Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint

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Preprint No. 59

Saarbrücken 2002

Universität des Saarlandes

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submitted: April 15, 2002

Preprint No. 59

Saarbrücken 2002

Edited by
FR 6.1 – Mathematik
Im Stadtwald
D–66041 Saarbrücken
Germany

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Entropy Decay of Discretized Fokker-Planck Equations I - Temporal Semi-Discretization

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AMS 2000 Subject Classification: 35K57, 35B40, 47D07, 26D10, 65M99
Key words: Fokker Planck equation, relative entropy, logarithmic Sobolev inequality, exponential decay rate

Abstract

In this paper we study the large time behavior of a fully implicit semi-discretization (in time) of parabolic Fokker–Planck type equations. Using logarithmic Sobolev inequalities exponential decay of the relative entropy (w.r.t. the steady state) is proved which yields convergence of the discrete scheme towards the unique steady state. The exponential decay rate recovers as $\Delta t \downarrow 0$ the decay rate of the original Fokker–Planck type equations.

1 Introduction

This paper is concerned with the behavior of temporal semi-discretizations of *Fokker–Planck type equations* for the real-valued function $\rho(x,t)$:

$$\begin{aligned} &\rho_t = \operatorname{div}\left[\mathbf{D}\cdot\left(\nabla\rho + \rho[\nabla A + \vec{F}]\right)\right], \quad x \in \mathbb{R}^d,\ t>0, \\ &\rho(t=0,x) = \rho^0(x) \in L^1_+(\mathbb{R}^d), \end{aligned} \tag{1}$$

with the confinement potential $A \in L^1_{loc}(\mathbb{R}^d)$ such that

$$\rho_\infty := e^{-A} \tag{2}$$

- which is a (formal) steady state of (1) - is in $L^1(\mathbb{R}^d)$. We assume that the symmetric diffusion matrix $\mathbf{D} = \mathbf{D}(x)$ is locally uniformly positive definite on $\mathbb{R}^d$ with $\mathbf{D} \in L^\infty_{loc}(\mathbb{R}^d;\mathbb{R}^{d\times d})$. The (possibly time-dependent) vector field $\vec{F}(x,t)$

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is assumed to satisfy $\vec{F}(.,t) \in L^1_{loc}(\mathbb{R}^d)$, $t \in (0,\infty)$, and (in an appropriate weak sense specified later on)

$$\operatorname{div}_x(\rho_\infty \mathbf{D} \cdot \vec{F}) = 0, \quad \text{on } \mathbb{R}^d \times (0,\infty). \tag{3}$$

We observe that (1) is (at least formally) mass-conserving, i.e. for all $t \in (0,\infty)$, $\int_{\mathbb{R}^d} \rho(x,t)dx = \int_{\mathbb{R}^d} \rho^0(x)dx$. Consequently, we shall assume in the sequel that A is gauged (by an additive constant) such that

$$\int_{\mathbb{R}^d} \rho_\infty(x)\, dx = \int_{\mathbb{R}^d} \rho^0(x)\, dx \tag{4}$$

holds.

In recent years the large time behavior of (1) and particularly the exponential convergence (in relative entropy) of $\rho(t)$ towards ρ_∞ as $t \to \infty$ has attracted lots of attention (cf. [1, 6] and references therein). Here we shall study whether the implicitly time-discretized Fokker–Planck equation

$$\begin{aligned} \frac{\rho^{n+1} - \rho^n}{\Delta t} &= \operatorname{div}\left[\mathbf{D} \cdot \left(\nabla \rho^{n+1} + \rho^{n+1}[\nabla A + \vec{F}^{n+1}]\right)\right] \\ &= \operatorname{div}\left[\rho_\infty \mathbf{D} \cdot \nabla \left(\frac{\rho^{n+1}}{\rho_\infty}\right)\right] + \operatorname{div}\left[\rho^{n+1}\mathbf{D} \cdot \vec{F}^{n+1}\right], \quad n \in \mathbb{N}_0, \quad (5) \\ \rho^0(x) &\in L^1_+(\mathbb{R}^d) \end{aligned}$$

(with the time step $\Delta t > 0$, and $\vec{F}^n(x) := \vec{F}(x, n\Delta t)$) preserves this exponential decay.

The large time behavior of such (semi-)discrete evolution equations is also interesting from a numerical point of view and has led to the construction of *entropy schemes* for dissipative Fokker–Planck-type models (cf. [4, 3, 2]). There the schemes guarantee a decay of the numerical entropy. However, exponential entropy decay rates of such numerical scheme have -to our knowledge- not yet been investigated.

