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Abstract

We consider the stationary flow of a generalized Newtonian fluid which is modelled by an anisotropic dissipative potential f. More precisely, we are looking for a solution u: $\Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, of the following system of nonlinear partial differential equations

$$\left.\begin{aligned} -\mathrm{div}\,\big\{T(\varepsilon(u))\big\} + u^k \frac{\partial u}{\partial x_k} + \nabla \pi = g \quad \text{in}\,\Omega\,, \\ \mathrm{div}\, u = 0 \ \ \text{in}\,\Omega\,, \qquad u = 0 \quad \text{on}\,\partial\Omega\,. \end{aligned}\right\} \tag{$*$}$$

Here π: $\Omega \to \mathbb{R}$ denotes the pressure, g is a system of volume forces, and the tensor T is the gradient of the potential f. Our main hypothesis imposed on f is the existence of exponents $1 < p \le q_0 < \infty$ such that

$$\lambda(1+|\varepsilon|^2)^{\frac{p-2}{2}}|\sigma|^2 \le D^2 f(\varepsilon)(\sigma,\sigma) \le \Lambda(1+|\varepsilon|^2)^{\frac{q_0-2}{2}}|\sigma|^2$$

holds with constants λ, $\Lambda > 0$. Under natural assumptions on p and q_0 we prove the existence of a weak solution u to the problem $(*)$, moreover we prove interior $C^{1,\alpha}$-regularity of u in the two-dimensional case. If $n = 3$, then interior partial regularity is established.

1 Introduction

We discuss the stationary flow of an incompressible generalized Newtonian fluid in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n = 2$ or $n = 3$. To be precise, we are looking for a velocity field u: $\Omega \to \mathbb{R}^n$ solving the following system of nonlinear partial differential equations

$$\left.\begin{aligned} -\mathrm{div}\,\big\{T(\varepsilon(u))\big\} + u^k \frac{\partial u}{\partial x_k} + \nabla \pi = g \quad \text{in}\,\Omega\,, \\ \mathrm{div}\, u = 0 \ \ \text{in}\,\Omega\,, \qquad u = u_0 \quad \text{on}\,\partial\Omega\,. \end{aligned}\right\} \tag{1.1}$$

Here π is the a priori unknown pressure function, g: $\Omega \to \mathbb{R}^n$ represents a system of volume forces and u_0: $\partial\Omega \to \mathbb{R}^n$ denotes a given boundary function. The tensor T is assumed to be the gradient of some (convex) potential f: $\mathbb{S}^n \to [0, \infty)$ which is of class C^2 on the space $\mathbb{S}^n$ of all symmetric matrices. We adopt the convention of summation over repeated indices running from 1 to n, moreover, for functions v: $\Omega \to \mathbb{R}^n$ we let

$$\varepsilon(v)(x) = \frac{1}{2}\big(\partial_i v^j + \partial_j v^i\big)(x) \in \mathbb{S}^n$$

whenever this expression makes sense.

In case $f(\varepsilon) = |\varepsilon|^2$ equation (1.1) reduces to the Dirichlet boundary value problem for the stationary Navier-Stokes system, for an overview on existence and regularity results we refer to the classical monograph [La] of Ladyzhenskaya or more recently to the monographs [Ga1], [Ga2] of Galdi where also the history of the problem is outlined in great detail.

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So-called power law models are investigated for example in [KMS]: for some exponent $1 < p < \infty$ f is assumed to satisfy

$$\lambda(1+|\varepsilon|^2)^{\frac{p-2}{2}}|\sigma|^2 \le D^2 f(\varepsilon)(\sigma,\sigma) \le \Lambda(1+|\varepsilon|^2)^{\frac{p-2}{2}}|\sigma|^2 \tag{1.2}$$

for all ε, $\sigma \in \mathbb{S}^n$ and with positive constants λ, Λ. Clearly (1.2) implies that f is of p-growth, moreover, the first inequality in (1.2) implies strict convexity of f. Then, if $u_0 \equiv 0$ and in addition $f(\varepsilon) = F(|\varepsilon|^2)$ (which is reasonable from the physical point of view) Kaplický, Málek and Stará discuss the two-dimensional case with the following results: if $p > 3/2$, then problem (1.1) admits a solution which is of class $C^{1,\alpha}$ up to the boundary, whereas for $p > 6/5$ (1.1) has a solution being $C^{1,\alpha}$-regular in the interior of Ω. Here of course the volume force term g is sufficiently smooth.

Suppose for the moment that the flow is also slow. Then in (1.1) the convective term $(\nabla u)u = u^k\partial_k u$ can be neglected, and (1.1) reduces to a generalized version of the classical Stokes problem. In the monograph [FS] a variational approach towards (1.1) for various classes of dissipative potentials f is described leading to existence and also (partial) regularity results in the absence of the convective term. Very recently these investigations were extended in [BF2] to the case of non-uniformly elliptic potentials which means that (1.2) is replaced by the condition

$$\lambda(1+|\varepsilon|^2)^{\frac{p-2}{2}}|\sigma|^2 \le D^2 f(\varepsilon)(\sigma,\sigma) \le \Lambda(1+|\varepsilon|^2)^{\frac{q_0-2}{2}}|\sigma|^2 \tag{1.3}$$

with exponents $1 < p \le q_0 < \infty$, $q_0 \ge 2$ and for all ε, $\sigma \in \mathbb{S}^n$. (Note that the validity of (1.3) with $q_0 < 2$ immediately implies (1.3) with q_0 replaced by 2 so that we may assume $q_0 \ge 2$.) From (1.3) it easily follows that f is of upper growth rate q_0, a lower bound for $f(\varepsilon)$ can be given in terms of $|\varepsilon|^p$. Examples of potentials f satisfying (1.3) are given in [BF2], moreover, it is shown in this paper that weak local solutions of (1.1) (with $(\nabla u)u = 0$!) under condition (1.3) are $C^{1,\alpha}$ in the interior of Ω, if $n = 2$, $q_0 = 2$, and partially $C^{1,\alpha}$, if $n = 3$, provided that we impose the bound $q_0 < p(1 + 2/n)$.

The objective of this note now is to study the anisotropic (w.r.t. the ellipticity condition) situation (1.3) for a non-vanishing convective term $(\nabla u)u$. To be more precise we assume that (1.3) holds with exponents p, q_0 such that

$$p > \begin{cases} \frac{6}{5} & \text{in case} \quad n = 2\,, \\ \frac{9}{5} & \text{in case} \quad n = 3\,, \end{cases} \qquad \text{together with} \qquad 2 \le q_0 < p\frac{n+2}{n}\,. \tag{1.4}$$

Moreover, let us assume that

$$u_0 = 0\,, \qquad g \in L^\infty(\Omega;\mathbb{R}^n)\,. \tag{1.5}$$

REMARK 1.1 *For the sake of technical simplicity we just assume that the volume forces are bounded functions. Of course our results are also valid under the weaker assumption $g \in L^{t(p)}(\Omega;\mathbb{R}^n)$, whenever $t(p)$ is chosen sufficiently large (for a definition of the Lebesgue and Sobolev spaces we refer the reader to [Ad]). For a discussion of the hypothesis on the boundary data we refer to Remark 2.1.*

