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Convex Variational Integrals with a Wide Range of Anisotropy. Part II: Mixed Linear/Superlinear Growth Conditions

Michael Bildhauer

Saarland University
Department of Mathematics
Postfach 15 11 50
D–66041 Saarbrücken
Germany
E-Mail: bibi@math.uni-sb.de

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Edited by
FR 6.1 – Mathematik
Im Stadtwald
D–66041 Saarbrücken
Germany

Fax:	+ 49 681 302 4443
e-mail:	preprint@math.uni-sb.de
WWW:	http://www.math.uni-sb.de/

Abstract

We propose the study of variational integrands with mixed anisotropic linear/superlinear growth conditions, i.e. of energy densities with mixed "plastic/elastic" behavior. A class of variational problems satisfying this new kind of growth condition is introduced, and some recent regularity results (see [Bi1] and [BF6]) are applied to prove uniqueness (up to a constant) and local $C^{1,\alpha}$-regularity of generalized minimizers.

1 Introduction

Suppose that we are given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, boundary data u_0 of the Sobolev class $W^1_\infty(\Omega)$ and a smooth strictly convex energy density $f \in C^2(\mathbb{R}^n)$. Then we consider the Dirichlet problem

$$J[w] := \int_\Omega f(\nabla w)\,dx \to \min \quad \text{in } \mathbb{K}, \tag{$\mathcal{P}$}$$

where $\mathbb{K}$ denotes a suitable energy class of comparison functions respecting the boundary values u_0. Our main concern is the study of smoothness properties of (maybe generalized) weak solutions of the variational problem $(\mathcal{P})$ in the case of anisotropic growth conditions, where we always concentrate on the scalar case.

Let us start with a short discussion of energy densities with anisotropic superlinear growth conditions. As a typical example one may take ($Z = (Z_1, Z_2) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, $1 \le k < n$)

$$f(Z) = (1 + |Z|^2)^{\frac{p}{2}} + (1 + |Z_2|^2)^{\frac{q}{2}} \tag{1}$$

with exponents $2 \le p < q$. In this (even superquadratic) situation the estimate

$$\lambda(1 + |Z|^2)^{\frac{p-2}{2}}|Y|^2 \le D^2 f(Z)(Y, Y) \le \Lambda(1 + |Z|^2)^{\frac{q-2}{2}}|Y|^2 \tag{2}$$

for all Z, $Y \in \mathbb{R}^n$ and with positive constants λ, Λ is obvious. The study of variational problems with this type of growth condition became more and more popular in the last years and was forced in particular by Marcellini (compare [Ma1]-[Ma3], see [Bi1] for a detailed overview). However, it is well known that we cannot expect regular solutions if p and q differ too much (compare, for instance, [Gi] for a counterexample), and we additionally have to assume that (see also [BFM])

$$q < p\frac{n+2}{n}. \tag{3}$$

Next we focus on the energy density (1) in the subquadratic situation $1 < p < q \le 2$. Alternatively, we can consider the completely anisotropic situation in the sense that

$$f(Z) = (1 + |Z_1|^2)^{\frac{p}{2}} + (1 + |Z_2|^2)^{\frac{q}{2}}, \quad 1 < p < q \le 2. \tag{4}$$

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In both cases, as an essential difference to (2), we now merely have the estimate

$$\lambda(1+|Z|^2)^{\frac{p-2}{2}}|Y|^2 \le D^2 f(Z)(Y,Y) \le \Lambda |Y|^2 \tag{5}$$

for all Z, $Y \in \mathbb{R}^n$ and with positive constants λ, Λ. With the structure condition (5), local $C^{1,\alpha}$-regularity of solutions of problem $(\mathcal{P})$ follows from [BFM] under the requirement that (as a counterpart to (3))

$$2 < p\frac{n+2}{n}\,. \tag{6}$$

Hence, if the variational integrand (4) is given with parameters $1 < p < q \le 2$ and with p close to one, then inequality (6) fails to be true even if $q-p$ becomes very small, and in general no regularity results are available.

Let us now assume in addition that $u_0 \in L^\infty(\Omega)$. With a refined study of [Ch], it turns out (see Chapter V of [Bi1], [BF6], compare also [BF5]) that in this case the structure condition

$$\lambda(1+|Z|^2)^{-\frac{\mu}{2}}|Y|^2 \le D^2 f(Z)(Y,Y) \le \Lambda(1+|Z|^2)^{\frac{q-2}{2}}|Y|^2 \tag{7}$$

(again for all Z, $Y \in \mathbb{R}^n$ and with positive constants λ, Λ) provides regular solutions whenever $\mu \in \mathbb{R}$, $q > 1$,

$$q < 4-\mu\,, \tag{8}$$

and f in addition is bounded from below by some (in particular superlinear) N-function F. Note that (8) reduces to the trivial inequality ($q := 2$, $\mu := 2-p$) $2 < 2+p$ in the case of the anisotropic subquadratic examples discussed above.

There is a completely analogous result in the linear growth situation if we formally let $q = 1$ in (7): it is proved in [Bi2] that the estimate

$$\lambda(1+|Z|^2)^{-\frac{\mu}{2}}|Y|^2 \le D^2 f(Z)(Y,Y) \le \Lambda(1+|Z|^2)^{-\frac{1}{2}}|Y|^2 \tag{9}$$

for all Z, $Y \in \mathbb{R}^n$ and with positive constants λ, Λ, gives a generalized (unique up to a constant) $C^{1,\alpha}_{loc}$-minimizer of problem $(\mathcal{P})$, if we assume that $1 < \mu < 3$ and $u_0 \in L^\infty(\Omega)$.

