Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint

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Preprint No. 68

Saarbrücken 2002

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submitted: August 14, 2002

Preprint No. 68

Saarbrücken 2002

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Abstract

The asymptotic normality of a class of estimators for extreme quantiles is established under mild structural conditions on the observed stationary β–mixing time series. Consistent estimators of the asymptotic variance are introduced, which render possible the construction of asymptotic confidence intervals for the extreme quantiles. Moreover, it is shown that many well-known time series models satisfy our conditions. Then the theory is applied to a time series of returns of a stock index. Finally, the finite sample behavior of the proposed confidence intervals is examined in a simulation study. It turns out that for most time series models under consideration the actual coverage probability is pretty close to the nominal level if the sample fraction used for estimation is chosen appropriately.

Key words and phrases: ARMA model, β–mixing, confidence interval, extreme quantiles, GARCH model, tail empirical quantile function, time series
AMS 2000 subject classification: primary 62G32, 62G30; secondary 60G70, 62P05
Running title: Extreme Quantile Estimation for Dependent Data

1 Introduction

Let S_i, $0 \leq i \leq n$, be a sequence of consecutive share prices. In the last couple of years, the Value at Risk (VaR), defined as a large quantile of the negative log-returns $X_i = -\log(S_i/S_{i-1})$ which are assumed stationary, has become a popular measure for the risk of an investment in these shares. It has been known for a long time past that the classical Gaussian models for the log-returns (like the famous Black-Scholes model) underestimate the risk of large losses and are hence not suitable as a basis for VaR-estimation. As alternatives, it has been proposed to model series of log-returns by independent random variables with heavy tails (Jansen and de Vries (1991) and Longin (1996), among others). To take into account the serial dependence which is usually observed in time series of log-returns, a large variety of more sophisticated ARCH-type models has been introduced since the seminal paper by Engle (1982).
Though some of these models describe real time series of log-returns reasonably well for specific purposes, none of them is able to capture all so-called 'stylized facts', i.e. features common in most of these financial data sets; see Mikosch and Stărică (2000) for a comprehensive discussion. In particular, it is questionable whether such a model can well describe both the central part

of the distribution and its tails. Therefore, it has been recently advocated to 'let the tails speak for themselves', i.e., to use merely the largest negative log-returns for the estimation of the VaR.
Statistical procedures of that type are provided by extreme value theory under rather mild structural assumptions on the tail of the marginal distributions of log-returns. Unfortunately, almost all results on the asymptotic behavior of extreme quantile estimators available by now are restricted to independent observations. For financial time series, however, it is rarely realistic to assume independence between consecutive observations. Thus the main aim of the present paper is to investigate the asymptotic behavior of quantile estimators based on large observations under mild assumptions on the serial dependence.
Of course, the results are not only relevant for the VaR-estimation but also for the tail analysis of any real-life time series if the assumption of independence seems inappropriate. For example, the interarrival times and lengths in teletraffic networks often exhibit heavy tails and serial dependence as well.
Denote the common distribution function (d.f.) of the stationary time series under consideration, X_i, $i \in \mathbb{N}$, by F. The basic assumption in the extreme value approach is

$$F^n(a_n x + b_n) \longrightarrow G(x), \qquad x \in \mathbb{R}, \tag{1}$$

for some $a_n > 0$, $b_n \in \mathbb{R}$, where G is a non-degenerate limit d.f. It is well known that then G must be one of the extreme value d.f.s (up to a scale and location parameter)

$$G_\gamma(x) = \exp\Big(- (1+\gamma x)^{-1/\gamma}\Big), \qquad 1+\gamma x \geq 0,\ \gamma \in \mathbb{R},$$

which is interpreted as $\exp(e^{-x})$ for $\gamma = 0$. In short we write $F \in D(G_\gamma)$. If the X_i are independent (or weakly dependent) then (1) is equivalent to the weak convergence of the d.f. of the standardized maximum of n observations to G. In general, though, the maximum of a stationary time series is stochastically smaller than the maximum of an i.i.d. sequence with the same marginal d.f., since the serial dependence leads to a clustering of large values. Indeed, under mild conditions on the dependence structure, $F \in D(G_\gamma)$ implies

$$\mathcal{L}\Big(a_n^{-1}(\max_{1\leq i\leq n} X_i - b_n)\Big) \longrightarrow G_\gamma^\theta \qquad \text{weakly} \tag{2}$$

for some $\theta \in [0,1]$; see Leadbetter et al. (1983), Section 3.7, for details. Typically the so-called extremal index θ can be interpreted as the reciprocal value of the asymptotic mean cluster size.

Note that by (1) and (2)

$$P\Big\{a_n^{-1}(\max_{1\le i\le n} X_i - b_n) \le x\Big\} \sim P\Big\{a_n^{-1}(\max_{1\le i\le [n\theta]} \tilde{X}_i - b_n) \le x\Big\}$$

as $n \to \infty$, where $\tilde{X}_i$, $i \in I\!N$, is an i.i.d. sequence with marginal d.f. F. (Here $c_n \sim d_n$ means $c_n/d_n \to 1$, and $[x]$ denotes the largest integer smaller than or equal to x.) Hence, as far as the behavior of the maximum is concerned, the serial dependence between large observations reduces the effective sample size by the factor θ. Since intuitively a cluster of large observations contains less information about F than the same number of independent large observations, one shall also expect an influence of the serial dependence on the precision of statistical extreme value procedures. More precisely, the dependence will lead to an increase of the estimation error. Thus it is important *not* to use the classical confidence intervals developed for i.i.d. settings if the serial dependence is not negligible. If, for example, the VaR of a financial investment is to be estimated, then an upper confidence bound obtained from the i.i.d.-theory will often indicate a risk much lower than the actual one.
To be more concrete, let $F^{-1}(1-p_n)$ be the extreme quantile that is to be estimated. We are mainly interested in the case $np_n = O(1)$, although our main result also holds if $np_n \to \infty$ not too fast.
Only estimators based on the k_n+1, say, largest order statistics $\max_{1\le i\le n} X_i = X_{n:n} \ge X_{n-1:n} \ge \ldots \ge X_{n-k_n:n}$ are considered. In order not to overload the paper, we will focus on heavy-tailed distributions, i.e. $\gamma > 0$, which is the most important case in financial applications. However, we will also indicate how to construct and analyze similar estimators in the general case $\gamma \in I\!R$.
To construct extreme quantile estimators, recall that the basic assumption $F \in D(G_\gamma)$ with $\gamma > 0$ is equivalent to

$$R(\lambda, t) := \frac{F^{-1}(1-\lambda t)}{F^{-1}(1-\lambda)} - t^{-\gamma} \longrightarrow 0, \qquad t > 0, \tag{3}$$

as $\lambda \downarrow 0$. Reading this convergence as an approximation for small λ, one obtains

$$\begin{aligned} x_{p_n} := F^{-1}(1-p_n) &\approx F^{-1}\Big(1-\frac{k_n}{n}\Big)\Big(\frac{np_n}{k_n}\Big)^{-\gamma} \\ &\approx X_{n-k_n:n}\Big(\frac{np_n}{k_n}\Big)^{-\hat{\gamma}_n} \\ &=: \hat{x}_{p_n}^{(k_n)} = \hat{x}_{p_n}, \end{aligned} \tag{4}$$

where $\hat{\gamma}_n$ denotes a suitable estimator of the extreme value index γ depending only on the k_n+1 largest order statistics. To justify the first approximation

k_n/n has to be small, while on the other hand k_n should be sufficiently large so that the empirical quantile $X_{n-k_n:n}$ estimates the intermediate quantile $F^{-1}(1-k_n/n)$ well. Thus in the sequel we assume that the natural numbers k_n form an intermediate sequence, i.e.

$$k_n \longrightarrow \infty, \qquad k_n/n \longrightarrow 0. \tag{5}$$

The extreme value index γ may be estimated, e.g., by the Hill estimator

$$\hat{\gamma}_n^{(H)} = \frac{1}{k_n} \sum_{i=1}^{k_n} \log \frac{X_{n-i+1:n}}{X_{n-k_n:n}}.$$

The consistency of the Hill estimator was proved in quite general time series models by Hsing (1991) and Resnick and Stărică (1998). Resnick and Stărică (1997) examined its asymptotic normality in specific models, while Novak (1999) proved asymptotic normality of a closely related estimator under suitable mixing conditions. Rootzén et al. (1992) established asymptotic normality for the quantile estimator $\hat{x}_{p_n}$ based on the Hill estimator under a rather complex set of conditions; among other things, they assumed that the von Mises condition is met, $np_n \to 0$, and the time series is strongly mixing. In contrast, Drees (2000) established the asymptotic normality of a much broader class of estimators for the extreme value index, including the maximum likelihood estimator, studied by Smith (1987) in an i.i.d. setting, the moment estimator by Dekkers et al. (1989) and Pickands' (1975) estimator. The main mathematical tool underlying these asymptotic results is a weighted approximation of the tail empirical quantile function (q.f.). This functional limit theorem will also enable us to derive the asymptotic normality of the quantile estimators $\hat{x}_{p_n}$ based on the general class of estimators $\hat{\gamma}_n$.

In the general case $\gamma \in I\!R$, a necessary and sufficient condition for $F \in D(G_\gamma)$ is

$$\frac{F^{-1}(1-\lambda t) - F^{-1}(1-\lambda)}{a(\lambda)} \longrightarrow \frac{t^{-\gamma}-1}{\gamma}, \qquad t > 0, \tag{6}$$

as $\lambda \downarrow 0$ for some normalizing function $a : (0,1) \to (0,\infty)$; for $\gamma = 0$ the right hand side is interpreted as $-\log t$. Hence the following extreme quantile estimator can be motivated in a similar fashion as $\hat{x}_{p_n}$ above:

$$\tilde{x}_{p_n} := X_{n-k_n:n} + \hat{a}(k_n/n) \frac{(np_n/k_n)^{-\hat{\gamma}_n} - 1}{\hat{\gamma}_n}. \tag{7}$$

Here $\hat{a}(k_n/n)$ denotes a suitable estimator for $a(k_n/n)$, e.g.

$$\hat{a}(k_n/n) := \frac{\hat{\gamma}_n}{2^{\hat{\gamma}_n} - 1} \Big(X_{n-[k_n/2]:n} - X_{n-k_n:n} \Big), \tag{8}$$

which is obtained by choosing $\lambda = k_n/n$ and $t = 1/2$ in (6) and replacing the unknown quantiles by their respective empirical counterparts. In an i.i.d.-setting, the limit distribution of particular estimators of that type was established by Dekkers et al. (1989) and de Haan and Rootzén (1993), among others. However, no general approach to construct estimators of a and hence of extreme quantiles, comparable to the broad class of statistical tail functionals for γ introduced in Drees (1998a), has been proposed so far. It should be emphasized that within a parametric model for the dependence structure, one may often construct more efficient estimators for extreme quantiles; see Section 3 for an example. However, these estimators will be very sensitive to deviations from the parametric model, while the estimators under consideration in the present paper yield reasonable results under mild structural assumptions.

The paper is organized as follows. In Section 2, first the approximation result for the tail empirical quantile function of absolutely regular time series established in Drees (2000) is specialized to the case $\gamma > 0$. Here we impose conditions which are more restrictive but often more easily checked. From this we derive the asymptotic normality of quantile estimators of type (4). Estimators of the asymptotic variance and resulting confidence intervals are also discussed. As examples of time series models satisfying our conditions, a particular class of nonlinear time series including ARCH(1) models and linear time series are considered in Section 3. Then the theory is applied to a time series of log-returns of the Nasdaq Composite index. It turns out that the classical i.i.d.-theory leads to confidence intervals that are much shorter than the new confidence intervals that take into account the serial dependence.

In Section 5 the finite sample performance of the statistical procedures is examined in a simulation study for several time series models with heavy tails. Again the confidence intervals proposed in the present paper usually have coverage probabilities that are much closer to the nominal level than those of classical confidence intervals.

Finally, we establish the asymptotic normality of a broad class of estimators for γ and a for general $\gamma \in I\!R$ and conclude the asymptotic normality of the resulting quantile estimators of type (7).

2 Asymptotics for $\gamma > 0$

In the sequel we assume that the sequence X_i, $i \in I\!N$, is strictly stationary, i.e., $\mathcal{L}((X_i)_{i\in I\!N}) = \mathcal{L}((X_{i+n})_{i\in I\!N})$ for all $n \in I\!N$. Since the quantile estimator of type (4) depends only on the $k_n + 1$ largest order statistics, it is essential to analyze the asymptotic behavior of the pertaining tail empirical quantile function

$$Q_{n,k_n}(t) = Q_n(t) := X_{n-[k_nt]:n}, \qquad 0 < t \le 1.$$

Drees (2000) gave a weighted approximation of this stochastic process for stationary β–mixing time series with a continuous marginal d.f. $F \in D(G_\gamma)$, $\gamma \in I\!R$. (In fact, the continuity assumption may be dropped; see Remark 2 of that paper.) Recall that X_i, $i \in I\!N$, is called β–mixing (or absolutely regular) if

$$\beta(l) := \sup_{m\in I\!N} E\Big(\sup_{A\in\mathcal{B}_{m+l+1}^\infty} |P(A|\mathcal{B}_1^m) - P(A)|\Big) \longrightarrow 0$$

as $l \to \infty$, where $\mathcal{B}_1^m$ and $\mathcal{B}_{m+l+1}^\infty$ denote the σ–fields generated by $(X_i)_{1\le i\le m}$ and $(X_i)_{m+l+1\le i}$, respectively. More precisely, it is assumed that there exists a sequence l_n, $n \in I\!N$, such that

(C1) $\displaystyle\lim_{n\to\infty} \frac{\beta(l_n)}{l_n} n + l_n k_n^{-1/2} \log^2 k_n = 0.$

Since the β–coefficients measure the influence of the past on future events, condition (C1) states that this influence vanishes sufficiently fast as past and future are separated by a time interval of increasing length. Typical examples are Harris recurrent Markov chains, for which the β–coefficients decrease geometrically; see Doukhan (1994), Section 2.4, for details. More specific, ARMA, ARCH and GARCH time series are geometrically β–mixing under rather mild conditions (Doukhan, 1994, Section 2.3). In these cases, condition (C1) is satisfied with $l_n = [C \log n]$ for a sufficiently large constant $C > 0$ and k_n satisfying

$$\log^2 n \log^4(\log n) = o(k_n). \tag{9}$$

Furthermore, we assume a regularity condition for the joint tail of (X_1, X_{1+m}):

(C2) There exist $\varepsilon > 0$ and functions c_m, $m \in I\!N$, such that

$$\lim_{n\to\infty} \frac{n}{k_n} P\Big\{X_1 > F^{-1}\Big(1 - \frac{k_n}{n}x\Big), X_{1+m} > F^{-1}\Big(1 - \frac{k_n}{n}y\Big)\Big\} \longrightarrow c_m(x,y) \qquad \forall\, m \in I\!N,\ 0 < x, y \le 1 + \varepsilon.$$

In addition, we need a uniform bound on the probability that both X_1 and X_{1+m} belong to an extreme interval:

(C3) There exist $D_1 \ge 0$ and a sequence $\tilde\rho(m)$, $m \in I\!N$, satisfying $\sum_{m=1}^\infty \tilde\rho(m) < \infty$ such that

$$\frac{n}{k_n} P\{X_1 \in I_n(x,y), X_{1+m} \in I_n(x,y)\} \le (y-x)\Big(\tilde\rho(m) + D_1 \frac{k_n}{n}\Big) \qquad \forall\, m \in I\!N,\ 0 < x, y \le 1 + \varepsilon$$

with $I_n(x,y) = (F^{-1}(1-yk_n/n), F^{-1}(1-xk_n/n)]$.

REMARKS.

(i) Condition (C2) is satisfied if all vectors (X_1, X_{1+m}) belong to the domain of attraction of a bivariate extreme value distribution, that is, if the suitably standardized coordinatewise maxima of n i.i.d. copies of (X_1, X_{1+m}) converge to a nontrivial limiting distribution as n tends to ∞. If the marginals of the limiting vector are independent, then $c_m(x,y) = 0$ for all $m \in I\!N$ and all $0 < x, y \le 1+\varepsilon$.