In order to get a weak form of (1.1) we multiply the first line of (1.1) with $\varphi \in C_0^\infty(\Omega;\mathbb{R}^n)$, $\operatorname{div}\varphi = 0$, and obtain after integration by parts (using $\operatorname{div}\varphi = 0$)

$$\int_\Omega Df(\varepsilon(u)) : \varepsilon(\varphi)\,\mathrm{d}x - \int_\Omega u \otimes u : \varepsilon(\varphi)\,\mathrm{d}x = \int_\Omega g \cdot \varphi\,\mathrm{d}x\,, \tag{1.6}$$

where $u \otimes v := (u^i v^k)$. Therefore we have to solve equation (1.6) together with $\operatorname{div} u = 0$ in Ω, $u = 0$ on $\partial\Omega$. A priori it is not clear to which space a weak solution should belong, therefore we give an existence proof by using an approximation procedure. To this purpose recall (1.4), hence it is possible to choose $q \geq q_0$ satisfying

$$p\frac{n+2}{n} > q > n\,. \tag{1.7}$$

We then let $(0 < \delta < 1)$

$$f_\delta(\varepsilon) := \delta(1+|\varepsilon|^2)^{\frac{q}{2}} + f(\varepsilon)\,, \qquad \varepsilon \in \mathbb{S}^n\,,$$

and consider the problem

$$\left.\begin{array}{l} \text{to find } u_\delta \in \overset{\circ}{W}{}^1_q(\Omega;\mathbb{R}^n),\ \operatorname{div} u_\delta = 0, \text{ such that} \\ \displaystyle\int_\Omega Df_\delta(\varepsilon(u_\delta)) : \varepsilon(\varphi)\,\mathrm{d}x - \int_\Omega u_\delta \otimes u_\delta : \varepsilon(\varphi)\,\mathrm{d}x = \int_\Omega g\cdot\varphi\,\mathrm{d}x \\ \text{for all } \varphi \in C_0^\infty(\Omega;\mathbb{R}^n),\ \operatorname{div}\varphi = 0. \end{array}\right\} \tag{1.6$_\delta$}$$

THEOREM 1.1 *Let f satisfy (1.3) with $1 < p < 2 \leq q_0 < \infty$ and choose q according to (1.7). Moreover, consider the data given in (1.5). Then (1.6$_\delta$) has at least one weak solution $u_\delta \in \overset{\circ}{W}{}^1_q(\Omega;\mathbb{R}^n)$. Moreover, we have*

$$\sup_{0<\delta<1} \|u_\delta\|_{W^1_p(\Omega;\mathbb{R}^n)} < \infty\,.$$

The proof of the existence result follows from a familiar fixed point argument (see [La]), the a priori estimate is shown in Lemma 2.1 for the sake of completeness. We remark that by construction u_δ turns out to be the minimizer of the energy

$$J_\delta[w] := \int_\Omega f_\delta(\varepsilon(w))\,\mathrm{d}x - \int_\Omega u_\delta \otimes u_\delta : \varepsilon(w)\,\mathrm{d}x - \int_\Omega g\cdot w\,\mathrm{d}x \tag{1.8}$$

within the class

$$\overset{\circ}{W}{}^1_q(\Omega;\mathbb{R}^n) \cap \operatorname{Kern}(\operatorname{div})\,.$$

Our first observation concerns uniform higher integrability properties of the sequence $\{u_\delta\}$.

THEOREM 1.2 *Suppose that the assumptions of Theorem 1.1 are satisfied. Then the functions u_δ are of class $W^1_{\tilde q,loc}(\Omega;\mathbb{R}^n)$ uniformly w.r.t. δ, where $\tilde q = 3p$ in case $n = 3$, and where we may choose any finite number $\tilde q$ in case $n = 2$.*

The proof of Theorem 1.2 will be given in Section 3. The main ingredient is a Caccioppoli-type inequality being valid for the approximative solutions u_δ. In Section 4 we will use this information to pass to the limit, more precisely we have

THEOREM 1.3 *Let the assumptions of Theorem 1.2 hold and fix any weak W^1_p-cluster point, i.e.*

$$u_\delta \overset{\delta\to 0}{\rightharpoondown:} \bar u \quad \text{in } W^1_p(\Omega;\mathbb{R}^n)$$

for some sequence $\delta = \delta_k$ going to zero. Then $\bar u$ is of class $W^1_{\tilde q,loc}(\Omega;\mathbb{R}^n)$, where $\tilde q$ is defined in Theorem 1.2. Moreover, $Df(\varepsilon(\bar u))$ locally is of class $W^1_{q/(q-1)}$ and $\bar u$ is a solution of problem (1.6) being Hölder continuous in the interior of Ω on account of Theorem 1.2.

REMARK 1.2 *In fact, $\bar{u}$ is a strong solution of class $W^2_{\alpha,loc}$ with some suitable α (compare Corollary 3.1).*

REMARK 1.3 *In [FMS] the Lipschitz truncation method leads to the existence of solutions for problem (1.6) in the case of power-law models provided that $p > 6/5$ (see also [FM]). Note that the convective term in (1.6) is well defined (in dimension $n = 3$) since the condition $p > 6/5$ implies the solution to be of class L^2. It is not evident, how to apply this method to the non-standard models under consideration (see, for instance, formula (42) of [FM]). Here we rely on the a priori estimates of Theorem 1.2.*

As to the regularity properties of the particular solution $\bar{u}$ from above we limit ourselves to the case $q_0 = 2$:

THEOREM 1.4 *Under the assumptions and with the notation of Theorem 1.3 we consider the case $n = 3$, $q_0 = 2$. Then $\bar{u}$ is partially of class $C^{1,\alpha}$, i.e. there is an open set Ω_0 of full Lebesgue measure, $|\Omega - \Omega_0| = 0$, such that $\bar{u} \in C^{1,\alpha}(\Omega_0; \mathbb{R}^3)$.*

REMARK 1.4 i) *As in [BF2] it is be possible to extend Theorem 1.4 to the case $q_0 > 2$ together with $q_0 < 5p/3$. We leave the details to the reader, some comments can be found in Section 5.*

ii) *We also like to remark that partial regularity in the setting of stationary electrorheological fluids has been recently established in the paper [AM] of Acerbi and Mingione.*

THEOREM 1.5 *Under the assumptions and with the notation of Theorem 1.3 we consider the case $n = 2$, $q_0 = 2$. Then $\bar{u}$ has locally Hölder continuous first derivatives, i.e. $\bar{u} \in C^{1,\alpha}(\Omega; \mathbb{R}^2)$.*

REMARK 1.5 *At this stage we remark that the regularization has to be defined with respect to an exponent larger than n in order to have the boundedness of u_δ. As a consequence, the proof of Lemma 2.2 below reduces more or less to the one given in [BF2]. Of course it would be directly possible to assume w.l.o.g. that we have $q_0 > n$ just by enlarging q_0 (keeping (1.4)!). As discussed in Remark 5.2 this would not affect the case $n = 3$ with the exception that the proof of Theorem 1.4 becomes more technical.*

But unfortunately the two-dimensional case actually requires the restriction $q_0 = 2$, i.e. up to Theorem 1.4 we can choose $q_0 > n$ (replacing also q by q_0 in the regularization) but, due to the method we use, the two-dimensional regularity Theorem 1.5 is limited to $q_0 = 2$. For this reason we made a distinction between the exponents q and q_0.

REMARK 1.6 i) *It is desirable to give global variants of our results, for example to prove higher integrability of $\nabla\bar{u}$ up to the boundary. Then, under suitable smallness conditions, some results on unique solvability extend to our non-uniformly elliptic problem. The idea for establishing a theorem of this kind is standard and, for instance, presented in [La], p. 118. The main difficulty in the case of non-uniform ellipticity is to handle potentials with lower growth rate $p < 2$. Again we leave the details to the reader.*

ii) *It should be noted that for stationary electrorheological fluids in two dimensions the existence of strong solutions has been obtained in [ER].*

Throughout this paper we use the following notation: for any $t \in [1,n)$ we denote by t^* the Sobolev exponent, i.e. $t^* = nt/(n-t)$. The conjugate of t^* is denoted by $\bar{t}$, i.e.

$$\bar{t} = \frac{nt}{(n+1)t-n}\,.$$

The symbol $u \odot v$ stands for the symmetric part of $u \otimes v$, $u \odot v = \frac{1}{2}(u \otimes v + v \otimes u)$. Uniform constants are just denoted by c without being relabelled in different occurrences. For tensors σ we use the notation $\nabla\sigma = (\partial_1\sigma, \dots, \partial_n\sigma)$ whenever this expression makes sense.

