Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint

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Preprint No. 78

Saarbrücken 2003

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submitted: March 13, 2003

Preprint No. 78

Saarbrücken 2003

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# Operator space structure and amenability for Figà-Talamanca–Herz algebras

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**Abstract**

Column and row operator spaces — which we denote by COL and ROW, respectively — over arbitrary Banach spaces were introduced by the first-named author; for Hilbert spaces, these definitions coincide with the usual ones. Given a locally compact group $G$ and $p, p' \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{p'} = 1$, we use the operator space structure on $\mathcal{CB}(\mathrm{COL}(L^{p'}(G)))$ to equip the Figà-Talamanca–Herz algebra $A_p(G)$ with an operator space structure, turning it into a quantized Banach algebra. Moreover, we show that, for $p \leq q \leq 2$ or $2 \leq q \leq p$ and amenable $G$, the canonical inclusion $A_q(G) \subset A_p(G)$ is completely bounded (with cb-norm at most $K_{\mathbb{G}}^2$, where $K_{\mathbb{G}}$ is Grothendieck's constant). As an application, we show that $G$ is amenable if and only if $A_p(G)$ is operator amenable for all — and equivalently for one — $p \in (1, \infty)$; this extends a theorem by Z.-J. Ruan.

*Keywords*: operator spaces, operator sequence spaces, column and row spaces, locally compact groups, Figà-Talamanca–Herz algebra, Fourier algebra, amenability, operator amenability.

2000 *Mathematics Subject Classification*: 43A15, 43A30, 46B70, 46J99, 46L07, 47L25 (primary), 47L50.

## Introduction

The Fourier algebra $A(G)$ of a locally compact group $G$ was introduced by P. Eymard in [Eym 1]. If $G$ is abelian with dual group $\Gamma$, then the Fourier

*Part of the research for this paper was done while the author held a postdoctoral fellowship of the Pacific Institute for the Mathematical Sciences (PIMS) at the University of Alberta; the support by PIMS is acknowledged with thanks.

†Research supported by NSERC under grant no. 227043-00.

transform induces an isometric isomorphism of $A(G)$ and $L^1(\Gamma)$. Although the Fourier algebra is an invariant for $G$ — like $L^1(G)$ —, its Banach algebraic amenability does not correspond well to the amenability of $G$ — very much unlike $L^1(G)$: The group $G$ is amenable if and only if $L^1(G)$ is amenable as a Banach algebra ([Joh 1]), but there are compact groups, among them $\mathrm{SO}(3)$, for which $A(G)$ fails to be even weakly amenable ([Joh 2]). In fact, the only locally compact groups $G$ for which $A(G)$ is an amenable Banach algebra in the sense of [Joh 1] are those with an abelian subgroup of finite index ([Run 3]).

Since $A(G)$ is the predual of the group von Neumann algebra $\mathrm{VN}(G)$, it is an operator space in a natural manner. In [Rua 1], Z.-J. Ruan introduced a variant of amenability for "quantized" Banach algebras — called operator amenability — that reflects the operator space structures of those algebras. He showed that a locally compact group $G$ is amenable if and only if $A(G)$ is operator amenable ([Rua 1, Theorem 3.6]). Further results by O. Yu. Aristov ([Ari]), P. J. Wood ([Woo]), and N. Spronk ([Spr]) lend additional support to the belief that homological properties of $A(G)$, such as amenability, biprojectivity or weak amenability, correspond to properties of $G$ much more naturally if the operator space structure is taken into account. Even if one is only interested in $A(G)$ as a Banach algebra, considering the canonical operator space structure can be valuable: Although the main result of [Run 3] is purely classical in its statement, its proof is operator space theoretic. (Further examples of classical results obtained with the help of operator space methods can be found in [F–K–L–S].)

In [Her 1], C. Herz introduced, for $p \in (1, \infty)$, an $L^p$-analogue of the Fourier algebra, denoted by $A_p(G)$. These algebras are called Figà-Talamanca–Herz algebras. It was asked by the third-named author if there was an analogue of Ruan's theorem for arbitrary Figà-Talamanca–Herz algebras ([Run 1, Problem 34]). The first obstacle towards a solution of this problem is that there is — at first glance — no natural operator space structure for $A_p(G)$ if $p \in (1, \infty) \setminus \{2\}$.

In [Run 2], the same author used the operator space structure introduced by G. Pisier on the $L^p$-spaces via complex interpolation ([Pis 1]) to define operator space analogues $OA_p(G)$ of the classical Figà-Talamanca–Herz algebras for all $p \in (1, \infty)$. These operator Figà-Talamanca–Herz algebras display properties similar to those of their classical counterparts (see, for example, [Run 2, Theorem 4.10]). Nevertheless, the construction from [Run 2] is unsatisfactory for two reasons:

1. Although we have a contractive inclusion $A_p(G) \subset OA_p(G)$ for all $p \in (1, \infty)$, the two algebras $A_p(G)$ and $OA_p(G)$ can fail to be identical

as Banach spaces for $p \neq 2$.

2. Even though $A(G) = OA_2(G)$ as Banach spaces, they need not have the same operator space structure.

The reason why $A(G)$ and $OA_2(G)$ need not coincide as operator spaces is that the operator space structure of $A(G)$ stems from $\mathrm{VN}(G)$ and thus from the column Hilbert space over $L^2(G)$, whereas $OA_2(G)$ is defined by means of the operator Hilbert space over $L^2(G)$. In order to get a satisfactory operator space structure for general Figà-Talamanca–Herz algebras, one should therefore strive for an extension of the notion of column space from Hilbert spaces to arbitrary $L^p$-spaces.
In his doctoral dissertation [Lam], the first-named author introduced such a notion; in fact, he defined column and row spaces over arbitrary Banach spaces. We first outline his construction and then use it to equip general Figà-Talamanca–Herz algebras with a canonical operator space structure, turning them into "quantized" Banach algebras. As an application, we extend [Rua 1, Theorem 3.6] to arbitrary Figà-Talamanca–Herz algebras.


## Acknowledgments

The authors would like to thank Gerd Wittstock for several valuable discussions on the topic of this paper. The first-named author wishes to thank him especially for his guidance and supervision that ultimately resulted in [Lam].


# 1 Preliminaries

## 1.1 Figà-Talamanca–Herz algebras

Let $G$ be a locally compact group. For any function $f : G \to \mathbb{C}$, we define $\check{f} : G \to \mathbb{C}$ by letting $\check{f}(x) := f(x^{-1})$ for $x \in G$. Let $p \in (1, \infty)$, and let $p' \in (1, \infty)$ be dual to $p$, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. The *Figà-Talamanca–Herz algebra* $A_p(G)$ consists of those functions $f : G \to \mathbb{C}$ such that there are sequences $(\xi_n)_{n=1}^\infty$ in $L^p(G)$ and $(\eta_n)_{n=1}^\infty$ in $L^{p'}(G)$ such that

$$\sum_{n=1}^{\infty} \|\xi_n\|_{L^p(G)} \|\eta_n\|_{L^{p'}(G)} < \infty \tag{1}$$

and

$$f = \sum_{n=1}^{\infty} \xi_n * \check{\eta}_n. \tag{2}$$

The norm on $A_p(G)$ is defined as the infimum over all sums (1) such that (2) holds. It is clear that $A_p(G)$ is a Banach space that embeds contractively into $\mathcal{C}_0(G)$. It was shown by C. Herz ([Her 1]) that $A_p(G)$ is closed under pointwise multiplication and, in fact, a Banach algebra. The case where $p = q = 2$ had previously been studied by P. Eymard ([Eym 1]); in this case $A(G) := A_2(G)$ is called the *Fourier algebra* of $G$.
Let $\lambda_{p'} : G \to \mathcal{B}(L^{p'}(G))$ be the regular left representation of $G$ on $L^{p'}(G)$. Via integration, $\lambda_{p'}$ extends to a representation of $L^1(G)$ on $L^{p'}(G)$. The algebra of *$p'$-pseudomeasures* $\mathrm{PM}_{p'}(G)$ is defined as the $w^*$-closure of $\lambda_{p'}(L^1(G))$ in $\mathcal{B}(L^{p'}(G))$. There is a canonical duality $\mathrm{PM}_{p'}(G) \cong A_p(G)^*$ via

$$\langle \xi * \check{\eta}, T \rangle := \langle T\eta, \xi \rangle \qquad (\xi \in L^{p'}(G),\, \eta \in L^p(G),\, T \in \mathrm{PM}_{p'}(G)).$$

If $p = 2$, then $\mathrm{VN}(G) := \mathrm{PM}_2(G)$ is a von Neumann algebra, the *group von Neumann algebra* of $G$.
For more information, see [Eym 1], [Eym 2], [Her 1], [Her 2], and [Pie].

## 1.2 Operator spaces and quantized Banach algebras

There is now a booklength monograph available on the subject of operator spaces ([E–R]), and a second one will soon appear ([Pis 2]); furthermore, a very accessible survey article is available on the internet ([Wit *et al.*]). We therefore refrain from repeating here the basic definitions of operator space theory. In our choice of notation, we mostly follow [E–R]. In particular, the projective and injective tensor product of Banach spaces are denoted by $\otimes_\gamma$ and $\otimes_\lambda$, respectively, whereas $\hat{\otimes}$ and $\check{\otimes}$ stand for the projective and injective tensor product of operator spaces.
We deviate from [E–R] with our notation in two points:

1. The minimal and maximal operator space over a given Banach space $E$ is denoted by $\mathrm{MIN}(E)$ and $\mathrm{MAX}(E)$, respectively.