The paper is organized as follows: In §2 we establish existence and uniqueness of a weak solution to (5) and prove that this iteration scheme preserves positivity (i.e. $\rho^n \geq 0$, $n \in \mathbb{N}$, provided $\rho^0 \geq 0$). Assuming that the evolution equation (1) corresponds to a convex Sobolev inequality (which is a generalization of Gross' *logarithmic Sobolev inequality* [5]) we prove in §3 exponential decay of ρ^n towards ρ_∞ in relative entropy and give a numerical illustration of this result. In particular, we show that the decay rate of the relative entropy converges to the exponential decay rate holding for the original Fokker-Planck-equation (1) as $\Delta t \downarrow 0$.

2 Wellposedness and positivity preservation of the iteration scheme

Equation (5) defines (at least formally) a sequence of functions $\{\rho^n\}_{n\in\mathbb{N}}$. In this section we shall give a weak formulation of it and prove its unique solvability for ρ^{n+1}.

To this end we shall need the following complex Hilbert spaces: The weighted L^2-space $L^2(\rho_\infty^{-1})$ is equipped with the inner product

$$\forall \rho, \phi \in L^2(\rho_\infty^{-1}) : \quad \langle \rho, \phi \rangle_{\rho_\infty^{-1}} = \int_{\mathbb{R}^d} \rho\,\overline{\phi}\,\rho_\infty^{-1}\,dx.$$

Furthermore, we introduce

$$H^1_{\rho_\infty,\mathbf{D}} := \left\{ \phi \in L^2(\rho_\infty^{-1}) : \int_{\mathbb{R}^d} \nabla^\top \left(\frac{\phi}{\rho_\infty}\right) \cdot \mathbf{D} \cdot \nabla \overline{\left(\frac{\phi}{\rho_\infty}\right)}\,\rho_\infty\,dx < \infty \right\},$$

which is equipped with the canonical inner product

$$\begin{aligned} \langle \rho, \phi \rangle_H &= \int_{\mathbb{R}^d} \frac{\rho}{\rho_\infty} \overline{\frac{\phi}{\rho_\infty}}\,\rho_\infty\,dx + \int_{\mathbb{R}^d} \nabla^\top \left(\frac{\rho}{\rho_\infty}\right) \cdot \mathbf{D} \cdot \nabla \overline{\left(\frac{\phi}{\rho_\infty}\right)}\,\rho_\infty\,dx \\ &=: \langle \rho, \phi \rangle_{\rho_\infty^{-1}} + \langle \rho, \phi \rangle_{\rho_\infty,\mathbf{D}}. \end{aligned}$$

For (5) we shall henceforth assume (further assumptions will follow):

A.1 The symmetric diffusion matrix $\mathbf{D} = \mathbf{D}(x)$ is locally uniformly positive definite on $\mathbb{R}^d$ with $\mathbf{D} \in L^\infty_{loc}(\mathbb{R}^d;\mathbb{R}^{d\times d})$.

A.2 $0 < \rho_\infty \in L^1(\mathbb{R}^d)$.

A.3 For each $n \in \mathbb{N}$, $\vec{F}^n \in L^1_{loc}(\mathbb{R}^d;\mathbb{R}^d)$ and $|\sqrt{\mathbf{D}} \cdot \vec{F}^n| \in L^\infty(\mathbb{R}^d)$.

A.4 $\rho^0 \in L^1_+(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} \rho^0\,dx = \int_{\mathbb{R}^d} \rho_\infty\,dx$.

We give

Definition 1. *A (real valued) function sequence $\{\rho^n\}_{n\in\mathbb{N}} \subseteq H^1_{\rho_\infty,\mathbf{D}}$ is a* weak solution of (5) *iff the following property holds for each $n \in \mathbb{N}_0$:*

$$\begin{aligned} \int_{\mathbb{R}^d} \rho^{n+1}\phi\,\rho_\infty^{-1}\,dx &= \int_{\mathbb{R}^d} \rho^n\phi\,\rho_\infty^{-1}\,dx - \Delta t \int_{\mathbb{R}^d} \nabla^\top \left(\frac{\rho^{n+1}}{\rho_\infty}\right) \cdot \mathbf{D} \cdot \nabla \left(\frac{\phi}{\rho_\infty}\right) \rho_\infty\,dx \\ &\quad -\Delta t \int_{\mathbb{R}^d} \rho^{n+1} \nabla^\top \left(\frac{\phi}{\rho_\infty}\right) \cdot \mathbf{D} \cdot \vec{F}^{n+1}\,dx \quad \forall \phi \in H^1_{\rho_\infty,\mathbf{D}}. \end{aligned} \tag{6}$$

Since $\rho_\infty \in H^1_{\rho_\infty,\mathbf{D}}$, choosing the test function $\phi = \rho_\infty$ in (6) yields mass conservation of a weak solution: $\int_{\mathbb{R}^d} \rho^n \, dx = \int_{\mathbb{R}^d} \rho^0 \, dx, \ n \in \mathbb{N}$.