2 Regularization: a priori energy estimates and weak differentiability properties

LEMMA 2.1 *Suppose that the hypotheses of Theorem 1.1 are satisfied. Then we have*

$$\sup_{0<\delta<1} \int_\Omega |\varepsilon(u_\delta)|^p \,\mathrm{d}x < \infty\,. \tag{2.1}$$

With the help of Korn's inequality (see e.g. [MM] or [Ko1], [Ko2], [Fi], [Fri], [St], [Ze]) we deduce from (2.1) the a priori bound of Theorem 1.1

$$\sup_{0<\delta<1} \|u_\delta\|_{W^1_p(\Omega;\mathbb{R}^n)} < \infty\,. \tag{2.2}$$

Proof of Lemma 2.1. In fact, u_δ is admissible in (1.6_δ) which shows that

$$\int_\Omega Df_\delta(\varepsilon(u_\delta)) : \varepsilon(u_\delta)\,\mathrm{d}x - \int_\Omega u_\delta \otimes u_\delta : \varepsilon(u_\delta)\,\mathrm{d}x = \int_\Omega g \cdot u_\delta \,\mathrm{d}x\,.$$

Assumption (1.3) implies $Df(Z) : Z \geq c_1|Z|^p - c_2$, $c_1 > 0$, hence

$$\begin{aligned}
\int_\Omega |\varepsilon(u_\delta)|^p \,\mathrm{d}x &\leq c_1^{-1}\left[c_2 + \int_\Omega Df_\delta(\varepsilon(u_\delta)) : \varepsilon(u_\delta)\,\mathrm{d}x\right] \\
&= c_1^{-1}\left[c_2 + \int_\Omega u_\delta \otimes u_\delta : \varepsilon(u_\delta)\,\mathrm{d}x + \int_\Omega g \cdot u_\delta\,\mathrm{d}x\right]. \qquad (2.3)
\end{aligned}$$

An integration by parts gives the well known relation

$$\begin{aligned}
\int_\Omega u_\delta \otimes u_\delta : \varepsilon(u_\delta)\,\mathrm{d}x &= \int_\Omega u_\delta^i u_\delta^j \partial_i u_\delta^j \,\mathrm{d}x = \int_\Omega \partial_i\left[u_\delta^i u_\delta^j u_\delta^j\right]\mathrm{d}x - \int_\Omega \partial_i\left[u_\delta^i u_\delta^j\right] u_\delta^j \,\mathrm{d}x \\
&= \int_{\partial\Omega} u_\delta^i u_\delta^j u_\delta^j \nu_i \,\mathrm{d}\mathcal{H}^{n-1} - \int_\Omega \underbrace{\partial_i u_\delta^i}_{=0} u_\delta^j u_\delta^j \,\mathrm{d}x - \int_\Omega u_\delta^i \partial_i u_\delta^j u_\delta^j \,\mathrm{d}x\,,
\end{aligned}$$

thus we have

$$\int_\Omega u_\delta \otimes u_\delta : \varepsilon(u_\delta)\,\mathrm{d}x = \frac{1}{2}\int_{\partial\Omega} u_\delta^i u_\delta^j u_\delta^j \nu_i \,\mathrm{d}\mathcal{H}^{n-1} = 0\,.$$

Note that the above expressions are well defined on account of $q \geq 2 > 3n/(n+2)$. Finally we have for any $\alpha > 0$ (with Sobolev's and Korn's inequality)

$$\int_\Omega |g|\,|u_\delta|\,\mathrm{d}x \leq \alpha \int_\Omega |\varepsilon(u_\delta)|^p\,\mathrm{d}x + c(\alpha)\,,$$

where the integral on the right-hand side can be absorbed on the left-hand side of (2.3) if α is sufficiently small, and the a priori estimates (2.1) and (2.2) are proved.

REMARK 2.1 *Let us give some short comments on the Dirichlet boundary data. Up to now we studied the so-called "no-slip boundary" which means that the fluid adheres to the boundary, i.e. $u = 0$ on $\partial\Omega$ (a detailled discussion of this kind of boundary data can be found in [FM]). Assume for the moment that $u = u_0 \neq 0$ on $\partial\Omega$.*

i) *Consider the general case $p < 2$: if $u_0 \neq 0$, $u_0 \in W^1_\infty(\Omega;\mathbb{R}^n)$, $\operatorname{div} u_0 = 0$, we need an additional uniform estimate (for the kinetic energy) $\int_\Omega |u_\delta|^2\,\mathrm{d}x \leq const$ in order to prove the a priori bounds (2.1) and (2.2). Now we have to take $u_\delta - u_0$ as a test function and obtain as additional term on the right-hand side of (2.3) the quantity*

$$\int_\Omega |u_\delta|^2 |\varepsilon(u_0)|\,\mathrm{d}x\,. \tag{2.4}$$

ii) *If $\nu(\varepsilon(u))$ denotes the kinematic viscosity of the fluid under consideration, then $Df(\varepsilon(u)) = \nu(\varepsilon(u))\varepsilon(u)$. If we merely assume that $Df(\varepsilon(u)) : \varepsilon(u) \geq c_1|\varepsilon(u)|^p - c_2$, then the viscosity $\nu(\varepsilon(u))$ may asymptotically vanish for large shear rates and it is not clear whether the Dirichlet boundary value problem is in accordance with the physical point of view – in general we just expect the normal component of u to vanish at the boundary (compare the inviscous Euler equations).*

iii) *Although, for example, many polymeric liquids are (at least in case of bounded shear rates) shear thinning (see [BAH]), i.e. the viscosity decreases if the shear rate increases, it is not evident that the viscosity asymptotically vanishes at high shear rates. This motivates the additional assumption $Df(\varepsilon(u)) : \varepsilon(u) \geq c_1|\varepsilon(u)|^s - c_2$ for some $s \geq 2$. Moreover, some fairly concentrated suspensions of small particles are shear thickening (see [BAH]). Hence, considering a generalized Newtonian fluid at very large shear rates in comparison to the particle sizes, it may even be reasonable to assume $s > 2$. Note that we do not restrict our considerations to power law models and that s is a free parameter which can be chosen independently from the exponent p occuring in (1.3) (compare the notion of (s,μ,q)-growth discussed for instance in [BFM] and [BF1]).*

iv) *In case $s \geq 2$, u_0 as in i), we have (by Korn's and Poincaré's inequality)*

$$\int_\Omega |u_\delta - u_0|^2\,\mathrm{d}x \leq c \int_\Omega |\varepsilon(u_\delta) - \varepsilon(u_0)|^2\,\mathrm{d}x\,.$$

Since the left-hand side of (2.3) is now of growth order $s \geq 2$, we may absorb the integral (2.4) under some suitable smallness condition on $\varepsilon(u_0)$. If $s > 2$, then we just have to assume $u_0 \in W^1_t(\Omega;\mathbb{R}^n)$, t sufficiently large, in order to obtain the a priori estimates (2.1) and (2.2) in the case of non-vanishing boundary data.

Next we are going to give some weak differentiability results for the regularizing sequence $\{u_\delta\}$ from Theorem 1.1 which are needed in the following.

LEMMA 2.2 *Let the assumptions of Theorem 1.1 hold. Then we have:*

a) $u_\delta \in W^2_{2,loc}(\Omega;\mathbb{R}^n)$;

b) $(1+|\varepsilon(u_\delta)|^2)^{q/4} \in W^1_{2,loc}(\Omega)$ *together with*

$$\nabla\big\{(1+|\varepsilon(u_\delta)|^2)^{q/4}\big\} = \frac{q}{2}(1+|\varepsilon(u_\delta)|^2)^{\frac{q}{4}-1}|\varepsilon(u_\delta)|\nabla|\varepsilon(u_\delta)|\,.$$

c) $Df_\delta(\varepsilon(u_\delta)) \in W^1_{q/(q-1),loc}(\Omega;\mathbb{S}^n)$ *and*

$$\partial_k\big\{Df_\delta(\varepsilon(u_\delta))\big\} = D^2 f_\delta(\varepsilon(u_\delta))(\partial_k\varepsilon(u_\delta),\cdot)\,,\qquad k=1,\dots,n\,.$$

