\nabla^s w \llcorner \partial B_\rho(0) = \big(v(\rho_1) - v(\rho_2) + 2a\big)\frac{x}{|x|}\, d\mathcal{H}^1\ .$$

Thus, (16) may be rewritten as

$$\begin{aligned} K[w] &= \int_\Omega f(r,\nabla^a w)\,\mathrm{d}x + 2\pi\rho\big|v(\rho_1) - v(\rho_2) + 2a\big| \\ &\le \int_\Omega f(r,\nabla v)\,\mathrm{d}x + 2\pi\rho\Big(\big|v(\rho_1)+a\big| + \big|v(\rho_2)-a\big|\Big)\ . \end{aligned} \tag{17}$$

Now choose γ sufficiently large such that for $i = 1, 2$

$$\frac{\rho}{\rho_i \alpha^{\frac{1}{k}}(\rho_i)} < 1 \ . \tag{18}$$

Then we obtain

$$K[w] < \int_\Omega f(r, \nabla v)\, \mathrm{d}x + \sum_{i=1}^{2} \alpha^{\frac{1}{k}}(\rho_i) \int_{\partial B_{\rho_i}(0)} |u_0 - v(\rho_i)|\, d\mathcal{H}^1 = K[v] \ ,$$

hence the desired contradiction since the characterization of K as stated in the introduction remains valid with an additional smooth x-dependence.
Step 3. Now we make use of the Euler equation for the generalized minimizer v which takes the standard form since v is assumed to be of class $W_1^1(\Omega)$, i.e. we have

$$\int_\Omega \nabla_P f(r, \nabla v) \cdot \nabla \psi \, \mathrm{d}x = 0 \quad \text{for all } \ \psi \in C_0^1(\Omega) \ . \tag{19}$$

In particular, this is true for test functions $\psi = \psi(r) \in C_0^1(\rho_1, \rho_2))$. In the following the derivative w.r.t. r is denoted by " $\dot{}$ ". Then, again using polar coordinates, $\nabla v = (\cos\varphi \dot{v}, \sin\varphi \dot{v})$ and with the notation $\nabla_P f(r, \nabla v) = g(r, |\dot{v}|)\nabla v$ we obtain from (19)

$$\int_{\rho_1}^{\rho_2} \int_0^{2\pi} g(r, |\dot{v}|)\dot{v}\dot{\psi} r \, \mathrm{d}r \, \mathrm{d}\varphi = 0 \quad \text{for all } \ \psi = \psi(r) \in C_0^1((\rho_1, \rho_2)) \ .$$

As a consequence, there is a real number $\lambda \in \mathbb{R}$ such that for all $r \in (\rho_1, \rho_2)$

$$g(r, |\dot{v}|)\dot{v} r = \lambda \ . \tag{20}$$

With the representation

$$g(r, |\dot{v}|) = \left\{ \begin{array}{rcl} \big(1 + \alpha(r)|\dot{v}|^k\big)^{\frac{1}{k}-1} \alpha(r) |\dot{v}|^{k-2} & \text{if} & |\dot{v}| > \varepsilon \ , \\ b\big(1 + \alpha(r)|\dot{v}|^l\big)^{\frac{1}{l}-1} \alpha(r) |\dot{v}|^{l-2} + 2c[\alpha(r)]^{\frac{2}{k}} & \text{if} & |\dot{v}| < \varepsilon \end{array} \right.$$

we have to distinguish two cases.
Case 1. If $|\dot{v}| < \varepsilon$, then using the formulas for b and c we immediately see that $g(r, |\dot{v}|) \leq c|\varepsilon|^{k-2}$, in particular

$$\varepsilon > |\dot{v}| = \frac{|\lambda|}{g(r, |\dot{v}|) r} \geq c|\lambda| \tag{21}$$

for some positive constant c.
Case 2. If $|\dot{v}| > \varepsilon$, then (20) implies by elementary calculations (note that $|\lambda| > 0$ in the case at hand)

$$|\dot{v}|^{-k} \alpha^{-\frac{k}{k-1}} + \alpha^{-\frac{1}{k-1}} = \big(|\lambda|/r\big)^{-\frac{k}{k-1}} \ . \tag{22}$$