2. The column and row space over a Hilbert space $\mathfrak{H}$ is denoted by $\mathrm{COL}(\mathfrak{H})$ and $\mathrm{ROW}(\mathfrak{H})$, respectively.

**Definition 1.1** A *quantized Banach algebra* is an algebra which is also an operator space such that multiplication is completely bounded.

*Remark* We do not require multiplication to be completely contractive (such algebras are called *completely contractive Banach algebras*; see [Rua 1]). In our choice of terminology, we follow [Ari].

*Examples* 1. For any Banach algebra $\mathfrak{A}$ — not necessarily with contractive multiplication — the maximal operator space $\mathrm{MAX}(\mathfrak{A})$ is a quantized Banach algebra.

2. If $\mathfrak{H}$ is a Hilbert space, then any closed subalgebra of $\mathcal{B}(\mathfrak{H})$ is completely contractive.

3. We denote the $W^*$-tensor product by $\bar{\otimes}$. A *Hopf–von Neumann algebra* is a pair $(\mathfrak{M}, \nabla)$, where $\mathfrak{M}$ is a von Neumann algebra, and $\nabla$ is a *co-multiplication*: a unital, $w^*$-continuous, and injective $^*$-homomorphism $\nabla \colon \mathfrak{M} \to \mathfrak{M}\bar{\otimes}\mathfrak{M}$ which is co-associative, i.e. the diagram

$$\begin{tikzcd} \mathfrak{M} \arrow[r, "\nabla"] \arrow[d, "\nabla"'] & \mathfrak{M}\bar{\otimes}\mathfrak{M} \arrow[d, "\nabla\otimes\mathrm{id}_{\mathfrak{M}}"] \\ \mathfrak{M}\bar{\otimes}\mathfrak{M} \arrow[r, "\mathrm{id}_{\mathfrak{M}}\otimes\nabla"'] & \mathfrak{M}\bar{\otimes}\mathfrak{M}\bar{\otimes}\mathfrak{M} \end{tikzcd}$$

commutes. Let $\mathfrak{M}_*$ denote the unique predual of $\mathfrak{M}$. By [E–R, Theorem 7.2.4], we have $\mathfrak{M}\bar{\otimes}\mathfrak{M} \cong (\mathfrak{M}_*\hat{\otimes}\mathfrak{M}_*)^*$. Thus $\nabla$ induces a complete contraction $\nabla_* \colon \mathfrak{M}_*\hat{\otimes}\mathfrak{M}_* \to \mathfrak{M}_*$ turning $\mathfrak{M}_*$ into a completely contractive Banach algebra.

4. Let $G$ be a locally compact group. Then the representation

$$G \mapsto \mathcal{B}(L^2(G \times G)), \quad x \mapsto \lambda_2(x) \otimes \lambda_2(x)$$

yields a co-multiplication $\nabla \colon \mathrm{VN}(G) \to \mathrm{VN}(G\times G) \cong \mathrm{VN}(G)\bar{\otimes}\,\mathrm{VN}(G)$. Hence, $A(G)$ with its canonical operator space structure is a completely contractive Banach algebra.

# 2 Operator sequence spaces

Operator sequence spaces were introduced by the first-named author in his doctoral dissertation ([Lam, Definition 1.1.1]). They form a category somewhere in between Banach and operator spaces. A full account of the theory of operator sequence spaces will be published elsewhere.
In this section, we content ourselves with outlining the concepts and results we'll require in the next section in order to define column and row spaces over arbitrary Banach spaces. We are also somewhat sketchy with our proofs — especially if they consist mainly of routine calculations or straightforward adaptations of proofs of the corresponding Banach or operator space results.

**Definition 2.1** A *sequential norm* over a Banach space $E$ is a sequence $(\|\cdot\|_{\widehat{n}})_{n=1}^{\infty}$ such that $\|\cdot\|_{\widehat{1}}$ is the given norm on $E$ and $\|\cdot\|_{\widehat{n}}$ is, for each $n \in \mathbb{N}$, a norm on $E^n$ such that

$$\begin{aligned} \left\| \begin{bmatrix} x \\ 0 \end{bmatrix} \right\|_{\widehat{m+n}} &= \|x\|_{\widehat{m}} \qquad (m, n \in \mathbb{N},\ x \in E^m), \\ \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|_{\widehat{m+n}}^2 &\leq \|x\|_{\widehat{m}}^2 + \|y\|_{\widehat{n}}^2 \qquad (m, n \in \mathbb{N},\ x \in E^m,\ y \in F^n) \end{aligned}$$

and

$$\|\alpha x\|_{\widehat{m}} \leq \|\alpha\| \|x\|_{\widehat{n}} \qquad (m, n \in \mathbb{N},\ x \in E^n,\ \alpha \in M_{m,n}).$$

For $n \in \mathbb{N}$, we write $E^{\widehat{n}}$ to denote $E^n$ equipped with $\|\cdot\|_{\widehat{n}}$. The space $E$ together with the sequential norm $(\|\cdot\|_{\widehat{n}})_{n=1}^{\infty}$ is called an *operator sequence space* (over $E$).

*Examples*

1. Letting $\mathbb{C}^{\widehat{n}} := \ell_n^2$ for $n \in \mathbb{N}$, we define *the* (unique) operator sequence space over $\mathbb{C}$.

2. Let $E$ be an operator space. Define an operator sequence space $C(E)$ over $M_1(E)$ by letting $C(E)^{\widehat{n}} := M_{n,1}(E)$ for $n \in \mathbb{N}$.

3. Let $E$ be any Banach space. The *minimal operator sequence space* $\min(E)$ over $E$ is defined by letting $\min(E)^{\widehat{n}} := \mathcal{B}(\ell_n^2, E)$ for $n \in \mathbb{N}$; the adjective minimal will be justified below.

4. Let $E$ be any Banach space. The *maximal operator sequence space* $\max(E)$ over $E$ is defined as follows: For $n \in \mathbb{N}$ and $x \in E^n$, define

   $$\|x\|_{\widehat{n}} := \inf\{\|\alpha\| \|y\|_{\ell_m^2(E)} : m \in \mathbb{N},\ \alpha \in M_{n,m},\ y \in E^m,\ x = \alpha y\}.$$

   As in the case of $\min(E)$, the use of the adjective maximal will soon become clear.

5. Let $E$ be an operator sequence space, and let $m \in \mathbb{N}$. We define $(E^{\widehat{m}})^{\widehat{n}} := E^{\widehat{mn}}$ for $n \in \mathbb{N}$. This turns $E^{\widehat{m}}$ into an operator sequence space.

Having introduced the objects of the category of operator sequence spaces, we now turn to defining its morphisms.

Given two operator sequence spaces $E$ and $F$, a linear map $T\colon E \to F$, and $n \in \mathbb{N}$, let

$$T^{\widehat{n}} : E^{\widehat{n}} \to F^{\widehat{n}}, \quad [x_j]_{j=1,\ldots,n} \mapsto [Tx_j]_{j=1,\ldots,n}$$

denote the $n$-th amplification of $T$.

**Definition 2.2** Let $E$ and $F$ be operator sequence spaces. Then $T \in \mathcal{B}\left(E^{\widehat{1}}, F^{\widehat{1}}\right)$ is called *sequentially bounded* if

$$\|T\|_{\mathrm{sb}} := \sup_{n\in\mathbb{N}} \left\|T^{\widehat{n}}\right\|_{\mathcal{B}\left(E^{\widehat{n}}, F^{\widehat{n}}\right)} < \infty.$$

If $\|T\|_{\mathrm{sb}} \leq 1$, we call $T$ a *sequential contraction*, and if $T^{\widehat{n}}$ is an isometry for each $n \in \mathbb{N}$, we call $T$ a *sequential isometry*. The collection of all sequentially bounded maps from $E$ to $F$ is denoted by $\mathcal{SB}(E, F)$.

*Remarks*

1. It is straightforward that $\|\cdot\|_{\mathrm{sb}}$ turns $\mathcal{SB}(E, F)$ into a Banach space.
2. We write $\mathcal{SB}(E)$ instead of $\mathcal{SB}(E, E)$.

*Examples*

1. Let $E$ be an operator sequence space, and let $F$ be a Banach space. Then

$$\mathcal{SB}(E, \min(F)) = \mathcal{B}\left(E^{\widehat{1}}, F\right) \tag{3}$$

holds isometrically. (This justifies the name minimal operator space.)

2. Let $E$ be a Banach space, and let $F$ be an operator sequence space. Then

$$\mathcal{SB}(\max(E), F) = \mathcal{B}\left(E, F^{\widehat{1}}\right) \tag{4}$$

holds isometrically. (This justifies the adjective maximal.)