Proof. The proof follows the line of Lemma 3.1 given in [BF2]. We fix a ball $B_R \Subset \Omega$, consider $\eta \in C^\infty_0(B_R)$, $\eta\equiv 1$ on B_r, $\eta \equiv 0$ outside of $B_{r'}$, $|\nabla\eta| \le c(r'-r)^{-1}$, where $0<r<r'<R$. We denote by Δ_h the difference quotient in direction e_k, $k=1,\dots,n$. Observe that by results of [La] or [Pi] (see also [Ga1], III, Theorem 3.2) there exists a function $\psi \in \overset{\circ}{W}{}^1_q(B_{r'},\mathbb{R}^n)$ with the property

$$\operatorname{div}\psi = \frac{1}{h}\nabla\eta^2\Delta_h u_\delta\,.$$

Therefore, taking into account the definition of u_δ and choosing the solenoidal test function φ in (1.6_δ) as

$$\varphi := h^{-1}\eta^2\Delta_h u_\delta - \psi\,,$$

we easily get for sufficiently small $|h|$

$$\begin{aligned}
&\int_{B_{r'}(x_0)} \Delta_h\Big\{Df_\delta(\varepsilon(u_\delta))\Big\} : \varepsilon(\Delta_h u_\delta)\eta^2\,\mathrm{d}x\\
&\quad= \int_{B_{r'}(x_0)} \Delta_h\Big\{Df_\delta(\varepsilon(u_\delta))\Big\} : \big(h\varepsilon(\psi) - \nabla\eta^2\odot\Delta_h u_\delta\big)\,\mathrm{d}x\\
&\qquad+\int_{B_{r'}(x_0)} \Delta_h(u_\delta\otimes u_\delta) : \big\{\varepsilon(\Delta_h u_\delta)\eta^2 + \Delta_h u_\delta\otimes\nabla\eta^2 - h\varepsilon(\psi)\big\}\,\mathrm{d}x\\
&\qquad+\int_{B_{r'}(x_0)} g\cdot\Delta_{-h}(\eta^2\Delta_h u_\delta + h\psi)\,\mathrm{d}x\,.
\end{aligned}\tag{2.5}$$

Arguing in the same way as in Lemma 3.1 of [BF2] we introduce the parameter-dependent bilinear form

$$\mathcal{B}_x := \int_0^1 D^2 f_\delta\big(\varepsilon(u_\delta)(x) + th\varepsilon(\Delta_h u_\delta)(x)\big)\,\mathrm{d}t$$

acting on pairs of symmetric matrices, and deduce from (2.5) the inequality

$$
\begin{aligned}
&\int_{B_{r'}(x_0)} \mathcal{B}_x(\varepsilon(\Delta_h u_\delta), \varepsilon(\Delta_h u_\delta))\,\mathrm{d}x \\
\leq\ & \frac{1}{2}\int_{B_{r'}(x_0)} \mathcal{B}_x(\varepsilon(\Delta_h u_\delta, \varepsilon(\Delta_h u_\delta))\,\mathrm{d}x \\
&+\frac{c}{(r'-r)^2}\left[\int_{B_{r'}(x_0)} |\Delta_h u_\delta|^q\,\mathrm{d}x\right]^{\frac{2}{q}}\left[\int_{B_{r'}(x_0)} (1+|\varepsilon(u_\delta)|^2+|h\varepsilon(\Delta_h u_\delta)|^2)^{\frac{q}{2}}\,\mathrm{d}x\right]^{1-\frac{2}{q}} \\
&+\left|\int_{B_{r'}(x_0)} \Delta_h(u_\delta\otimes u_\delta) : \Big\{\varepsilon(\Delta_h u_\delta)\eta^2 + \Delta_h u_\delta \odot \nabla\eta^2 - h\varepsilon(\psi)\Big\}\,\mathrm{d}x\right| \\
&+\left|\int_{B_{r'}(x_0)} g\cdot \Delta_{-h}(\eta^2\Delta_h u_\delta + h\psi)\,\mathrm{d}x\right| . \qquad (2.6)
\end{aligned}
$$

Note also that the ellipticity condition for $D^2 f_\delta$ guarantees

$$
\frac{1}{4}\int_{B_{r'}(x_0)} \mathcal{B}_x(\varepsilon(\Delta_h u_\delta), \varepsilon(\Delta_h u_\delta))\eta^2\,\mathrm{d}x \geq c(q,\delta)\int_{B_{r'}(x_0)} |\varepsilon(\Delta_h u_\delta)|^2\eta^2\,\mathrm{d}x\,. \qquad (2.7)
$$

Further we consider the convective term from the right-hand side of (2.6). Successive application of Young's and Hölder's inequalities immediately gives (recall the choice (1.7) of q which implies that u_δ is bounded)

$$
\begin{aligned}
&\left|\int_{B_{r'}(x_0)} \Delta_h(u_\delta\otimes u_\delta) : \big\{\varepsilon(\Delta_h u_\delta)\eta^2 + \Delta_h u_\delta\odot\nabla\eta^2 - h\varepsilon(\psi)\big\}\,\mathrm{d}x\right| \qquad (2.8)\\
\leq\ & c\int_{B_{r'}(x_0)} \big(|\Delta_h u_\delta||\varepsilon(\Delta_h u_\delta)|\eta^2 + |\Delta_h u_\delta|^2|\nabla\eta| + |\Delta_h u_\delta||\varepsilon(\psi)|h\big)\,\mathrm{d}x \\
\leq\ & \alpha\int_{B_{r'}(x_0)} |\varepsilon(\Delta_h u_\delta)|^2\eta^2\,\mathrm{d}x + \frac{c(\alpha)}{(r'-r)^2}\left[\int_{B_{r'}(x_0)} |\Delta_h u_\delta|^q\,\mathrm{d}x\right]^{\frac{2}{q}}\left[\int_{B_{r'}(x_0)} 1\,\mathrm{d}x\right]^{1-\frac{2}{q}},
\end{aligned}
$$

where $\alpha > 0$ is chosen sufficiently small such that the first integral on the right-hand side of (2.8) can be absorbed on the left-hand side of (2.6). It remains to discuss the last integral from the right-hand side of (2.6) involving the volume forces. We have the identity

$$
\partial_i\partial_k v^i = \partial_j\varepsilon_{ik}(v) + \partial_k\varepsilon_{ij}(v) - \partial_i\varepsilon_{jk}(v)\,,
$$

hence

$$
|\nabla^2 v| \leq c|\nabla\varepsilon(v)|\,.
$$

The L^∞ bound of g and standard application of Young's and Hölder's inequalities obviously give

$$\begin{aligned}
&\left|\int_{B_{r'}(x_0)} g\Delta_{-h}(\eta^2\Delta_h u_\delta + h\psi)\,\mathrm{d}x\right| \\
&\quad\le\; c\int_{B_{r'}(x_0)} \big(|\varepsilon(\Delta_h u_\delta)|\eta^2 + |\Delta_h u_\delta||\nabla\eta| + |\varepsilon(\psi)|h\big)\,\mathrm{d}x \\
&\quad\le\; \alpha\int_{B_{r'}(x_0)} |\varepsilon(\Delta_h u_\delta)|^2\eta^2\,\mathrm{d}x \\
&\qquad + \frac{c(\alpha)}{(r'-r)^2}\left\{1 + \left[\int_{B_{r'}(x_0)} |\Delta_h u_\delta|^q\,\mathrm{d}x\right]^{\frac{2}{q}}\left[\int_{B_{r'}(x_0)} 1\,\mathrm{d}x\right]^{1-\frac{2}{q}}\right\}. \qquad (2.9)
\end{aligned}$$

Again, as in (2.8), the first integral on the right-hand side of (2.9) may be absorbed on the left-hand side of (2.6) provided that $\alpha > 0$ is chosen sufficiently small. Now, combining (2.6), (2.7), (2.8) and (2.9), chosing α sufficiently small and taking into account the properties of η we arrive at

$$\begin{aligned}
&\int_{B_r(x_0)} \mathcal{B}_x(\varepsilon(\Delta_h u_\delta), \varepsilon(\Delta_h u_\delta))\,\mathrm{d}x \qquad (2.10) \\
&\le\; \frac{2}{3}\int_{B_{r'}(x_0)} \mathcal{B}_x(\varepsilon(\Delta_h u_\delta), \varepsilon(\Delta_h u_\delta))\,\mathrm{d}x + \frac{c}{(r'-r)^2}\Bigg\{1 + \\
&\left[\int_{B_{r'}(x_0)} |\Delta_h u_\delta|^q\,\mathrm{d}x\right]^{\frac{2}{q}}\left[\int_{B_{r'}(x_0)} (1 + |\varepsilon(\Delta_h u_\delta)h|^2 + |\varepsilon(u_\delta)|^2)^{\frac{q}{2}}\,\mathrm{d}x\right]^{1-\frac{2}{q}}\Bigg\}.
\end{aligned}$$

It remains only to observe that (2.10) is completely analogous to inequality (3.8) from Lemma 3.1 of [BF2]. Now the rest of the proof follows by verbatim repetition of the arguments of [BF2] given after (3.8). ■

3 Caccioppoli-type inequalities and uniform higher integrability

We start by proving a Caccioppoli-type inequality for the functions u_δ.