(ii) It is readily seen that condition (C3) is met if the ρ-mixing coefficients of the time series are finitely summable, that is, $\sum_{l=1}^{\infty} \rho(l) < \infty$ with

$$\rho(l) := \sup_{m \in I\!N} \sup_{U \in L_2(\mathcal{B}_1^m), V \in L_2(\mathcal{B}_{m+l+1}^{\infty})} \frac{|\mathrm{Cov}(U,V)|}{(\mathrm{Var}(U)\mathrm{Var}(V))^{1/2}} \tag{10}$$

and $L_2(\mathcal{A})$ denoting the space of square integrable $\mathcal{A}, I\!B$–measurable functions.

□

The conditions (C2) and (C3) ensure that the suitably standardized covariance of the numbers of exceedances over different high quantiles of F converges to a limit covariance function as the sample size increases. Moreover, they imply a bound on the second moment of the number of observations in an extreme interval.

Proposition 2.1 *Suppose that $l_n = o(n/k_n)$ and that the conditions (C2) and (C3) are met. Then, for all $0 < x, y \le 1+\varepsilon$,*

$$\begin{aligned} \lim_{n\to\infty} \frac{n}{l_n k_n} &\mathrm{Cov}\Big(\sum_{i=1}^{l_n} 1_{\{X_i > F^{-1}(1-\frac{k_n}{n}x)\}}, \sum_{i=1}^{l_n} 1_{\{X_i > F^{-1}(1-\frac{k_n}{n}y)\}} \Big) \\ = \ & c(x,y) \end{aligned} \tag{11}$$

with

$$c(x,y) := x \wedge y + \sum_{m=1}^{\infty} (c_m(x,y) + c_m(y,x)) \in I\!R \tag{12}$$

and $x \wedge y := \min(x,y)$.
Moreover, there exists $D > 0$ such that for all $0 < x, y \le 1+\varepsilon$ and all $n \in I\!N$

$$\frac{n}{l_n k_n} E\Big(\sum_{i=1}^{l_n} 1_{\{X_i \in I_n(x,y)\}} \Big)^2 \le D\,(y-x) \tag{13}$$

Proof. In (C3) choose $y = 1+\varepsilon$ and let x tend to 0 to obtain

$$\begin{aligned} &\frac{n}{k_n} P\Big\{X_1 > F^{-1}\Big(1 - \frac{k_n}{n}(1+\varepsilon)\Big), X_{1+m} > F^{-1}\Big(1 - \frac{k_n}{n}(1+\varepsilon)\Big)\Big\} \\ &\leq \; (1+\varepsilon)\Big(\tilde\rho(m) + D_1 \frac{k_n}{n}\Big). \end{aligned}$$

Because of (C2), $\lim_{n\to\infty} \sum_{m=1}^{l_n} (\tilde\rho(m) + (D_1 + (1+\varepsilon)^2) k_n/n) = \sum_{m=1}^\infty \tilde\rho(m) < \infty$ and $l_n k_n/n \to 0$, Pratt's (1960) lemma yields

$$\begin{aligned} &\lim_{n\to\infty} \frac{n}{k_n} \sum_{m=1}^{l_n} \mathrm{Cov}\Big(1_{\{X_1 > F^{-1}(1-\frac{k_n}{n}x)\}}, 1_{\{X_{1+m} > F^{-1}(1-\frac{k_n}{n}y)\}}\Big) \\ &= \; \sum_{m=1}^\infty c_m(x,y) \in I\!R. \end{aligned}$$

Hence, by the stationarity of the time series,

$$\begin{aligned} &\frac{n}{k_n l_n} \sum_{1\le i<j\le l_n} \mathrm{Cov}\Big(1_{\{X_i > F^{-1}(1-\frac{k_n}{n}x)\}}, 1_{\{X_j > F^{-1}(1-\frac{k_n}{n}y)\}}\Big) \\ &= \; \frac{1}{l_n}\sum_{i=1}^{l_n} \frac{n}{k_n} \sum_{j=i+1}^{i+l_n-1} \mathrm{Cov}\Big(1_{\{X_i > F^{-1}(1-\frac{k_n}{n}x)\}}, 1_{\{X_j > F^{-1}(1-\frac{k_n}{n}y)\}}\Big) \\ &\quad -\frac{1}{l_n}\sum_{i=2}^{l_n} \frac{n}{k_n} \sum_{j=l_n+1}^{i+l_n-1} \mathrm{Cov}\Big(1_{\{X_i > F^{-1}(1-\frac{k_n}{n}x)\}}, 1_{\{X_j > F^{-1}(1-\frac{k_n}{n}y)\}}\Big) \\ &\longrightarrow \; \sum_{m=1}^\infty c_m(x,y) \end{aligned}$$

since the second term can be bounded by $\sum_{m=1}^{l_n-1} m(\tilde\rho(m) + (D_1 + (1+\varepsilon)^2)k_n/n)/l_n$ which tends to 0. Now (11) is obvious.
Likewise, one obtains

$$\frac{n}{l_n k_n} E\Big(\sum_{i=1}^{l_n} 1_{\{X_i \in I_n(x,y)\}}\Big)^2 \le (y-x)\Big(1 + 2\sum_{m=1}^{l_n-1}(\tilde\rho(m) + D_1\frac{k_n}{n})\Big)$$

so that (13) follows from the summability of $\tilde\rho(m)$ and $l_n k_n/n \to 0$. □

Remark. Using Theorem 1.1 of Shao (1995), in (13) one may even replace the second moment with the fourth moment. These moment conditions can be interpreted in terms of moments of cluster sizes of exceedances; see Drees (2000) for details. □

Finally, we need a condition on the rate of convergence of $k_n \to \infty$ to ensure that the extreme value approximation used in (4) is sufficiently accurate. For the sake of simplicity, we assume that the quantile function admits the following representation:

(C4) $\quad F^{-1}(1-t) = dt^{-\gamma}\Big(1 + r(t)\Big) \quad$ with $\quad |r(t)| \le \Phi(t)$

for some constant $d > 0$ and a function Φ which is τ-varying at 0 for some $\tau > 0$, or $\tau = 0$ and Φ is nondecreasing with $\lim_{t\downarrow 0}\Phi(t) = 0$.

Then we assume that k_n is an intermediate sequence such that

(C5) $\quad \lim_{n\to\infty} k_n^{1/2}\Phi(k_n/n) = 0.$

(However, see remark (ii) below the proof of Theorem 2.2 for more general conditions.)

Theorem 2.1 *Under the conditions (C1)–(C5) with $l_n = o(n/k_n)$ there exist versions of the tail empirical q.f. Q_n and a centered Gaussian process e with covariance function c defined by (12) such that*

$$\sup_{t\in(0,1]} t^{\gamma+1/2}(1+|\log t|)^{-1/2}\left|k_n^{1/2}\Big(\frac{Q_n(t)}{F^{-1}(1-k_n/n)} - t^{-\gamma}\Big) - \gamma t^{-(\gamma+1)}e(t)\right| \longrightarrow 0 \tag{14}$$

in probability.

Proof. In view of (C4), the remainder term defined in (3) equals $R(\lambda,t) = t^{-\gamma}\,O(\Phi(\lambda t) + \Phi(\lambda))$ uniformly for bounded t. Thus, because of the τ-variation of Φ with $\tau > 0$ respectively the monotonicity of Φ, $k_n^{1/2}\Phi(k_n/n) \to 0$ implies

$$\lim_{n\to\infty} k_n^{1/2} \sup_{0<t\le 1+\varepsilon} t^{\gamma+1/2}(1+|\log t|)^{-1/2}|R(k_n/n,t)| = 0.$$

Combining this with Proposition 2.1 and the subsequent remark, we see that the conditions of Theorem 3.1 of Drees (2000) are satisfied, from which the assertion is obvious. □

Remark. The conditions (C2) and (C3) are only needed to verify (11) and the analog to (13) for the fourth moment. Hence Theorem 2.1 holds under these considerably weaker (but somewhat more complex) conditions. If (C2) and (C3) are replaced with (11) and (13), then (14) holds with weight function $t^{\gamma+1/2}$ replaced with $t^{\gamma+3/4}$. See Drees (2002) for a more detailed discussion of these conditions. □

It is worth mentioning that for independent observations the Gaussian process e is a standard Brownian motion. Hence in that case Theorem 2.1 is essentially equivalent to Theorem 2.1(i) of Drees (1998b).
In the next step, we deduce the asymptotic normality of estimators of the extreme value index that use only the $k_n + 1$ largest order statistics, and of the pertaining quantile estimators of type (4). In Drees (1998a,b) it has been observed that almost every estimator $\hat{\gamma}_n$ of this type can be represented as a so-called statistical tail functional, i.e., as a smooth functional applied to the tail empirical q.f.: $\hat{\gamma}_n = T(Q_n)$.
To establish asymptotic normality for this class of estimators we impose the following regularity conditions on T:

(T0) T is a Borel-measurable real-valued functional on the set of functions $z \in D(0,1]$ satisfying $t^{\gamma+1/2}|\log t|^{-1/2} z(t) \to 0$ as $t \downarrow 0$.

(T1) T is scale-invariant: $\qquad T(az) = T(z) \quad$ for all $a > 0$.

(T2) $T\Big((t^{-\gamma})_{0<t\le 1}\Big) = \gamma$

(T3) There exists a signed measure $\nu_{T,\gamma}$ on (0,1] with $\int_{(0,1]} t^{-\gamma-1/2}(1+|\log t|)^{1/2}\,|\nu_{T,\gamma}|(dt) < \infty$ such that

$$\varepsilon_n^{-1}\Big(T\Big((t^{-\gamma}+\varepsilon_n z_n(t))_{0<t\le 1}\Big) - T\Big((t^{-\gamma})_{0<t\le 1}\Big)\Big) \longrightarrow \int_{(0,1]} z(t)\,\nu_{T,\gamma}(dt)$$

for all $\varepsilon_n \downarrow 0$ and z_n satisfying

$$\sup_{0<t\le 1} t^{\gamma+1/2}(1+|\log t|)^{-1/2}|z_n(t) - z(t)| \longrightarrow 0$$

for some continuous function z as described in (T0).

Condition (T2) means that exactly the true extreme value index is obtained if one plugs in the limiting Pareto q.f. instead of the tail empirical q.f. Condition (T3) can be interpreted as T being Hadamard-differentiable at $(t^{-\gamma})_{0<t\le 1}$ in a suitable function space. Refer to Drees (1998a,b) for a thorough discussion of these regularity conditions. In particular, there it is shown that the Hill estimator and the maximum likelihood estimator in a generalized Pareto model satisfy these conditions with signed measures

$$\nu_{H,\gamma}(dt) = t^{\gamma}\,dt - \varepsilon_1(dt)$$

and

$$\nu_{ML,\gamma}(dt) = \frac{(\gamma+1)^2}{\gamma^2}\Big(t^{\gamma} - (2\gamma+1)t^{2\gamma}\Big)\,dt + \frac{\gamma+1}{\gamma}\varepsilon_1(dt),$$

when ε_1 denotes the Dirac measure with mass 1 at 1. Other examples are the Pickands estimator (Pickands, 1975), the moment estimator proposed by Dekkers et al. (1989) and generalized probability weighted moment estimators.

In addition to (C5) we need the following assumption about the relationship between the number of order statistics used for estimation and the expected number of exceedances over the extreme quantile to be estimated:

$$\lim_{n\to\infty} k_n^{-1/2} \log(np_n) = 0, \qquad \lim_{n\to\infty} np_n/k_n = 0 \tag{15}$$

The first assumption is very weak; it is satisfied if, e.g., $n^{-m} = o(p_n)$ for some $m > 0$ and $\log^2 n = o(k_n)$. Notice that we allow np_n to tend to infinity, but the whole extreme value approach only makes sense if $np_n = o(k_n)$.

Theorem 2.2 *Suppose that the conditions of Theorem 2.1 and condition (15) are met. If $\hat{\gamma}_n = T(Q_n)$ with T fulfilling conditions (T0)–(T3), then*

$$\frac{k_n^{1/2}}{\log(k_n/(np_n))} \log \frac{\hat{x}_{p_n}}{x_{p_n}} \sim \frac{k_n^{1/2}}{\log(k_n/(np_n))} \Big(\frac{\hat{x}_{p_n}}{x_{p_n}} - 1 \Big) \quad \sim \quad k_n^{1/2}(\hat{\gamma}_n - \gamma) \longrightarrow \mathcal{N}(0, \sigma^2_{T,\gamma}) \tag{16}$$

weakly with

$$\sigma^2_{T,\gamma} = \gamma^2 \int_{(0,1]} \int_{(0,1]} (st)^{-(\gamma+1)} c(s,t)\, \nu_{T,\gamma}(ds)\, \nu_{T,\gamma}(dt).$$

Proof. The weak convergence of $k_n^{1/2}(\hat{\gamma}_n - \gamma)$ follows from the following calculation:

$$\begin{aligned} \hat{\gamma}_n &\overset{(T1)}{=} T\Big(\frac{Q_n}{F^{-1}(1 - k_n/n)} \Big) \\ &=^d T\Big(\Big(t^{-\gamma} + k_n^{-1/2} \gamma t^{-(\gamma+1)} e(t) + o_P(k_n^{-1/2}) \Big)_{0<t\le 1} \Big) \\ &\overset{(T3)}{=} T\Big((t^{-\gamma})_{0<t\le 1} \Big) + k_n^{-1/2} \gamma \int_{(0,1]} t^{-(\gamma+1)} e(t)\, \nu_{T,\gamma}(dt) + o_P(k_n^{-1/2}). \end{aligned}$$

Hence by (T2)

$$k_n^{1/2}(\hat{\gamma}_n - \gamma) \longrightarrow \gamma \int_{(0,1]} t^{-(\gamma+1)} e(t)\, \nu_{T,\gamma}(dt),$$

which proves the assertion; we refer to Drees (1998a) for technical details.

Because of $\log(1+x) \sim x$ as $x \to 0$, it remains to verify that

$$\frac{1}{\log(k_n/(np_n))}\Big(\frac{\hat{x}_{p_n}}{x_{p_n}} - 1\Big) = \hat{\gamma}_n - \gamma + o_P(k_n^{-1/2}). \tag{17}$$

To this end, check that

$$\begin{aligned}
&\hat{x}_{p_n} - x_{p_n} \\
&= (Q_n(1) - F^{-1}(1-k_n/n))\Big(\frac{np_n}{k_n}\Big)^{-\hat{\gamma}_n} \\
&\quad + F^{-1}(1-k_n/n)\bigg(\Big(\frac{np_n}{k_n}\Big)^{-\hat{\gamma}_n} - \Big(\frac{np_n}{k_n}\Big)^{-\gamma}\bigg) \\
&\quad + \Big(F^{-1}(1-k_n/n)\Big(\frac{np_n}{k_n}\Big)^{-\gamma} - F^{-1}(1-p_n)\Big) \\
&=: I + II + III.
\end{aligned}$$

As in the proof of Theorem 2.1, (C4) and (C5) imply

$$\begin{aligned}
\bigg|\frac{F^{-1}(1-p_n)}{F^{-1}(1-k_n/n)}\Big(\frac{np_n}{k_n}\Big)^{\gamma} - 1\bigg| &= \bigg|R\Big(\frac{k_n}{n}, \frac{np_n}{k_n}\Big)\bigg|\Big(\frac{np_n}{k_n}\Big)^{\gamma} \\
&= O(\Phi(k_n/n)) \\
&= o(k_n^{-1/2}).
\end{aligned}$$

Hence Theorem 2.1, condition (15) and $\hat{\gamma}_n - \gamma = O_P(k_n^{-1/2})$ yield

$$\begin{aligned}
&\frac{I}{x_{p_n}\log(k_n/(np_n))} \\
&= k_n^{-1/2}(e(1) + o_P(1))\frac{F^{-1}(1-k_n/n)}{F^{-1}(1-p_n)}\frac{1}{\log(k_n/(np_n))}\Big(\frac{np_n}{k_n}\Big)^{-\hat{\gamma}_n} \\
&= o_P(k_n^{-1/2}).
\end{aligned}$$

Likewise

$$\frac{III}{x_{p_n}\log(k_n/(np_n))} = \bigg(\frac{F^{-1}(1-k_n/n)}{F^{-1}(1-p_n)}\Big(\frac{np_n}{k_n}\Big)^{-\gamma} - 1\bigg)\Big/\log\frac{k_n}{np_n} = o(k_n^{-1/2}).$$

Finally, because of $\partial/(\partial\tau)x^\tau = x^\tau \log x$, using the mean value theorem and (15) one obtains

$$\begin{aligned}
\frac{II}{x_{p_n}\log(k_n/(np_n))} &= \frac{F^{-1}(1-k_n/n)}{F^{-1}(1-p_n)}\Big(\frac{np_n}{k_n}\Big)^{-\gamma}\Big(\frac{np_n}{k_n}\Big)^{\vartheta(\gamma-\hat{\gamma}_n)}(\hat{\gamma}_n - \gamma) \\
&= (\hat{\gamma}_n - \gamma)(1 + o_P(1))
\end{aligned}$$

for some $\vartheta \in (0,1)$. Adding the expressions for I,II and III, one arrives at (17). $\square$

Remarks.