Observe that, as a consequence of (22),

$$\big(|\lambda|/r\big)^k \leq \alpha(r) \ . \tag{23}$$

Now, again with some simple computations, (22) gives

$$|\dot v| = \frac{\big(|\lambda|/r\big)^{\frac{1}{k-1}}}{\alpha^{\frac{1}{k}}\Big(\alpha^{\frac{1}{k-1}} - \big(|\lambda|/r\big)^{\frac{k}{k-1}}\Big)^{\frac{1}{k}}} \, . \tag{24}$$

Summarizing both cases we have the formulas (21) and (24), respectively, for $|\dot v|$.
We then choose $\lambda_0 > 0$ sufficiently small such that (24) (which is independent of the parameter $\varepsilon > 0$) implies $|\dot v| \le 1$ and assume that $|\lambda| < \lambda_0$. Then, by (21) and (24) we see that $|\dot v| \le 1$ for all $r \in (\rho_1, \rho_2)$. On the other hand, v takes its boundary data and $v(r)$ is of class $W^1_1((\rho_1, \rho_2))$, hence $v(r)$ is an absolutely continuous function and we may write

$$a = \frac{1}{2}|v(\rho_1) - v(\rho_2)| \le \frac{1}{2}\int_{\rho_1}^{\rho_2} |\dot v(r)| \, \mathrm{d}r \le \frac{\rho_2 - \rho_1}{2} \, . \tag{25}$$

This gives a contradiction if a is sufficiently large and we may assume $|\lambda| \ge \lambda_0$ which was chosen independent of ε. Hence, if Case 1 holds true, then (21) yields

$$\varepsilon \ge c|\lambda_0| \, , \tag{26}$$

and we choose ε sufficiently small such that this is not possible. Once it is established that Case 2 holds for all $r \in (\rho_1, \rho_2)$, we obtain from (23)

$$|\lambda| \le \inf_{r\in(\rho_1,\rho_2)} \alpha^{\frac{1}{k}}(r) r \, . \tag{27}$$

Moreover, (24) gives the right representation and using (23), (27), $\alpha \ge 1$ and $k > 2$ we estimate

$$\begin{aligned} |\dot v| \;&\le\; \frac{\alpha^{\frac{1}{k(k-1)}}}{\alpha^{\frac{1}{k}}\Big(\alpha^{\frac{1}{k-1}} - \big(|\lambda|/r\big)^{\frac{k}{k-1}}\Big)^{\frac{1}{k}}} \le \frac{\alpha^{\frac{2-k}{k(k-1)}}}{\Big(\alpha^{\frac{1}{k-1}} - \big(|\lambda|/r\big)^{\frac{k}{k-1}}\Big)^{\frac{1}{k}}} \\ &\le\; \left[\alpha^{\frac{1}{k-1}} - \Big(r^{-1} \inf_{r\in(\rho_1,\rho_2)} [\alpha^{1/k}(r) r]\Big)^{\frac{k}{k-1}}\right]^{-\frac{1}{k}} =: \big[h(r)\big]^{-\frac{1}{k}} \, . \end{aligned} \tag{28}$$

Here we first note that $h(r)$ is independent of λ, in particular $h(r)$ does not depend on the boundary values u_0 given in terms of a. Moreover, $h(r) \ge 0$ is evident by definition. Finally, the zeros of $h(r)$ are of finite number and simultaneously (by (18) interior) minima of $h(r)$. Thus, with Taylor's formula we see that

$$h(r) \approx c(r - r_0)^2 \quad \text{near the zeros } r_0 \text{ of } h(r) \, ,$$

and that we may choose $a < \infty$ such that

$$\int_{\rho_1}^{\rho_2} \big[h(r)\big]^{-\frac{1}{k}} \, \mathrm{d}r < a \, . \tag{29}$$