To analyze the solvability of (6) we shall now introduce a sequence of (complex) quadratic forms, $\{q^n\}_{n\in\mathbb{N}}$ with the common (n-independent) *form domain*

$$Q(q^n) = Q := H^1_{\rho_\infty,\mathbf{D}}.$$

We define $q^n : Q \times Q \to \mathbb{C}$ with

$$\begin{aligned} q^n(\rho,\phi) \ &:= \ \frac{1}{2}\int_{\mathbb{R}^d} \frac{\rho}{\rho_\infty} \overline{\frac{\phi}{\rho_\infty}} \rho_\infty \, dx + \Delta t \int_{\mathbb{R}^d} \nabla^\top \left(\frac{\rho}{\rho_\infty}\right) \cdot \mathbf{D} \cdot \nabla \overline{\left(\frac{\phi}{\rho_\infty}\right)} \rho_\infty \, dx \\ &\quad + \Delta t \int_{\mathbb{R}^d} \frac{\rho}{\rho_\infty} \nabla^\top \overline{\left(\frac{\phi}{\rho_\infty}\right)} \cdot \mathbf{D} \cdot \vec{F}^n \, \rho_\infty \, dx \\ &=: \ \frac{1}{2}\langle \rho,\phi\rangle_{\rho_\infty^{-1}} + \Delta t \, \langle \rho,\phi\rangle_{\rho_\infty,\mathbf{D}} + q^n_{im}(\rho,\phi). \end{aligned} \tag{7}$$

To make (3) precise we shall henceforth assume:

A.5 For each $n \in \mathbb{N}$, $\operatorname{div}(\rho_\infty \mathbf{D} \cdot \vec{F}^n) = 0$ in the following weak sense:

$$\begin{aligned} &\forall \text{ real valued } \rho,\phi \in H^1_{\rho_\infty,\mathbf{D}} : \quad q^n_{im}(\rho,\phi) + q^n_{im}(\phi,\rho) \\ &= \Delta t \left(\int_{\mathbb{R}^d} \frac{\rho}{\rho_\infty} \nabla^\top \left(\frac{\phi}{\rho_\infty}\right) \cdot \mathbf{D} \cdot \vec{F}^n \rho_\infty \, dx + \int_{\mathbb{R}^d} \frac{\phi}{\rho_\infty} \nabla^\top \left(\frac{\rho}{\rho_\infty}\right) \cdot \mathbf{D} \cdot \vec{F}^n \rho_\infty \, dx \right) = 0. \end{aligned}$$

Remark 2. *(a) Due to the following estimate q^n is well-defined on $Q \times Q$:*

$$\begin{aligned} |q^n_{im}(\rho,\phi)| \ &= \ \left| \Delta t \int_{\mathbb{R}^d} \frac{\rho}{\rho_\infty} \nabla^\top \overline{\left(\frac{\phi}{\rho_\infty}\right)} \cdot \mathbf{D} \cdot \vec{F}^n \, \rho_\infty \, dx \right| \\ &\leq \ \Delta t \, \left\| |\sqrt{\mathbf{D}} \cdot \vec{F}^n| \right\|_{L^\infty(\mathbb{R}^d)} \|\rho\|_{\rho_\infty^{-1}} \, \|\phi\|_{\rho_\infty,\mathbf{D}}. \end{aligned} \tag{8}$$

(b) Since $\operatorname{div}(\rho_\infty \mathbf{D} \cdot \vec{F}^n)$ *vanishes in the sense of* $H^1_{\rho_\infty,\mathbf{D}}$*, we have for all* $\rho \in H^1_{\rho_\infty,\mathbf{D}}$*:* $\mathsf{Re}(q^n_{im}(\rho,\rho)) = 0$.
(c) From part (b) we have for all (complex valued) functions $\rho \in H^1_{\rho_\infty,\mathbf{D}}$*:*

$$\mathsf{Re}(q^n(\rho,\rho)) = \frac{1}{2}\|\rho\|^2_{\rho_\infty^{-1}} + \Delta t \, \|\rho\|^2_{\rho_\infty,\mathbf{D}} \geq 0, \tag{9a}$$

$$\mathsf{Im}(q^n(\rho,\rho)) = -i \, q^n_{im}(\rho,\rho). \tag{9b}$$

We easily prove

Proposition 3. *Assume* A.1-A.5. *Then each quadratic form q^n, $n \in \mathbb{N}$ is strictly m-accretive (in the sense of the definition on p. 281 in [7]).*

Proof. First, the form domain Q is dense in $L^2(\rho_\infty^{-1})$.