(i) If $np_n \to \infty$ one may estimate x_{p_n} consistently also by the empirical quantile $X_{n-[np_n]:n}$, i.e., $X_{n-[np_n]:n}/x_{p_n} \to 1$ in probability. However, typically the relative estimation error will be of the order $(np_n)^{-1/2}$. Particularly this holds if conditions (C1)–(C5) are fulfilled with k_n replaced by $[np_n]+1$. So the quantile estimator $\hat{x}_{p_n}$ is asymptotically more efficient, provided $np_n = o(k_n)$. Of course, this higher efficiency is achieved only under considerably stronger model assumptions than necessary to ensure consistency of the empirical quantile.

(ii) Often time series models are described implicitly as stationary solutions of certain equations involving innovations with a given distribution (see Section 3 for examples). Then usually no analytical expression for the distribution function F of the time series at any time t is known. In this situation it might be difficult to verify condition (C4), but for $F \in D(G_\gamma)$ the following milder condition replacing (C4) and (C5) is *always* satisfied for some intermediate sequence k_n tending to infinity not too fast:

$$\lim_{n\to\infty} k_n^{1/2} \sup_{0<t\le 1+\varepsilon} t^{\gamma+1/2}(1+|\log t|)^{-1/2}|R(k_n/n,t)| = 0 \tag{18}$$

with R defined by (3); cf. Drees (1998a,2000). Under the conditions (C1)–(C3) and (18) the assertion of Theorem 2.1 holds. To prove asymptotic normality of the quantile estimators, in addition to (15) we need

$$\lim_{n\to\infty} \frac{k_n^{1/2}}{\log(k_n/(np_n))}\Big(\frac{np_n}{k_n}\Big)^\gamma R\Big(\frac{k_n}{n},\frac{np_n}{k_n}\Big) = 0. \tag{19}$$

This convergence implies

$$\frac{F^{-1}(1-p_n)}{F^{-1}(1-k_n/n)}\Big(\frac{np_n}{k_n}\Big)^\gamma - 1 = \Big(\frac{np_n}{k_n}\Big)^\gamma R\Big(\frac{k_n}{n},\frac{np_n}{k_n}\Big) = o\Big(k_n^{-1/2}\log\frac{k_n}{np_n}\Big).$$

Hence the proof of Theorem 2.2 shows that convergence (16) holds under these milder conditions.

$\square$

In simulations it turned out that in most cases the normal approximation is more accurate for $\log(\hat{x}_{p_n}/x_{p_n})$ than for $\hat{x}_{p_n}/x_{p_n}-1$. Heuristically, this may be explained by the fact that $\log \hat{x}_{p_n}$ is a linear function of $\hat{\gamma}_n$, whose estimation error determines the dominating part of the error of the quantile estimator. So if the distribution of $\hat{\gamma}_n$ is well approximated by a normal distribution (which is usually true for the Hill estimator), this often also holds for $\log \hat{x}_{p_n}$ but not necessarily for $\hat{x}_{p_n}$ which, according to the δ–method, is only locally linear in $\hat{\gamma}_n$.

In order to construct confidence intervals based on (16), one has to estimate the asymptotic variance $\sigma^2_{T,\gamma}$, which depends not only on γ but also on the unknown limiting covariance function c. Instead of trying to estimate this function nonparametrically, it seems more reasonable to employ (16) for the estimation of $\sigma^2_{T,\gamma}$.

In a blocks approach, one would split the time series in blocks of constant length m_n, say, and estimate an extreme quantile or γ for each block separately. If m_n is not too small, by condition (C1) these estimates are almost independent. Hence one may estimate $\sigma^2_{T,\gamma}$ by the suitably standardized sample variance of the block estimates. In practice, however, this procedure will be rather cumbersome, because one must find not only a suitable block length m_n, but also a number $\tilde{k}_n < m_n$ such that in every block it is reasonable to use the $\tilde{k}_n + 1$ largest order statistics for estimation. Given that it is often not easy to choose k_n for one fixed sample, this may be a delicate task.

As an alternative, we propose an approach which uses a process version of convergence (16). For this, note that under very weak conditions the covariance function c is homogeneous:

$$c(\lambda x, \lambda y) = \lambda c(x, y), \qquad \lambda, x, y \in [0, 1], \tag{20}$$

and that hence the Gaussian process e is self-similar, i.e.

$$e(\lambda \cdot) =^d \lambda^{1/2} e(\cdot), \qquad \lambda \in [0, 1]. \tag{21}$$

For example, if condition (11) holds for k_n and two sequences $k_{n,\lambda_j} \sim \lambda_j k_n$

with $\lambda_j \in (0,1)$, $j = 1,2$, then by the continuity of c

$$\begin{aligned}
& c(x,y) \\
\to\; & \frac{n}{l_n k_{n,\lambda_j}} \mathrm{Cov}\Big(\sum_{i=1}^{l_n} 1\{X_i > F^{-1}(1 - \tfrac{k_{n,\lambda_j}}{n} x)\}, \sum_{i=1}^{l_n} 1\{X_i > F^{-1}(1 - \tfrac{k_{n,\lambda_j}}{n} y)\}\Big) \\
\sim\; & \frac{n}{l_n k_n \lambda_j} \mathrm{Cov}\Big(\sum_{i=1}^{l_n} 1\{X_i > F^{-1}(1 - \tfrac{k_n}{n}\tfrac{k_{n,\lambda_j}}{k_n} x)\}, \\
& \qquad\qquad \sum_{i=1}^{l_n} 1\{X_i > F^{-1}(1 - \tfrac{k_n}{n}\tfrac{k_{n,\lambda_j}}{k_n} y)\}\Big) \\
\to\; & \frac{1}{\lambda_j} c(\lambda_j x, \lambda_j y).
\end{aligned}$$

If $\log \lambda_1 / \log \lambda_2$ is irrational then, by Theorem 1.4.3 of Bingham et al. (1987), this in turn implies (20). Likewise, if convergence (14) holds for k_n and k_{n,λ_j} then from the regular variation of $F^{-1}(1-\cdot)$ it follows that $e(\cdot) =^d \lambda_j^{-1/2} e(\lambda_j \cdot)$ and thus (21) and (20).

Now one may argue heuristically as follows. Denote by $\hat\gamma_n^{(i)}$ the estimator for γ that uses the $i+1$ largest order statistics: $\hat\gamma_n^{(i)} = T(Q_n(i/k_n\cdot))$. Under a slightly stronger differentiability condition as (T3), one obtains as in Theorem 2.2 the approximation

$$k_n^{1/2}(\hat\gamma_n^{([k_n s])} - \gamma) \approx \frac{\gamma}{s} \int_{(0,1]} t^{-(\gamma+1)} e(st)\, \nu_{T,\gamma}(dt) =: Z_{T,\gamma}(s) \tag{22}$$

for $1/k_n \le s \le 1$. Notice that, by the homogeneity property (20) of c,

$$\tilde Z_{T,\gamma}(u) := \mathrm{e}^{u/2} Z_{T,\gamma}(\mathrm{e}^u), \qquad u \in (-\infty, 0], \tag{23}$$

defines a strictly stationary centered Gaussian process with covariance function

$$\begin{aligned}
& \mathrm{Cov}(\tilde Z_{T,\gamma}(u), \tilde Z_{T,\gamma}(v)) \\
&= \gamma^2 \exp\Big(-\frac{u+v}{2}\Big) \int_{(0,1]}\int_{(0,1]} (st)^{-(\gamma+1)} c(\mathrm{e}^u s, \mathrm{e}^v t)\, \nu_{T,\gamma}(ds)\, \nu_{T,\gamma}(dt) \\
&= \gamma^2 \exp\Big(\frac{u-v}{2}\Big) \int_{(0,1]}\int_{(0,1]} (st)^{-(\gamma+1)} c(s, \mathrm{e}^{v-u} t)\, \nu_{T,\gamma}(ds)\, \nu_{T,\gamma}(dt)
\end{aligned}$$

depending only on $u-v$. Using the ergodic theorem one can show that

$$\begin{aligned}
&\Big(\log \frac{k_n}{j_n}\Big)^{-1} \sum_{i=j_n}^{k_n} (\hat\gamma_n^{(i)} - \hat\gamma_n^{(k_n)})^2 \\
&\quad\approx \quad \Big(\log \frac{k_n}{j_n}\Big)^{-1} \int_{j_n/k_n}^1 (Z_{T,\gamma}(s) - Z_{T,\gamma}(1))^2 \, ds \qquad (24)\\
&\quad\sim \quad \Big(\log \frac{k_n}{j_n}\Big)^{-1} \int_{\log(j_n/k_n)}^0 \tilde Z_{T,\gamma}^2(u)\, du \\
&\quad\longrightarrow \quad E(\tilde Z_{T,\gamma}^2(0)) \;=\; \sigma_{T,\gamma}^2,
\end{aligned}$$

provided $k_n/j_n \to \infty$ (refer to the proof of Theorem 2.3 for details). Unfortunately, from Theorem 2.1 it can only be shown that approximation (24) is sufficiently accurate for some sequence $j_n = o(k_n)$; for a more precise assertion about j_n one would need the rate of convergence in (14).

Theorem 2.3 *Suppose that (20) (or, equivalently, (21)) holds and that T is Fréchet differentiable at $(t^{-\gamma})_{0<t\le 1}$:*

$$\varepsilon^{-1}\Big(T\Big((t^{-\gamma} + \varepsilon z(t))_{0<t\le1}\Big) - T\Big((t^{-\gamma})_{0<t\le1}\Big)\Big) \longrightarrow \int_{(0,1]} z(t)\, \nu_{T,\gamma}(dt) \quad (25)$$

as $\varepsilon \downarrow 0$ uniformly for all z satisfying

$$\sup_{0<t\le1} t^{\gamma+1/2}(1+|\log t|)^{-1/2}|z(t)| \le 1.$$

Then, under the conditions of Theorem 2.2, there exists a sequence $j_n = o(k_n)$ such that

$$\hat\sigma_{T,\gamma,1}^2 \;:=\; \Big(\log \frac{k_n}{j_n}\Big)^{-1} \sum_{i=j_n}^{k_n} (\hat\gamma_n^{(i)} - \hat\gamma_n^{(k_n)})^2 \longrightarrow \sigma_{T,\gamma}^2 \qquad (26)$$

$$\hat\sigma_{T,\gamma,2}^2 \;:=\; \Big(\log \frac{k_n}{j_n}\Big)^{-1} \sum_{i=j_n}^{k_n} \Big(\frac{\log(\hat x_{p_n}^{(i)}/\hat x_{p_n}^{(k_n)})}{\log(i/(np_n))}\Big)^2 \longrightarrow \sigma_{T,\gamma}^2 \qquad (27)$$

in probability. Here $\hat\gamma_n^{(i)} = T(Q_n(i/k_n\cdot))$ and $\hat x_{p_n}^{(i)}$ is defined as in (4) with $\hat\gamma_n$ replaced by $\hat\gamma_n^{(i)}$ and k_n by i.

PROOF. By (T1), (T2) and the differentiability assumption (25)

$$
\begin{aligned}
\hat{\gamma}_n^{(i)} &= T\Big(Q_n(\frac{i}{k_n}\cdot)/F^{-1}(1-k_n/n)\Big) \\
&=^d T\bigg(\bigg(\Big(\frac{i}{k_n}t\Big)^{-\gamma} + \gamma k_n^{-1/2}\Big(\frac{i}{k_n}t\Big)^{-(\gamma+1)} e\Big(\frac{i}{k_n}t\Big) \\
&\qquad + o_P\Big(k_n^{-1/2}\Big(\frac{i}{k_n}t\Big)^{-(\gamma+1)+j/8}\Big(1+\Big|\log\Big(\frac{i}{k_n}t\Big)\Big|\Big)^{1/2}\Big)\bigg)_{0<t\le 1}\bigg) \\
&= T\bigg(\bigg(t^{-\gamma} + \gamma\frac{k_n^{1/2}}{i}t^{-(\gamma+1)}e\Big(\frac{i}{k_n}t\Big) + \\
&\qquad o_P\Big(k_n^{-1/2}t^{-(\gamma+1)+j/8}(1+|\log t|)^{1/2}\Big)\bigg)_{0<t\le 1}\bigg) \\
&= \gamma + \gamma\frac{k_n^{1/2}}{i}\int_{(0,1]} t^{-(\gamma+1)}e\Big(\frac{i}{k_n}t\Big)\,\nu_{T,\gamma}(dt) + o_P(k_n^{-1/2})
\end{aligned}
$$

uniformly for $s_0 k_n \le i \le k_n$ and all $s_0 > 0$. Hence, by the continuity of e and a standard diagonal argument, there exists a sequence $s_n \downarrow 0$ such that

$$
\sup_{s_n \le s \le 1}\Big|k_n^{1/2}(\hat{\gamma}_n^{([k_n s])} - \gamma) - Z_{T,\gamma}(s)\Big| \longrightarrow 0
$$

in probability with $Z_{T,\gamma}$ defined in (22).
Therefore, for $j_n := [s_n k_n] + 1$ and $\tilde{Z}_{T,\gamma}$ defined by (23),

$$
\begin{aligned}
\sum_{i=j_n}^{k_n}(\hat{\gamma}_n^{(i)} - \hat{\gamma}_n^{(k_n)})^2 &= \int_{j_n/k_n}^{1}\Big(Z_{T,\gamma}(s) - Z_{T,\gamma}(1)\Big)^2 ds\cdot(1+o_P(1)) \qquad (28) \\
&= \int_{\log(j_n/k_n)}^{0}\Big(\tilde{Z}_{T,\gamma}(u) - \mathrm{e}^{u/2}\tilde{Z}_{T,\gamma}(0)\Big)^2 du\cdot(1+o_P(1)).
\end{aligned}
$$

By the stationarity of $\tilde{Z}_{T,\gamma}$ and the ergodic theorem (see, e.g., Cramér and Leadbetter, 1967, p. 151) one has

$$
\Big(\log\frac{k_n}{j_n}\Big)^{-1}\int_{\log(j_n/k_n)}^{0}\tilde{Z}_{T,\gamma}^2(u)\,du \longrightarrow E\tilde{Z}_{T,\gamma}^2(0) = \sigma_{T,\gamma}^2 \quad \text{a.s.}
$$

Hence assertion (26) follows from $\int_{\log(j_n/k_n)}^{0}(\mathrm{e}^{u/2}\tilde{Z}_{T,\gamma}(0))^2\,du = O(1)$.
Similarly, one can show that for some $s_n \downarrow 0$

$$
\sup_{s_n\le s\le 1}\Big|\frac{k_n^{1/2}}{\log([k_n s]/(np_n))}\log\frac{\hat{x}_{p_n}^{([k_n s])}}{x_{p_n}} - Z_{T,\gamma}(s)\Big| \longrightarrow 0
$$

in probability. Without loss of generality one may assume that $\log(s_n) = o(\log(k_n/(np_n)))$. Hence

$$
\begin{aligned}
&\sum_{i=j_n}^{k_n}\Big(\frac{\log(\hat{x}_{p_n}^{(i)}/\hat{x}_{p_n}^{(k_n)})}{\log(i/(np_n))}\Big)^2 \\
&= \int_{j_n/k_n}^1 \Big(Z_{T,\gamma}(s) - \frac{\log(k_n/(np_n))}{\log([k_n s]/(np_n))} Z_{T,\gamma}(1)\Big)^2\, ds\cdot(1+o_P(1)) \qquad (29)\\
&= \int_{\log(j_n/k_n)}^0 \Big(\tilde{Z}_{T,\gamma}(u) - \frac{\log(k_n/(np_n))}{\log([k_n \mathrm{e}^u]/(np_n))} \mathrm{e}^{u/2}\tilde{Z}_{T,\gamma}(0)\Big)^2\, du\cdot(1+o_P(1)).
\end{aligned}
$$