3. Let $E$ be an infinite-dimensional Banach space. Then $\mathrm{id}_E\colon \mathrm{MIN}(E) \to \mathrm{MAX}(E)$ is not completely bounded ([Pau 1, Theorem 2.12]). Interestingly, the situation for operator sequence spaces and sequentially bounded maps is different: For example, if $\mathfrak{A}$ is a $C^*$-algebra, then $\mathfrak{A}$ is subhomogeneous if and only if $\mathrm{id}_{\mathfrak{A}} \in \mathcal{SB}(\min(\mathfrak{A}), \max(\mathfrak{A}))$ ([Lam, Satz 2.2.25]).

Next, we sketch the duality theory for operator sequence spaces. We first introduce a canonical operator sequence space structure over the Banach space of all sequentially bounded maps between operator sequence spaces:

**Proposition 2.3** *Let $E$ and $F$ be operator sequence spaces. Letting*

$$\mathcal{SB}(E,F)^{\widehat{n}} := \mathcal{SB}\left(E, F^{\widehat{n}}\right) \qquad (n \in \mathbb{N})$$

*defines an operator sequence space over $\mathcal{SB}(E,F)$.*

We skip the proof which parallels the one of the corresponding result for operator spaces.
We require an analogue of Smith's lemma ([E–R, Proposition 2.2.2]) for operator sequence spaces:

**Lemma 2.4 (Smith's lemma for operator sequence spaces)** *Let $E$ and $F$ be operator sequence spaces such that $\dim F = m < \infty$. Then $\mathcal{B}\left(E^{\widehat{1}}, F^{\widehat{1}}\right) = \mathcal{SB}(E,F)$ holds with $\|T\|_{\mathrm{sb}} = \left\|T^{\widehat{m}}\right\|$ for each $T \in \mathcal{B}(E,F)$.*

*Proof* Let $T \in \mathcal{B}(E,F)$, and let $n \in \mathbb{N}$. Let $x = [x_j]_{j=1,\ldots,n} \in E^{\widehat{n}}$, and let $y_1, \ldots, y_m$ be a basis for $F$. Then there are $\alpha_{j,k} \in \mathbb{C}$ for $j = 1, \ldots, n$ and $k = 1, \ldots, m$ such that

$$Tx_j = \sum_{k=1}^{m} \alpha_{j,k} y_k.$$

Let $y := [y_k]_{k=1,\ldots,m} \in F^{\widehat{m}}$ and $\alpha := [\alpha_{j,k}]_{\substack{j=1,\ldots,n \\ k=1,\ldots,m}} \in M_{n,m}$. It follows that

$$T^{\widehat{n}} x = \alpha y.$$

Let $v \in M_{n,m}$ be a partial isometry such that $\alpha = v|\alpha|$, where $|\alpha| := (\alpha^* \alpha)^{\frac{1}{2}}$. It follows that

$$\|\alpha y\|_{F^{\widehat{n}}} = \|v|\alpha|y\|_{F^{\widehat{n}}} \leq \||\alpha|y\|_{F^{\widehat{m}}} \tag{5}$$

and

$$\||\alpha|y\|_{F^{\widehat{m}}} = \|v^* \alpha y\|_{F^{\widehat{m}}} \leq \|\alpha y\|_{F^{\widehat{n}}} \tag{6}$$

so that in (5) and (6) equality holds. Consequently, we have

$$\begin{aligned}\left\|T^{\widehat{n}}x\right\|_{F^{\widehat{n}}} &= \|\alpha y\|_{F^{\widehat{n}}} \\ &= \|v^*\alpha y\|_{F^{\widehat{m}}} \\ &= \left\|v^*T^{\widehat{n}}x\right\|_{F^{\widehat{m}}} \\ &= \left\|T^{\widehat{m}}(v^*x)\right\|_{F^{\widehat{m}}} \\ &\leq \left\|T^{\widehat{m}}\right\| \|v^*x\|_{F^{\widehat{m}}} \\ &\leq \left\|T^{\widehat{m}}\right\| \|x\|_{F^{\widehat{n}}},\end{aligned}$$

so that $\left\|T^{\widehat{n}}\right\| \leq \left\|T^{\widehat{m}}\right\|$. Since $n \in \mathbb{N}$ was arbitrary, this yields $\|T\|_{\mathrm{sb}} \leq \left\|T^{\widehat{m}}\right\|$. The converse inequality is trivial. $\square$

*Remark* In view of [E–R, Proposition 2.2.2], one might have suspected that Lemma 2.4 holds true for $F = \ell^2_m$ only. The fact that there is a stronger version of Smith's lemma for operator sequence spaces than for operator spaces has interesting consequences. For example, the principle of local reflexivity, which is a cornerstone of the local theory of Banach spaces ([D–F, 6.6]), but fails to have an analogue for operator spaces ([E–R, Corollary 14.3.8]), still works in the category of operator sequence spaces ([Lam, Satz 1.3.26])

**Corollary 2.5** *Let $E$ be an operator sequence space. Then $E^* = \mathcal{SB}(E, \mathbb{C})$ holds isometrically.*

With Corollary 2.5 at hand, we can now equip the (Banach space) dual of an operator sequence space with a canonical operator sequence space structure. In analogy with the situation for operator spaces, the two natural operator sequence space structures on the bidual of an operator sequence space are compatible ([Lam, Satz 1.3.19]):

**Theorem 2.6 (compatibility of biduals)** *Let $E$ be an operator sequence space. Then*

$$(E^{**})^{\widehat{n}} = \left(E^{\widehat{n}}\right)^{**} \qquad (n \in \mathbb{N}) \tag{7}$$

*holds isometrically.*

*Remark* We would like to emphasize that the left hand side of (7) denotes the $n$-th sequence level of the bidual operator sequence space $E^{**}$, whereas the right hand side stands for the second (Banach space) dual of the Banach space $E^{\widehat{n}}$.

Taking the adjoint of a sequentially bounded operator yields again a sequentially bounded operator. But more is true:

**Theorem 2.7** *Let $E$ and $F$ be operator sequence spaces. Then*

$$\left\|(T^*)^{\widehat{n}}\right\| = \left\|T^{\widehat{n}}\right\| \qquad (n \in \mathbb{N},\ T \in \mathcal{SB}(E,F))$$

*holds. Moreover,*

$$\mathcal{SB}(E,F) \to \mathcal{SB}(F^*,E^*), \quad T \mapsto T^* \tag{8}$$

*is a sequential isometry.*

*Proof* The first part of the theorem is [Lam, Satz 1.3.14] and has a proof similar to its operator space analogue [E–R, Proposition 3.2.2]. In particular, (8) is an isometry.
To see that (8) is in fact a sequential isometry, fix $n \in \mathbb{N}$ and note that, we have a (sequential) isometric canonical isomorphism

$$\mathcal{SB}\left(\left(F^{\widehat{n}}\right)^*, E^*\right) \cong \mathcal{SB}\left(F^*, (E^*)^{\widehat{n}}\right) \tag{9}$$

by [Lam, Satz 1.3.10 and Satz 1.3.12]. Hence, we have the following canonical isometries:

$$\begin{aligned}
\mathcal{SB}(E,F)^{\widehat{n}} &= \mathcal{SB}\left(E, F^{\widehat{n}}\right) \\
&\hookrightarrow \mathcal{SB}\left(\left(F^{\widehat{n}}\right)^*, E^*\right), && \text{by [Lam, Satz 1.3.14]}, \\
&\cong \mathcal{SB}\left(F^*, (E^*)^{\widehat{n}}\right), && \text{by (9)}, \\
&= \mathcal{SB}(F^*,E^*)^{\widehat{n}}.
\end{aligned}$$

This completes the proof. $\square$

We conclude this section with the analogue of the MIN-MAX duality ([Lam, Satz 2.1.11]):

**Theorem 2.8** (min-max **duality)** *For any Banach space $E$, we have the sequentially isometric isomorphisms*

$$\min(E)^* = \max(E^*) \qquad \textit{and} \qquad \max(E)^* = \min(E^*)$$

*Proof* In view of the compatibility of biduals for operator sequence spaces, a proof of this result can be given that more or less follows the proof of the MIN-MAX duality in [Ble]. $\square$

# 3 Column and row spaces over arbitrary Banach spaces

In [Math], B. Mathes characterized the column operator space $\mathrm{COL}(\mathfrak{H})$ over a Hilbert space $\mathfrak{H}$ through the isometries

$$M_{n,1}(\mathrm{COL}(\mathfrak{H})) = M_{n,1}(\mathrm{MAX}(\mathfrak{H}))$$

and

$$M_{1,n}(\mathrm{COL}(\mathfrak{H})) = M_{1,n}(\mathrm{MIN}(\mathfrak{H}))$$

for all $n \in \mathbb{N}$. A similar characterization holds for $\mathrm{ROW}(\mathfrak{H})$.
In this section, we use these characterizations of column and row spaces over a Hilbert space, to define column and row spaces over arbitrary Banach spaces. As in the previous section, the material is from [Lam] and will appear elsewhere in fuller detail.

**Definition 3.1** Let $E$ be an operator sequence space. Then the *minimal operator space* $\mathrm{Min}(E)$ over $E$ is defined by letting $M_n(\mathrm{Min}(E)) := \mathcal{B}\left(\ell^2_n, E^{\widehat{n}}\right)$.