Secondly, we shall prove that q^n is a closed form: Let the sequence $\{\phi_j\}_{j\in\mathbb{N}} \subseteq Q$ converge to ϕ in $L^2(\rho_\infty^{-1})$ and be a Cauchy sequence with respect to q^n, i.e.

$$\lim_{(j,k)\uparrow(\infty,\infty)} q^n(\phi_j - \phi_k, \phi_j - \phi_k) = 0.$$

Then, due to (9a), $\{\phi_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $H^1_{\rho_\infty,\mathbf{D}}$. Since the embedding $H^1_{\rho_\infty,\mathbf{D}} \subseteq L^2(\rho_\infty^{-1})$ is continuous, the limit of $\{\phi_j\}_{j\in\mathbb{N}}$ in $H^1_{\rho_\infty,\mathbf{D}}$ equals ϕ. Thus $\phi \in Q$. Since

$$\frac{1}{2}\|\rho\|^2_{\rho_\infty^{-1}} + \Delta t\,\|\rho\|^2_{\rho_\infty,\mathbf{D}} \le \max\left\{\frac{1}{2}, \Delta t\right\}\|\rho\|^2_H, \quad \forall \rho \in Q,$$

and via (8), we also have $\lim_{j\uparrow\infty} q^n(\phi_j - \phi, \phi_j - \phi) = 0$. Hence, q^n is closed.

Finally, we estimate $|\arg[q^n(\rho,\rho)]|$ for $\rho \in Q$: For $\rho \ne 0$ (8) and (9a) imply:

$$\left|\frac{\mathsf{Im}(q^n(\rho,\rho))}{\mathsf{Re}(q^n(\rho,\rho))}\right| \le \frac{\Delta t\,\left\||\sqrt{\mathbf{D}}\cdot\vec{F}^n|\right\|_{L^\infty(\mathbb{R}^d)}\|\rho\|_{\rho_\infty^{-1}}\|\rho\|_{\rho_\infty,\mathbf{D}}}{\frac{1}{2}\|\rho\|^2_{\rho_\infty^{-1}} + \Delta t\,\|\rho\|^2_{\rho_\infty,\mathbf{D}}}$$
$$\le \sqrt{\frac{\Delta t}{2}}\left\||\sqrt{\mathbf{D}}\cdot\vec{F}^n|\right\|_{L^\infty(\mathbb{R}^d)},$$

and hence q^n is strictly m-accretive. $\square$

According to Theorem VIII.16 in [7] the form q^n corresponds to a strictly m-accretive operator T^n:

Theorem 4. *Assume* A.1-A.5.
Then, $\forall n \in \mathbb{N}$ *there exists a unique closed operator* $T^n : D(T^n) \to L^2(\rho_\infty^{-1})$, *with* $D(T^n) \subseteq Q$, *such that* $q^n(\rho,\phi) = \langle T^n(\rho), \phi\rangle_{\rho_\infty^{-1}}$ *holds for all* $\rho, \phi \in D(T^n)$. *Furthermore,* $D(T^n)$ *is* $H^1_{\rho_\infty,\mathbf{D}}$*-dense in* $H^1_{\rho_\infty,\mathbf{D}}$, *and for each* $\lambda \in \mathbb{C}$ *with* $\mathsf{Re}(\lambda) < 0$, *the operator* $T^n - \lambda$ *has a bounded inverse with* $\|(T^n-\lambda)^{-1}\| \le \frac{1}{|\mathsf{Re}(\lambda)|}$.

With the aid of Theorem 4 we easily prove:

Theorem 5. *Assume* A.1-A.5.
Then there is for each $\rho^0 \in L^2(\rho_\infty^{-1})$ *exactly one weak solution of* (5).

Proof. Using (7) we re-write the recursion (6) in the equivalent form

$$\forall n \in \mathbb{N}, \phi \in H^1_{\rho_\infty,\mathbf{D}} : q^{n+1}(\rho^{n+1},\phi) = -\frac{1}{2}\langle\rho^{n+1},\phi\rangle_{\rho_\infty^{-1}} + \langle\rho^n,\phi\rangle_{\rho_\infty^{-1}}. \tag{10}$$

Using Theorem 4 we re-write the l.h.s. of (10): For each $n \in \mathbb{N}$ and for each $\rho \in D(T^{n+1})$ there is a unique $T^{n+1}(\rho) \in L^2(\rho_\infty^{-1})$ such that

$$\forall \phi \in D(T^{n+1}) : q^{n+1}(\rho,\phi) = \langle T^{n+1}(\rho),\phi\rangle_{\rho_\infty^{-1}}.$$

If $\rho^n \in L^2(\rho_\infty^{-1})$ (for some $n \in \mathbb{N}_0$), then there is a unique $\rho^{n+1} \in D(T^{n+1})$ such that $(T^{n+1} + \frac{1}{2})(\rho^{n+1}) = \rho^n$, since $T^{n+1} + \frac{1}{2}$ has a bounded inverse. Furthermore, $\|\rho^{n+1}\|_{\rho_\infty^{-1}} = \|(T^{n+1} + \frac{1}{2})^{-1}(\rho^n)\|_{\rho_\infty^{-1}} \leq 2\|\rho^n\|_{\rho_\infty^{-1}}$.