This proves the theorem since (28) and (29) contradict (25). ∎

Remark 4 *Let us again concentrate on the regularizing sequence $\{u_\delta\}$ with L^1-cluster point u^* as studied in Sect. 3. Then it is not difficult to locate* $\mathrm{spt}\nabla^s u^*$ *in the situation at hand. To this purpose denote by $\rho_{0,i}$, $i = 1, \dots, M$, the minima (of finite number, lying in the interior of (ρ_1, ρ_2) by (18)) of the function $\alpha^{\frac{1}{k}}(r)r$ on (ρ_1, ρ_2). We then have*

$$\mathrm{spt}\, \nabla^s u^* \subset \bigcup_{i=1}^{M} \partial B_{\rho_{0,i}}(0) \, .$$

In fact, the sequence of radially symmetric functions $\{u_\delta\} = \{u_\delta(r)\}$ yields a minimizing sequence of the one-dimensional energy ($\Omega = I = (\rho_1, \rho_2)$)

$$J_I[w] := \int_I f(r, |\dot{w}(r)|) r \, \mathrm{d}r$$

with respect to $W^1_1(I)$-comparison functions $w(\rho_1) = -a$, $w(\rho_2) = a$. In this sense, u^* provides a generalized J_I-minimizer. Now we again extend the ideas of [10] and let $\dot{u}^*_a$, $\dot{u}^*_s$ denote the Lebesgue-decomposition of $\dot{u}^*$ in absolutely continuous and singular part, respectively. Moreover, Corollary 3.33, [1], p. 140, on the decomposition of functions of bounded variation defined on intervalls allows us to choose $\tilde{v}(r) \in W^1_1(I)$ such that for almost all $r \in I$

$$\dot{\tilde{v}}(r) = \dot{u}^*_a(r) \quad \text{and} \quad \tilde{v}(\rho_1) = u^*(\rho_1) \, .$$

Next let $v(r)$ differ from $\tilde{v}(r)$ just by additional jumps at the points $\rho_{0,i}$ such that

$$\dot{v}_s(r) = \frac{1}{M} \sum_{i=1}^{M} \delta_{\rho_{0,i}} \int_I \dot{u}^*_s \, ,$$

where $\delta_{\rho_{0,i}}$ denotes the Dirac-measure centered at $\rho_{0,i}$, $i = 1, \dots, M$. Note that

$$\begin{aligned} \int_I \dot{u}^*_a(t) \, \mathrm{d}t + \int_I \dot{u}^*_s &= u^*(\rho_2) - u^*(\rho_1) \, , \\ \int_I \dot{v}_a(t) \, \mathrm{d}t + \left(\int_I \dot{u}^*_s \right) \frac{1}{M} \sum_{i=1}^{M} \int_I \delta_{\rho_{0,i}} &= v(\rho_2) - v(\rho_1) \end{aligned}$$

also implies $v(\rho_2) = u^*(\rho_2)$. Thus we obtain

$$\begin{aligned} &\int_I f(r, |\dot{u}^*_a(r)|) r \, \mathrm{d}r + \int_I \alpha^{\frac{1}{k}}(r) r \, \mathrm{d}|\dot{u}^*_s|(r) \\ &\leq \int_I f(r, |\dot{v}_a(r)|) r \, \mathrm{d}r + \int_I \alpha^{\frac{1}{k}}(r) r \, \mathrm{d}|\dot{v}_s|(r) \\ &\leq \int_I f(r, |\dot{u}^*_a(r)|) r \, \mathrm{d}r + \min_{r \in (\rho_1, \rho_2)} \alpha^{\frac{1}{k}}(r) r \int_I |\dot{u}^*_s| \leq K[u^*] \end{aligned}$$

and our claim is proved. ■

Remark 5 *Although u^* as discussed above is not of class $W_1^1(\Omega)$ and although we do not know whether u^* is of class $C^{2,\alpha}$ on the complement of $\mathrm{spt}\nabla^s u^*$, we might conjecture that there exist analogous examples in the case $\mu = 3$ providing W_1^1-minimizers of*