*Remarks* 1. By [Lam, Satz 4.1.2], $\mathrm{Min}(E)$ is an operator space for any operator sequence space $E$.

2. For any operator space $E$, and for any operator sequence space $F$,

$$\mathcal{CB}(E, \mathrm{Min}(F)) = \mathcal{SB}(C(E), F)$$

holds isometrically ([Lam, Satz 4.1.6]).

3. Let $F$ be a Banach space. Then the previous remark and the isometric identity (3) combined yield that

$$\mathcal{CB}(E, \mathrm{Min}(\min(F))) = \mathcal{B}(M_1(E), F)$$

holds isometrically for each operator space $E$, so that $\mathrm{Min}(\min(F)) = \mathrm{MIN}(F)$ holds.

The following definition generalizes V. I. Paulsen's formula for the maximal operator space norm over a Banach space ([Pau 2, Theorem 2.1]):

**Definition 3.2** Let $E$ be an operator sequence space. Then the *maximal operator space* $\mathrm{Max}(E)$ over $E$ is defined by letting, for $x \in M_n(E)$,

$$\|x\|_{M_n(\mathrm{Max}(E))} := \inf\{\|\alpha\|\|\beta\| : x = \alpha\,\mathrm{diag}(v_1, \ldots, v_k)\beta\},$$

where the infimum is taken over all $k, l \in \mathbb{N}$, $\alpha \in M_{n,kl}$, $\beta \in M_{k,n}$, and $v_1, \ldots, v_k$ in the closed unit ball of $E^{\widehat{l}}$.

*Remarks* 1. By [Lam, Satz 4.1.10], $\mathrm{Max}(E)$ is an operator space for any operator sequence space $E$.

2. For any operator sequence space $E$, and for any operator space $F$,

$$\mathcal{CB}(\mathrm{Max}(E), F) = \mathcal{SB}(E, C(F))$$

holds isometrically ([Lam, Satz 4.1.12]).

3. Let $E$ be a Banach space. Then the previous remark and the isometric identity (4) combined yield that

$$\mathcal{CB}(\mathrm{Max}(\max(E)), F) = \mathcal{B}(E, M_1(F))$$

holds isometrically for each operator space $F$, i.e. $\mathrm{Max}(\max(E)) = \mathrm{MAX}(E)$.

There is a duality between Min and Max as between min and max and MIN and MAX ([Lam, Satz 4.2.1]):

**Theorem 3.3 (**Min-Max **duality)** *For any operator sequence space $E$, we have the completely isometric isomorphisms*

$$\mathrm{Min}(E)^* = \mathrm{Max}(E^*) \qquad \textit{and} \qquad \mathrm{Max}(E)^* = \mathrm{Min}(E^*)$$

We can now define the column space $\mathrm{COL}(E)$ and the row space $\mathrm{ROW}(E)$ over an *arbitrary* Banach space $E$:

**Definition 3.4** Let $E$ be a Banach space.

(a) The *column space* over $E$ is defined as $\mathrm{COL}(E) := \mathrm{Min}(\max(E))$.

(b) The *row space* over $E$ is defined as $\mathrm{ROW}(E) := \mathrm{Max}(\min(E))$.

Recall that an operator space $E$ is called *homogeneous* if $\mathcal{CB}(E) = \mathcal{B}(M_1(E))$ holds isometrically.

**Theorem 3.5** *Let $E$ be a Banach space. Then* $\mathrm{COL}(E)$ *and* $\mathrm{ROW}(E)$ *are homogeneous operator spaces such that*

$$\mathrm{COL}(E)^* = \mathrm{ROW}(E^*) \qquad \textit{and} \qquad \mathrm{ROW}(E)^* = \mathrm{COL}(E^*). \tag{10}$$

*Proof* Since

$$\mathcal{B}(E) = \mathcal{SB}(\max(E)) = \mathcal{CB}(\mathrm{Min}(\max(E))),$$

the homogeneity of $\mathrm{COL}(E)$ is clear (and similarly for $\mathrm{ROW}(E)$).
The dualities (10) follow immediately from Theorems 2.8 and 3.3. □

*Remark* It follows from [Math] that, for a Hilbert space $\mathfrak{H}$, Definition 3.4 yields the same operator spaces as the usual one ([E–R, 3.4]).

# 4 Amplifying operators on $L^p$-spaces

The following definition is from [Her 1]:

**Definition 4.1** Let $p \in (1,\infty)$. A Banach space $E$ is called a $p$-space if, for any two measure spaces $X$ and $Y$, the amplification map

$$\mathcal{B}(L^p(X), L^p(Y)) \to \mathcal{B}(L^p(X,E), L^p(Y,E)), \quad T \mapsto T \otimes \mathrm{id}_E \tag{11}$$

is an isometry.

*Remarks* 1. Let $q \in [1,\infty]$. Then an $L^q$-space is a $p$-space if $p \leq q \leq 2$ or $2 \leq q \leq p$ ([Her 1, Theorem 1]). In particular, any Hilbert space is a $p$-space for any $p \in (1,\infty)$.

2. A Banach space $E$ is a $p$-space if and only if $E^*$ is a $p'$-space ([Her 1, Proposition 4]).

In this section, we shall see that (11) is even a complete isometry — provided that all Banach spaces involved are equipped with their respective column space structures.
We start with a proof that (11) is a sequential isometry if the spaces involved are both equipped with their minimal or maximal operator sequence space structure, respectively.

**Lemma 4.2** *Let $E$, $F$, and $X$ be Banach spaces. For $x \in X$, define $\pi_x^F : \mathcal{B}(E,\mathcal{B}(X,F)) \to \mathcal{B}(E,F)$ by letting*

$$\pi_x^F(T)(y) := (Ty)(x) \qquad (T \in \mathcal{B}(E,\mathcal{B}(X,F)),\, y \in E,\, x \in X).$$

*Then the following are true:*

(i) *The equality*

$$\|T\| = \sup\left\{\left\|\pi_x^F(T)\right\| : x \in X,\, \|x\| \leq 1\right\} \qquad (T \in \mathcal{B}(E,\mathcal{B}(X,F)))$$

*holds.*

(ii) *For $x \in X$ with norm one, $\pi_x$ is a quotient map.*

*Proof* We have

$$\begin{aligned} \|T\| &= \sup\{\|Ty\| : y \in E,\, \|y\| \leq 1\} \\ &= \sup\{\|(Ty)(x)\| : y \in E,\, \|y\| \leq 1,\, x \in X,\, \|x\| \leq 1\} \\ &= \sup\left\{\left\|\pi_x^F(T)\right\| : x \in X,\, \|x\| \leq 1\right\}, \end{aligned}$$

which proves (i).
To prove (ii), we define an isometric right inverse of $\pi_x$ in case $\|x\| = 1$. Fix $\phi \in X^*$ with $\|\phi\| = \langle x, \phi \rangle = 1$. For $T \in \mathcal{B}(E, F)$, define $\tilde{T} \in \mathcal{B}(E, \mathcal{B}(X, F))$ by letting

$$\tilde{T}y := \phi \otimes Ty \in X^* \otimes_\lambda F \subset \mathcal{B}(X, F) \qquad (y \in E).$$

The map $\mathcal{B}(E, F) \ni T \mapsto \tilde{T}$ is then the desired right inverse of $\pi_x$. $\square$

**Corollary 4.3** *Let $p \in (1, \infty)$, let $X$ and $Y$ be measure spaces, let $E$ be a $p$-space, and let $n \in \mathbb{N}$. Then the amplification map*

$$\begin{aligned} \mathcal{B}\left(L^p(X), \min(L^p(Y))^{\widehat{n}}\right) &\to \mathcal{B}\left(L^p(X, E), \min(L^p(Y, E))^{\widehat{n}}\right), \\ T &\mapsto T \otimes \mathrm{id}_E \end{aligned}$$

*is an isometry.*

*Proof* Let $T \in \mathcal{B}(L^p(X), \mathcal{B}(\ell^2_n, L^p(Y)))$, and fix $\xi \in \ell^2_n$ with $\|\xi\| \leq 1$. It follows that $\pi_\xi^{L^p(Y)} \circ T \in \mathcal{B}(L^p(X), L^p(Y))$. Since $E$ is a $p$-space, we have the norm equalities

$$\left\| \pi_\xi^{L^p(Y)} \circ T \right\| = \left\| \left( \pi_\xi^{L^p(Y)} \circ T \right) \otimes \mathrm{id}_E \right\| = \left\| \pi_\xi^{L^p(Y,E)} \circ (T \otimes \mathrm{id}_E) \right\|.$$

The claim then follows from Lemma 4.2(i). $\square$

**Proposition 4.4** *Let $p \in (1, \infty)$, let $X$ and $Y$ be measure spaces, and let $E$ be a $p$-space. Then*

$$\left\| T^{\widehat{m}} \right\|_{\mathcal{B}\left(\min(L^p(X))^{\widehat{m}}, \min(L^p(Y))^{\widehat{mn}}\right)} = \left\| (T \otimes \mathrm{id}_E)^{\widehat{m}} \right\|_{\mathcal{B}\left(\min(L^p(X,E)^{\widehat{m}}), \min(L^p(Y,E))^{\widehat{mn}}\right)}$$

*holds for all $m, n \in \mathbb{N}$ and for all $T \in \mathcal{SB}\left(\min(L^p(X)), \min(L^p(Y))^{\widehat{n}}\right)$. In particular, the amplification map*

$$\begin{aligned} \mathcal{SB}(\min(L^p(X)), \min(L^p(Y))) &\to \mathcal{SB}(\min(L^p(X, E)), \min(L^p(Y, E))), \\ T &\mapsto T \otimes \mathrm{id}_E \end{aligned}$$

*is a sequential isometry.*

*Proof* Clearly, the first assertion implies the second one.