By a standard density argument one readily proves that ρ^{n+1} satisfies (10). A standard energy estimate using (9a) ensures that (10) has indeed at most one solution ρ^{n+1}. Furthermore, if ρ^n is real-valued, ρ^{n+1} will also be real-valued. $\square$

Next, we show that the iteration scheme (5) preserves positivity:

Theorem 6. *Assume* A.1-A.5 *and let* $\rho^0 \in L^2_+(\rho_\infty^{-1})$. $\{\rho^n\}_{n\in\mathbb{N}}$, *the weak solution of* (5) *then satisfies*

$$\rho^n(x) \geq 0 \quad \forall n \in \mathbb{N}.$$

Proof. It suffices to prove: If $\rho^n \geq 0$, then $\rho^{n+1} \geq 0$. We observe: If a real-valued function $\rho \in H^1_{\rho_\infty,\mathbf{D}}$, then its negative part, $[\rho]^- \in H^1_{\rho_\infty,\mathbf{D}}$. Now, using $[\rho^{n+1}]^- \in H^1_{\rho_\infty,\mathbf{D}}$ as test function, (6) reads:

$$\langle \rho^{n+1}, [\rho^{n+1}]^- \rangle_{\rho_\infty^{-1}} = \langle \rho^n, [\rho^{n+1}]^- \rangle_{\rho_\infty^{-1}} - \Delta t \langle \rho^{n+1}, [\rho^{n+1}]^- \rangle_{\rho_\infty,\mathbf{D}} - q_{im}^{n+1}(\rho^{n+1}, [\rho^{n+1}]^-).$$

Since $\rho^n \geq 0$ we readily verify

$$\langle \rho^{n+1}, [\rho^{n+1}]^- \rangle_{\rho_\infty^{-1}} = -\langle [\rho^{n+1}]^-, [\rho^{n+1}]^- \rangle_{\rho_\infty^{-1}} \leq 0,$$

$$\langle \rho^n, [\rho^{n+1}]^- \rangle_{\rho_\infty^{-1}} \geq 0,$$

$$-\langle \rho^{n+1}, [\rho^{n+1}]^- \rangle_{\rho_\infty,\mathbf{D}} = \langle [\rho^{n+1}]^-, [\rho^{n+1}]^- \rangle_{\rho_\infty,\mathbf{D}} \geq 0.$$

And since ρ^{n+1} is real-valued Remark 2(b) gives:

$$q_{im}^{n+1}(\rho^{n+1}, [\rho^{n+1}]^-) = -q_{im}^{n+1}([\rho^{n+1}]^-, [\rho^{n+1}]^-) = 0.$$

Hence we deduce $\langle [\rho^{n+1}]^-, [\rho^{n+1}]^- \rangle_{\rho_\infty^{-1}} = 0$. $\square$

3 Decay of the relative entropy

We define the *relative entropy* of ρ with respect to ρ_∞ as

$$e(\rho|\rho_\infty) := \int_{\mathbb{R}^d} \Psi\left(\frac{\rho}{\rho_\infty}\right) \rho_\infty \, dx, \tag{11}$$

where $\Psi \in C^2(\mathbb{R}^+)$ is a given non-negative convex function satisfying $\Psi(1) = 0$. Typical examples of such entropies are (cf. [1])

$$\Psi_1(\sigma) := \sigma \ln \sigma - (\sigma - 1), \quad \text{and } \Psi_p(\sigma) := \sigma^p - 1 - p(\sigma - 1),\ 1 < p \leq 2. \tag{12}$$

Our main assumption for this section is the validity of a *convex Sobolev inequality* in $H^1_{\rho_\infty,\mathbf{D}}$ for a fixed entropy that is generated by the function Ψ. More precisely, we assume:

A.6 There is a positive constant λ such that

$$e(\rho|\rho_\infty) \le \frac{1}{\lambda}\int_{\mathbb{R}^d} \Psi''\left(\frac{\rho}{\rho_\infty}\right) \nabla^\top\left(\frac{\rho}{\rho_\infty}\right)\cdot \mathbf{D}\cdot\nabla\left(\frac{\rho}{\rho_\infty}\right)\rho_\infty\,dx \quad \text{holds}$$
$$\forall \rho\in H^1_{\rho_\infty,\mathbf{D}} \text{ with } 0\le\rho \text{ and } \int_{\mathbb{R}^d}\rho\,dx = \int_{\mathbb{R}^d}\rho_\infty\,dx. \quad (13)$$

Note that (13) is equivalent to the exponential decay (with rate λ) of the relative entropy for the solution of (1).