$$\int_\Omega f(\nabla w)\,\mathrm{d}x + \int_{\partial\Omega} f_\infty\big((u_0 - w)\nu\big)\,\mathrm{d}\mathcal{H}^{n-1} \to \min ,$$

which are smooth on the complement of a finite number of interior spheres. However, if solutions of this kind exist, then they are caused by the non-convexity of Ω. In fact, consider a smooth convex domain Ω, assume that $n \geq 2$, $N = 1$, and suppose that there is a $W_1^1(\Omega)$-solution which is of class $C^{2,\alpha}$ near the boundary $\partial\Omega$. Then, on account of the uniqueness of solutions (up to a constant), we apply Hilbert-Haar arguments (compare [14]) to see that the singular set is empty. In this sense, as the typical behaviour, singularities must concentrate near the boundary.

Remark 6 *In order to show rigorously that our regularity theory breaks down if $\mu > 3$, we have to ensure that the energy density f studied in Theorem 5 satisfies Assumption 1 (of course now with ellipticity exponent $\mu = k+1$). Here it is clearly sufficient to consider ($P \in \mathbb{R}^n$)*

$$\tilde{f}(P) = (1 + |P|^k)^{\frac{1}{k}} , \quad k > 2 ,$$

and to study Assumption 1 w.r.t.

$$f(x,P) = \tilde{f}(\alpha(x)P) , \quad \alpha(x) = (1 + |x|^2)^{\frac{1}{k}} ,$$

whenever $|P| > 1$ and $x \in B_1(0) \subset \mathbb{R}^n$. To this purpose we first observe that direct calculations yield in the case $|P| > 1$

$$D^2\tilde{f}(P)\cdot P = (1 + |P|^k)^{\frac{1}{k}-2}(k-1)|P|^{k-2}P \tag{30}$$

and, as a direct consequence,

$$\big|D^3\tilde{f}(P)(P,U,V)\big| \leq c\big|D^2\tilde{f}(P)(U,V)\big| + c(1+|P|^2)^{-\frac{1+k}{2}}|U||V| \tag{31}$$

for all U, $V \in \mathbb{R}^n$. For the discussion of f we just have to verify v), vi) and vii) of Assumption 1, where v) immediately follows from (30). Now note that for $1 \leq \gamma \leq n$ and $|P| > 1$

$$\begin{aligned}
\partial_\gamma\partial_\gamma\nabla_P f(x,P) &= \partial_\gamma\partial_\gamma\alpha(x)\nabla\tilde{f}(\alpha(x)P)\\
&\quad +2\partial_\gamma\alpha(x)D^2\tilde{f}(\alpha(x)P)\cdot P\partial_\gamma\alpha(x)\\
&\quad +\alpha(x)D^3\tilde{f}(\alpha(x)P)(P,P)\big[\partial_\gamma\alpha(x)\big]^2\\
&\quad +\alpha(x)D^2\tilde{f}(\alpha(x)P)\cdot P\partial_\gamma\partial_\gamma\alpha(x) =: \sum_{i=1}^4 I_i .
\end{aligned}$$

Clearly I_1 is uniformly bounded and the same follows for I_2 and I_4 from (30). I_3 is estimated with the help of (31)

$$\begin{aligned} \big|D^3\tilde{f}(\alpha(x)P)(P,P)\big| &\le \frac{1}{\alpha(x)}\big|D^3\tilde{f}(\alpha(x)P)(\alpha(x)P,P)\big| \\ &\le c\big|D^2\tilde{f}(\alpha(x)P)\big||P| + c(1+|P|^2)^{-\frac{k}{2}} \le c\,, \end{aligned}$$

hence we have vi). Finally vii) is established by observing

$$\begin{aligned} \partial_\gamma D_P^2 f(x,P)(U,V) &= 2\alpha(x)\partial_\gamma\alpha(x)D^2\tilde{f}(\alpha(x)P)(U,V) \\ &\quad +\big[\alpha(x)\big]^2 D^3\tilde{f}(\alpha(x)P)(P,U,V)\partial_\gamma\alpha(x) \end{aligned}$$

if we once more recall (31). ∎

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