Remark 1 *Despite of the corresponding results, the superlinear and the linear growth situation are essentially different:*

on one hand, problem $(\mathcal{P})$ in general fails to have solutions if f merely is of linear growth. As a consequence, one has to introduce a suitable notion of generalized minimizers or one must pass to the dual variational problem.

On the other hand, the upper bound of (9) corresponds to the linear growth of f itself, i.e. (9) (in contrast to the anisotropic examples) describes a "balanced" problem.

Following the discussion of [Bi1] and [BF4], we finally note that the results of [Bi2] at least are sharp if some additional x-dependence of the energy density is admitted, whereas this question is completely open in the superlinear case.

The above discussion shows that under the additional assumption $u_0 \in L^\infty(\Omega)$ it is possible to establish regularity results for variational problems with a wide range of anisotropy. In particular, the improvements are substantial and open up new vistas if the (upper) growth rate of the energy density f is of subquadratic order.

Now, the main idea of our paper can be summarized in a short observation, where we again assume that $u_0 \in L^\infty(\Omega)$: for the consideration of *anisotropic superlinear* examples with lower growth rate close to one, we (at least) have to take the upper exponent 2 on the right-hand side of (7) (compare the right-hand side of (5)). Then, by condition (8), μ-elliptic energy densities (in the sense that the left-hand side of (7) holds true) still are admissible if $\mu < 2$.

On the other hand, μ-elliptic variational integrands of *linear growth* are discussed in [Bi2] even up to the case $1 < \mu < 3$.

Thus, there is the hope to include variational problems with *mixed anisotropic linear/superlinear growth conditions* (see Assumption 1 below) in our studies – at least up to a certain extend.

Remark 2 *We like to emphasize that this new kind of growth condition also seems to be interesting from the viewpoint of material sciences: as an application one may think of* anisotropic materials with mixed plastic and elastic behavior. *Note that f_1 (as considered in Assumption 1 below) really is of linear growth, i.e. we do not assume a "superlinear hardening" similar to the isotropic $L \log L$ case studied in [FrS], [FS], [EM] and [MS].*

Of course we cannot expect a one-to-one correspondence to the results obtained in the anisotropic superlinear growth situation: we have already discussed this in Remark 1. Note that some of our main examples even do not admit a dual solution – this is caused by an additional anisotropic behavior of the superlinear part (compare Remark 5 and Remark 7 below).

Nevertheless, the a priori estimates given in [Bi1] and [BF6] (compare also [BF5]) are strong enough to ensure local $C^{1,\alpha}$-regularity and uniqueness (up to a constant) of generalized minimizers.

Before going into details, let us shortly sketch the line of our paper. The general hypotheses and the main result are made precise in the next section, where we also give some typical examples. Some facts on a suitable regularizing sequence are summarized in Section 3, higher integrability properties are then discussed in Section 4. This enables us to introduce a dual limit (some kind of "local stress tensor") in Section 5, and finally the proof of our main theorem is completed in Section 6.

2 General hypotheses and main results

In the following it is always supposed that we have

Assumption 1 *The energy density $f\colon \mathbb{R}^n \to \mathbb{R}$ is of class $C^2(\mathbb{R}^n)$ and admits the decomposition*

$$f(Z) = f_1(Z_1) + f_2(Z_2)\,, \qquad Z = (Z_1, Z_2) \in \mathbb{R}^k \times \mathbb{R}^{n-k}\,, \tag{10}$$

for some $k \in \mathbb{N}$, $1 \le k < n$. Here $f_1 \in C^2(\mathbb{R}^k)$ is a function of linear growth (see Remark 3, ii) below) such that

$$|\nabla f_1(Z_1)| \le A \tag{11}$$

holds for any $Z_1 \in \mathbb{R}^k$ with some constant A. The function $f_2 \in C^2(\mathbb{R}^{n-k})$ is supposed to satisfy for some $1 < p \le 2$

$$c_1 |Z_2|^p - c_2 \le f_2(Z_2)\,, \tag{12}$$

now for any $Z_2 \in \mathbb{R}^{n-k}$ *and with constants* c_1, c_2. *Our assumption on the second derivative of* $f = f_1 + f_2$ *is given by*

$$\lambda(1+|Z|^2)^{-\frac{\mu}{2}}|Y|^2 \le D^2 f(Z)(Y,Y) \le \Lambda |Y|^2 \tag{13}$$

for all Z, $Y \in \mathbb{R}^n$, *with constants* λ, Λ *and with an exponent of ellipticity*

$$1 < \mu < 2\,. \tag{14}$$

Remark 3 i) *W.l.o.g. we will assume in Sections 4–6 that* $f_i(0) = 0$, $i = 1, 2$, *and* $\nabla f(0) = 0$ *(to verify the second claim replace* f *by* $f - \nabla f(0) \cdot Z$*).*

ii) *The ellipticity condition on the left-hand side of (13) shows that* $\nabla f(Z) \cdot Z$ *is at least of linear growth, which in turn implies*

$$a|Z| - b \le f(Z) \quad \text{for all } Z \in \mathbb{R}^n$$

and with some real numbers $a > 0$, b.

iii) *From the right-hand side of (13) we see that* f *is at most of quadratic growth and therefore (compare [Da], Lemma 2.2, p. 156)*

$$|\nabla f(Z)| \le c(1+|Z|) \quad \text{for all } Z \in \mathbb{R}^n,$$

where c *is another positive constant.*

At this point we should give some examples to describe the class of energy densities which we have in mind.