Now assertion (27) follows by the above arguments and

$$
\frac{\log(k_n/(np_n))}{\log([k_n \mathrm{e}^u]/(np_n))} \le \frac{1}{1+\log s_n/\log(k_n/(np_n))} \longrightarrow 1
$$

for all $\log(j_n/k_n) \le u \le 0$. $\square$

The proof shows that the left-hand sides of (26) and (27) are consistent estimators of $\sigma_{T,\gamma}^2$ for all sequences $(j_n)_{n\in I\!N}$ such that j_n/k_n converges to 0 not too fast. In practice usually one may choose j_n rather small. Indeed, even the smallest number for which the estimator is defined will often do the job; cf. Sections 4 and 5.
In the proof it was also shown that in (28) and in (29) the terms pertaining to $Z_{T,\gamma}(1)$ are asymptotically negligible. This suggests the approximation

$$
E\sum_{i=j_n}^{k_n}\Big(\frac{\log(\hat{x}_{p_n}^{(i)}/\hat{x}_{p_n}^{(k_n)})}{\log(i/(np_n))}\Big)^2 \approx E\int_{j_n/k_n}^1 Z_{T,\gamma}^2(s)\, ds = \int_{j_n/k_n}^1 \frac{\sigma_{T,\gamma}^2}{s}\, ds = \sigma_{T,\gamma}^2 \log\frac{k_n}{j_n},
$$

and likewise for the estimator $\hat{\sigma}_{T,\gamma,1}^2$, which leads to the normalizing factor $\log(k_n/j_n)$ in the definition of $\hat{\sigma}_{T,\gamma,1}^2$ and $\hat{\sigma}_{T,\gamma,2}^2$. For moderate sample sizes, however, this approximation is too crude, that is, it overestimates the left-hand side considerably and hence yields too short confidence intervals. More appropriate would be the normalizing factor

$$
\begin{aligned}
&\frac{1}{k_n\sigma_{T,\gamma}^2} E\sum_{i=j_n}^{k_n}\Big(Z_{T,\gamma}\Big(\frac{i}{k_n}\Big) - \frac{\log(k_n/(np_n))}{\log(i/(np_n))} Z_{T,\gamma}(1)\Big)^2 \\
&= \sum_{i=j_n}^{k_n} \frac{1}{i} - \frac{2}{\sigma_{T,\gamma}^2}\frac{\log(k_n/(np_n))}{\log(i/(np_n))}\frac{\mathrm{Cov}\Big(Z_{T,\gamma}(i/k_n), Z_{T,\gamma}(1)\Big)}{k_n} \\
&\qquad + \Big(\frac{\log(k_n/(np_n))}{\log(i/(np_n))}\Big)^2\frac{1}{k_n}. \qquad (30)
\end{aligned}
$$

Unfortunately, the covariance of $Z_{T,\gamma}(i/k_n)$ and $Z_{T,\gamma}(1)$ depends on the unknown limiting covariance function c, so (30) cannot be used directly for the estimation of the asymptotic variance. Instead we propose to use the lower bound

$$\sum_{i=j_n}^{k_n} \frac{1}{i} - \frac{2}{\sigma^2_{T,\gamma}} \cdot \frac{\log(k_n/(np_n))}{\log(i/(np_n))} \frac{\Big(\mathrm{Var} Z_{T,\gamma}(i/k_n) \cdot \mathrm{Var} Z_{T,\gamma}(1)\Big)^{1/2}}{k_n}$$
$$+ \Big(\frac{\log(k_n/(np_n))}{\log(i/(np_n))}\Big)^2 \frac{1}{k_n}$$
$$= \sum_{i=j_n}^{k_n} \Big(i^{-1/2} - \frac{\log(k_n/(np_n))}{\log(i/(np_n))} k_n^{-1/2}\Big)^2 \tag{31}$$

as normalizing factor substituting $\log(k_n/j_n)$ in (27). Note that (31) is asymptotically equivalent to $\log(k_n/j_n)$. Thus the resulting modified estimator

$$\hat{\sigma}^2_{T,\gamma,3} := \Bigg(\sum_{i=j_n}^{k_n} \Big(i^{-1/2} - \frac{\log(k_n/(np_n))}{\log(i/(np_n))} k_n^{-1/2}\Big)^2 \Bigg)^{-1} \sum_{i=j_n}^{k_n} \Big(\frac{\log(\hat{x}^{(i)}_{p_n}/\hat{x}^{(k_n)}_{p_n})}{\log(i/(np_n))}\Big)^2 \tag{32}$$

is also consistent for the asymptotic variance, yet for finite sample sizes it yields substantially more conservative confidence intervals. For example, the pertaining two-sided asymptotic confidence interval to the nominal coverage probability $1-\alpha \in (0,1)$ is given by

$$\Big[\hat{x}_{p_n} \exp\Big(- z_{\alpha/2}\hat{\sigma}_{T,\gamma,3} k_n^{-1/2} \log\frac{k_n}{np_n}\Big), \hat{x}_{p_n} \exp\Big(z_{\alpha/2}\hat{\sigma}_{T,\gamma,3} k_n^{-1/2} \log\frac{k_n}{np_n}\Big)\Big], \tag{33}$$

with $z_{\alpha/2}$ denoting the $(1-\alpha/2)$–quantile of the standard normal distribution. For that reason, we will mainly use $\hat{\sigma}^2_{T,\gamma,3}$ to construct confidence intervals in our application and the simulation study.
Note that one may also modify the estimator $\hat{\sigma}^2_{T,\gamma,1}$ in a similar way, but it seems more natural to use a variance estimator that is based on quantile estimators if one is interested in confidence intervals for extreme quantiles.

3 Time Series Models

Here we demonstrate the applicability of the theory outlined in the previous section to specific time series models: first we consider solutions of certain stochastic recurrence equations, including ARCH(1) time series, and then linear time series.

3.1 Solutions of a Stochastic Difference Equation

Consider the stochastic recursion

$$X_i = A_i X_{i-1} + B_i, \qquad i \in I\!N, \tag{34}$$

where $(A_i, B_i), i \in I\!N$, denote i.i.d. $I\!R^2$–valued random vectors. Such stochastic difference equations occur in many contexts. For example, X_i describes the balance of an account at time i if A_i denotes the inverse of the stochastic discount factor for the time interval from $i-1$ to i and B_i a random deposit made just before time i; see Embrechts et al. (1997, Section 8.4.1) for details. Closely related is the first order autoregressive conditional heteroscedastic (ARCH(1)) time series, which is a popular simple model for returns on a risky investment:

$$Y_i = \Big(\alpha_0 + \alpha_1 Y_{i-1}^2\Big)^{1/2} Z_i, \qquad i \in I\!N, \tag{35}$$

where Z_i are i.i.d. innovations with mean zero and variance 1. Then $X_i = Y_i^2$ satisfies equation (34) with $A_i = \alpha_1 Z_i^2$ and $B_i = \alpha_0 Z_i^2$. Further applications of model (34) were discussed by Vervaat (1979).
In the sequel, we assume that A_1 and B_1 have an absolute continuous d.f. Kesten (1973) proved that a stationary solution of (34) with heavy-tailed marginals exists if

(D1) $A_1, B_1 > 0$ and there exists $\kappa > 0$ such that

$$EA_1^\kappa = 1, \quad E\Big(A_1^\kappa \max(\log A_1, 0)\Big) < \infty \quad \text{and} \quad EB_1^\kappa \in (0,\infty).$$

Then the d.f. F of X_1 belongs to the domain of attraction of G_γ with extreme value index $\gamma = 1/\kappa$. Indeed, F satisfies $1 - F(x) \sim cx^{-\kappa}$ as $x \to \infty$ for some $c > 0$. Note that one obtains heavy tails for X_i even if the 'random coefficients' A_i and B_i have light tails.
In Drees (2000), Corollary 4.1, it is shown that the conditions of Theorem 2.1 are satisfied and hence (14) holds with covariance function

$$c(x,y) = x \wedge y + \sum_{j=1}^\infty \left(x \int_0^{y/x} P\Big\{\prod_{i=1}^j A_i > t^\gamma\Big\}\, dt + y \int_0^{x/y} P\Big\{\prod_{i=1}^j A_i > t^\gamma\Big\}\, dt \right) \tag{36}$$

if, in addition, the following conditions hold:

(D2) There exists $\xi > 0$ such that $\quad EA_1^{\kappa+\xi} < \infty \quad$ and $\quad EB_1^{\kappa+\xi} < \infty$.

(D3) $\log^2 n \log^4(\log n) = o(k_n)$ and $k_n = o(n^{2\tau/(2\tau+1)})$ where $\tau > 0$ is such that

$$1 - F(x) = dx^{-1/\gamma}\Big(1 + O(x^{-\tau/\gamma})\Big). \tag{37}$$

Goldie (1989) proved that indeed, under conditions (D1) and (D2), there always exists a $\tau > 0$ satisfying (37), which is a special case of (C4), while the upper bound on k_n required in (D3) is equivalent to (C5). Therefore, under the additional condition (15), we obtain the asymptotic normality of the statistical tail functionals and the pertaining quantile estimators as well. Likewise, one may check the conditions of Theorem 2.1 for the ARCH(1) model (35). However, if the distribution of the innovations Z_i is symmetric, then (C1)–(C5) follow immediately from the corresponding conditions for Y_i^2 and thus from the aforementioned result established in Drees (2000). For example, (C2) for Y_i^2 combined with the relationship

$$\begin{aligned} & P\Big\{Y_i > F_Y^{-1}(1 - \frac{k_n}{n}x),\, Y_j > F_Y^{-1}(1 - \frac{k_n}{n}y)\Big\} \\ = \;& \frac{1}{4}P\Big\{Y_i^2 > F_{Y^2}^{-1}(1 - \frac{k_n}{n}2x),\, Y_j^2 > F_{Y^2}^{-1}(1 - \frac{k_n}{n}2y)\Big\} \end{aligned}$$

implies (C2) for the ARCH(1) time series Y_i with

$$c_{m,Y}(x,y) = \frac{1}{4}c_{m,Y^2}(2x,2y) = \frac{1}{2}c_{m,Y^2}(x,y)$$

and F_Y^{-1} and $F_{Y^2}^{-1}$ denoting the q.f. of Y_i and Y_i^2, respectively. Hence the analogous relation also holds for the limiting covariance functions c_Y and c_{Y^2} given by (36). Note that in general this covariance function cannot be calculated analytically, but Stărică (1999) proposed a method to compute it by simulation.

3.2 Linear Time Series

Here we examine classical linear time series

$$X_i = \sum_{j=0}^{\infty} \psi_j Z_{i-j}, \qquad i \in I\!N, \tag{38}$$

with i.i.d. innovations Z_i. Without loss of generality we assume $\psi_0 = 1$. For simplicity, we confine ourselves to geometrically decreasing coefficients, that is,

$$|\psi_j| = O(\tau^j) \tag{39}$$

as $j \to \infty$ for some $\tau \in (0,1)$; in particular, finite order autoregressive moving average (ARMA) models are included. However, the results given below hold true under much weaker summability conditions on the coefficients (cf. Datta and McCormick, 1998, Lemma 5.2, or Mikosch and Samorodnitsky, 2000, Lemma A.3).

In model (38) the variables X_i are heavy-tailed if and only if the innovations have heavy tails. Hence the stochastic behavior of the linear time series (38) is very different from that of the nonlinear time series considered in the last subsection.

In the sequel we assume that the d.f. F_Z of Z_1 has balanced heavy tails, i.e.,

$$F_Z \in D(G_\gamma) \quad \text{and} \quad \lim_{x\to\infty} \frac{1-F_Z(x)}{F_Z(-x)} = \frac{p}{q} \quad \text{for some } p = 1-q \in (0,1) \tag{40}$$

(or, equivalently, $1-F_Z(x) \sim px^{-1/\gamma}l(x)$ and $F_Z(-x) \sim qx^{-1/\gamma}l(x)$ as $x \to \infty$ for some slowly varying function l). Then, by Lemma 5.2 of Datta and McCormick (1998), the d.f. F of X_1 satisfies

$$\frac{1-F(x)}{1-F_Z(x)} \longrightarrow \sum_{j=0}^{\infty} \Big(p\psi_j^{1/\gamma} 1_{\{\psi_j > 0\}} + q|\psi_j|^{1/\gamma} 1_{\{\psi_j < 0\}} \Big) =: d_\psi \tag{41}$$

as $x \to \infty$. In particular, $F \in D(G_\gamma)$, too.

If, in addition, F_Z has a Lebesgue density f_Z which is L_1-Lipshitz continuous, i.e.,

$$\int |f_Z(z+u) - f_Z(z)|\, dz = O(u) \tag{42}$$

as $u \downarrow 0$, then the time series $X_i, i \in I\!N$, is geometrically β–mixing (Doukhan, 1994, Theorem 2.3.2). (For a finite order ARMA process the mere existence of a Lebesgue density is sufficient; see Doukhan, 1994, Theorem 2.4.6.) Hence condition (C1) is satisfied with $l_n = [const \cdot \log n]$ provided k_n satisfies (9).

In the same way as in Lemma 5.1 of Datta and McCormick (1998) one can show that

$$\frac{P\{X_1 > u,\, X_{1+m} > uv\}}{1-F_Z(u)} \longrightarrow \sum_{j=0}^{\infty} \Big(|\psi_j|^{1/\gamma} \wedge (v^{-1/\gamma}|\psi_{j+m}|^{1/\gamma}) \Big)$$

as $u \to \infty$, for $v > 0$ and $m > 1$. Combining this with (41) and $F^{-1}(1 - yk_n/n)/F^{-1}(1-xk_n/n) \to (y/x)^{-\gamma}$, one obtains (C2) with

$$c_m(x,y) = \frac{1}{pd_\psi} \sum_{j=0}^{\infty} \Big((x|\psi_j|^{1/\gamma}) \wedge (y|\psi_{j+m}|^{1/\gamma}) \Big) \Big(p1_{\{\psi_j \wedge \psi_{j+m} > 0\}} + q1_{\{\psi_j \vee \psi_{j+m} < 0\}} \Big). \tag{43}$$

It is more complicated to check (C3) for general linear time series. Of course, any finite order moving average meets this condition. More interesting is the example given by Bosq (1998, p. 18): if the innovations Z_i have finite variance (which is ensured by $\gamma < 1/2$), then the time series X_i, $i \in I\!N$, is geometrically ρ–mixing and hence remark (ii) below (C3) applies.
Though in practice it is often realistic to assume a finite variance, this condition is a bit disturbing in an extreme value setting. As a simple example of a time series that is neither m-dependent for a finite m nor has necessarily finite variance, we consider a first order autoregressive (AR(1)) process

$$X_i = \theta X_{i-1} + Z_i$$

for some $\theta \in (-1, 1)$. This time series has representation (38) with $\psi_j = \theta^j$, so that (C2) holds if the d.f. of the innovations has a Lebesgue density and satisfies (40).
Next we verify condition (C3). We restrict ourselves to the case $\theta \geq 0$, the other case can be treated in the same way. Then the representation $X_{1+m} = \theta^m X_1 + \sum_{k=2}^{1+m} \theta^{1+m-k} Z_k$ shows that

$$\begin{aligned}
&P\{X_1 \in I_n(x,y),\, X_{1+m} \in I_n(x,y)\} \\
&\quad \leq \; P\{X_1 \in I_n(x,y),\, \theta^m X_1 > \theta^{1/2} F^{-1}(1-\frac{k_n}{n}y)\} \\
&\qquad + P\Big\{X_1 \in I_n(x,y),\, \sum_{k=2}^{1+m} \theta^{1+m-k} Z_k > (1-\theta^{1/2}) F^{-1}(1-\frac{k_n}{n}y)\Big\} \\
&\quad \leq \; \Big(1 - F\Big(\theta^{1/2-m} F^{-1}(1-\frac{k_n}{n}y)\Big) - \frac{k_n}{n}x\Big)^+ \\
&\qquad + \frac{k_n}{n}(y-x) \cdot P\Big\{\sum_{j=0}^{\infty} \theta^j |Z_j| > (1-\theta^{1/2}) F^{-1}(1-\frac{k_n}{n}(1+\varepsilon))\Big\}. \qquad (44)
\end{aligned}$$

Here the second term is of the order $(k_n/n)^2(y-x)$. By the Potter bounds (Bingham et al., 1987, Theorem 1.5.6) we have

$$\begin{aligned}
\Big(1 - F\Big(\theta^{1/2-m} F^{-1}(1-\frac{k_n}{n}y)\Big) - \frac{k_n}{n}x\Big)^+ \; &\leq \; \Big(\theta^{(m-1)/(2\gamma)} \frac{k_n}{n} y - \frac{k_n}{n} x\Big)^+ \\
&\leq \; \frac{k_n}{n} \theta^{(m-1)/(2\gamma)} (y-x).
\end{aligned}$$

Combine this with (44) to obtain (C3).
To sum up, if X_i allows representation (38) with coefficients satisfying (39) and F_Z satisfying (40) and (42), and if k_n meets conditions (18), (9) and $k_n = O(n/\log n)$, then the approximation (14) of the tail empirical quantile

function Q_n holds with limiting covariance function given by (12) and (43), provided that $\gamma < 1/2$, or $\psi_j = 0$ for all but finitely many j, or $\psi_j = \theta^j$ for some $\theta \in (-1, 1)$. Hence, under the additional conditions (19) and (15) on k_n, the asymptotic normality of the quantile estimator $\hat{x}_{p_n}$ follows.
Notice that for the AR(1) model the asymptotic variance is particularly simple if one uses the Hill or the maximum likelihood estimator for the estimation of the extreme value index, since

$$c(1,1) = 1 + 2 \cdot \begin{cases} \theta^{1/\gamma}/(1-\theta^{1/\gamma}) & \text{if} \quad \theta \geq 0, \\ |\theta|^{1/\gamma}/(1-|\theta|^{2/\gamma}) & \text{if} \quad \theta < 0. \end{cases}$$

Hence, within this model, one may construct confidence intervals without using the variance estimators discussed in Section 2. Instead one may define an estimator of $c(1,1)$ using, e.g., the same estimator for γ as for the quantile estimation and the sample autocorrelation function at lag 1 as an estimator of θ.
Resnick and Stărică (1997) demonstrated that, if one trusts in the simple AR(1) model, one gets more accurate estimates of the extreme value index by first estimating θ and then the extreme value index based on the resulting residuals $X_i - \hat{\theta} X_{i-1}$. By fitting a Pareto distribution to the tails of the residuals and then using relation (41), one might also get an accurate estimate of extreme quantiles of F. The main advantage of the approach presented here is its robustness, as it does not rely on a specific model but yields reasonable estimates under mild structural assumptions.