For the entropies (12) the convex Sobolev inequality (13) was derived in [1] under the condition that $A\in W^{2,\infty}_{loc}(\mathbb{R}^d)$, and $\mathbf{D}=D(x)\mathbf{I}$, $D\in W^{2,\infty}_{loc}(\mathbb{R}^d)$ satisfy

$$\begin{aligned}&\left(\frac{1}{2}-\frac{d}{4}\right)\frac{1}{D}\nabla D\otimes\nabla D+\frac{1}{2}(\Delta D-\nabla D\cdot\nabla A)\mathbf{I}\\ &+D\frac{\partial^2 A}{\partial x^2}+\frac{\nabla A\otimes\nabla D+\nabla D\otimes\nabla A}{2}-\frac{\partial^2 D}{\partial x^2} \;\ge\; \frac{\lambda}{2}\mathbf{I}\end{aligned}$$

(in the sense of positive definite matrices) $\forall x\in\mathbb{R}^d$. If $\mathbf{D}=\mathbf{I}$, this condition means uniform strict convexity of the potential A, i.e. $\left(\frac{\partial^2 A(x)}{\partial x_i\partial x_j}\right)_{i,j=1,\dots,d}\ge\frac{\lambda}{2}\mathbf{I}$.

Our main result is

Theorem 7. *Assume* A.1-A.5, *and assume the validity of the convex Sobolev inequality* (13) *for some convex function* $\Psi\in C^2(\mathbb{R}^+)$ *with* $\Psi(1)=\Psi'(1)=0$. *Then,* $\{\rho^n\}_{n\in\mathbb{N}}$, *the weak solution of* (5) *satisfies*

$$e(\rho^n|\rho_\infty)\le(1+\lambda\Delta t)^{-n}\,e(\rho^0|\rho_\infty),\quad \forall n\in\mathbb{N}.$$

Proof. Since there is nothing to prove in the case $e(\rho^0|\rho_\infty)=\infty$, we assume $e(\rho^0|\rho_\infty)<\infty$ henceforth.

Our line of argument involves several integrations by parts. Since the involved test functions do not necessarily belong to $H^1_{\rho_\infty,\mathbf{D}}$ we shall need an approximation for the entropy functional: Let $\{\eta_k\}_{k\in\mathbb{N}}$ be a sequence in $C(\mathbb{R}^+)$ such that for all $k\in\mathbb{N}$, η_k is compactly supported, $0\le\eta_k\le\eta_{k+1}\le 1$ and $\eta_k=1$ on $[2^{-k},2^k]$. We set $\Psi_k''=\Psi''.\eta_k$, $k\in\mathbb{N}$, and define

$$\Psi_k'(s)=\int_1^s\Psi_k''(\sigma)\,d\sigma,\quad \Psi_k(s)=\int_1^s\Psi_k'(\sigma)\,d\sigma,\quad s\in\mathbb{R}^+.$$

Then, for each $k\in\mathbb{N}$, $0\le\Psi_k''\le\Psi_{k+1}''\le\Psi''$ and $0\le\Psi_k\le\Psi_{k+1}\le\Psi$. Consequently, by Lebesgue's Monotone Convergence Theorem for each non-negative $\rho\in H^1_{\rho_\infty,\mathbf{D}}$,

$$\lim_{k\uparrow\infty}e_k(\rho|\rho_\infty)=\lim_{k\uparrow\infty}\int_{\mathbb{R}^d}\Psi_k\left(\frac{\rho}{\rho_\infty}\right)\rho_\infty\,dx=\int_{\mathbb{R}^d}\Psi\left(\frac{\rho}{\rho_\infty}\right)\rho_\infty\,dx=e(\rho|\rho_\infty),$$

where all terms have values in $[0,\infty]$. Certainly, for each $k\in\mathbb{N}$, $\Psi_k\left(\frac{\rho}{\rho_\infty}\right)\rho_\infty\in H^1_{\rho_\infty,\mathbf{D}}$. Thus, setting $\rho=\rho_\infty$, $\phi=\Psi_k\left(\frac{\rho}{\rho_\infty}\right)\rho_\infty$ in A.5 we have

$$\int_{\mathbb{R}^d}\nabla^\top\Psi_k\left(\frac{\rho}{\rho_\infty}\right)\cdot\mathbf{D}\cdot\vec{F}^n\rho_\infty\,dx=0\quad\forall k,\,n\in\mathbb{N}. \tag{14}$$