Example 1 *With the notation of Assumption 1 we may take:*

i) *A linear growth integrand with* μ*-ellipticity,* $1 < \mu$, *was introduced in [BFM] (compare [BF2] and [Bi2] for a more detailed discussion). Here we suppose that* $1 < \mu < 2$ *and let*

$$\varphi(r) = \int_0^r \int_0^s (1+t^2)^{-\frac{\mu}{2}}\, dt\, ds\,, \qquad r \in \mathbb{R}_0^+\,.$$

Then we may choose $f_1(Z_1) = \varphi(|Z_1|)$.

ii) *The most elementary superlinear part is of power growth, i.e.*

$$f_2(Z_2) = (1+|Z_2|^2)^{p/2}\,, \qquad 1 < p \le 2\,.$$

iii) *Anisotropic behavior of* f_2 *itself is not excluded: if* $Z_2 = (P, Q) \in \mathbb{R}^{n-k}$, $n - k \ge 2$ *then*

$$f_2(Z_2) = (1+|P|^2)^{p/2} + (1+|Q|^2)^{q/2}\,, \qquad 1 < p < q \le 2\,,$$

is an admissible choice.

iv) *With the notation of iii), there is no need to assume the above additive decomposition of* f_2, *i.e. one may also think of*

$$f_2(Z_2) = \Big[1 + |P|^2 + (1+|Q|^2)^{\frac{q}{p}}\Big]^{\frac{p}{2}}$$

as another example.

We now come to a precise formulation of our main results: we consider the Dirichlet problem

$$J[w] := \int_\Omega f(\nabla w)\, dx \to \min \quad \text{in } u_0 + \overset{\circ}{W}{}^1_1(\Omega)\,, \tag{$\mathcal{P}$}$$

where the boundary values u_0 are supposed to be of class $W^1_\infty(\Omega)$ (compare Remark 8 below for this choice). Since f_i, $i = 1$, 2, is at least of linear growth, J-minimizing sequences are uniformly bounded in the space of functions with bounded variation. Hence, we define generalized minimizers $u^* \in BV(\Omega)$ of problem ($\mathcal{P}$) by

$$\begin{aligned} u^* \in \mathcal{M} \;:=\; & \Big\{ u \in BV(\Omega) :\, u \text{ is the } L^1\text{-limit of a } J\text{-minimizing} \\ & \quad \text{sequence from } u_0 + \overset{\circ}{W}{}^1_1(\Omega) \Big\}\,. \end{aligned}$$

Remark 4 *In the case of variational problems with linear growth, the elements of $\mathcal{M}$ are in one-to-one correspondence with the solutions of any relaxed version of problem ($\mathcal{P}$) (see [BF3]).*

Then the smoothness properties of generalized minimizers are summarized in the following main

Theorem 1 *Let f satisfy the general hypotheses, i.e. Assumption 1, and suppose that $u_0 \in W^1_\infty(\Omega)$. Then any generalized minimizer $u^* \in \mathcal{M}$ of problem ($\mathcal{P}$) is of class $C^{1,\alpha}(\Omega)$ for any $0 < \alpha < 1$. Moreover, the elements of $\mathcal{M}$ are uniquely determined up to a constant.*

Before going into the proof of Theorem 1, we recall the notions from Assumption 1, in particular the decomposition $Z = (Z_1, Z_2) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, $1 \le k < n$, and $f(Z) = f_1(Z_1) + f_2(Z_2)$, and introduce the notation

$$\nabla w = (\nabla_{x^1} w, \nabla_{x^2} w) \in \mathbb{R}^k \times \mathbb{R}^{n-k}\,,$$

for any weakly differentiable function w: $\Omega \to \mathbb{R}$, $w(x) = w(x^1, x^2)$, $x = (x^1, x^2) \in \Omega \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$. Moreover, on account of the growth rate of f, the domain $u_0 + \overset{\circ}{W}{}^1_1(\Omega)$ of the energy J is replaced in the following by the anisotropic space

$$u_0 + \overset{\circ}{W}{}^1_{1,p}(\Omega) := \Big\{ w \in u_0 + \overset{\circ}{W}{}^1_1(\Omega) :\, \nabla_{x^2} u \in L^p(\Omega; \mathbb{R}^{n-k}) \Big\}\,.$$

Finally, various facts from duality theory enter the proof of Theorem 1. We only give some short remarks on this topic – the book of Ekeland and Temam ([ET]) serves as a general reference.

Let f_i^*, $i = 1$, 2, denote the conjugate function of f_i, respectively. For instance, f_1^*: $\mathbb{R}^k \to \mathbb{R}$ is defined through the formula

$$f_1^*(Q_1) := \sup_{Z_1 \in \mathbb{R}^k} \big\{ Q_1 \cdot Z_1 - f_1(Z_1) \big\} \quad \text{for all } Q_1 \in \mathbb{R}^k\,.$$

Note that the conditions $f_i \ge 0$ and $f_i(0) = 0$, $i = 1$, 2, yield the same properties for the conjugate functions.

Following [ET] we obtain the representation formula ($p' = p/(p-1)$)

$$J[w] = \sup_{\varkappa \in L^{\infty,p'}(\Omega;\mathbb{R}^n)} \left\{ \int_\Omega \varkappa \cdot \nabla w \, dx - \int_\Omega f_1^*(\varkappa_1) \, dx - \int_\Omega f_2^*(\varkappa_2) \, dx \right\} \tag{15}$$

for any $w \in u_0 + \overset{\circ}{W}{}^1_{1,p}(\Omega)$, where for $1 \le s, t \le \infty$ the notation

$$\varkappa \in L^{s,t}(\Omega;\mathbb{R}^n) \Leftrightarrow \varkappa = (\varkappa_1, \varkappa_2) \in L^s(\Omega;\mathbb{R}^k) \times L^t(\Omega;\mathbb{R}^{n-k})$$

is used. Note that the assumptions of Proposition 2.1, [ET], p. 271, are clearly satisfied without a further specification of the upper growth rate of f_2, thus (15) really follows from our standard reference [ET] without essential modifications (see also [Bi1]). We next define the Lagrangian $l(w, \varkappa)$ for all $(w, \varkappa)$ in the class $(u_0 + \overset{\circ}{W}{}^1_{1,p}(\Omega)) \times L^{\infty,p'}(\Omega;\mathbb{R}^n)$ by the formula

$$l(w,\varkappa) := \int_\Omega \varkappa \cdot \nabla w \, dx - \sum_{i=1}^{2} \int_\Omega f_i^*(\varkappa_i) \, dx \, .$$