Let $m,n \in \mathbb{N}$. First, note that for all $T \in \mathcal{SB}\left(\min(L^p(X)), \min(L^p(Y))^{\widehat{n}}\right)$:

$$
\begin{aligned}
&\left\| T^{\widehat{m}} \right\|_{\mathcal{B}(\mathcal{B}(\ell^2_m, L^p(X)), \mathcal{B}(\ell^2_m \otimes_\gamma \ell^2_n, L^p(Y)))} \\
&\quad = \quad \|\mathrm{id}_{\ell^2_m} \otimes T\|_{\mathcal{B}(\ell^2_m \otimes_\lambda L^p(X), \ell^2_m \otimes_\lambda \mathcal{B}(\ell^2_n, L^p(Y)))}, \\
&\quad = \quad \|T\|_{\mathcal{B}(L^p(X), \mathcal{B}(\ell^2_n, L^p(Y)))}, \qquad \text{by the mapping property of } \otimes_\lambda, \\
&\quad = \quad \|T \otimes \mathrm{id}_E\|_{\mathcal{B}(L^p(X,E), \mathcal{B}(\ell^2_n, L^p(Y,E)))}, \qquad \text{by Corollary 4.3}, \\
&\quad = \quad \|\mathrm{id}_{\ell^2_m} \otimes T \otimes \mathrm{id}_E\|_{\mathcal{B}(\ell^2_m \otimes_\lambda L^p(X,E), \ell^2_m \otimes_\lambda \mathcal{B}(\ell^2_n, L^p(Y,E)))}, \\
&\qquad\qquad \text{again by the mapping property of } \otimes_\lambda, \\
&\quad = \quad \left\| (T \otimes \mathrm{id}_E)^{\widehat{m}} \right\|_{\mathcal{B}(\mathcal{B}(\ell^2_m, L^p(X,E)), \mathcal{B}(\ell^2_m \otimes_\gamma \ell^2_n, L^p(Y,E)))}.
\end{aligned}
$$

Since

$$
\pi_\xi^{L^p(Y)}\left(T^{\widehat{m}}\right) = 0 \quad \Longleftrightarrow \quad \pi_\xi^{L^p(Y,E)}\left((T \otimes \mathrm{id}_E)^{\widehat{m}}\right) = 0
$$

for all $\xi \in \ell^2_m \otimes \ell^2_n$ and for all $T \in \mathcal{SB}\left(\min(L^p(X)), \min(L^p(Y))^{\widehat{n}}\right)$, we conclude from Lemma 4.2(ii) that

$$
\left\| \pi_\xi^{L^p(Y)}\left(T^{\widehat{m}}\right) \right\| = \left\| \pi_\xi^{L^p(Y,E)}\left((T \otimes \mathrm{id}_E)^{\widehat{m}}\right) \right\| \tag{12}
$$

for all $T \in \mathcal{SB}\left(\min(L^p(X)), \min(L^p(Y))^{\widehat{n}}\right)$ and for all $\xi \in \ell^2_m \otimes \ell^2_n$ with $\|\xi\|_{\ell^2_m \otimes_\gamma \ell^2_n} = 1$ — and hence for all $\xi \in \ell^2_m \otimes \ell^2_n$. It follows, for $T \in \mathcal{SB}\left(\min(L^p(X)), \min(L^p(Y))^{\widehat{n}}\right)$, that

$$
\begin{aligned}
&\left\| T^{\widehat{m}} \right\|_{\mathcal{B}(\mathcal{B}(\ell^2_m, L^p(X)), \mathcal{B}(\ell^2_{mn}, L^p(Y)))} \\
&\quad = \quad \sup\left\{ \left\| \pi_\xi^{L^p(Y)}\left(T^{\widehat{m}}\right) \right\| : \xi \in \ell^2_m \otimes \ell^2_n,\ \|\xi\|_{\ell^2_{mn}} \le 1 \right\}, \\
&\qquad\qquad \text{by Lemma 4.2(i)}, \\
&\quad = \quad \sup\left\{ \left\| \pi_\xi^{L^p(Y,E)}\left((T \otimes \mathrm{id}_E)^{\widehat{m}}\right) \right\| : \xi \in \ell^2_m \otimes \ell^2_n,\ \|\xi\|_{\ell^2_{mn}} \le 1 \right\}, \\
&\qquad\qquad \text{by (12)}, \\
&\quad = \quad \left\| (T \otimes \mathrm{id}_E)^{\widehat{m}} \right\|_{\mathcal{B}(\mathcal{B}(\ell^2_m, L^p(X,E)), \mathcal{B}(\ell^2_{mn}, L^p(Y,E)))}, \\
&\qquad\qquad \text{again by Lemma 4.2(i)}.
\end{aligned}
$$

This completes the proof. $\square$

Together with [Her 1, Proposition 4], Theorem 2.7, and the min-max duality, Proposition 4.4 yields:

**Corollary 4.5** *Let $p \in (1,\infty)$, let $X$ and $Y$ be measure spaces, and let $E$ be a $p$-space. Then*

$$\begin{aligned} & \left\| T^{\widehat{m}} \right\|_{\mathcal{B}\left(\max(L^p(X))^{\widehat{m}}, \max(L^p(Y))^{\widehat{mn}}\right)} \\ = & \left\| (T \otimes \mathrm{id}_E)^{\widehat{m}} \right\|_{\mathcal{B}\left(\max(L^p(X,E)^{\widehat{m}}), \max(L^p(Y,E))^{\widehat{mn}}\right)} \end{aligned}$$

*holds for all $m,n \in \mathbb{N}$ and for all $T \in \mathcal{B}\left(\max(L^p(X)), \max(L^p(Y))^{\widehat{n}}\right)$. In particular, the amplification map*

$$\begin{aligned} \mathcal{SB}(\max(L^p(X)), \max(L^p(Y))) & \to \mathcal{SB}(\max(L^p(X,E)), \max(L^p(Y,E))), \\ T & \mapsto T \otimes \mathrm{id}_E \end{aligned}$$

*is a sequential isometry.*

We can now state and prove the main result of this section:

**Theorem 4.6** *Let $p \in (1,\infty)$, let $X$ and $Y$ be measure spaces, and let $E$ be a $p$-space. Then the amplification map*

$$\begin{aligned} \mathcal{CB}(\mathrm{COL}(L^p(X)), \mathrm{COL}(L^p(Y))) & \to \mathcal{CB}(\mathrm{COL}(L^p(X,E)), \mathrm{COL}(L^p(Y,E))), \\ T & \mapsto T \otimes \mathrm{id}_E \end{aligned} \tag{13}$$

*is a complete isometry.*

*Proof* Let $m,n \in \mathbb{N}$, and let $T \in \mathcal{CB}(\mathrm{COL}(L^p(X)), M_n(\mathrm{COL}(L^p(Y))))$. We can amplify $T$ to an operator $T^{\widehat{m}} \colon \max(L^p(X))^{\widehat{m}} \to \mathcal{B}\left(\ell^2_n, \max(L^p(Y))^{\widehat{mn}}\right)$. From Lemma 4.2 and the first part of Corollary 4.5, we conclude that

$$\begin{aligned} & \left\| T^{\widehat{m}} \right\|_{\mathcal{B}\left(\max(L^p(X))^{\widehat{m}}, \mathcal{B}\left(\ell^2_n, \max(L^p(Y))^{\widehat{mn}}\right)\right)} \\ = & \left\| (T \otimes \mathrm{id}_E)^{\widehat{m}} \right\|_{\mathcal{B}\left(\max(L^p(X,E))^{\widehat{m}}, \mathcal{B}\left(\ell^2_n, \max(L^p(Y,E))^{\widehat{mn}}\right)\right)}. \end{aligned}$$

An almost verbatim copy of the argument used to prove Proposition 4.4 yields that

$$\begin{aligned} & \left\| T^{(m)} \right\|_{\mathcal{B}\left(\mathcal{B}\left(\ell^2_m, \max(L^p(X))^{\widehat{m}}\right), \mathcal{B}\left(\ell^2_{mn}, \max(L^p(Y))^{\widehat{mn}}\right)\right)} \\ = & \left\| (T \otimes \mathrm{id}_E)^{(m)} \right\|_{\mathcal{B}\left(\mathcal{B}\left(\ell^2_m, \max(L^p(X))^{\widehat{m}}\right), \mathcal{B}\left(\ell^2_{mn}, \max(L^p(Y))^{\widehat{mn}}\right)\right)}. \end{aligned}$$

Consequently,

$$\|T\|_{\mathrm{cb}} = \|T \otimes \mathrm{id}_E\|_{\mathrm{cb}}$$

holds, so that (13) is a complete isometry. $\square$

Even though we won't need it in the sequel, we note the following analogue of Theorem 4.6 for row spaces: It follows immediately from the theorem due to the COL-ROW duality.