4 The Nasdaq Composite Index: a Case Study

In this section we analyze the 'risk' of a large hike of the Nasdaq Composite index. (In fact, it is a risk for investors betting on a decrease of the index, which may seem a reasonable strategy given the huge losses observed in the last months of the period considered here.) More precisely, we examine the (log) returns $X_i = \log(S_i/S_{i-1})$, $1 \leq i \leq n$, with S_i denoting the index calculated at the end of the ith trading day in the years 1997 to 2000, amounting to a sample size $n = 1007$. We do not consider negative returns, i.e. the left tail of the returns, because, somewhat surprisingly, there is only very weak evidence for a positive extreme value index there. Hence for the analysis of the left tail one must apply estimators for general $\gamma \in \mathbb{R}$, which we discuss in less detail in Section 6.
A scatterplot of these returns is given in Figure 1. To some extent, there is an increasing trend in the volatility, which seems to contradict stationarity of the time series. On the other hand, bursts of volatility have also been observed in

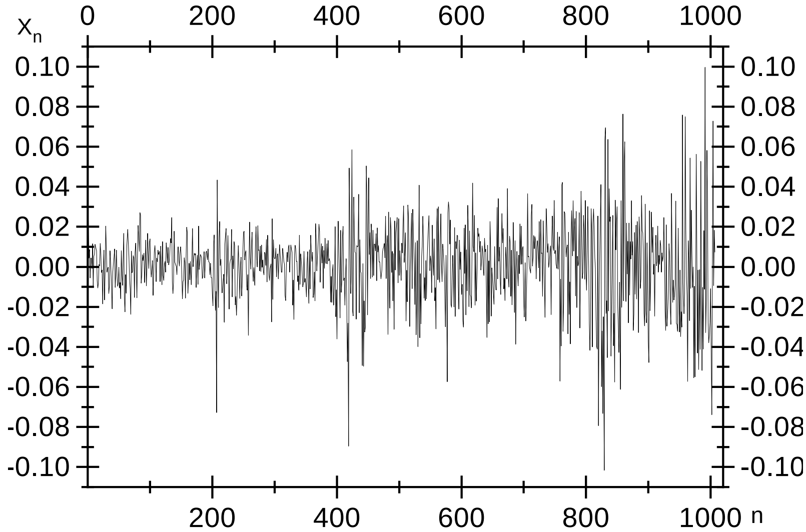

Figure 1: Log returns of the Nasdaq Composite index, 1997–2000

the first half of the observation period (for i about 200 and 400). Moreover, the observed increase may also be due to a persistence in the volatility after these random shocks. All in all, given the moderate length of the period under consideration and the fact that there was no obvious structural change in the economic environment during this period, stationarity may be regarded as a reasonable approximation to reality. (Contrary to that judgement, Stărică and Granger (2001) argue for shorter periods of stationarity of the S&P500 index in the second half of the 1990's.)

In the sequel, we aim at estimating the upper $p_n = 1/1000$ quantile $x_{p_n} = F^{-1}(0.999)$ under the assumption of stationarity. Note that np_n is about 1 so that we are actually looking for an extreme quantile.

The left plot of Figure 2 displays the graphs of the Hill estimator (dashed curve), the maximum likelihood estimator (solid line) and the moment estimator proposed by Dekkers et al. (1989) (dotted line) as a function of k, the number of largest order statistics reduced by 1. All estimates are positive for k ranging from about 50 to 460, so that we may assume a heavy tailed distribution. However, the values obtained by the different estimation methods differ quite a lot. In particular, the Hill estimator shows a clear upward trend starting from $k = 100$, whereas the curve pertaining to the maximum likelihood estimator is much more stable. This may indicate that the Pareto approximation (3) becomes sufficiently accurate only after a suitable shift of the data, i.e., $F^{-1}(1-t) \approx dt^{-\gamma} + \mu$ for some $\mu \neq 0$, because it is well known that a non-vanishing location parameter leads to a large bias of the

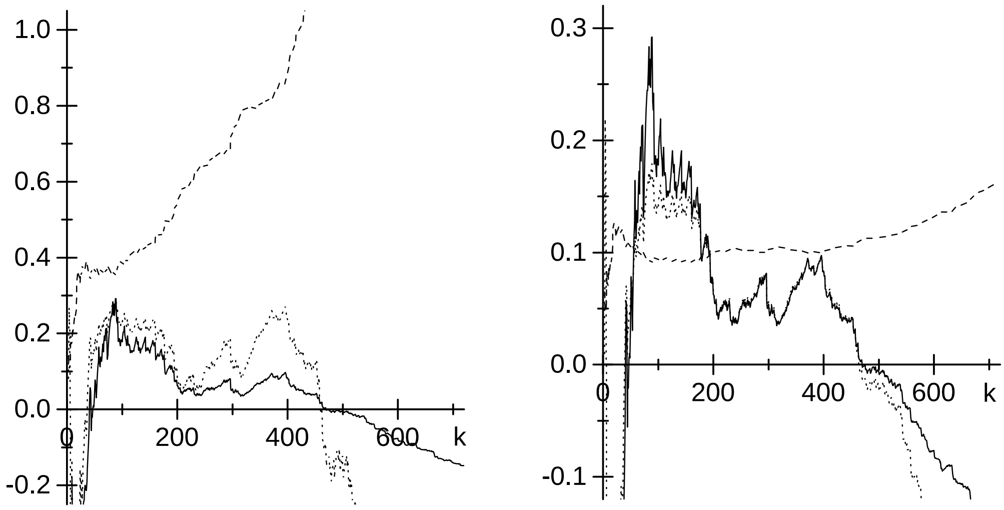

Figure 2: Hill (dashed line), moment (dotted) and maximum likelihood estimator (solid) for original returns (left plot) and for returns shifted into positive real halfline (right plot)

Hill estimator, whereas the maximum likelihood estimator is invariant under a shift transformation and the moment estimator is less sensitive to shifts than the Hill estimator.
This hypothesis can be checked by subtracting a suitable constant from the data. A choice suggesting itself is the smallest observation, since after this shift all transformed data points are non-negative, thus allowing to use (almost) up to the full sample for the Hill and the moment estimator. The right plot in Figure 2 shows the resulting estimates for the extreme value index based on these shifted data. Now the behavior of the Hill estimator has changed completely, yielding an almost flat line for k ranging from 100 to 400. (Note that the scale of the y-axis has been changed to magnify the relevant range of y-values.) Even more strikingly, the moment and the maximum likelihood estimator (which is not influenced by the transformation) are now almost identical for k between 180 and 470.
According to our experience with several data sets, as a rule of thumb this similarity indicates that the pure Pareto approximation without a location parameter is particularly accurate, i.e. $F_s^{-1}(1-t) \approx dt^{-\gamma}$ with F_s denoting the d.f. of the shifted random variables. To check this for the shifted data set under consideration, in Figure 3 we have plotted a linearly interpolated version of the tail empirical q.f. $Q_{n,k}$ for $k = 400$ based on the transformed data (solid line) together with the estimated Pareto approximation $X_{n-k:n}t^{-\hat{\gamma}_n^{(H)}}$ using the Hill estimator $\hat{\gamma}_n^{(H)} \approx 0.10$ (dashed line). The fit is convincing for

the whole unit interval and almost perfect for its upper half.

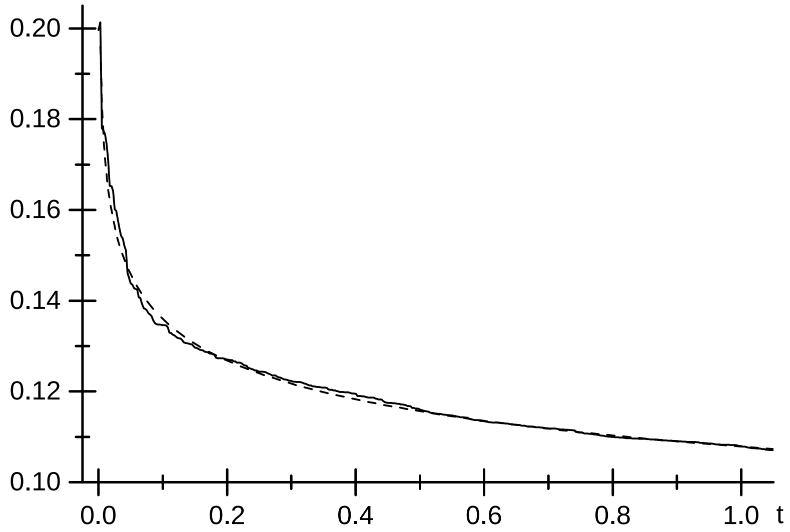

Figure 3: Continuous version of $Q_{n,k}$ for shifted log returns with $k = 400$ (solid line) and estimated Pareto approximation (dashed)

Encouraged by this fit, we carry on with the statistical analysis of the shifted returns. Figure 4 displays the estimator $\hat{\sigma}_{T,\gamma,3}$ whose square is defined in (32). Here we have used the Hill estimator and $j_n = 2$, the smallest integer exceeding np_n, so that $\hat{\sigma}_{T,\gamma,3}$ is well defined. (Different small values for $j_n > np_n$ lead to similar results, but if j_n is chosen too large then the performance of the variance estimator deteriorates.)

After large fluctuations when only few order statistics are used for estimation, the curve stabilizes at a value slightly below 0.2. Then, starting with about $k = 400$, there is a strong upward trend in the curve suggesting that a non-negligible bias shows up. Note that the kink in the curve at $k = 400$ is much more pronounced in this plot than in the graph of the estimators for the extreme value index or the extreme quantile (Figure 5). Hence to plot $\hat{\sigma}_{T,\gamma,3}$ against k might be a useful data analytic tool for choosing a suitable sample fraction, even in case of i.i.d. data where such an estimate of the variance is not needed for the construction of confidence intervals.

After these preparations, we arrive at our final plot in Figure 5. Here the quantile estimator $\hat{x}_{p_n}$ is plotted against k (solid line) together with the 99% confidence intervals (33) (dashed line) and the confidence intervals

$$\Big[\hat{x}_{p_n} \exp\Big(- z_{\alpha/2}\hat{\gamma}_n^{(H)} k_n^{-1/2} \log(k_n/(np_n))\Big), \\ \hat{x}_{p_n} \exp\Big(z_{\alpha/2}\hat{\gamma}_n^{(H)} k_n^{-1/2} \log(k_n/(np_n))\Big)\Big], \tag{45}$$

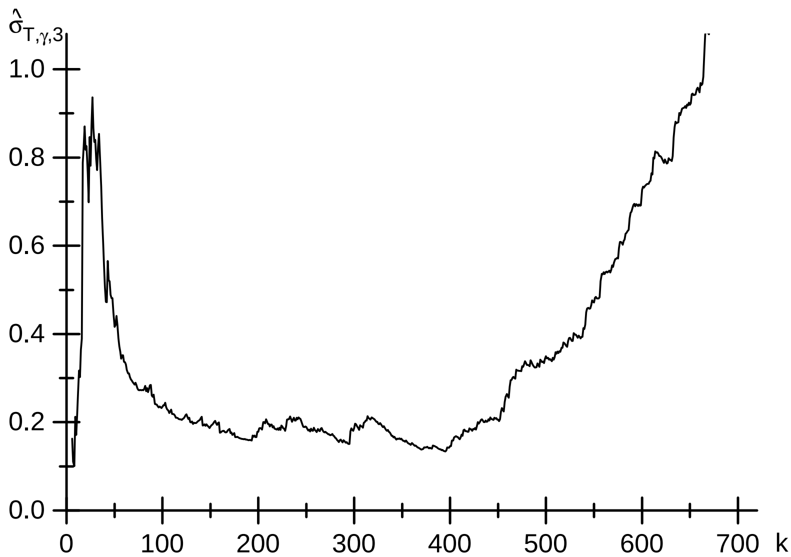

Figure 4: Estimated asymptotic standard deviation $\hat{\sigma}_{T,\gamma,3}$

suggested by the theory for i.i.d. data (dotted line). The estimators are calculated from the shifted data and then the shift has been corrected, so that the graphs show the estimates for the original distribution of the returns. For $k = 400$ one obtains a quantile estimate of about 0.096 with confidence interval (33) equal to $[0.075, 0.119]$. As expected, the intervals ignoring the serial dependence are much shorter than the intervals obtained by the new approach presented here, indicating that perhaps the former pretend a much higher estimation accuracy than it is actually achieved. Despite this fact, in the literature about the statistical analysis of financial series with a clear serial dependence often confidence intervals are displayed which are motivated by the classical extreme value theory for independent data; see, e.g., Longin (1996), Caserta et al. (1998) and Müller et al. (1998). This, of course, does not mean that the standard confidence intervals are necessarily too short because in some cases the dependence may be negligible for large observations, but the theoretical justification for these confidence intervals is very weak.
It is also worth noting that, unlike the intervals (45), the confidence intervals based on the estimator $\hat{\sigma}_{T,\gamma,3}$ automatically widens for large k where the bias kicks in. Hence they actually reflect not only the variance of the quantile estimator but also the bias, thus avoiding an empty intersection of confidence intervals based on different sample fractions. In contrast, the standard confidence intervals are completely misleading if too many order statistics are used for estimation. In fact, in our simulation study it turned out that in some cases the actual coverage probability of the new intervals comes quite

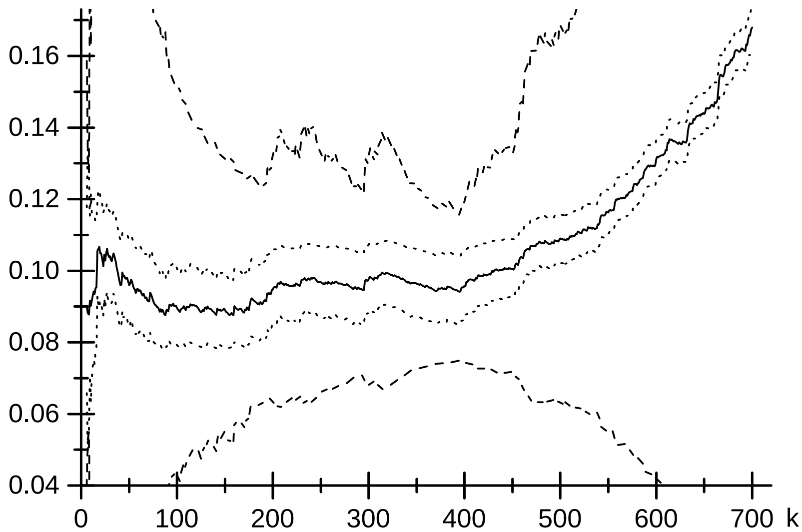

Figure 5: Estimated 0.999-quantile of the original return distribution (solid line) with 99% confidence intervals (33) (dashed) and (45) (dotted)

close to the nominal probability even for large k, though we do not offer any theoretical explanation for this effect.