Remark 5 *If f_2 is of standard p growth, i.e.*

$$f_2(Z_2) = (1 + |Z_2|^2)^{\frac{p}{2}} \, , \qquad 1 < p \le 2 \, , \tag{16}$$

for all $Z_2 \in \mathbb{R}^{n-k}$, then we may follow the lines of [ET], define the dual functional R: $L^{\infty,p'}(\Omega;\mathbb{R}^n) \to \overline{\mathbb{R}}$,

$$R[\varkappa] := \inf_{w \in u_0 + \overset{\circ}{W}{}^1_{1,p}(\Omega)} l(w,\varkappa) \, ,$$

and obtain the dual variational problem

$$\text{to maximize } R \text{ among all functions } \varkappa \in L^{\infty,p'}(\Omega;\mathbb{R}^n) \, . \tag{$\mathcal{P}^*$}$$

Note that for any $\varkappa \in L^{\infty,p'}(\Omega;\mathbb{R}^n)$

$$R[\varkappa] = \left\{ \begin{array}{ll} -\infty & \text{if } \operatorname{div} \varkappa \neq 0 \, , \\ l(u_0,\varkappa) & \text{if } \operatorname{div} \varkappa = 0 \, . \end{array} \right.$$

With (16), the existence proof for a dual solution also is standard (see [ET], Theorem 4.1, p. 59), moreover we have

$$\inf_{w \in u_0 + \overset{\circ}{W}{}^1_{1,p}(\Omega)} J[w] = \sup_{\varkappa \in L^{\infty,p'}(\Omega;\mathbb{R}^n)} R[\varkappa] \, .$$

Here we do not want to restrict to the consideration of energy densities satisfying (16) (see the discussion of Example 1). Then, due to the lack of continuity of J on $W^1_{1,p}(\Omega)$, the existence Theorem 4.1 of [ET], p. 59, is no longer applicable. Nevertheless, assuming

$$\operatorname{div} \varkappa = 0 \, , \qquad \varkappa \in L^{\infty,p}(\Omega;\mathbb{R}^n) \, , \qquad f_i^*(\varkappa^i) \in L^1(\Omega) \, , \qquad i = 1,\, 2, \tag{17}$$

we again let

$$R[\varkappa] := l(u_0,\varkappa) \, .$$

Note that on account of $u_0 \in W^1_\infty(\Omega)$ and since we have (17), we do not have to suppose $\varkappa_2 \in L^{p'}(\Omega;\mathbb{R}^{n-k})$.

3 Regularization

Now we introduce a (standard) regularization of problem $(\mathcal{P})$, i.e. for $\delta \in (0,1)$ we let u_δ denote the unique solution of the variational problem

$$J_\delta[w] := \int_\Omega f_\delta(\nabla w)\, dx \to \min \quad \text{in } u_0 + \overset{\circ}{W}{}^1_2(\Omega)\,, \tag{$\mathcal{P}_\delta$}$$

where we have set

$$f_\delta(Z) := f(Z) + \frac{\delta}{2}|Z|^2\,.$$

The definition of u_δ immediately gives

Lemma 1 i) *There is a positive number c, independent of δ, such that we have for any $\delta \in (0,1)$*

$$\delta \int_\Omega |\nabla u_\delta|^2\, dx \le c\,, \qquad \int_\Omega f_i(\nabla_{x^i} u_\delta)\, dx \le c\,, \quad i = 1, 2\,.$$

ii) *For any $\delta \in (0,1)$, u_δ is a solution of*

$$\int_\Omega \nabla f_\delta(\nabla u_\delta) \cdot \nabla\varphi\, dx = 0 \quad \textit{for all } \varphi \in C^1_0(\Omega)\,,$$

hence $\operatorname{div}\sigma_\delta = 0$ *if we let*

$$\sigma_\delta := \nabla f_\delta(\nabla u_\delta)\,.$$

iii) *For any $\delta \in (0,1)$, u_δ is of class $W^2_{2,loc} \cap W^1_{\infty,loc}(\Omega)$.*

iv) *For any $\delta \in (0,1)$ and for any $\gamma = 1, \dots n$ we also have*

$$\int_\Omega D^2 f_\delta(\partial_\gamma \nabla u_\delta, \nabla\varphi)\, dx = 0 \quad \textit{for all } \varphi \in C^1_0(\Omega)\,.$$

Proof. The proof of i) is immediate by $J_\delta[u_\delta] \le J_\delta[u_0] \le J_1[u_0]$, ii) is the Euler equation for u_δ. The third claim follows from [LU], Theorem 5.2 of Chapter 4, hence we may differentiate the Euler equation with iv) as a result. ■

Remark 6 *As a main corollary of the differentiated Euler equation, two Caccioppoli-type inequalities are proved in Lemma 3 of [BF6] (compare also Lemma V.2.8 of [Bi1]), where also the case of obstacle problems is included. Here we immediately obtain the corresponding results by taking (for some cut-off function η) $\varphi = \eta^2 \partial_\gamma u_\delta (1 + |\nabla u_\delta|^2)^s$, $s \ge 0$, and $\varphi = \eta^2 \partial_\gamma u_\delta \max[(1 + |\nabla u_\delta|^2) - k, 0]$, $k > 0$, as a test-function, respectively.*

In particular, the choice $\varphi = \eta^2 \partial_\gamma u_\delta$, $\eta \in C^\infty_0(\Omega)$, $0 \le \eta \le 1$, gives

$$\int_\Omega D^2 f_\delta(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\eta^2\, dx \le c \int_\Omega |D^2 f_\delta(\nabla u_\delta)||\nabla u_\delta|^2 |\nabla\eta|^2\, dx\,, \tag{18}$$

where the constant c is independent of δ and where we always take the sum w.r.t. repeated Greek indices $\gamma = 1, \dots, n$.