Furthermore, for each $k\in\mathbb{N}$, $\Psi_k'\left(\frac{\rho}{\rho_\infty}\right)\rho_\infty\in H^1_{\rho_\infty,\mathbf{D}}$, such that via A.5 and due to (14) for all $n\in\mathbb{N}$,

$$\begin{aligned}\int_{\mathbb{R}^d}\frac{\rho}{\rho_\infty}\nabla^\top\Psi_k'\left(\frac{\rho}{\rho_\infty}\right)\cdot\mathbf{D}\cdot\vec{F}^n\rho_\infty\,dx&=-\int_{\mathbb{R}^d}\Psi_k'\left(\frac{\rho}{\rho_\infty}\right)\nabla^\top\left(\frac{\rho}{\rho_\infty}\right)\cdot\mathbf{D}\cdot\vec{F}^n\rho_\infty\,dx\\&=-\int_{\mathbb{R}^d}\nabla^\top\Psi_k\left(\frac{\rho}{\rho_\infty}\right)\cdot\mathbf{D}\cdot\vec{F}^n\rho_\infty\,dx=0.\end{aligned}\tag{15}$$

We deduce from (15) for all $k,n\in\mathbb{N}$,

$$\int_{\mathbb{R}^d}\Psi_k''\left(\frac{\rho}{\rho_\infty}\right)\rho\,\nabla^\top\left(\frac{\rho}{\rho_\infty}\right)\cdot\mathbf{D}\cdot\vec{F}^n\,dx=0. \tag{16}$$

Exploiting the convexity of Ψ_k we calculate for all $k\in\mathbb{N}$ and for all $n\in\mathbb{N}_0$:

$$\begin{aligned}e(\rho^n|\rho_\infty)&\geq e_k(\rho^n|\rho_\infty)=\int_{\mathbb{R}^d}\Psi_k\left(\frac{\rho^{n+1}+(\rho^n-\rho^{n+1})}{\rho_\infty}\right)\rho_\infty\,dx\\&\geq\int_{\mathbb{R}^d}\Psi_k\left(\frac{\rho^{n+1}}{\rho_\infty}\right)\rho_\infty\,dx+\int_{\mathbb{R}^d}\Psi_k'\left(\frac{\rho^{n+1}}{\rho_\infty}\right)(\rho^n-\rho^{n+1})\,dx\\&=e_k(\rho^{n+1}|\rho_\infty)+\int_{\mathbb{R}^d}\left(\Psi_k'\left(\frac{\rho^{n+1}}{\rho_\infty}\right)\rho_\infty\right)(\rho^n-\rho^{n+1})\,\rho_\infty^{-1}\,dx.\end{aligned}\tag{17}$$

Setting $\phi=\Psi_k'\left(\frac{\rho^{n+1}}{\rho_\infty}\right)\rho_\infty\in H^1_{\rho_\infty,\mathbf{D}}$ in (6) gives:

$$\begin{aligned}&\int_{\mathbb{R}^d}\left(\Psi_k'\left(\frac{\rho^{n+1}}{\rho_\infty}\right)\rho_\infty\right)(\rho^n-\rho^{n+1})\,\rho_\infty^{-1}\,dx\\&=\Delta t\int_{\mathbb{R}^d}\nabla^\top\left(\frac{\rho^{n+1}}{\rho_\infty}\right)\cdot\mathbf{D}\cdot\nabla\Psi_k'\left(\frac{\rho^{n+1}}{\rho_\infty}\right)\rho_\infty\,dx+\Delta t\int_{\mathbb{R}^d}\rho^{n+1}\,\nabla^\top\Psi_k'\left(\frac{\rho^{n+1}}{\rho_\infty}\right)\cdot\mathbf{D}\cdot\vec{F}^{n+1}\,dx\\&\qquad=\Delta t\int_{\mathbb{R}^d}\Psi_k''\left(\frac{\rho^{n+1}}{\rho_\infty}\right)\nabla^\top\left(\frac{\rho^{n+1}}{\rho_\infty}\right)\cdot\mathbf{D}\cdot\nabla\left(\frac{\rho^{n+1}}{\rho_\infty}\right)\rho_\infty\,dx,\end{aligned}\tag{18}$$

where we used (16) in the last step. Combining (17) and (18) gives for all $k\in\mathbb{N}$ and for all $n\in\mathbb{N}_0$:

$$e(\rho^n|\rho_\infty)\geq e_k(\rho^{n+1}|\rho_\infty)+\Delta t\int_{\mathbb{R}^d}\Psi_k''\left(\frac{\rho^{n+1}}{\rho_\infty}\right)\nabla^\top\left(\frac{\rho^{n+1}}{\rho_\infty}\right)\cdot\mathbf{D}\cdot\nabla\left(\frac{\rho^{n+1}}{\rho_\infty}\right)\rho_\infty\,dx.$$