LEMMA 3.1 *Consider a ball $B_R(x_0) \Subset \Omega$ and choose radii $0 < r < r' < R$. Then there exists a local constant $c(r, r')$, $c(r, r') = c(r'-r)^{-\beta}$ for some suitable positive exponent β, such that for any $\eta \in C_0^\infty(B_{r'}(x_0))$, $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_r(x_0)$, $|\nabla\eta| \le c(r'-r)^{-1}$*

$$\begin{aligned}
\int_{B_{r'}(x_0)} \eta^2\Gamma_\delta^{\frac{p-2}{2}}|\nabla\varepsilon(u_\delta)|^2\,\mathrm{d}x \;&\le\; c(r,r')\Bigg[1 + \int_{B_{r'}(x_0)} \Gamma_\delta^{\frac{q}{2}}\,\mathrm{d}x + \int_{B_{r'}(x_0)} |u_\delta||\nabla u_\delta|^2\,\mathrm{d}x \\
&\qquad + \left|\int_{B_{r'}(x_0)} \partial_k u_\delta^j\partial_j u_\delta^i\partial_k u_\delta^i\eta^2\,\mathrm{d}x\right|\Bigg],
\end{aligned}$$

where the last integral on the right-hand side vanishes in the two-dimensional case. Here we have set $\Gamma_\delta := 1 + |\varepsilon(u_\delta)|^2$.

Proof. Again we follow ideas of [BF2]. Let $\sigma_\delta = Df_\delta(\varepsilon(u_\delta))$. The growth of f implies $\sigma_\delta \in L^{q/(q-1)}(B_R(x_0);\mathbb{S}^n)$, $|u_\delta \otimes u_\delta| \in L^{q/(q-1)}$ is immediate and we recall that $g \in L^\infty(\Omega;\mathbb{R}^n)$. As a result

$$\Phi:\ \overset{\circ}{W}{}^1_q(B_R(x_0);\mathbb{R}^n) \ni \varphi \mapsto \int_{B_R(x_0)} \Big[\sigma_\delta - u_\delta \otimes u_\delta\Big] : \varepsilon(\varphi)\,\mathrm{d}x - \int_{B_R(x_0)} g\cdot\varphi\,\mathrm{d}x$$

belongs to the dual space $\overset{\circ}{W}{}_q^1(B_R(x_0);\mathbb{R}^n)^*$. On account of (1.6_δ) we have $\Phi(\varphi) = 0$ whenever $\operatorname{div}\varphi = 0$. This ensures the existence of a pressure function $p_\delta \in L^{q/(q-1)}(B_R)$, $\int_{B_R(x_0)} p_\delta\,\mathrm{d}x = 0$, (see, for instance, [Ga1], p. 180, Lemma 1.1, or [La], [LS]) such that

$$\int_{B_R(x_0)} \Big[\sigma_\delta - u_\delta \otimes u_\delta\Big] : \varepsilon(\varphi)\,\mathrm{d}x - \int_{B_R(x_0)} g\cdot\varphi\,\mathrm{d}x = \int_{B_R(x_0)} p_\delta \operatorname{div}\varphi\,\mathrm{d}x \tag{3.1}$$

for all $\varphi \in \overset{\circ}{W}{}_q^1(B_R(x_0);\mathbb{R}^n)$. In particular we have $p_\delta \in W^1_{q/(q-1)}(B_R(x_0))$. Now fix η as above and denote by Δ_h the difference quotient in direction e_k, $k = 1,\dots,n$. Moreover, we choose $\varphi = \Delta_{-h}\{\eta^2\Delta_h u_\delta\}$, h sufficiently small, in (3.1). This gives

$$\begin{aligned}\int_{B_{r'}(x_0)} \Delta_h\Big[\sigma_\delta - u_\delta\otimes u_\delta\Big] &: \varepsilon(\eta^2\Delta_h u_\delta)\,\mathrm{d}x\\ &+ \int_{B_{r'}(x_0)} g\Delta_{-h}\{\eta^2\Delta_h u_\delta\} = \int_{B_{r'}(x_0)} \Delta_h p_\delta \operatorname{div}(\eta^2\Delta_h u_\delta)\,\mathrm{d}x\,. \end{aligned}\tag{3.2}$$

Note that it is not immediate whether in (3.2) the difference quotients may be replaced by the corresponding derivatives. Let us consider the convective term: by Lemma 2.2 we know that u_δ is of class $W^2_{2,loc}$, hence $|\nabla u_\delta|^2$ is locally of class L^3 and we find real numbers $2 < s_1 < 3$ and $1 < s_2 < 2$ such that

$$\begin{aligned}\Big|\Delta_h(u_\delta\otimes u_\delta) : \varepsilon\Big\{\eta^2\Delta_h u_\delta\Big\}\Big| &\le c\Big\{|\Delta_h(u_\delta\otimes u_\delta)|^{s_1}\\ &\qquad + |\varepsilon(\Delta_h u_\delta)|^{s_2} + c(\eta)|\Delta_h u_\delta|^{s_2}\Big\}\,,\end{aligned}$$

thus we have equi-integrability which together with the almost everywhere convergence

$$\Delta_h(u_\delta\otimes u_\delta) : \varepsilon(\eta^2\Delta_h u_\delta) \overset{h\to 0}{\to} \partial_k(u_\delta\otimes u_\delta) : \varepsilon(\eta^2\partial_k u_\delta)$$

implies by Vitali's Theorem (see, e.g. [AFP], p. 38)

$$\int_{B_{r'}(x_0)} \Delta_h(u_\delta\otimes u_\delta) : \varepsilon(\eta^2\Delta_h u_\delta)\,\mathrm{d}x \overset{h\to 0}{\to} \int_{B_{r'}(x_0)} \partial_k(u_\delta\otimes u_\delta) : \varepsilon(\eta^2\partial_k u_\delta)dx\,. \tag{3.3}$$

Moreover, the L^∞ bound on g obviously gives

$$\int_{B_{r'}(x_0)} g\Delta_{-h}\Big\{\eta^2\Delta_h u_\delta\Big\}\,\mathrm{d}x \overset{h\to 0}{\to} \int_{B_{r'}(x_0)} g\partial_k\Big\{\eta^2\partial_k u_\delta\Big\}\,\mathrm{d}x\,. \tag{3.4}$$

With (3.3) and (3.4) we follow the lines of [BF2] to obtain

$$\begin{aligned}\int_{B_{r'}(x_0)} \eta^2 \partial_k \sigma_\delta : \partial_k \varepsilon(u_\delta)\,\mathrm{d}x \quad &\leq \quad -2\int_{B_{r'}(x_0)} \eta \partial_k \sigma_\delta : (\nabla \eta \odot \partial_k u_\delta)\,\mathrm{d}x \\ &\quad + \int_{B_{r'}(x_0)} \partial_k (u_\delta \otimes u_\delta) : \varepsilon(\eta^2 \partial_k u_\delta)\,\mathrm{d}x \\ &\quad - \int_{B_{r'}(x_0)} g \partial_k \Big\{ \eta^2 \partial_k u_\delta \Big\}\,\mathrm{d}x \\ &\quad -2\int_{B_{r'}(x_0)} \eta \partial_k p_\delta \mathbf{1} : (\nabla \eta \odot \partial_k u_\delta)\,\mathrm{d}x\,. \end{aligned} \tag{3.5}$$