4 Integrability results

In this section we are going to discuss some global and local uniform integrability results of the sequence $\{(u_\delta, \sigma_\delta)\}$ and its weak derivatives. We have

Lemma 2 i) *The sequence* $\{\sigma_\delta\}$ *is uniformly bounded in* $L^{2,p}(\Omega;\mathbb{R}^n)$.

ii) *The sequence* $\{\nabla u_\delta\}$ *is uniformly bounded in* $L^\infty_{loc}(\Omega;\mathbb{R}^n)$.

iii) *The sequence* $\{\sigma_\delta\}$ *is uniformly bounded in* $W^1_{2,loc}(\Omega;\mathbb{R}^n)$.

iv) *There is a local constant* c*, independent of* δ*, such that*

$$\|\nabla^2 u_\delta\|_{L^2(\widehat{\Omega};\mathbb{R}^{nn})} \le c(\widehat{\Omega})$$

for any domain $\widehat{\Omega} \Subset \Omega$.

Proof. ad i) We have by definition

$$\sigma_\delta = (\sigma_\delta^1, \sigma_\delta^2) = \delta(\nabla_{x^1} u_\delta, \nabla_{x^2} u_\delta) + (\nabla f_1(\nabla_{x^1} u_\delta), \nabla f_2(\nabla_{x^2} u_\delta))\,,$$

and the uniform bounds of Lemma 1, i), even give

$$\|\delta \nabla u_\delta\|_{L^2(\Omega;\mathbb{R}^n)} \to 0 \quad \text{as } \delta \to 0\,.$$

Hence, since we also have $p \le 2$, the estimates for the δ-terms are immediate.

Now the assertion for σ_δ^1 follows from (11), whereas (12) (together with i) of Lemma 1) gives a uniform bound for $\|\nabla_{x^2} u_\delta\|_{L^p(\Omega;\mathbb{R}^n)}$. As a consequence, we may apply Remark 3, iii) with the result

$$\int_\Omega \big|\nabla f_2(\nabla_{x^2} u_\delta)\big|^p \, dx \le c\bigg\{1 + \int_\Omega |\nabla_{x^2} u_\delta|^p \, dx\bigg\} \le c\,,$$

which establishes the first claim.

ad ii). Local uniform gradient bounds for a regularizing sequence are proved in Theorem 3 of [BF6] (compare Theorem V.2.10 of [Bi1]). In order to apply this theorem in the case of anisotropic linear/superlinear growth, we first observe that we again have a similar Caccioppoli-type inequality (this was already outlined in Remark 6).

Moreover, (14) gives (q=) $2 < 4 - \mu$, hence assumption (4) of [BF6] (again compare [Bi1], assumption (4) of Section V.1).

Under this assumption, uniform local higher integrability of the gradients is established in [BF6], Theorem 2 (Theorem V.2.9 of [Bi1]), where the choice $\alpha_0 = 0$ has to be admissible in order to start the iteration procedure at the end of the proof. This means that we need a uniform local bound for

$$\int (1 + |\nabla u_\delta|^2)^{\frac{2-\mu}{2}} \, dx\,,$$

which is obvious by $(2-\mu)/2 < 1/2$ (again recall (14) and the lower growth rate 1 of f).

Once uniform local higher integrability in the sense of [BF6], Theorem 2 ([Bi1], Theorem V.2.9), is established, our claim ii) exactly follows as in [BF6], Theorem 3 ([Bi1], Theorem V.2.9).

ad iii). The Cauchy-Schwarz inequality implies that a.e.

$$\begin{aligned} |\nabla\sigma_\delta|^2 &= D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \sigma_\delta) \\ &\le \Big[D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\Big]^{\frac{1}{2}} \Big[D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \sigma_\delta, \partial_\gamma \sigma_\delta)\Big]^{\frac{1}{2}}, \end{aligned}$$

and since $|D^2 f_\delta|$ is uniformly bounded there exists a constant, independent of δ, such that a.e.

$$|\nabla\sigma_\delta|^2 \le c D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\,.$$

Hence, the assertion immediately follows from ii) and (18).

ad iv) Recalling the ellipticity condition (13), we again observe on account of ii) and (18) that for any domain $\widehat{\Omega} \Subset \Omega$

$$\begin{aligned} \int_{\widehat{\Omega}} (1+|\nabla u_\delta|^2)^{-\frac{\mu}{2}} |\partial_\gamma \nabla u_\delta|^2\, dx &\le \int_{\widehat{\Omega}} D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\, dx \\ &\le c(\widehat{\Omega}) < \infty\,, \end{aligned}$$

and if we once more apply ii) on the left-hand side, then iv) is proved as well. ■

Remark 7 *Note that we have a lack of integrability in i) which is due to the anisotropic behavior of the function f_2. If we restrict ourselves to the consideration of linear/p-growth energy densities as discussed in Remark 5, then ∇f_2 has a growth rate $p-1$. Hence $\nabla f_2(\nabla_{x^2} u_\delta)$ is uniformly bounded in $L^{p'}(\Omega;\mathbb{R}^n)$, $p' = p/p-1$. Moreover, f_2^* is of power p' and $f_2^*(\nabla f_2(\nabla_{x^2} u_\delta))$ is uniformly bounded in $L^1(\Omega)$. Besides the existence of a dual solution, this essentially enlarges the class of admissible boundary values (see Remark 8 below).*