**Corollary 4.7** *Let $p \in (1, \infty)$, let $X$ and $Y$ be measure spaces, and let $E$ be a $p$-space. Then the amplification map*

$$\begin{aligned} & \mathcal{CB}(\mathrm{ROW}(L^p(X)), \mathrm{ROW}(L^p(Y))) \\ & \quad \to \ \mathcal{CB}(\mathrm{ROW}(L^p(X,E)), \mathrm{ROW}(L^p(Y,E))), \\ T & \ \mapsto \ T \otimes \mathrm{id}_E \end{aligned}$$

*is a complete isometry.*

# 5 Column and row space norms on tensor products

Let $p \in [1, \infty]$, let $X$ be a measure space, and let $E$ be a Banach space. Then the algebraic tensor product $L^p(X) \otimes E$ embeds canonically into $L^p(X, E)$. The norm of $L^p(X, E)$ restricted to $L^p(X) \otimes E$ is a cross norm.
In this section, we want to prove an operator space analogue of this fact for column and row spaces.

**Definition 5.1** Let $E$ and $F$ be operator spaces, and let $C \geq 0$. A matricial norm on $E \otimes F$ is called a *matricial $C$-subcross norm* if

$$\begin{aligned} \|x \otimes y\|_{M_{mn}(E \otimes F)} & \leq C \|x\|_{M_m(E)} \|y\|_{M_n(F)} \\ & \qquad (m, n \in \mathbb{N},\ x \in M_m(E),\ y \in M_n(F)). \end{aligned}$$

If $C = 1$, we simply speak of a *matricial subcross norm.*

There is an analogue of Definition 5.1 in the category of operator sequence spaces ([Lam, Chapter 3]):

**Definition 5.2** Let $E$ and $F$ be operator sequence spaces, and let $C \geq 0$. A sequential norm on $E \otimes F$ is called a *sequential $C$-subcross norm* if

$$\|x \otimes y\|_{(E \otimes F)^{\widehat{mn}}} \leq C \|x\|_{E^{\widehat{m}}} \|y\|_{F^{\widehat{n}}} \qquad \left(m, n \in \mathbb{N},\ x \in E^{\widehat{m}},\ y \in F^{\widehat{n}}\right).$$

If $C = 1$, we simply speak of a *sequential subcross norm.*

For the definition of Grothendieck's constant, which we will denote by $K_{\mathbb{G}}$, see [D–F, 14.4].
Our next lemma is a consequence of [D–F, 26.3, Proposition 1] (see the following remark on [D–F, p. 347]):

**Lemma 5.3** *Let $p, q \in [1, \infty]$, let $X$ and $Y$ be measure spaces, and let $\mathfrak{H}$ be a Hilbert space. Then the amplification map*

$$\mathcal{B}(L^p(X), L^q(Y)) \to \mathcal{B}(L^p(X, \mathfrak{H}), L^q(Y, \mathfrak{H}))), \quad T \mapsto T \otimes \mathrm{id}_{\mathfrak{H}}$$

*is bounded and has norm at most $K_{\mathbb{G}}$.*

**Proposition 5.4** *Let $p \in (1, \infty)$, let $X$ be a measure space, and let $E$ be a Banach space. Then $\min(L^p(X, E))$ yields a sequential $K_{\mathbb{G}}$-subcross norm on $L^p(X) \otimes E$.*

*Proof* Let $m, n \in \mathbb{N}$, let $x \in \min(L^p(X))^{\widehat{m}}$ and let $y \in \min(E)^{\widehat{n}}$. Let $S \in \mathcal{B}(\ell^2_m, L^p(X))$ and $T \in \mathcal{B}(\ell^2_n, E)$ represent $x$ and $y$, respectively. We have to show that

$$\|S \otimes T\|_{\mathcal{B}(\ell^2_{mn}, L^p(X,E))} \leq K_{\mathbb{G}} \|S\|_{\mathcal{B}(\ell^2_m, L^p(X))} \|T\|_{\mathcal{B}(\ell^2_n, E)}.$$

By Lemma 5.3, we have

$$\|S \otimes \mathrm{id}_{\ell^2_n}\|_{\mathcal{B}(\ell^2_{mn}, L^p(X, \ell^2_n))} \leq K_{\mathbb{G}} \|S\|_{\mathcal{B}(\ell^2_m, L^p(X))},$$

and, by [D–F, 7.3],

$$\|\mathrm{id}_{L^p(X)} \otimes T\|_{\mathcal{B}(L^p(X, \ell^2_n), L^p(X,E))} = \|T\|_{\mathcal{B}(\ell^2_n, E)}$$

holds. Consequently,

$$\begin{aligned} & \|S \otimes T\|_{\mathcal{B}(\ell^2_{mn}, L^p(X,E))} \\ \leq \ & \|S \otimes \mathrm{id}_{\ell^2_n}\|_{\mathcal{B}(\ell^2_{mn}, L^p(X, \ell^2_n))} \|\mathrm{id}_{L^p(X)} \otimes T\|_{\mathcal{B}(L^p(X, \ell^2_n), L^p(X,E))} \\ \leq \ & K_{\mathbb{G}} \|S\|_{\mathcal{B}(\ell^2_m, L^p(X))} \|T\|_{\mathcal{B}(\ell^2_n, E)} \end{aligned}$$

holds, which completes the proof. $\square$

*Remark* For $p \geq 2$, we even obtain a sequential subcross norm on $L^p(X, E)$: This follows from [D–F, 7.2, Proprosition, and 7.3].

**Theorem 5.5** *Let $p \in (1, \infty)$, let $X$ be a measure space, and let $E$ be a Banach space. Then $\mathrm{ROW}(L^p(X, E))$ yields a matricial $K_{\mathbb{G}}$-subcross norm on $L^p(X) \otimes E$.*

*Proof* Let $m, n \in \mathbb{N}$, and let $x \in M_m(L^p(X))$ and $y \in M_n(E)$. Let $k_x, l_x, k_y \in \mathbb{N}$, and let:

- $\alpha_x \in M_{m,k_x l_x}$, $\beta_x \in M_{k_x,m}$, and $v_1, \ldots, v_{k_x}$ belonging to the closed unit ball of $\min(L^p(X,E))^{\widehat{l_x}}$ such that

$$x = \alpha_x \operatorname{diag}(v_1, \ldots, v_{k_x})\beta_x; \tag{14}$$

- $\alpha_y \in M_{n,k_y l_y}$, $\beta_y \in M_{k_y,n}$, and $u_1, \ldots, u_{k_y}$ belonging to the closed unit ball of $\min(L^p(X,E))^{\widehat{l_y}}$ such that

$$y = \alpha_y \operatorname{diag}(u_1, \ldots, u_{k_y})\beta_y. \tag{15}$$

It follows that $\alpha_x \otimes \alpha_y \in M_{mn,(k_x k_y)(\ell_x \ell_y)}$ and that $\beta_x \otimes \beta_y \in M_{k_x k_y, mn}$. From Proposition 5.4, it follows that $K_{\mathbb{G}}^{-1}(v_\mu \otimes u_\nu)$ belong to the closed unit ball of $\min(L^p(X,E))^{\widehat{l_x l_y}}$ for $\mu = 1, \ldots, k_x$ and $\nu = 1, \ldots, k_y$. Consequently,

$$\begin{aligned} & K_{\mathbb{G}}^{-1}(x \otimes y) \\ = \ & (\alpha_x \otimes \alpha_y) \operatorname{diag}\left(K_{\mathbb{G}}^{-1}(v_\mu \otimes u_\nu) : \mu = 1, \ldots, k_x,\ \nu = 1, \ldots, k_y\right)(\beta_x \otimes \beta_y) \end{aligned}$$

is a representation of $K_{\mathbb{G}}^{-1}(x \otimes y)$ as in the definition of $\operatorname{Max}(\min(L^p(X,E)))$, i.e. of $\operatorname{ROW}(L^p(X,E))$, so that

$$\|K_{\mathbb{G}}^{-1}(x \otimes y)\|_{M_{mn}(\operatorname{ROW}(L^p(X,E)))} \leq |\alpha_x \otimes \alpha_y||\beta_x \otimes \beta_y| = |\alpha_x||\beta_x||\alpha_y||\beta_y|$$

holds. Since (14) and (15) are representations of $x$ and $y$ as the occur in the definition of $\operatorname{Max}(\min(L^p(X,E)))$, we conclude that

$$\|x \otimes y\|_{M_{mn}(\operatorname{ROW}(L^p(X,E)))} \leq K_{\mathbb{G}} \|x\|_{M_m(\operatorname{ROW}(L^p(X)))} \|y\|_{M_n(\operatorname{ROW}(E))}.$$

This yields the claim. $\square$

We now turn to proving the analogue of Theorem 5.5 for column spaces. First, we need two more lemmas:

**Lemma 5.6** *Let $p, q \in [1,\infty]$, let $X$ and $Y$ be a measures space, and let $\mathfrak{H}$ be a Hilbert space. Then the amplification map*

$$\begin{aligned} \mathcal{SB}(\max(L^p(X)), \max(L^q(Y))) &\to \mathcal{SB}(\max(L^p(X,\mathfrak{H})), \max(L^q(Y,\mathfrak{H}))), \\ T &\mapsto T \otimes \mathrm{id}_{\mathfrak{H}} \end{aligned}$$

*is sequentially bounded with* sb*-norm at most $K_{\mathbb{G}}$.*

*Proof* As the proof of Corollary 4.5, except that Lemma 5.3 instead of Definition 4.1 is invoked. $\square$

**Lemma 5.7** *Let $p \in (1,\infty)$, let $X$ be a measure space, and let $E$ and $F$ be Banach spaces. Then the amplification map*

$$\begin{aligned} \mathcal{SB}(\max(E),\max(F)) &\to \mathcal{SB}(\max(L^p(X,E)),\max(L^p(X,F))), \\ T &\mapsto \mathrm{id}_{L^p(X)} \otimes T \end{aligned}$$

*is a sequential isometry.*

*Proof* As the proof of Corollary 4.5, except that [D–F, 7.3] instead of Definition 4.1 is invoked. □

**Theorem 5.8** *Let $p \in (1,\infty)$, let $X$ be a measure space, and let $E$ be a Banach space. Then $\mathrm{COL}(L^p(X,E))$ yields a matricial $K_{\mathbb{G}}$-subcross norm on $L^p(X) \otimes E$.*

*Proof* Let $m,n \in \mathbb{N}$, and let $x \in M_m(L^p(X))$ and $y \in M_n(E)$. Let the operators $S \in \mathcal{B}\left(\ell^2_m, \max(L^p(X))^{\widehat{m}}\right)$ and $T \in \mathcal{B}\left(\ell^2_n, \max(E)^{\widehat{n}}\right)$ represent $x$ and $y$, respectively. We need to show that

$$\|S \otimes T\|_{\mathcal{B}\left(\ell^2_{mn},\max(L^p(X,E))^{\widehat{mn}}\right)} \leq K_{\mathbb{G}} \|S\|_{\mathcal{B}\left(\ell^2_m,\max(L^p(X))^{\widehat{m}}\right)} \|T\|_{\mathcal{B}\left(\ell^2_n,\max(E)^{\widehat{n}}\right)}. \tag{16}$$

First note that

$$\begin{aligned} & K_{\mathbb{G}} \|S\|_{\mathcal{B}\left(\ell^2_m,\max(L^p(X))^{\widehat{m}}\right)} \\ &= K_{\mathbb{G}} \|S\|_{\mathcal{SB}\left(\max(\ell^2_m),\max(L^p(X))^{\widehat{m}}\right)}, \qquad \text{by (4)}, \\ &\geq \|S \otimes \mathrm{id}_{\ell^2_n}\|_{\mathcal{SB}\left(\max(\ell^2_m(\ell^2_n)),\max(L^p(X,\ell^2_n))^{\widehat{m}}\right)}, \qquad \text{by Lemma 5.6}, \\ &= \|S \otimes \mathrm{id}_{\ell^2_n}\|_{\mathcal{B}\left(\ell^2_{mn},\max(L^p(X,\ell^2_n))^{\widehat{m}}\right)}, \qquad \text{again by (4)}. \end{aligned} \tag{17}$$

On the other hand, the following holds:

$$\begin{aligned} & \|T\|_{\mathcal{B}\left(\ell^2_n,\max(E)^{\widehat{n}}\right)} \\ &= \|T\|_{\mathcal{SB}\left(\max(\ell^2_n),\max(E)^{\widehat{n}}\right)} \\ &= \|\mathrm{id}_{L^p(X)} \otimes T\|_{\mathcal{SB}\left(\max(L^p(X,\ell^2_n)),\max(L^p(X,E))^{\widehat{n}}\right)}, \qquad \text{by Lemma 5.7} \\ &= \left\|(\mathrm{id}_{L^p(X)} \otimes T)^{\widehat{m}}\right\|_{\mathcal{B}\left(\max(L^p(X,\ell^2_n)^{\widehat{m}}),\max(L^p(X,E))^{\widehat{mn}}\right)}. \end{aligned} \tag{18}$$

Combined, (17) and (18) yield (16). □

# 6 An operator space structure for Figà-Talamanca–Herz algebras

We now use the work done in the previous sections in order to define, for $p \in (1, \infty)$ and a locally compact group $G$, a canonical operator space structure on $A_p(G)$.

**Definition 6.1** Let $G$ be a locally compact group, and let $p \in (1, \infty)$.

(a) The canonical operator space structure on $\mathrm{PM}_{p'}(G)$ is the one it inherits as a closed subspace of $\mathcal{CB}(\mathrm{COL}(L^{p'}(G)))$.

(b) The canonical operator space structure on $A_p(G)$ is the one it inherits as the predual of $\mathrm{PM}_{p'}(G)$.

*Remark* From this definition, it is immediate that

$$\mathrm{ROW}(L^p(G))\hat{\otimes}\,\mathrm{COL}(L^{p'}(G)) \to A_p(G), \quad \xi \otimes \eta \mapsto \xi * \check{\eta} \tag{19}$$

is a complete quotient map.

Let $G$ be a locally compact group, and let $p, q \in (1, \infty)$. Then

$$\lambda_{p',q'} : G \to \mathcal{B}(L^{p'}(G, L^{q'}(G))), \quad x \mapsto \lambda_{p'}(x) \otimes \lambda_{q'}(x)$$

is a strongly continuous representation of $G$ and thus yields a representation of $L^1(G)$, which we denote likewise by $\lambda_{p',q'}$. Let the $w^*$-closure of $\lambda_{p',q'}(L^1(G))$ in $\mathcal{B}(L^{p'}(G, L^{q'}(G)))$ be denoted by $\mathrm{PM}_{p',q'}(G \times G)$. Then $\mathrm{PM}_{p',q'}(G \times G)$ inherits a canonical operator space structure from $\mathcal{CB}(\mathrm{COL}(L^{p'}(G, L^{q'}(G)))$. Consequently, its predual, which we denote by $A_{p,q}(G \times G)$ has a canonical operator space structure as well. In analogy with (19), we have a complete quotient map

$$\begin{aligned} \mathrm{ROW}(L^p(G, L^q(G)))\hat{\otimes}\,\mathrm{COL}(L^{p'}(G, L^{q'}(G))) &\to A_{p,q}(G \times G), \\ \xi \otimes \eta &\mapsto \xi * \check{\eta}. \end{aligned}$$

**Lemma 6.2** *Let $G$ be a locally compact group, and let $p, q \in (1, \infty)$. Then there is a canonical completely bounded map from $A_p(G)\hat{\otimes}A_q(G)$ into $A_{p,q}(G \times G)$ with* cb-*norm at most $K_{\mathbb{G}}^2$.*

*Proof* We have a completely isometric isomorphism

$$\begin{aligned}&(\mathrm{ROW}(L^p(G))\hat{\otimes}\,\mathrm{COL}(L^{p'}(G)))\hat{\otimes}(\mathrm{ROW}(L^q(G))\hat{\otimes}\,\mathrm{COL}(L^{q'}(G)))\\ \cong\ &(\mathrm{ROW}(L^p(G))\hat{\otimes}\,\mathrm{ROW}(L^q(G)))\hat{\otimes}(\mathrm{COL}(L^{p'}(G))\hat{\otimes}\,\mathrm{COL}(L^{q'}(G)))\end{aligned} \tag{20}$$

by [E–R, Proposition 7.1.4]. Let the left hand side of (20) be denoted by $E$, and consider the diagram

$$\begin{array}{ccc} E & \longrightarrow & \mathrm{ROW}(L^p(G, L^q(G)))\hat{\otimes}\,\mathrm{COL}(L^{p'}(G), L^{q'}(G))) \\ \big\downarrow & & \big\downarrow \\ A_p(G)\hat{\otimes}A_q(G) & \longrightarrow & A_{p,q}(G\times G), \end{array}$$

where the top row is the composition of (20) with the canonical completely bounded maps

$$\mathrm{ROW}(L^p(G))\hat{\otimes}\,\mathrm{ROW}(L^q(G)) \to \mathrm{ROW}(L^p(G, L^q(G))) \tag{21}$$

and

$$\mathrm{COL}(L^{p'}(G))\hat{\otimes}\,\mathrm{COL}(L^{q'}(G)) \to \mathrm{COL}(L^{p'}(G, L^{q'}(G))), \tag{22}$$

which exist according to Theorems 5.5 and 5.8 and the universal property of $\hat{\otimes}$. From Theorems 5.5 and 5.8, it is also clear that both (21) and (22) have cb-norm at most $K_{\mathbb{G}}$.