5 Simulations

In this section we study the actual coverage probabilities of the two-sided confidence interval (33) derived in the present paper in comparison with those of the confidence interval (45) suggested by the theory for i.i.d. samples. Here both types of confidence intervals are calculated for the nominal coverage probability of 95% and they are based on the Hill estimator for γ. In the definition of $\hat{\sigma}_{T,\gamma,3}$ used in (33) we choose j_n equal to 2 for $p_n \leq 1/n$ and $j_n = 3$ for $p_n = 2/n$ so that $\log(j_n/(np_n))$ is strictly positive. All simulations were carried out using the programming language StatPascal which is part of the software package XTREMES (see Reiss and Thomas, 2001).
As examples of linear time series we consider four ARMA(1,1) models

$$X_i - \phi X_{i-1} = Z_i + \theta Z_{i-1}. \tag{46}$$

Here the i.i.d. innovations Z_i have a two-sided Pareto d.f. with extreme value index $\gamma = 1/3$, that is,

$$1 - F_Z(x) = F_Z(-x) = \frac{1}{2}x^{-3}, \qquad x \geq 1,$$

and

(i) $\phi = 0.95, \quad \theta = 0.9,$

(ii) $\phi = 0.95, \quad \theta = -0.6,$

(iii) $\phi = 0.95, \quad \theta = -0.9,$

(iv) $\phi = 0.3, \quad \theta = 0.9,$

respectively. Observe that the innovations have finite variance and thus, according to Subsection 3.2, these models satisfy the conditions of Theorem 2.2.

In models (i)–(iii) the dependence is mainly due to the autoregressive part and, roughly speaking, the degree of dependence is decreasing from model (i) to model (iii) as the effect of the large autoregressive parameter $\phi = 0.95$ is partly compensated by the negative moving average parameter θ. (Note that for $\theta = -\phi$ one has $X_i = Z_i$, that is, independent random variables are observed.) In model (iv) the dependence is locally strong due to the large moving average parameter θ, but it has a very short memory because ϕ is small.

In addition we consider two non-linear (G)ARCH time series

$$X_i = \sigma_i Z_i$$

with i.i.d. standard normal innovations Z_i and

(v) $\sigma_i^2 = 0.0001 + 0.9X_{i-1}^2,$

(vi) $\sigma_i^2 = 0.0001 + 0.4X_{i-1}^2 + 0.5\sigma_{i-1}^2,$

respectively. For the ARCH(1) model (v) our conditions have been checked in Subsection 3.1. GARCH(1,1) time series like (vi) are widely used in finance to model returns of risky assets. It is known that such time series are geometrically β–mixing (Doukhan, 1994, Section 2.4.2.3), but conditions (C2) and (C3) have not been verified yet. The choice of the parameters describing the influence of X_{i-1}^2 and σ_{i-1}^2 on σ_i^2 is motivated by the observation that in financial applications typically the sum of these parameters is close to, but less than 1.

Finally, we also simulate i.i.d. sequences of Fréchet random variables with d.f.

(vii) $F(x) = \exp(-x^{-3}), \quad x > 0,$

in order to examine the performance of the confidence interval (33) in a situation when the interval (45) is appropriate.

model	x_{p_n}			
	$p_n = 0.0005$		$p_n = 0.0001$	
(i)	41.88,	[41.75,41.96]	63.77,	[63.35,64.28]
(ii)	11.74,	[11.71,11.76]	19.03,	[18.94,19.13]
(iii)	10.02,	[10.00,10.03]	17.13,	[17.08,17.17]
(iv)	14.59,	[14.56,14.61]	24.38,	[24.32,24.47]
(v)	0.2479,	[0.2462,0.2494]	0.4940,	[0.4854,0.5029]
(vi)	0.2114,	[0.2109,0.2117]	0.3450,	[0.3435,0.3466]

Table 1: Estimated quantiles for models (i)–(vi) with 95%–confidence intervals

The quantiles $x_{p_n} = F^{-1}(1 - p_n)$ are to be estimated for $p_n = 1/n$ and $p_n = 1/(5n)$. Since the quantiles are not known exactly for models (i)–(vi), they are determined by simulations. For this, recall from Theorem 2.1 that an empirical intermediate quantile is asymptotically normal with median equal to the pertaining true quantile. Thus we simulate $m = 1000$ time series of length $5 \cdot 10^6$ and estimate x_{p_n} by the median of the empirical $(1 - p_n)$–quantiles. Table 1 gives the resulting estimates and 95%–confidence intervals $[Y_{[(1-z_{0.025}m^{-1/2})m/2]:m}, Y_{[(1+z_{0.025}m^{-1/2})m/2]:m}]$ with Y_i, $1 \le i \le m$, denoting the observed empirical quantiles.

Next, $m = 10\,000$ time series of length $n = 2000$ are simulated from each of the above models and the relative frequency of samples is determined for which the true quantile lies outside the confidence intervals (33) and (45), respectively. In Figure 6 the resulting empirical noncoverage probabilities of (33) (solid line) and (45) (dashed line) are plotted against k, the number of order statistics reduced by 1, for models (i)–(vi) and $p_n = 1/n = 0.0005$. The nominal level 5% is indicated by the dotted line. The maximal k–values are chosen such that in (almost) all samples the kth largest order statistic is still positive, so that the Hill estimator is well defined.

The confidence interval (45) derived from the theory for i.i.d. samples yields an acceptable level of noncoverage only for the ARMA(1,1) model (iii), which is close to an i.i.d.-model. In all other cases, the noncoverage probability is always larger than 13% and it is typically larger than 20%. Moreover, if k is taken too large such that a non-negligible bias enters, then the probability of noncoverage increases rapidly, as the confidence interval (45) does not take into account any bias.

In contrast to that behavior, the confidence interval (33) is unbiased in all cases if one uses at least 40 order statistics for estimation, except in the

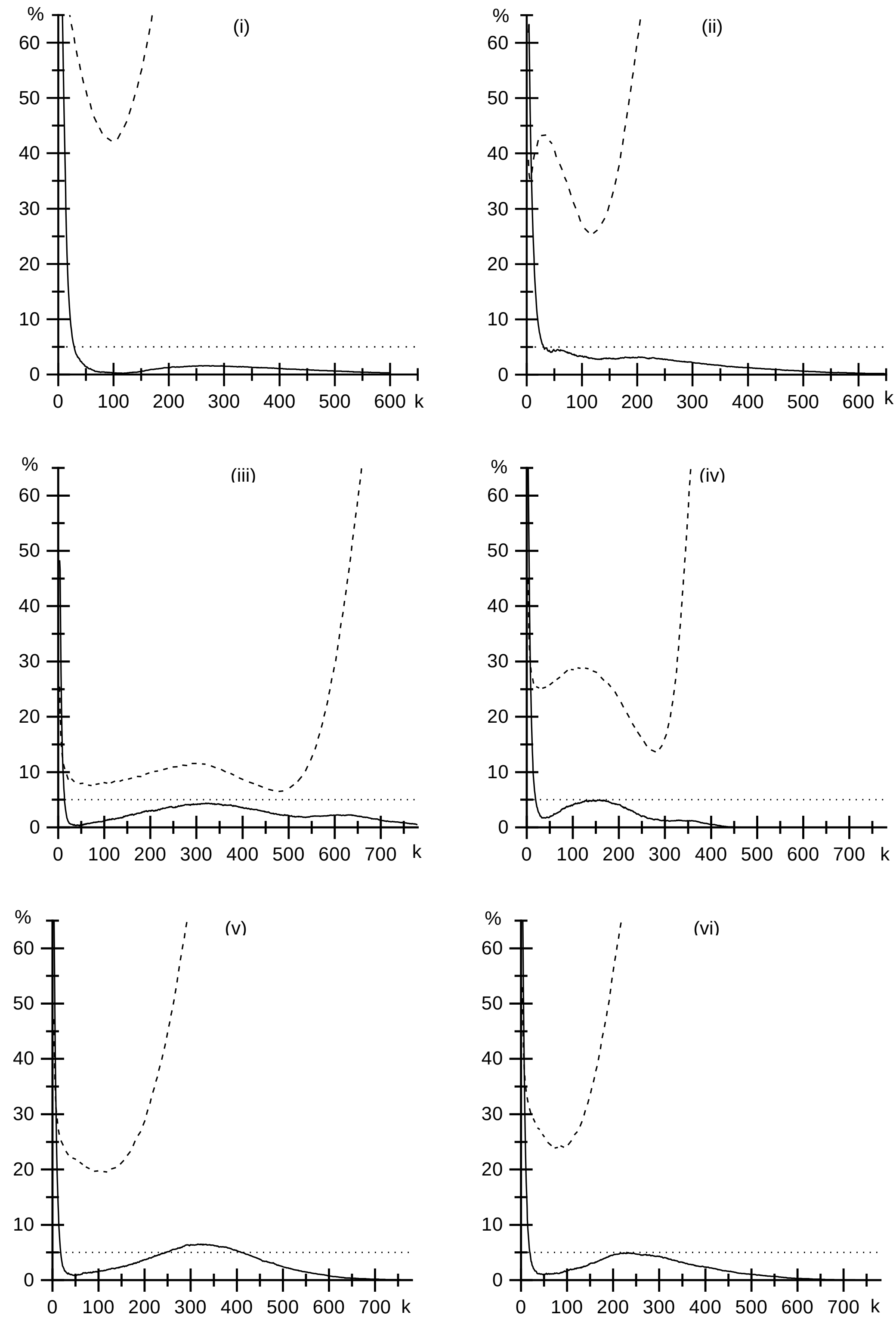

Figure 6: Empirical noncoverage probabilities of (33) (solid line) and (45) (dashed line) for $p_n = 1/n$; the nominal probability 5% is indicated by the dotted line

ARCH(1) model (v) with k between 240 and 410 when the nominal level is exceeded by less than 2%. At first glance, it is somewhat surprising that the confidence interval is most conservative for the ARMA(1,1) model (i) which exhibits the strongest dependence. This, however, is due to the rather poor fit of the tail of the stationary distribution by a Pareto distribution if one uses more than 200 order statistics. As a result, the quantile estimator has a large bias if k is much bigger than 200 (as it can be seen from the quickly increasing actual noncoverage probability of the i.i.d. confidence interval (45)). This in turn leads to an overestimation of the asymptotic variance by $\hat{\sigma}^2_{T,\gamma,3}$ and hence to too a wide confidence interval (33).

Figure 7 is the analog to Figure 6 for $p_n = 1/(5n) = 0.0001$. By and large, the performance of the confidence interval assuming i.i.d. observations is the same as for $p_n = 1/n$. In contrast, the noncoverage probabilities of the confidence interval (33) are considerably higher than in Figure 6. This is particularly true for the non-linear time series models (v) and (vi), where the actual probability is much larger than the nominal level for most k. This problem is mainly due to the large estimation error of the estimator $\hat{\sigma}^2_{T,\gamma,3}$ for the asymptotic variance, which is based on the quantile estimates $\hat{x}^{(i)}_{1/(5n)}$. The dash-dotted line in Figure 7 shows the empirical noncoverage probability when this variance estimator is replaced with the one based on the quantile estimates $\hat{x}^{(i)}_{1/n}$, i.e., the same estimator as used in Figure 6. Indeed, now the nominal 5%–levels is exceeded only for very small k and, for the models (iv) and (v) with k between 90 and 240 resp. between 230 and 430, by merely a few percentage points. So apparently the estimates for $x_{1/(5n)}$ are not reliable enough to be used for the estimation of the asymptotic variance. This, of course, is not completely surprising, since it is much more delicate to estimate the quantile $x_{1/(5n)}$ which lies far outside the range of observations than to estimate the quantile $x_{1/n}$ on the boundary of that range.

Next, we consider model (vii) of i.i.d. Fréchet observations (Figure 8). Not surprisingly, the confidence interval (45) derived from the theory for i.i.d. observations does a very good job if one uses an appropriate number of order statistics, while the confidence interval (33) is often too conservative. On the other hand, the latter is less sensitive to a misspecification of the sample fraction used for estimation, albeit the nominal level is exceeded if k is chosen much too large.
As usual in extreme value theory, the choice of the sample fraction used for estimation is crucial for the performance of the quantile estimators and the pertaining confidence intervals. Given a fixed level $1-\alpha$, one often aims at

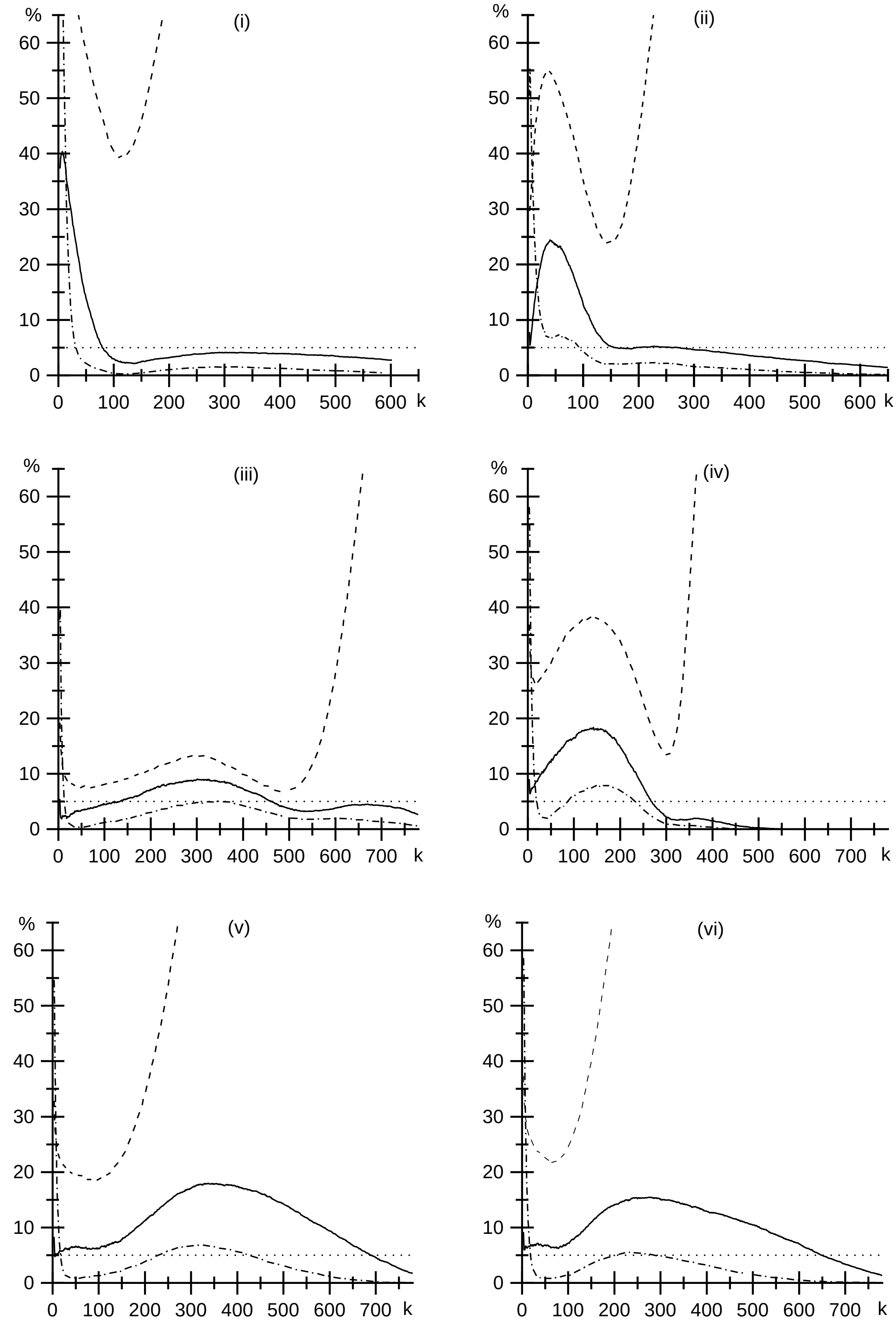

Figure 7: Empirical noncoverage probabilities of (33) (solid line), (33) with variance estimator $\hat{\sigma}^2_{T,\gamma,3}$ based on $\hat{x}^{(i)}_{1/n}$ 34 (dash-dotted line) and (45) (dashed line) for $p_n = 1/(5n)$; the nominal probability 5% is indicated by the dotted line