Corollary 1 i) *If u^* denotes a L^1-cluster point of the sequence $\{u_\delta\}$, then, after passing to a subsequence,*

$$\nabla u_\delta \to \nabla u^* \quad \text{a.e. in } \Omega \text{ as } \delta \to 0\,.$$

ii) *If σ denotes the corresponding $L^{2,p}$-cluster point of $\{\sigma_\delta\}$, then the limit version of the duality relation, i.e.*

$$\sigma = \nabla f(\nabla u^*)$$

holds true a.e.

iii) *In particular, σ is a mapping into the open set $U := \mathrm{Im}(\nabla f)$.*

Proof. Since $\{u_\delta\}$ is uniformly bounded in $L^\infty(\Omega)$ and on account of Lemma 2, ii), we have as $\delta \to 0$

$$u_\delta \overset{*}{\rightharpoondown} u^* \quad \text{in } W^1_{\infty,loc}(\Omega)\,, \tag{19}$$

at least for a subsequence. Moreover, from Lemma 2, ii), iv), and Sobolev's embedding theorem we deduce the existence of a subsequence such that

$$\nabla u_\delta \overset{\delta\to 0}{\longrightarrow}: w \quad \text{in } L^t_{loc}(\Omega;\mathbb{R}^n)\,, \qquad t < \frac{2n}{n-2}\,.$$

With (19) we have $w = \nabla u^*$, and passing to another subsequence, if necessary, the first claim is proved. The second one immediately follows from $\sigma_\delta = \nabla f_\delta(\nabla u_\delta)$. It remains to show that U is an open set. In fact, we have for $Z \neq Y \in \mathbb{R}^n$

$$\begin{aligned}&\big[\nabla f(Z) - \nabla f(Y)\big]\cdot(Z-Y)\\ &= \int_0^1 D^2 f(sZ+(1-s)Y)\big((Z-Y),(Z-Y)\big)\,ds \geq 0\,. \end{aligned} \tag{20}$$

Setting $g(s) := f(sZ+(1-s)Y)$, equality would give $g''(s) \equiv 0$ for all $s \in (0,1)$ which contradicts the strict convexity. Thus, the strict inequality holds in (20) and ∇f is one-to-one. Then we may apply the Theorem on Domain Invariance (compare [Sch], Corollary 3.22, p. 77) to see that U is an open set. ■

5 A dual limit

From now on a weak $L^{2,p}$-cluster point σ of the sequence $\{\sigma_\delta\}$ is fixed and (without relabelling) we always pass to subsequences if necessary. Let

$$\tau^i_\delta = \nabla f_i(\nabla_{x^i} u_\delta)\,, \qquad i = 1,\,2\,,$$

hence τ_δ is uniformly bounded in $L^{\infty,p}(\Omega;\mathbb{R}^n)$ and we may assume as $\delta \to 0$

$$\tau^1_\delta \overset{*}{\rightharpoondown} \tau^1 \quad \text{in } L^\infty(\Omega;\mathbb{R}^k)\,, \qquad \tau^2_\delta \rightharpoondown \tau^2 \quad \text{in } L^p(\Omega;\mathbb{R}^{n-k})\,.$$

Note that Corollary 1 immediately gives $\tau = \sigma$. The main properties of σ are summarized in the following lemma, where we should keep Remark 7 (compare again Remark 5) in mind.

Lemma 3 i) *The limit σ is R-admissible in the sense that* $\operatorname{div}\sigma = 0$.

ii) *σ also satisfies the remaining assumptions of (17).*

iii) $\inf\big\{J[w]:\ w \in u_0 + \overset{\circ}{W}{}^1_{1,p}(\Omega)\big\} \leq R[\sigma]$.

Proof. Passing to the limit $\delta \to 0$, i) follows from $\operatorname{div}\sigma_\delta = 0$.

ad ii). The duality relation for the conjugate function (once more see [ET]) reads as

$$\tau^i_\delta \cdot \nabla_{x^i} u_\delta - f^*_i(\tau^i_\delta) = f_i(\nabla_{x^i} u_\delta)\,, \qquad i = 1,\,2\,.$$

At this point let us shortly discuss the global starting integrabilities: since u_δ is of class $W^1_2(\Omega)$ and since ∇f is at most of linear growth (see Remark 3, iii)), τ_δ is of class $L^2(\Omega;\mathbb{R}^n)$ and the same is true for σ_δ. Moreover, $f(\nabla u_\delta)$ is of class $L^1(\Omega)$ (even uniformly) and the duality relation proves L^1-integrability (of course not uniformly) for $f_i^*(\tau_\delta^i)$, $i=1$, 2, as well. As a consequence, the expressions of inequality (21) below are well defined. We write (using the definition of σ_δ together with $\operatorname{div}\sigma_\delta=0$)

$$
\begin{aligned}
J[u_\delta] &= \sum_{i=1}^2 \int_\Omega \Big[\tau_\delta^i\cdot\nabla_{x^i}u_\delta - f_i^*(\tau_\delta^i)\Big]\,dx\\
&= -\delta\int_\Omega |\nabla u_\delta|^2\,dx + \sum_{i=1}^2\int_\Omega\Big[\sigma_\delta^i\cdot\nabla_{x^i}u_\delta - f_i^*(\tau_\delta^i)\Big]\,dx\\
&= -\delta\int_\Omega |\nabla u_\delta|^2\,dx + \sum_{i=1}^2\int_\Omega\Big[\sigma_\delta^i\cdot\nabla_{x^i}u_0 - f_i^*(\tau_\delta^i)\Big]\,dx\,. \qquad (21)
\end{aligned}
$$