Clearly, going along the top row and down the second column is a completely bounded map with cb-norm at most $K_{\mathbb{G}}^2$ that factors through the kernel of the first column. Since the first column is a complete quotient map by [E–R, Proposition 7.1.7], we obtain the bottom row, which yields a completely bounded map whose cb-norm is at most $K_{\mathbb{G}}^2$ and which makes the diagram commutative. $\square$

**Lemma 6.3** *Let $G$ be a locally compact group, and let $p, q \in (1,\infty)$ be such that $p \leq q \leq 2$ or $2 \leq q \leq p$. Then restricting functions on $G \times G$ to the diagonal subgroup yields a complete quotient map from $A_{p,q}(G\times G)$ onto $A_p(G)$.*

*Proof* Due to the particular choice of $p$ and $q$, the space $L^{q'}(G)$ is a $p'$-space by [Her 1, Theorem 1]. Consequently, by Theorem 4.6, the amplification map

$$\mathcal{CB}(\mathrm{COL}(L^{p'}(G))) \to \mathcal{CB}(\mathrm{COL}(L^{p'}(G, L^{q'}(G)))), \quad T \mapsto T\otimes \mathrm{id}_{L^{q'}(G)}$$

is a complete isometry. Define $W\colon L^{p'}(G, L^{q'}(G)) \to L^{p'}(G, L^{q'}(G))$ by letting

$$(W\xi)(x,y) := \xi(x,xy) \qquad (\xi \in L^{p'}(G, L^{q'}(G)),\ x,y \in G).$$

Then $W$ is an invertible isometry whose inverse is given by

$$(W^{-1}\xi)(x,y) := \xi(x,x^{-1}y) \qquad (\xi \in L^{p'}(G,L^{q'}(G)),\, x,y \in G).$$

Since $\mathrm{COL}(L^{p'}(G,L^{q'}(G)))$ is a homogeneous operator space, both $W$ and $W^{-1}$ are complete isometries on $\mathrm{COL}(L^{p'}(G,L^{q'}(G)))$. Consequently,

$$\begin{aligned} \nabla\colon \mathcal{CB}(\mathrm{COL}(L^{p'}(G))) &\to \mathcal{CB}(\mathrm{COL}(L^{p'}(G,L^{q'}(G)))), \\ T &\mapsto W^{-1}(T\otimes \mathrm{id}_{L^{q'}(G)})W \end{aligned}$$

is a complete isometry. A routine calculation reveals that

$$\nabla(\lambda_{p'}(x)) = \lambda_{p'}(x)\otimes\lambda_{q'}(x) = \lambda_{p',q'}(x,x). \tag{23}$$

It follows that $\nabla(\mathrm{PM}_{p'}(G)) \subset \mathrm{PM}_{p',q'}(G\times G)$. Moreover, $\nabla|_{\mathrm{PM}_{p'}(G)}$ is clearly $w^*$-continuous, and thus has a preadjoint $\Delta\colon A_{p,q}(G\times G)\to A_p(G)$. From (23), is is immediate that $\Delta$ is the restriction to the diagonal. Finally, since $\nabla|_{\mathrm{PM}_{p'}(G)}$ is a complete isometry, $\Delta$ is a complete quotient map. $\square$

Forming the compositon of the completely bounded maps in Lemmas 6.2 and 6.3, we obtain the main result of this section:

**Theorem 6.4** *Let $G$ be a locally compact group, and let $p,q\in(1,\infty)$ be such that $p\le q\le 2$ or $2\le q\le p$. Then pointwise multiplication induces a completely bounded map of* cb-*norm at most $K_{\mathbb{G}}^2$ from $A_p(G)\hat{\otimes}A_q(G)$ into $A_p(G)$.*

Letting $q=p$ in Theorem 6.4, we obtain immediately:

**Corollary 6.5** *Let $G$ be a locally compact group, and let $p\in(1,\infty)$. Then $A_p(G)$ is a quantized Banach algebra.*

If $G$ is amenable, then $A_p(G)$ has an approximate identity bounded by one ([Pie, Theorem 4.10]). Hence, we obtain the operator space version of [Her 1, Theorem C]:

**Corollary 6.6** *Let $G$ be an amenable, locally compact group, and let $p,q\in(1,\infty)$ be such that $p\le q\le 2$ or $2\le q\le p$. Then $A_q(G)\subset A_p(G)$ such that the inclusion is completely bounded with* cb-*norm at most $K_{\mathbb{G}}^2$.*

## 7 Operator amenability for Figà-Talamanca–Herz algebras

We conclude this paper with an extension of [Rua 1, Theorem 3.6] to Figà-Talamanca–Herz algebras.
Let $\mathfrak{A}$ be a quantized Banach algebra. A *quantized Banach $\mathfrak{A}$-bimodule* is an $\mathfrak{A}$-bimodule equipped with an operator space structure such that the module operations are completely bounded. Let $E$ be a quantized Banach $\mathfrak{A}$-bimodule. Then the dual space $E^*$ of $E$ is a quantized Banach $\mathfrak{A}$-bimodule in a canonical fashion via

$$\langle x, a \cdot \phi \rangle := \langle x \cdot a, \phi \rangle \quad \text{and} \quad \langle x, \phi \cdot a \rangle := \langle a \cdot x, \phi \rangle \qquad (a \in \mathfrak{A},\ \phi \in E^*,\ x \in E).$$

A *derivation* from a quantized Banach algebra $\mathfrak{A}$ into a quantized Banach $\mathfrak{A}$-bimodule $E$ is a completely bounded map $D \colon \mathfrak{A} \to E$ such that

$$D(ab) = a \cdot Db + (Da) \cdot b \qquad (a, b \in \mathfrak{A}).$$

The derivation is called *inner* if there is $x \in E$ such that

$$Da = a \cdot x - x \cdot a \qquad (a \in \mathfrak{A}).$$

The following definition was introduced by Z.-J. Ruan in [Rua 1] (for completely contractive Banach algebras) and adds operator space overtones to B. E. Johnson's definition of an amenable Banach algebra ([Joh 1]):

**Definition 7.1** A quantized Banach algebra $\mathfrak{A}$ is called *operator amenable* if, for every quantized Banach $\mathfrak{A}$-bimodule $E$, every (completely bounded) derivation $D \colon \mathfrak{A} \to E^*$ is inner.

The classical analogue of the following lemma is well known ([Run 1, Proposition 2.3.1]), and the proof carries over to the quantized setting with only standard modifications. (It was formulated for completely contractive Banach algebras as [Rua 2, Propositon 2.2], but nowhere in the proof, complete contractivity of the multiplication is actually required.)

**Lemma 7.2** *Let $\mathfrak{A}$ and $\mathfrak{B}$ be quantized Banach algebras such that $\mathfrak{A}$ is operator amenable, and let $\theta \colon \mathfrak{A} \to \mathfrak{B}$ be a completely bounded algebra homomorphism with dense range. Then $\mathfrak{B}$ is operator amenable.*

For Figà-Talamanca–Herz algebras with their canonical operator space structure, we eventually obtain:

**Theorem 7.3** *The following are equivalent for a locally compact group $G$:*

(i) *$G$ is amenable.*

(ii) *$A(G)$ is operator amenable.*

(iii) *$A_p(G)$ is operator amenable for each $p \in (1, \infty)$.*

(iv) *There is $p \in (1, \infty)$ such that $A_p(G)$ is operator amenable.*

*Proof* (i) $\Longleftrightarrow$ (ii) is [Rua 1, Theorem 3.6].
(ii) $\Longrightarrow$ (iii): Let $p \in (1, \infty)$. If $A(G)$ is operator amenable, then $G$ is amenable, so that $A(G) \subset A_p(G)$, where the inclusion is completely bounded and has dense range (by Corollary 6.6). By Lemma 7.2, this yields the operator amenability of $A_p(G)$.
(iii) $\Longrightarrow$ (iv) is trivial.
(iv) $\Longrightarrow$ (i): Let $p \in (1, \infty)$ be such that $A_p(G)$ is operator amenable. By [Rua 1, Proposition 2.3], $A_p(G)$ then has a bounded approximate identity. This is enough to guarantee the amenability of $G$ ([Pie, Theorem 4.10]). $\square$

*Remarks* 1. Virtually all concepts from Banach homology can be provided with operator space overtones. For the Fourier algebra $A(G)$ of a locally compact group, this quantized Banach homology seems to be the appropriate one when it comes to characterizing properties of $G$ in terms of cohomological properties of $A(G)$:

- $A(G)$ is always operator weakly amenable ([Spr]);
- $A(G)$ is operator biprojective if and only if $G$ is discrete ([Ari], [Woo]);
- For many locally compact groups — among them all [SIN]-groups —, $A(G)$ is operator biflat ([R–X]). The operator biflatness of $A(G)$ is systematically investigated in [A–R–S].

It would be interesting to know which of these results extend to general Figà-Talamanca–Herz algebras.

2. In [Run 3], the third-named author showed that the Fourier algebra of a locally compact group $G$ is amenable (in the classical sense) if and only if $G$ has an abelian subgroup of finite index. The proof is mostly operator space theoretic. It is easy to see that, if $G$ has an abelian subgroup of finite index, then $A_p(G)$ is amenable for each $p \in 91, \infty)$. In view of [Run 3] and Theorem 7.3, it is plausible to conjecture that

$A_p(G)$ can be amenable for some $p \in (1, \infty)$ only for such $G$. It is an intriguing question, whose answer seems to be far from obvious, whether the canonical operator space structure on $A_p(G)$ — combined with the methods from [Run 3] — can be used to affirm this conjecture.

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