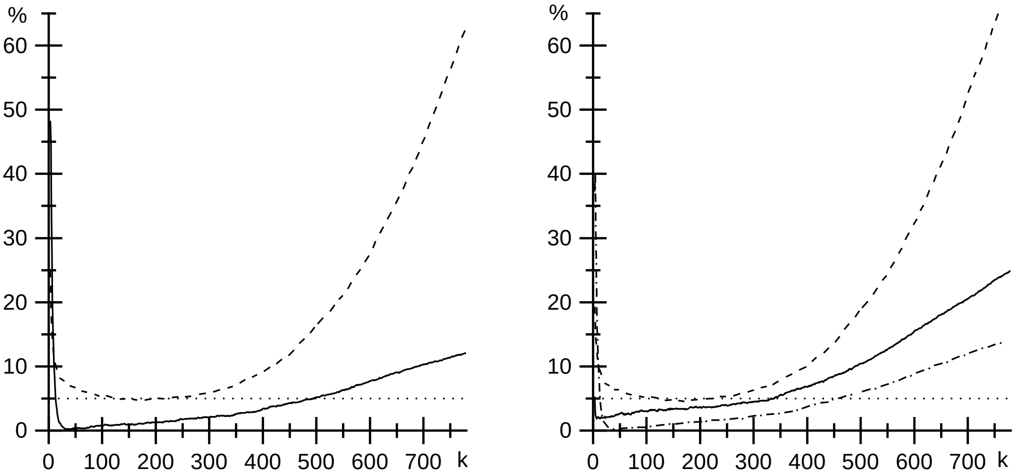

Figure 8: Empirical noncoverage probabilities of (33) (solid line), (33) with variance estimator $\hat{\sigma}^2_{T,\gamma,3}$ based on $\hat{x}^{(i)}_{1/n}$ (dash-dotted line) and (45) (dashed line) for the Fréchet model (vii) and $p_n = 1/n$ (left) resp. $p_n = 1/(5n)$ (right); the nominal probability 5% is indicated by the dotted line

a confidence interval as short as possible such that the coverage probability is at least $1 - \alpha$. Hence it seems natural to choose k such that the estimate of the asymptotic variance is minimized. Obviously, this approach relies on good variance estimates. Therefore, as mentioned above, the estimator $\hat{\sigma}^2_{T,\gamma,3}$ should be based on quantile estimators $\hat{x}^{(i)}_{\tilde{p}_n}$ for a quantile $x_{\tilde{p}_n}$ that lies inside the range of observations. On the other hand, $\tilde{p}_n$ must be sufficiently small to justify the use of extreme value theory. As a compromise between these conditions, we choose $\tilde{p}_n = 2/n$. (Taking $\tilde{p}_n = 1/n$ as in Figure 7 leads to slightly worse results, with noncoverage probabilities being about 1–2% higher than reported in Table 2.) In addition, one has to rule out that k is too small, since then the variance estimates are not reliable. Here we restrict k to values larger than or equal to 80, that is, at least 4% of the sample is used for estimation. In addition, we exclude unrealistic small variance estimates by requiring that the estimate is at least $(\hat{\gamma}^{(k)}_n)^2$, the estimated variance in the case of independent observations. To sum up, we choose

$$\hat{k} := \arg\min \left\{ \hat{\sigma}^{(k)}_{T,\gamma,3} \,\middle|\, k \geq 80,\ \hat{\sigma}^{(k)}_{T,\gamma,3} \geq \hat{\gamma}^{(k)}_n \right\} \tag{47}$$

where $\hat{\sigma}^{(k)}_{T,\gamma,3}$ is the estimator of the asymptotic standard deviation defined analogously to (32), but based on the estimators $\hat{x}^{(i)}_{2/n}$, $3 \leq i \leq k$, instead of $\hat{x}^{(i)}_{p_n}$.

model	$p_n = 0.0005$	$p_n = 0.0001$
(i)	2.5%	2.2%
(ii)	5.3%	6.6%
(iii)	6.1%	6.7%
(iv)	10.1%	14.1%
(v)	7.7%	8.6%
(vi)	5.5%	6.3%
(vii)	5.4%	6.0%

Table 2: Empirical noncoverage probabilities for models (i)–(vi) with k chosen according to (47)

The resulting empirical probabilities of noncoverage are reported in Table 2. With the exception of the ARMA(1,1) model (iv), the method works pretty well: the nominal level is at most exceeded by just a narrow margin in models (i)–(iii), (vi) and (vii), and by about 2.5–3.5% for the ARCH(1) model (v). In contrast to that performance, the actual probability of noncoverage is 2–3 times as large as the nominal one in model (iv), which exhibits a strong local but very short-ranged dependence. Nevertheless, the approach to minimize the estimated asymptotic variance seems very promising if reliable variance estimates are at hand.

Finally, we discuss the effect observed in the analysis of the Nasdaq Composite index that a shift of the data can improve a lot the fit of the extremes by a Pareto distribution and consequently also the estimation accuracy. In the present study this particularly holds for the non-linear ARCH(1) and GARCH(1,1) time series used to model the returns of risky financial assets. For example, Figure 9 shows the Pareto Q-Q plot $(\log((n+1)/i), \log X_{n-i+1:n})_{1\le i\le n^+}$ for a GARCH time series of size $n = 50\,000$ drawn from model (vi) and for the sample shifted by 0.035. (Here n^+ denotes the number of positive observations; we use simulated data to get an estimate for the unknown d.f.'s.) Clearly the Q-Q plot for the shifted data set can be well approximated by a line over a much wider range than the plot for the original data, thus indicating that a larger sample fraction of extremes can be fitted well by a Pareto distribution. Here the amount by which the data set is shifted does not depend on the particular sample (but, of course, on the model). Indeed, the value 0.035 was chosen such that the Hill plot for a different sample from model (vi) seems flat and the moment estimator and the maximum likelihood estimator yield similar results over a wide range of k-values; cf. the discussion in Section 4.

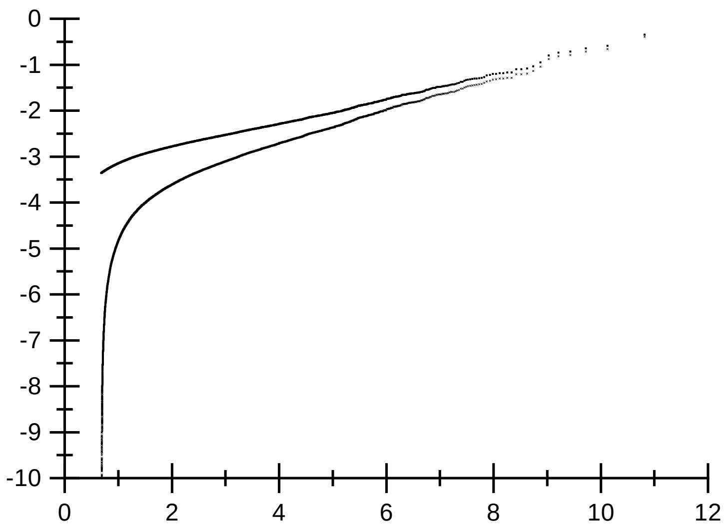

Figure 9: Pareto Q-Q plot for GARCH(1,1) model (vi) (crosses, lower plot) and for model shifted by 0.035 (squares, upper plot)

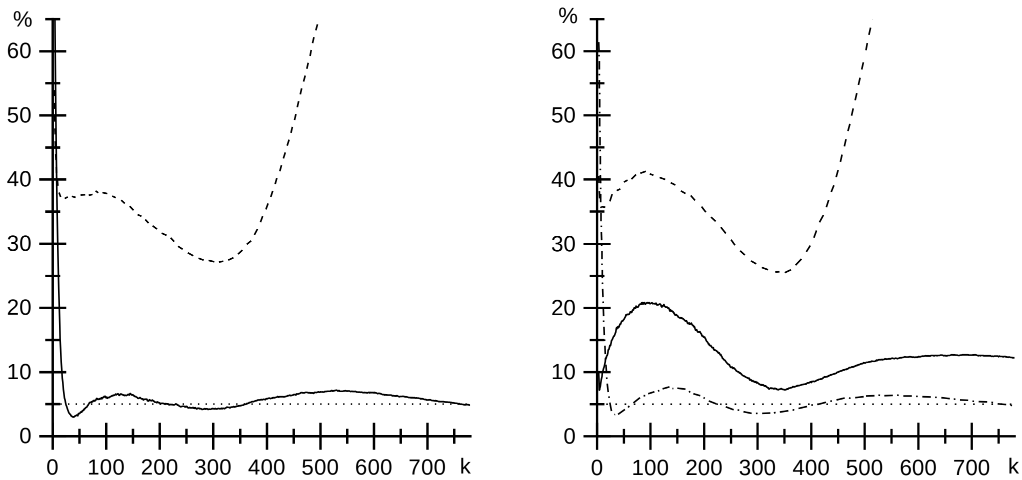

Figure 10: Empirical noncoverage probabilities of (33) (solid line), (33) with variance estimator $\hat{\sigma}^2_{T,\gamma,3}$ based on $\hat{x}^{(i)}_{1/n}$ (dash-dotted line) and (45) (dashed line) for the GARCH(1,1) model (vi) shifted by 0.035 and $p_n = 1/n$ (left) resp. $p_n = 1/(5n)$ (right); the nominal probability 5% is indicated by the dotted line

Figure 10 displays the empirical noncoverage probabilities of the confidence intervals (33) and (45) for the GARCH(1,1) model (vi) shifted by 0.035. From the curves corresponding to the interval (45) one can see that a significant bias occurs only if k is taken larger than 400, whereas for the original model this happens for about $k \geq 150$. Hence in the average one obtains much shorter confidence intervals by choosing k between 250 and 350, say, leading to noncoverage probabilities of about 4–5% for $p_n = 1/n$ and about 3.5–5% for $p_n = 1/(5n)$. More precisely, although these probabilities are smaller than those reported in Table 2 for the original GARCH model when almost shortest confidence intervals are used, there the average length of the confidence intervals is more than 3 times as large as in the shifted model for $p_n = 1/n$ and about 6 times as large for $p_n = 1/(5n)$. This demonstrates the huge improvement of the estimation accuracy achieved by an appropriate shift of the data. In addition, the confidence interval becomes less sensitive to a misspecification of the sample fraction used for estimation, as far as the coverage probability is concerned.

6 Asymptotics: the general case

In this section we analyze the asymptotic behavior of quantile estimators of type (7) when $F \in D(G_\gamma)$ for some $\gamma \in I\!R$. Unlike in the special case $\gamma > 0$, there is no simple unifying representation of the quantile function that is sufficient for $F \in D(G_\gamma)$ for all $\gamma \in I\!R$. Therefore we replace (C4) and (C5) with an analog to condition (18) based on convergence (6):

$(\widetilde{C4})$ $\lim_{n\to\infty} k_n^{1/2} \sup_{0<t\le 1+\varepsilon} t^{\gamma+1/2}(1+|\log t|)^{-1/2}|\tilde{R}(k_n/n,t)| = 0$

with

$$\tilde{R}(\lambda,t) := \frac{F^{-1}(1-\lambda t) - F^{-1}(1-\lambda)}{a(\lambda)} - \frac{t^{-\gamma}-1}{\gamma}.$$

Then we have the following counterpart to Theorem 2.1 (see Drees (2000)):

Theorem 6.1 *Under conditions (C1)–(C3) and $(\widetilde{C4})$ for some $l_n = o(n/k_n)$ there exist versions of the tail empirical q.f. Q_n, random variables D_n and a centered Gaussian process e with covariance function c defined by (12) such that*

$$\sup_{t\in(0,1]} t^{\gamma+1/2}(1+|\log t|)^{-1/2}\left| k_n^{1/2}\Big(\frac{Q_n(t)-D_n}{a(k_n/n)} - \frac{t^{-\gamma}-1}{\gamma}\Big) - t^{-(\gamma+1)}e(t)\right| \longrightarrow 0 \tag{48}$$

in probability.

Remarks.

(i) For $\gamma \ge -1/2$ the r.v.'s D_n may be replaced with $F^{-1}(1-k_n/n)$, while for $\gamma < -1/2$ one merely has $D_n - F^{-1}(1-k_n/n) = o_P(k_n^{-1/2})$ (see Drees (1998a) for details about D_n).

(ii) The remark following Theorem 2.1 also applies in the present case.

□

As in the case $\gamma > 0$, the extreme value index may be estimated by a statistical tail functional $T(Q_n)$. However, in the conditions (T1)–(T3) $t^{-\gamma}$ must be replaced with $(t^{-\gamma}-1)/\gamma$ and, in addition to the scale invariance of T, we also need location invariance to deal with the random shift by D_n in (48). This leads to the following modified conditions:

($\widetilde{T1}$) $T(az+b) = T(z) \quad$ for all $a > 0$ and $b \in I\!R$.

($\widetilde{T2}$) $T\Big(\Big(\dfrac{t^{-\gamma}-1}{\gamma}\Big)_{0<t\le 1}\Big) = \gamma.$

($\widetilde{T3}$) There exists a signed measure $\nu_{T,\gamma}$ on (0,1] with $\int_{(0,1]} t^{-\gamma-1/2}(1+|\log t|)^{1/2}\,|\nu_{T,\gamma}|(dt) < \infty$ such that

$$\varepsilon_n^{-1}\Big(T\Big(\Big(\frac{t^{-\gamma}-1}{\gamma}+\varepsilon_n z_n(t)\Big)_{0<t\le 1}\Big) - T\Big(\Big(\frac{t^{-\gamma}-1}{\gamma}\Big)_{0<t\le 1}\Big)\Big) \longrightarrow \int_{(0,1]} z(t)\,\nu_{T,\gamma}(dt)$$

for all $\varepsilon_n \downarrow 0$ and z_n satisfying

$$\sup_{0<t\le 1} t^{\gamma+1/2}(1+|\log t|)^{-1/2}|z_n(t)-z(t)| \longrightarrow 0$$

for some continuous function z as described in (T0).

Note that, for $\gamma > 0$, here $\nu_{T,\gamma}$ has a slightly different meaning than in (T3), since here we consider a derivative of T at $(t^{-\gamma}-1)/\gamma$.
Next we need an estimator of the scale function a. To this end, one can employ a similar approach, that is, one estimates a by a smooth functional $S(Q_n)$. Like T, the functional S should be invariant under shifts but it must be *equivariant* under scale transformations. Moreover, S should give the value 1 when applied to the standard generalized Pareto q.f. Hence we impose the following conditions:

(S0) S is a Borel-measurable real-valued functional on the set of functions $z \in D(0,1]$ satisfying $t^{\gamma+1/2}|\log t|^{-1/2}z(t) \to 0$ as $t \downarrow 0$.

(S1) $S(az+b) = aS(z) \quad$ for all $a > 0$ and $b \in I\!R$.

(S2) $S\Big(\Big(\frac{t^{-\gamma}-1}{\gamma}\Big)_{0<t\le 1}\Big) = 1.$

(S3) There exists a signed measure $\mu_{S,\gamma}$ on (0,1] with $\int_{(0,1]} t^{-\gamma-1/2}(1+|\log t|)^{1/2}\,|\mu_{S,\gamma}|(dt) < \infty$ such that

$$\begin{aligned} \varepsilon_n^{-1}\bigg(S\Big(\Big(\frac{t^{-\gamma}-1}{\gamma}+\varepsilon_n z_n(t)\Big)_{0<t\le 1}\Big) - S\Big(\Big(\frac{t^{-\gamma}-1}{\gamma}\Big)_{0<t\le 1}\Big)\bigg) \\ \longrightarrow \int_{(0,1]} z(t)\,\mu_{S,\gamma}(dt) \end{aligned}$$

for all $\varepsilon_n \downarrow 0$ and z_n satisfying

$$\sup_{0<t\le 1} t^{\gamma+1/2}(1+|\log t|)^{-1/2}|z_n(t)-z(t)| \longrightarrow 0$$

for some continuous function z as described in (S0).