Let us have a closer look at (21): obviously $J_\delta[u_\delta]+\delta\int_\Omega|\nabla u_\delta|^2\,dx$ is uniformly bounded and

$$
\left|\int_\Omega \sigma_\delta\cdot\nabla u_0\,dx\right| \le c
$$

(independent of δ) follows from Lemma 2, i), and the smoothness assumptions on u_0. Moreover, by the duality relation, by (11) and Lemma 1, i),

$$
\int_\Omega f_1^*(\tau_\delta^1)\,dx = \int_\Omega\big(\tau_\delta^1\cdot\nabla_{x^1}u_\delta - f_1(\nabla_{x^1}u_\delta)\big)\,dx \le c\,, \qquad (22)
$$

again with a constant which does not depend on δ. Thus the representation formula (21) gives a positive number c such that for any $\delta>0$

$$
\int_\Omega f_2^*(\tau_\delta^2)\,dx \le c\,. \qquad (23)
$$

Note that the integrability established in Lemma 2, i), is not sufficient to imply (23). Once (22) and (23) are established, the second claim follows form Fatou's Lemma and Corollary 1, i).

ad iii). Now that the limit σ is seen to be R-admissible, the third claim is proved exactly as in [BF1] using (21) (compare [SE1]–[SE3] for the case of integrands depending on the modulus of ∇u):

$$
\begin{aligned}
&\inf\Big\{J[w]:\ w\in u_0+\overset{\circ}{W}{}^1_{1,p}(\Omega)\Big\}\le J[u_\delta]\\
&\le\ -\delta\int_\Omega|\nabla u_\delta|^2\,dx+\sum_{i=1}^2\int_\Omega\Big[\sigma_\delta^i\cdot\nabla_{x^i}u_0 - f_i^*(\tau_\delta^i)\Big]\,dx\,.
\end{aligned}
$$

Passing to the limit $\delta\to0$ we obviously have (recall Lemma 2, i))

$$
\sum_{i=1}^2\int_\Omega\sigma_\delta^i\cdot\nabla_{x^i}u_0\,dx\to\sum_{i=1}^2\int_\Omega\sigma^i\cdot\nabla_{x^i}u_0\,dx\,,
$$

moreover,

$$
\limsup_{\delta\to0} -\sum_{i=1}^2\int_\Omega f_i^*(\tau_\delta^i)\,dx\le-\sum_{i=1}^2\liminf_{\delta\to0}\int_\Omega f_i^*(\tau_\delta^i)\,dx\,.
$$

As in ii), Fatou's lemma proves the last assertion. ■

Remark 8 *Let us shortly discuss the choice of u_0. The assumption $u_0 \in L^\infty(\Omega)$ is needed to obtain uniformly bounded solutions u_δ, hence we may apply the results of [BF6] (Chapter V of [Bi1]) and obtain uniform local a priori gradient bounds.*

If the boundary values u_0 are not supposed to be in addition of class $W^1_\infty(\Omega)$, then we may pass to an approximating sequence u_0^m and regularize w.r.t. these boundary values. With the obvious changes in notation, it is to verify in this case that ($i = 1, 2$)

$$\int_\Omega \sigma^i_{\delta(m)} \cdot \nabla_{x^i} u_0^m \, dx \to \int_\Omega \sigma^i \cdot \nabla_{x^i} u_0 \, dx \qquad \text{as } m \to \infty \, ,$$

where $\delta(m)$ is chosen sufficiently small. By assumption (11) we know that $|\nabla f_1| \leq A$, and it is sufficient to suppose $\nabla_{x^1} u_0 \in L^1(\Omega; \mathbb{R}^k)$ in order to prove the above convergence for $i = 1$. In the case $i = 2$ we recall Remark 7, hence we at least have to assume that $\nabla_{x^2} u_0 \in L^{p'}(\Omega; \mathbb{R}^{n-k})$.

Up to now it is not proved that $\{u_\delta\}$ is a J-minimizing sequence from $u_0 + \overset{\circ}{W}{}^1_{1,p}(\Omega)$. Anyhow, we have the Euler equation of Lemma 3, i), hence

Corollary 2 *The limit σ fixed as above is of class $C^{0,\alpha}(\Omega; \mathbb{R}^n)$ for any $0 < \alpha < 1$.*

Proof. With the notation of Corollary 1 we obtain

$$\int_\Omega \nabla f(\nabla u^*) \cdot \nabla \varphi \, dx = 0 \qquad \text{for all } \varphi \in C^1_0(\Omega) \, .$$

Then the standard difference quotient technique shows that we may differentiate the Euler equation (recall that u^* is Lipschitz), hence letting $v = \partial_\gamma u^*$

$$\int_\Omega D^2 f(\nabla u^*)(\nabla v, \nabla \varphi) \, dx = 0 \qquad \text{for all } \varphi \in C^1_0(\Omega) \, .$$

Since the coefficients $\frac{\partial^2 f}{\partial z_\alpha \partial z_\beta}(\nabla u^*)$ are uniformly elliptic on $\Omega' \Subset \Omega$, Theorem 8.22 of [GT] proves Hölder continuity of v, and the same is true for $\sigma = \nabla f(\nabla u^*)$. ■

6 Proof of Theorem 1

Roughly speaking it remains to derive an appropriate minimax inequality as given in [SE3] (compare [BF2]) although we do not know σ to be a solution of the dual problem.