As shown in [BF2] we have

$$|\nabla \sigma_\delta| \Gamma_\delta^{\frac{2-q}{4}} \leq c\big(\partial_k \sigma_\delta : \partial_k \varepsilon(u_\delta)\big)^{\frac{1}{2}}\,,$$

hence the first integral on the right-hand side of (3.5) is estimated by

$$\begin{aligned}&\left| \int_{B_{r'}(x_0)} \eta \partial_k \sigma_\delta : (\nabla \eta \odot \partial_k u_\delta)\,\mathrm{d}x \right| \\ &\quad \leq \quad \rho \int_{B_{r'}(x_0)} \eta^2 |\nabla \sigma_\delta|^2 \Gamma_\delta^{\frac{2-q}{2}}\,\mathrm{d}x + \rho^{-1} \int_{B_{r'}(x_0)} |\nabla \eta|^2 \Gamma_\delta^{\frac{q-2}{2}} |\nabla u_\delta|^2\,\mathrm{d}x \\ &\quad \leq \quad c\rho \int_{B_{r'}(x_0)} \eta^2 \partial_k \sigma_\delta : \partial_k \varepsilon(u_\delta)\,\mathrm{d}x + \rho^{-1} \int_{B_{r'}(x_0)} |\nabla \eta|^2 \Gamma_\delta^{\frac{q-2}{2}} |\nabla u_\delta|^2\,\mathrm{d}x\,,\end{aligned}$$

where $\rho > 0$ is chosen sufficiently small such that the first integral on the right-hand side can be absorbed on the left-hand side of (3.5). The pressure term on the right-hand side of (3.5) is handled with the equation

$$g + \operatorname{div}(\sigma_\delta - u_\delta \otimes u_\delta) = \nabla p_\delta\,.$$

This gives

$$\begin{aligned}&\left| \int_{B_{r'}(x_0)} \eta \partial_k p_\delta \mathbf{1} : (\nabla \eta \odot \partial_k u_\delta)\,\mathrm{d}x \right| \\ &\quad \leq \quad c \int_{B_{r'}(x_0)} \eta |\partial_k \sigma_\delta| |\nabla \eta| |\nabla u_\delta|\,\mathrm{d}x \\ &\qquad + c \int_{B_{r'}(x_0)} \eta |\nabla (u_\delta \otimes u_\delta)| |\nabla \eta| |\nabla u_\delta|\,\mathrm{d}x \\ &\qquad + c \int_{B_{r'}(x_0)} \eta |g| |\nabla \eta| |\nabla u_\delta|\,\mathrm{d}x =: T_1 + T_2 + T_3\,.\end{aligned}$$

T_2 is bounded as claimed above, i.e.

$$T_2 \leq c(r, r') \int_{B_{r'}(x_0)} |u_\delta| |\nabla u_\delta|^2\,\mathrm{d}x\,.$$

The last integral T_3 is estimated from above by $\int_{B_{r'}(x_0)} \Gamma_\delta^{q/2}\,\mathrm{d}x$ and it remains to discuss T_1:

$$T_1 \le c\rho \int_{B_{r'}(x_0)} \eta^2 |\nabla \sigma_\delta|^2 \Gamma_\delta^{\frac{2-q}{2}}\,\mathrm{d}x + c(\rho)\int_{B_{r'}(x_0)} |\nabla\eta|^2|\nabla u_\delta|^2\Gamma_\delta^{\frac{q-2}{2}}\,\mathrm{d}x\,,$$

hence we obtain the same upper bound as given for the first integral on the right-hand side of (3.5). Summarizing these results and recalling the ellipticity condition for $D^2 f$ we get

$$\begin{aligned}
\int_{B_{r'}(x_0)} \eta^2\Gamma_\delta^{\frac{p-2}{2}}|\nabla\varepsilon(u_\delta)|^2\,\mathrm{d}x \;&\le\; c\int_{B_{r'}(x_0)} \eta^2\partial_k\sigma_\delta : \partial_k\varepsilon(u_\delta)\,\mathrm{d}x\\
&\le\; c\Bigg[\int_{B_{r'}(x_0)} |\nabla\eta|^2\Gamma_\delta^{\frac{q-2}{2}}|\nabla u_\delta|^2\,\mathrm{d}x + \int_{B_{r'}(x_0)} |u_\delta||\nabla u_\delta|^2\,\mathrm{d}x\\
&\quad + \Bigg|\int_{B_{r'}(x_0)} \partial_k(u_\delta\otimes u_\delta) : \varepsilon(\eta^2\partial_k u_\delta)\,\mathrm{d}x\Bigg|\\
&\quad + \Bigg|\int_{B_{r'}(x_0)} g\partial_k[\eta^2\partial_k u_\delta]\,\mathrm{d}x\Bigg|\Bigg]\,. \qquad (3.6)
\end{aligned}$$

The first integral on the right-hand side of (3.6) is estimated with the help of Korn's inequality, where we use the fact that $q < p^*$

$$\begin{aligned}
\int_{B_{r'}(x_0)} |\nabla\eta|^2\Gamma_\delta^{\frac{q-2}{2}}|\nabla u_\delta|^2\,\mathrm{d}x \;&\le\; c(r,r')\Bigg[1+\int_{B_{r'}(x_0)} |\nabla u_\delta|^q\,\mathrm{d}x\Bigg]\\
&\le\; c(r,r')\Bigg[1+\int_{B_{r'}(x_0)} |u_\delta|^q\,\mathrm{d}x + \int_{B_{r'}(x_0)} |\varepsilon(u_\delta)|^q\,\mathrm{d}x\Bigg]\\
&\le\; c(r,r')\Bigg[1+\int_{B_{r'}(x_0)} \Gamma_\delta^{\frac{q}{2}}\,\mathrm{d}x\Bigg]\,.
\end{aligned}$$

For the last integral on the right-hand side of (3.6) we have

$$\Bigg|\int_{B_{r'}(x_0)} g\partial_k[\eta^2\partial_k u_\delta]\,\mathrm{d}x\Bigg| \le c\Bigg[\int_{B_{r'}(x_0)} \eta|\nabla\eta||\nabla u_\delta|\,\mathrm{d}x + \int_{B_{r'}(x_0)} \eta^2|\nabla^2 u_\delta|\,\mathrm{d}x\Bigg]\,.$$

weiter hence with $\alpha > 0$ sufficiently small

$$\int_{B_{r'}(x_0)} \eta^2|\nabla^2 u_\delta|\,\mathrm{d}x \le \alpha\int_{B_{r'}(x_0)} \eta^2\Gamma_\delta^{\frac{p-2}{2}}|\nabla\varepsilon(u_\delta)|^2\,\mathrm{d}x + c(\alpha)\int_{B_{r'}(x_0)} \eta^2\Gamma_\delta^{\frac{2-p}{2}}\,\mathrm{d}x\,.$$

Here the first integral on the right-hand side may be absorbed on the left-hand side of (3.6) provided that $\alpha > 0$ is chosen sufficiently small. Summing up, it is proved at this point

$$\begin{aligned}
\int_{B_{r'}(x_0)} \eta^2\Gamma_\delta^{\frac{p-2}{2}}|\nabla\varepsilon(u_\delta)|^2\,\mathrm{d}x \;&\le\; c(r,r')\Bigg[\int_{B_{r'}(x_0)} \Gamma_\delta^{\frac{q}{2}}\,\mathrm{d}x + \int_{B_{r'}(x_0)} |u_\delta||\nabla u_\delta|^2\,\mathrm{d}x\Bigg]\\
&\quad + c\Bigg[1+\Bigg|\int_{B_{r'}(x_0)} \partial_k(u_\delta\otimes u_\delta) : \varepsilon(\eta^2\partial_k u_\delta)\,\mathrm{d}x\Bigg|\Bigg] \qquad (3.7)
\end{aligned}$$

and it remains to discuss the integral resulting from the convective term on the right-hand side of (3.7). This is done with a standard observation (see, for instance, [MNRR]):

$$
\begin{aligned}
\int_{B_{r'}(x_0)} \partial_k(u_\delta \otimes u_\delta) : \varepsilon(\eta^2 \partial_k u_\delta)\,\mathrm{d}x &= \int_{B_{r'}(x_0)} \partial_k(u^i_\delta u^j_\delta)\partial_j\{\eta^2 \partial_k u^i_\delta\}\,\mathrm{d}x \\
&= \int_{B_{r'}(x_0)} \partial_j(u^i_\delta u^j_\delta)\partial_k\{\eta^2 \partial_k u^i_\delta\}\,\mathrm{d}x \\
&= \int_{B_{r'}(x_0)} u^j_\delta \partial_j u^i_\delta \partial_k\{\eta^2 \partial_k u^i_\delta\}\,\mathrm{d}x \\
&= -\int_{B_{r'}(x_0)} \partial_k u^j_\delta \partial_j u^i_\delta \partial_k u^i_\delta \eta^2\,\mathrm{d}x \\
&\quad -\frac{1}{2}\int_{B_{r'}(x_0)} u^j_\delta |\nabla u_\delta|^2 \partial_j \eta^2\,\mathrm{d}x\,.
\end{aligned}
$$