Example. The estimator (8) is of that type with

$$S(z) = (z(1/2)-z(1))\frac{T(z)}{2^{T(z)}-1}$$

if $\hat\gamma_n = T(Q_n)$ for some T satisfying (T0) and $(\widetilde{T1})$–$(\widetilde{T3})$.
Conditions (S0)–(S2) are readily verified. To check (S3) note that, with $y_\gamma(t) := (t^{-\gamma}-1)/\gamma$, condition $(\widetilde{T3})$ and a Taylor expansion of $x \mapsto x/(2^x-1)$ at γ yield

$$\frac{T(y_\gamma+\varepsilon_n z_n)}{2^{T(y_\gamma+\varepsilon_n z_n)}-1} = \frac{\gamma}{2^\gamma-1} + \varepsilon_n \frac{2^\gamma-1-\gamma 2^\gamma \log 2}{(2^\gamma-1)^2}\int_{(0,1]} z(t)\,\nu_{T,\gamma}(dt) + o(\varepsilon_n),$$

which for $\gamma=0$ is to be interpreted as the limit for $\gamma\to 0$. Hence

$$\begin{aligned} &\varepsilon_n^{-1}\Big(S(y_\gamma+\varepsilon_n z_n) - S(y_\gamma)\Big) \\ &= (z_n(1/2)-z_n(1))\frac{\gamma}{2^\gamma-1} \\ &\quad +(y_\gamma(1/2)-y_\gamma(1))\varepsilon_n^{-1}\bigg(\frac{T(y_\gamma+\varepsilon_n z_n)}{2^{T(y_\gamma+\varepsilon_n z_n)}-1} - \frac{\gamma}{2^\gamma-1}\bigg) + o(1) \\ &\longrightarrow (z(1/2)-z(1))\frac{\gamma}{2^\gamma-1} + \frac{2^\gamma-1-\gamma 2^\gamma\log 2}{\gamma(2^\gamma-1)}\int_{(0,1]} z(t)\,\nu_{T,\gamma}(dt), \end{aligned}$$

that is, (S3) with

$$\mu_{S,\gamma} = \frac{\gamma}{2^\gamma-1}(\varepsilon_{1/2}-\varepsilon_1) + \frac{2^\gamma-1-\gamma 2^\gamma\log 2}{\gamma(2^\gamma-1)}\nu_{T,\gamma}.$$

□

Theorem 6.2 *Suppose that the conditions of Theorem 6.1 are met. If $\hat{\gamma}_n = T(Q_n)$ and $\hat{a}(k_n/n) = S(Q_n)$ with T and S satisfying (T0), $(\widetilde{T1})$–$(\widetilde{T3})$ and (S0)–(S3), respectively, then*

$$k_n^{1/2}(\hat{\gamma}_n - \gamma) \longrightarrow \mathcal{N}(0, \sigma^2_{T,\gamma}) \tag{49}$$

and

$$k_n^{1/2}\Big(\frac{\hat{a}(k_n/n)}{a(k_n/n)} - 1\Big) \longrightarrow \mathcal{N}(0, \sigma^2_{S,\gamma}) \tag{50}$$

weakly with

$$\begin{aligned} \sigma^2_{T,\gamma} &= \int_{(0,1]}\int_{(0,1]} (st)^{-(\gamma+1)}\, \nu_{T,\gamma}(ds)\, \nu_{T,\gamma}(dt), \\ \sigma^2_{S,\gamma} &= \int_{(0,1]}\int_{(0,1]} (st)^{-(\gamma+1)}\, \mu_{S,\gamma}(ds)\, \mu_{S,\gamma}(dt). \end{aligned}$$

Suppose, in addition, condition (15) holds and

$$\lim_{n\to\infty} d_n \tilde{R}\Big(\frac{k_n}{n}, \frac{np_n}{k_n}\Big) = 0 \tag{51}$$

with

$$d_n := k_n^{1/2} \frac{\gamma}{(np_n/k_n)^{-\gamma} - 1} \begin{cases} (\log(k_n/(np_n)))^{-1} & \text{if } \gamma \geq 0, \\ 1 & \phantom{\text{if }} \gamma < 0. \end{cases}$$

Then the estimator $\tilde{x}_{p_n}$ defined by (7) satisfies

$$\frac{d_n}{a(k_n/n)}(\tilde{x}_{p_n} - x_{p_n}) \longrightarrow \mathcal{N}(0, \sigma^2_{S,T,\gamma}) \tag{52}$$

and $\sigma^2_{S,T,\gamma} = \sigma^2_{T,\gamma}$ if $\gamma > 0$, $\sigma^2_{S,T,\gamma} = \sigma^2_{T,\gamma}/4$ if $\gamma = 0$, and

$$\begin{aligned} \sigma^2_{S,T,\gamma} &= \gamma^2 c(1,1) - 2\gamma \int_{(0,1]} t^{-(\gamma+1)} c(1,t)\, \mu_{S,\gamma}(dt) \\ &\quad + 2\int_{(0,1]} t^{-(\gamma+1)} c(1,t)\, \nu_{T,\gamma}(dt) \\ &\quad + \int_{(0,1]}\int_{(0,1]} (st)^{-(\gamma+1)} c(s,t)\, \mu_{S,\gamma}(ds)\, \mu_{S,\gamma}(dt) \\ &\quad - \frac{2}{\gamma}\int_{(0,1]}\int_{(0,1]} (st)^{-(\gamma+1)} c(s,t)\, \mu_{S,\gamma}(ds)\, \nu_{T,\gamma}(dt) \\ &\quad + \frac{1}{\gamma^2}\int_{(0,1]}\int_{(0,1]} (st)^{-(\gamma+1)} c(s,t)\, \nu_{T,\gamma}(ds)\, \nu_{T,\gamma}(dt) \end{aligned}$$

if $\gamma < 0$.

REMARKS.

(i) In view of the proof of Theorem 6.2, condition (51) is a natural generalization of (19).

(ii) Note that for $\gamma \geq 0$ the choice of the estimator for the scale function a does not matter asymptotically. For $\gamma < 0$, though, both the estimators of γ and of a influence the asymptotic behavior of the quantile estimator, leading to a considerably more complicated expression for the asymptotic variance.

$\square$

PROOF. According to Skorohod's representation theorem there exist versions of Q_n, D_n and e such that the convergence (48) holds almost surely. Let $y_\gamma(t) := (t^{-\gamma}-1)/\gamma$ and $z_n := k_n^{1/2}((Q_n-D_n)/a(k_n/n)-y_\gamma)$. Since the process e has almost surely continuous sample paths (see Drees (2000)), (S1)–(S3) combined with (48) gives

$$\begin{aligned} \frac{\hat{a}(k_n/n)}{a(k_n/n)} &= S\Big(\frac{Q_n - D_n}{a(k_n/n)}\Big) = S(y_\gamma + k_n^{-1/2} z_n) \\ &= 1 + k_n^{-1/2} \int_{(0,1]} t^{-(\gamma+1)} e(t)\, \mu_{S,\gamma}(dt) + o(k_n^{-1/2}) \end{aligned} \tag{53}$$

a.s., from which (50) is obvious.
Likewise, one can show that

$$k_n^{1/2}(\hat{\gamma}_n - \gamma) \longrightarrow \int_{(0,1]} t^{-(\gamma+1)} e(t)\, \nu_{T,\gamma}(dt) \quad a.s. \tag{54}$$

which implies (49) (see proof of Theorem 2.2).
To prove (52) check that

$$\begin{aligned} \frac{\tilde{x}_{p_n} - x_{p_n}}{a(k_n/n)} &= \frac{Q_n(1) - F^{-1}(1-k_n/n)}{a(k_n/n)} - \Bigg(\frac{x_{p_n} - F^{-1}(1-k_n/n)}{a(k_n/n)} \\ &\quad - y_\gamma\Big(\frac{np_n}{k_n}\Big)\Bigg) \\ &\quad + \Big(\frac{\hat{a}(k_n/n)}{a(k_n/n)} - 1\Big) y_{\hat{\gamma}_n}\Big(\frac{np_n}{k_n}\Big) + \Big(y_{\hat{\gamma}_n}\Big(\frac{np_n}{k_n}\Big) - y_\gamma\Big(\frac{np_n}{k_n}\Big)\Big) \\ &=: I + II + III + IV. \end{aligned}$$

Theorem 6.1 in combination with the subsequent remark shows

$$k_n^{1/2} I \longrightarrow e(1) \qquad a.s.$$

By condition (51) we have

$$k_n^{1/2} II = \tilde{R}\Big(\frac{k_n}{n}, \frac{np_n}{k_n}\Big) = o(y_\gamma(np_n/k_n)) \qquad a.s.$$

Condition (15) ensures that $y_{\hat{\gamma}_n}(np_n/k_n) = y_\gamma(np_n/k_n)(1+o(1))$. Hence, in view of (53),

$$k_n^{1/2} III = y_\gamma\Big(\frac{np_n}{k_n}\Big)\int_{(0,1]} t^{-(\gamma+1)} e(t)\,\mu_{S,\gamma}(dt)(1+o(1)) \qquad a.s.$$

where

$$y_\gamma\Big(\frac{np_n}{k_n}\Big) = (1+o(1))\cdot\begin{cases} ((np_n)/k_n)^{-\gamma}/\gamma & \gamma>0, \\ \log(k_n/(np_n)) \quad \text{if} & \gamma=0, \\ -1/\gamma & \gamma<0. \end{cases}$$

Finally, similar as in the proof of Theorem 2.2 and the example given above, a Taylor expansion of $x \mapsto y_x(np_n/k_n)$ at γ in combination with (15) and (54) yields

$$\begin{aligned} k_n^{1/2} IV &= k_n^{1/2}(\hat{\gamma}_n - \gamma)\frac{1}{\gamma^2}\Big(1 - \Big(1+\gamma\log\frac{np_n}{k_n}\Big)\Big(\frac{np_n}{k_n}\Big)^{-\gamma}\Big)(1+o(1)) \\ &= \int_{(0,1]} t^{-(\gamma+1)} e(t)\,\nu_{T,\gamma}(dt)(1+o(1)) \\ &\qquad \cdot\begin{cases} \log(k_n/(np_n))(np_n/k_n)^{-\gamma}/\gamma & \gamma>0, \\ \log^2(k_n/(np_n))/2 \quad \text{if} & \gamma=0, \\ 1/\gamma^2 & \gamma<0. \end{cases} \end{aligned}$$

Because, for $\gamma \geq 0$, $I + II + III = o(IV)$, assertion (52) follows readily in that case. For $\gamma < 0$ we obtain

$$\begin{aligned} &\frac{d_n}{a(k_n/n)}(\tilde{x}_{p_n} - x_{p_n}) \\ &\quad \longrightarrow \quad -\gamma e(1) + \int_{(0,1]} t^{-(\gamma+1)} e(t)\,\mu_{S,\gamma}(dt) - \frac{1}{\gamma}\int_{(0,1]} t^{-(\gamma+1)} e(t)\,\nu_{T,\gamma}(dt) \end{aligned}$$

from which (52) follows by straightforward calculations. □

Based on Theorem 6.2, one may construct confidence intervals along the lines given in Section 2. In the present situation one uses different estimators of the asymptotic variance depending on the estimated extreme value index $\hat{\gamma}_n$.

For example, if $\gamma < 0$ then one can show by similar arguments as in the proofs of Theorems 6.2 and 2.3 that, for all $s > 0$,

$$\begin{aligned}
\hat{\gamma}_n^{(i)} &:= T(Q_{n,i}) = \gamma + \frac{k_n^{1/2}}{i} \int_{(0,1]} t^{-(\gamma+1)} e\Big(\frac{i}{k_n} t\Big)\, \nu_{T,\gamma}(dt) + o_P(k_n^{-1/2}) \\
\frac{\hat{a}(i/n)}{a(k_n/n)} &:= \frac{S(Q_{n,i})}{a(k_n/n)} \\
&= \Big(\frac{i}{k_n}\Big)^{-\gamma} \Big(1 + \frac{k_n^{1/2}}{i} \int_{(0,1]} t^{-(\gamma+1)} e\Big(\frac{i}{k_n} t\Big)\, \mu_{S,\gamma}(dt) + o_P(k_n^{-1/2})\Big)
\end{aligned}$$

uniformly for $sk_n \le i \le k_n$. From this one may conclude the existence of a sequence $s_n \downarrow 0$ such that

$$\sup_{s_n \le s \le 1} \left| k_n^{1/2} \frac{\tilde{x}_{p_n}^{([k_n s])} - x_{p_n}}{\hat{a}([k_n s]/n)} - Z_{S,T,\gamma}(s) \right| \longrightarrow 0$$

in probability with

$$\tilde{x}_{p_n}^{(i)} := X_{n-i:n} + \hat{a}\Big(\frac{i}{n}\Big) \frac{(np_n/i)^{\hat{\gamma}_n^{(i)}} - 1}{\hat{\gamma}_n^{(i)}}$$

and

$$Z_{S,T,\gamma}(s) := \frac{e(s)}{\gamma} - \frac{1}{\gamma s} \int_{(0,1]} t^{-(\gamma+1)} e(st)\, \mu_{S,\gamma}(dt) + \frac{1}{\gamma^2 s} \int_{(0,1]} t^{-(\gamma+1)} e(st)\, \nu_{t,\gamma}(dt).$$

In view of the proof of Theorem 2.3, this in turn implies, with $j_n := [k_n s_n] + 1$ and $\tilde{Z}_{S,T,\gamma}(u) := e^{u/2} Z_{S,T,\gamma}(e^u)$,

$$\begin{aligned}
\tilde{\sigma}_n^2 &:= \frac{1}{\log(k_n/j_n)} \sum_{i=j_n}^{k_n} \Big(\frac{\tilde{x}_{p_n}^{(i)} - \tilde{x}_{p_n}}{\hat{a}(i/n)} \Big)^2 \\
&= \frac{1}{\log(k_n/j_n)} \int_{j_n/k_n}^1 \Big(Z_{S,T,\gamma}(s) - \frac{\hat{a}(k_n/n)}{\hat{a}(i/n)} Z_{S,T,\gamma}(1) \Big)^2 ds (1 + o_P(1)) \\
&= \frac{1}{\log(k_n/j_n)} \int_{j_n/k_n}^1 \Big(Z_{S,T,\gamma}(s) - s^{-\gamma} Z_{S,T,\gamma}(1) \Big)^2 ds (1 + o_P(1)) \\
&= \frac{1}{\log(k_n/j_n)} \int_{\log(j_n/k_n)}^0 \Big(\tilde{Z}_{S,T,\gamma}(u) - \mathrm{e}^{(1/2-\gamma)u} \tilde{Z}_{S,T,\gamma}(0) \Big)^2 du (1 + o_P(1)) \\
&\longrightarrow E\Big(\tilde{Z}_{S,T,\gamma}^2(0) \Big) = \frac{\sigma_{S,T,\gamma}^2}{\gamma^2}
\end{aligned}$$

because

$$\frac{1}{\log(k_n/j_n)} E\Big(\int_{\log(j_n/k_n)}^{0} \Big(\mathrm{e}^{(1/2-\gamma)u} \tilde{Z}_{S,T,\gamma}(0) \Big)^2 \, du \longrightarrow \; 0.$$

Therefore, one may use the asymptotic $(1-\alpha)$–confidence interval

$$\Big[\tilde{x}_{p_n} - k_n^{-1/2} \tilde{\sigma}_n z_{\alpha/2}, \tilde{x}_{p_n} + k_n^{-1/2} \tilde{\sigma}_n z_{\alpha/2} \Big]$$

if $\hat{\gamma}_n < 0$, since $d_n \sim -\gamma k_n^{1/2}$ if $\gamma < 0$. In order not to overload the paper, we do not discuss the case $\gamma \geq 0$ and all the ramifications considered in Section 2.

Acknowledgment: The author was partly supported by a DFG Heisenberg grant. He thanks an anonymous referee for valuable remarks that helped to improve the presentation of the results.

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