To this purpose we fix $u \in \mathcal{M}$, i.e. we consider a J-minimizing sequence $\{u_m\}$ in $u_0 + \overset{\circ}{W}{}^1_{1,p}(\Omega)$ such that as $m \to \infty$

$$u_m \overset{L^{n/(n-1)}}{\rightharpoondown} u \, , \qquad u_m \overset{L^1}{\to} u \, .$$

For the sake of notational simplicity assume now that $\Omega = B = B_1(0)$, the general case is handled by an additional covering argument. Moreover, we still consider σ as discussed in Section 5 and let $\sigma_\rho := \sigma(\rho x)$ for $0 < \rho < 1$. Finally, for $\lambda \in C_0^\infty(\Omega;\mathbb{R}^n)$ and a real number t, $|t|$ sufficiently small, we let $\chi_\rho = \sigma_\rho + t\lambda(\rho x)$. Then χ_ρ is admissible to obtain (recalling the representation formula (15) and the definition of the Lagrangian $l(u,\varkappa)$)

$$J[u_m] = \sup_{\varkappa \in L^{\infty,p'}(\Omega;\mathbb{R}^n)} l(u_m,\varkappa) \geq l(u_m,\chi_\rho)\,, \tag{24}$$

and we may write ($\operatorname{div}\sigma_\rho = 0$)

$$l(u_m,\chi_\rho) = t\int_\Omega \operatorname{div}\lambda(\rho x)(u_0 - u_m)\,dx - \sum_{i=1}^2 \int_\Omega f_i^*(\chi_\rho^i)\,dx + \int_\Omega \chi_\rho\cdot\nabla u_0\,dx\,. \tag{25}$$

Passing to the limit $\rho \to 1$ we immediately obtain the convergence of the first integral on the right-hand side of (25). Letting $\chi = \sigma + t\lambda$, the last integral is seen to converge with limit $\int_\Omega \chi\cdot\nabla u_0\,dx$. The convergence of the remaining integrals follows from $f_i^*(\chi^i) \in L^1(\Omega)$, $i = 1$, 2 (which is proved in Lemma 3, ii)), together with a standard reasoning (compare [Al], Lemma 1.16, p. 18).

Next we combine (24), (25) and pass to the limit $m \to \infty$ with the result

$$\inf_{w\in u_0 + \overset{\circ}{W}{}^1_{1,p}(\Omega)} J[w] \geq t\int_\Omega \operatorname{div}\lambda(u_0 - u)\,dx - \sum_{i=1}^2 \int_\Omega f_i^*(\chi^i)\,dx + \int_\Omega \chi\cdot\nabla u_0\,dx\,.$$

If we additionally observe that (recall Lemma 3, iii))

$$\inf_{w\in u_0 + \overset{\circ}{W}{}^1_{1,p}(\Omega)} J[w] \leq R[\sigma] = \int_\Omega \sigma\cdot\nabla u_0\,dx - \sum_{i=1}^2 \int_\Omega f_i^*(\sigma^i)\,dx\,,$$

then we obtain the variational inequality

$$t\int_\Omega \operatorname{div}\lambda(u_0 - u)\,dx - \sum_{i=1}^2 \int_\Omega f_i^*(\chi^i)\,dx + \int_\Omega \chi\cdot\nabla u_0\,dx \leq R[\sigma]\,.$$

Inserting the definition of χ, we finally arrive at

$$-\int_{\operatorname{spt}\lambda} t(\operatorname{div}\lambda)u\,dx \leq \sum_{i=1}^2 \int_{\operatorname{spt}\lambda} \big(f_i^*(\sigma^i + t\lambda^i) - f_i^*(\sigma^i)\big)\,dx\,.$$

Hence, dividing through $t > 0$ and passing to the limit $t \to 0$ one gets

$$-\int_{\operatorname{spt}\lambda} (\operatorname{div}\lambda)u\,dx \leq \sum_{i=1}^2 \int_{\operatorname{spt}\lambda} \nabla f_i^*(\sigma^i)\cdot\lambda^i\,dx\,,$$

i.e., by definition, the first weak derivative of u is given by $(\nabla f_1^*(\sigma^1), \nabla f_2^*(\sigma^2))$. Since σ is of class $C^{0,\alpha}$ (see Corollary 2) and since σ is a mapping into the open set $\operatorname{Im}(\nabla f)$ (this was proved in Corollary 1) we obtain Hölder continuity of the derivatives of u. Moreover, each $v \in \mathcal{M}$ satisfies $\nabla v = \nabla f^*(\sigma)$ and uniqueness (up to a constant) of generalized minimizers is proved as well. ∎

Remark 9 *Note that in Lemma 3 it is actually proved that (after passing to another subsequence)*

$$\lim_{\delta \to 0} J[u_\delta] \leq R[\sigma] \,.$$

Moreover, choosing $\lambda \equiv 0$ *in the above considerations, we see that*

$$R[\sigma] \leq \inf_{w \in u_0 + \overset{\circ}{W}{}^1_{1,p}(\Omega)} J[w] \leq \lim_{\delta \to 0} J[u_\delta] \leq R[\sigma] \,,$$

thus $\{u_\delta\}$ *in fact provides a* J*-minimizing sequence.*

Remark 10 *We like to finish with the remark that our results are somewhat surprising in the following sense: since the function* f_1 *is merely of linear growth, we expect difficulties while studying the deformation gradient* ∇u*, whereas the stress tensor usually admits a clear interpretation as the unique solution of the dual problem. In contrast to this expection, generalized minimizers are uniquely determined up to a constant, and it is not clear whether the dual problem does even admit a solution (which is due to the possible anisotropy of* f_2*). Nevertheless, we have found a dual limit which satisfies the stress-strain relation and such that* $R[\sigma]$ *realizes* $\inf J$*, i.e. in some sense we have found a "local stress tensor". If global higher integrability of* σ *holds true, then this terminology coincides with the usual notion of a dual solution.*

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