In the two-dimensional case we recall (again compare [MNRR]) that the first integral on the right-hand side vanishes on account of the divergence-free condition and the lemma is shown. ■

Combining some arguments of [KMS] and [BF2] we are now going to give a **Proof of Theorem 1.2.** To this purpose we first claim

LEMMA 3.2 *Suppose that the hypotheses of Theorem 1.2 hold. Then, for any $\Omega' \Subset \Omega$ there is a constant $c(\Omega')$ (independent of δ) such that*

$$
\int_{\Omega'} |\nabla u_\delta|^{\frac{np}{n-p}}\,\mathrm{d}x \le c(\Omega')\,.
$$

Proof of Lemma 3.2. *Step 1.* Given η as above we have by Korn's and Hölder's inequality (see [MNRR], p. 227)

$$
\begin{aligned}
\|\eta^2 D^2 u_\delta\|^2_{L^p(B_{r'}(x_0))} &\le c(1 + \|\nabla u_\delta\|_{L^p(B_{r'}(x_0))})^{2-p} \int_{B_{r'}(x_0)} \Gamma_\delta^{\frac{p-2}{2}} |\nabla \varepsilon(u_\delta)|^2 \eta^2\,\mathrm{d}x \\
&\le c(r,r')\Bigg[1 + \int_{B_{r'}(x_0)} \Gamma_\delta^{\frac{q}{2}}\,\mathrm{d}x + \left|\int_{B_{r'}(x_0)} |u_\delta||\nabla u_\delta|^2\,\mathrm{d}x\right| \\
&\quad + \left|\int_{B_{r'}(x_0)} \partial_k u^j_\delta \partial_j u^i_\delta \partial_k u^i_\delta \eta^2\,\mathrm{d}x\right|\Bigg]\,, \qquad (3.8)
\end{aligned}
$$

where we also used Lemma 3.1. Note that on account of the a priori estimate (2.2) an analogous inequality holds if $\|D^2 u_\delta \eta^2\|^2_p$ on the left-hand side of (3.8) is replaced by $\|\nabla(\nabla u_\delta \eta^2)\|^2_p$. Sobolev's Embedding Theorem then implies

$$
\|\nabla u_\delta \eta^2\|^2_{\frac{np}{n-p}} \le \text{right-hand side of (3.8).} \qquad (3.9)
$$

Step 2. Let us choose some real numbers $a \in (0,1)$, $\gamma > 1$ such that

$$
qa\gamma = \frac{np}{n-p}\,, \qquad q(1-a)\frac{\gamma}{\gamma-1} = p\,. \qquad (3.10)
$$

Note that (3.10) holds for

$$a = \frac{n(q-p)}{pq} \in (0,1)\,, \qquad \gamma = \frac{p^2}{(n-p)(q-p)} > 1\,,$$

where the range of a and γ follows from our assumptions on q and p. Hölder's inequality now implies

$$\begin{aligned}\int_{B_{r'}(x_0)} |\nabla u_\delta|^q \,\mathrm{d}x &\leq \left[\int_{B_{r'}(x_0)} |\nabla u_\delta|^{qa\gamma}\,\mathrm{d}x\right]^{\frac{1}{\gamma}} \left[\int_{B_{r'}(x_0)} |\nabla u_\delta|^{q(1-a)\frac{\gamma}{\gamma-1}}\,\mathrm{d}x\right]^{\frac{\gamma-1}{\gamma}} \\ &\leq c\left[\int_{B_{r'}(x_0)} |\nabla u_\delta|^{\frac{np}{n-p}}\,\mathrm{d}x\right]^{\frac{1}{\gamma}}.\end{aligned}$$

We finally observe that $q < p(n+2)/n$ gives $\gamma^{-1} < 2(n-p)/np$, hence for some sufficiently small parameter $\alpha > 0$:

$$\int_{B_{r'}(x_0)} |\nabla u_\delta|^q\,\mathrm{d}x \leq c(\alpha) + \alpha\left[\int_{B_{r'}(x_0)} |\nabla u_\delta|^{\frac{np}{n-p}}\,\mathrm{d}x\right]^{\frac{2(n-p)}{np}}. \tag{3.11}$$

Step 3. Following [KMS] we find a bound for the second integral on the right-hand side of (3.8): choose real numbers $\tilde{a} \in (0,1)$, $\tilde{\gamma} > 1$ such that

$$2\bar{p}\tilde{a}\tilde{\gamma} = \frac{np}{n-p}\,, \qquad 2\bar{p}(1-\tilde{a})\frac{\tilde{\gamma}}{\tilde{\gamma}-1} = p\,. \tag{3.12}$$

Note that (3.12) holds for

$$\tilde{a} = \frac{3n-(n+1)p}{2p} \in (0,1)\,, \qquad \tilde{\gamma} = \frac{p((n+1)p-n)}{(n-p)(3n-(n+1)p)} > 1\,,$$

where the conditions $\tilde{a} \in (0,1)$ and $\tilde{\gamma} > 1$ follow from $p > 3n/(n+3)$. The same condition implies

$$\frac{1}{\tilde{\gamma}\bar{p}} < \frac{2(n-p)}{np}$$

and we obtain for $\tilde{\alpha} > 0$ sufficiently small

$$\begin{aligned}\left|\int_{B_{r'}(x_0)} |u_\delta||\nabla u_\delta|^2\,\mathrm{d}x\right| &\leq c\|u_\delta\|_{\frac{np}{n-p}}\left[\int_{B_{r'}(x_0)} |\nabla u_\delta|^{2\bar{p}}\,\mathrm{d}x\right]^{\frac{1}{\bar{p}}} \\ &\leq c\left[\int_{B_{r'}(x_0)} |\nabla u_\delta|^{2\bar{p}\tilde{a}\tilde{\gamma}}\,\mathrm{d}x\right]^{\frac{1}{\bar{p}\tilde{\gamma}}} \left[\int_{B_{r'}(x_0)} |\nabla u_\delta|^{2\bar{p}(1-\tilde{a})\frac{\tilde{\gamma}}{(\tilde{\gamma}-1)}}\,\mathrm{d}x\right]^{\frac{\tilde{\gamma}-1}{\tilde{\gamma}\bar{p}}} \\ &\leq c\left[\int_{B_{r'}(x_0)} |\nabla u_\delta|^{\frac{np}{n-p}}\,\mathrm{d}x\right]^{\frac{1}{\bar{p}\gamma}} \\ &\leq c(\alpha) + \alpha\left[\int_{B_{r'}(x_0)} |\nabla u_\delta|^{\frac{np}{n-p}}\,\mathrm{d}x\right]^{\frac{2(n-p)}{np}}.\end{aligned} \tag{3.13}$$

Step 4. In the three-dimensional case $n = 3$ it remains to discuss the third integral on the right-hand side of (3.8). The discussion is the same as in Step 2. whenever the choice $q = 3$ is admissible. This leads to the requirement $p > 9/5$ in the three-dimensional case.
Step 5. It is proved in (3.9), (3.11), (3.13) that we can find a sufficiently small number $\rho > 0$ such that

$$\left[\int_{B_r(x_0)} |\nabla u_\delta|^{\frac{np}{n-p}}\right]^{\frac{2(n-p)}{np}} \leq c(\rho)c(r,r') + \rho\left[\int_{B_{r'}(x_0)} |\nabla u_\delta|^{\frac{np}{n-p}}\,\mathrm{d}x\right]^{\frac{2(n-p)}{np}} .$$

This gives by Lemma 3.1, p. 161 of [Gi] the uniform bound

$$\int_{B_r(x_0)} |\nabla u_\delta|^{\frac{np}{n-p}}\,\mathrm{d}x \leq const\,,$$

and the proof of Lemma 3.2 is finished. ■

In a next step we make use of Lemma 3.2 in order to improve the Caccioppoli-type inequality from Lemma 3.1.