We recall $e(\rho^n|\rho_\infty) < \infty$ and all integrands on the right-hand side of this inequality are non-negative. We deduce from Lebesgue's Dominated Convergence Theorem via $\Psi_k \uparrow \Psi$ and $\Psi_k'' \uparrow \Psi''$ as $k \uparrow \infty$,

$$e(\rho^n|\rho_\infty) \geq e(\rho^{n+1}|\rho_\infty) + \Delta t \int_{\mathbb{R}^d} \Psi'' \left(\frac{\rho^{n+1}}{\rho_\infty} \right) \nabla^\top \left(\frac{\rho^{n+1}}{\rho_\infty} \right) \cdot \mathbf{D} \cdot \nabla \left(\frac{\rho^{n+1}}{\rho_\infty} \right) \rho_\infty \, dx.$$

Applying the convex Sobolev inequality (13) finally gives for all $n \in \mathbb{N}_0$,

$$e(\rho^n|\rho_\infty) \geq e(\rho^{n+1}|\rho_\infty) + \lambda \Delta t \, e(\rho^{n+1}|\rho_\infty). \qquad \square$$

Finally, we shall numerically illustrate the discussed entropy behavior for the 1D test case with $A = x^2/2$, $\mathbf{D} = \mathbf{I}$, $\vec{F} = 0$, and $\Delta t = 0.003$. We use a finite difference discretization ($\Delta x = 0.08$) of (5) and zero-flux boundary conditions on a sufficiently large computational interval. Fig. 1 shows the exponential decay of the logarithmic and quadratic relative entropies (corresponding to Ψ_1 and Ψ_2, resp.). In the first example we chose the initial condition $\rho_1^0 = \exp[-x^2/2 + 2x]$, which is an *extremal function* for Ψ_1, i.e. $\rho = \rho_1^0$ makes (13) an equality. Hence we observe in Fig. 1a the predicted exponential decay of the logarithmic entropy with rate 2.0091 (for $\Delta t \to 0$ one recovers the rate $\lambda = 2$ of the continuous equation (1)) and an initially faster decay of the quadratic entropy. After $n \approx 2500$ time steps effects of the spatial discretization become visible.

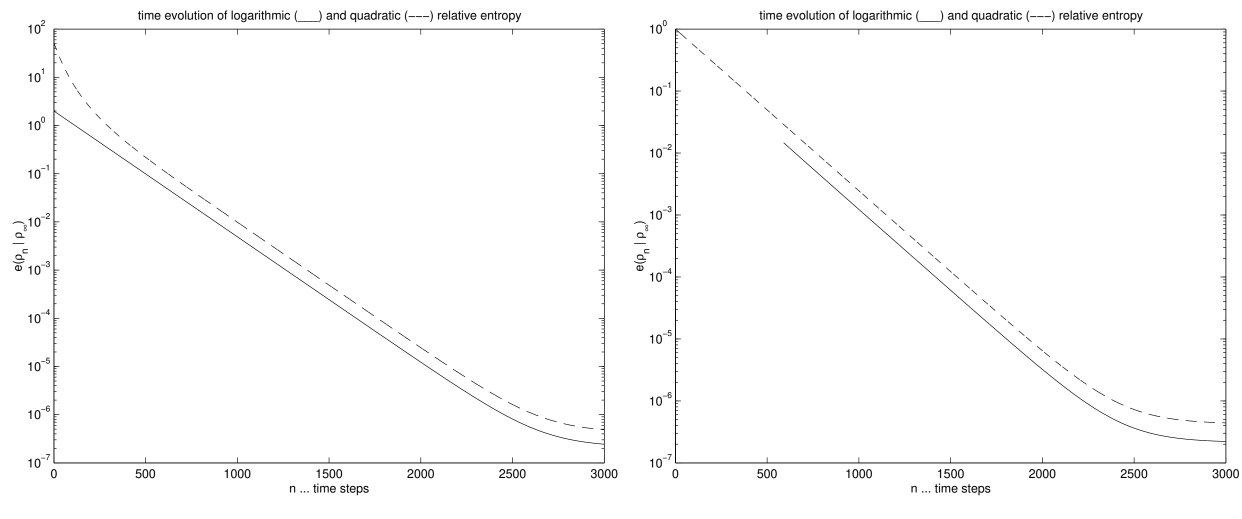

Figure 1: Exponential decay of the logarithmic and quadratic entropies. The initial condition was chosen as an extremal function for (a - left) the logarithmic entropy and (b - right) the quadratic entropy.

In the second example we chose $\rho_2^0 = (1+x)\exp[-x^2/2]$, which is an *extremal function* for Ψ_2. Hence Fig. 1b shows the predicted exponential decay of the quadratic entropy. The logarithmic entropy is only plotted for $n \geq 600$, as $\rho^n(x)$ takes negative values for smaller times.

Acknowledgments

The first author was supported by the grants ERBFMRXCT970157 (TMR-Network) from the EU and the DFG under Grant-No. AR 277/3-1.

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