LEMMA 3.3 *Under the hypotheses of Theorem 1.2 and with the notation of Lemma 3.1 there is a positive exponent $\gamma > 0$ such that*

$$\int_{B_r(x_0)} \Gamma_\delta^{\frac{p-2}{2}} |\nabla\varepsilon(u_\delta)|^2\,\mathrm{d}x \leq c_1(r'-r)^{-2}\int_{B_{r'}(x_0)} \Gamma_\delta^{\frac{q-2}{2}} |\nabla u_\delta - Q|^2\,\mathrm{d}x + c_2(r')^\gamma\,.$$

Here c_1, c_2 are uniform positive constants and $Q \in \mathbb{R}^{n\times n}$ is an arbitrary matrix.

Let us assume for the moment that the lemma is true and let us first finish the proof of Theorem 1.2. We choose

$$\chi = \begin{cases} \frac{n}{n-2} = 3 & \text{if} \quad n = 3\,, \\ \text{any number} > \frac{2p}{2p-q} & \text{if} \quad n = 2\,, \end{cases}$$

and let $\alpha = p\chi$. Following the proof of Corollary 4.1 in [BF2], we note that an appropriate choice of Q in Lemma 3.3 gives

$$\int_{B_{r'}(x_0)} \Gamma_\delta^{\frac{q-2}{2}} |\nabla u_\delta - Q|^2\,\mathrm{d}x \leq c\int_{B_{r'}(x_0)} \Gamma_\delta^{\frac{q}{2}}\,\mathrm{d}x\,.$$

Moreover, the quantity $c_2(r')^\gamma$ can be interpreted as a local constant. Then the interpolation arguments presented in Lemma 4.4, [BF2], give with some appropriate exponent β ($B_{r'}(x_0) \Subset \Omega' \Subset \Omega$)

$$\left[\int_{B_r(x_0)} \Gamma_\delta^{\frac{\alpha}{2}}\,\mathrm{d}x\right]^{\frac{1}{\chi}} \leq \frac{1}{2}\left[\int_{B_{r'}(x_0)} \Gamma_\delta^{\frac{\alpha}{2}}\,\mathrm{d}x\right]^{\frac{1}{\chi}} + c(r,r')\left[\int_{\Omega'} \Gamma_\delta^{\frac{p}{2}}\,\mathrm{d}x\right]^{\beta} + c\,,$$

and with Lemma 3.1, p. 161, of [Gi] Theorem 1.2 is established. ■
It remains to give a **Proof of Lemma 3.3.** It is known by Lemma 3.2 that

$$\|u_\delta\|_{L^\infty_{loc}(\Omega;\mathbb{R}^n)} \leq c\,, \tag{3.14}$$

where the local constant is independent of δ. If $Q \in \mathbb{R}^{n\times n}$ is fixed, then the arguments leading to (3.6) give with u_δ replaced by $u_\delta - Qx$

$$\begin{aligned}\int_{B_{r'}(x_0)} \eta^2 \Gamma_\delta^{\frac{p-2}{2}} |\nabla\varepsilon(u_\delta)|^2 \,\mathrm{d}x \;&\leq\; c_1 \int_{B_{r'}(x_0)} |\nabla\eta|^2 \Gamma_\delta^{\frac{q-2}{2}} |\nabla u_\delta - Q|^2 \,\mathrm{d}x \\ &\quad + c_2 \Bigg\{ \Bigg| \int_{B_{r'}(x_0)} \partial_k[u_\delta \otimes u_\delta] : \varepsilon(\eta^2 \partial_k [u_\delta - Qx]) \,\mathrm{d}x \Bigg| \\ &\quad + \Bigg| \int_{B_{r'}(x_0)} g \cdot \partial_k[\eta^2 \partial_k (u_\delta - Qx)] \,\mathrm{d}x \Bigg| \Bigg\} . \qquad (3.15)\end{aligned}$$

The second integral on the right-hand-side of (3.15) is handled with the help of (3.14):

$$\begin{aligned}&\Bigg| \int_{B_{r'}(x_0)} \partial_k[u_\delta \otimes u_\delta] : \varepsilon(\eta^2 \partial_k [u_\delta - Qx]) \,\mathrm{d}x \Bigg| \\ &\leq c \left[\int_{B_{r'}(x_0)} |\nabla u_\delta| |\nabla \eta^2| |\nabla u_\delta - Q| \,\mathrm{d}x + \int_{B_{r'}(x_0)} |\nabla u_\delta| \eta^2 |\nabla\varepsilon(u_\delta)| \,\mathrm{d}x \right] \\ &\leq c \Bigg[\int_{B_{r'}(x_0)} |\nabla u_\delta|^2 \,\mathrm{d}x + (r'-r)^{-2} \int_{B_{r'}(x_0)} |\nabla u_\delta - Q|^2 \,\mathrm{d}x \\ &\quad + \int_{B_{r'}(x_0)} \eta^2 |\nabla u_\delta| \Gamma_\delta^{\frac{2-p}{4}} |\nabla\varepsilon(u_\delta)| \Gamma_\delta^{\frac{p-2}{4}} \,\mathrm{d}x \Bigg]\end{aligned}$$

From Young's inequality we deduce

$$\begin{aligned}\int_{B_{r'}(x_0)} \eta^2 \Gamma_\delta^{\frac{p-2}{2}} |\nabla\varepsilon(u_\delta)|^2 \,\mathrm{d}x \;&\leq\; c \Bigg[(r'-r)^{-2} \int_{B_{r'}(x_0)} \Gamma_\delta^{\frac{q-2}{2}} |\nabla u_\delta - Q|^2 \,\mathrm{d}x \qquad (3.16) \\ &\quad + \int_{B_{r'}(x_0)} \Gamma_\delta^{\frac{4-p}{2}} \,\mathrm{d}x + \Bigg| \int_{B_{r'}(x_0)} g \cdot \partial_k[\eta^2 \partial_k (u_\delta - Qx)] \,\mathrm{d}x \Bigg| \Bigg] .\end{aligned}$$

The integral involving the volume forces is estimated by

$$\begin{aligned}\Bigg| \int_{B_{r'}(x_0)} g \cdot \partial_k[\eta^2 \partial_k (u_\delta - Qx)] \,\mathrm{d}x \Bigg| \;&\leq\; c \Bigg[\int_{B_{r'}(x_0)} \eta |\nabla\eta| |\nabla u_\delta - Q| \,\mathrm{d}x \\ &\quad + \int_{B_{r'}(x_0)} \eta^2 |\nabla^2 u_\delta| \,\mathrm{d}x \Bigg] =: c\{T_1 + T_2\} .\end{aligned}$$

Here we have for T_1

$$\begin{aligned}T_1 \;&\leq\; c \left[(r'-r)^{-2} \int_{B_{r'}(x_0)} |\nabla u_\delta - Q|^2 \,\mathrm{d}x + |B_{r'}(x_0)| \right] \\ &\leq\; c \left[(r'-r)^{-2} \int_{B_{r'}(x_0)} \Gamma_\delta^{\frac{q-2}{2}} |\nabla u_\delta - Q|^2 \,\mathrm{d}x + (r')^n \right] , \qquad (3.17)\end{aligned}$$

and for T_2

$$
\begin{aligned}
T_2 &\leq c \int_{B_{r'}(x_0)} \eta^2 |\nabla \varepsilon(u_\delta)| \, \mathrm{d}x \\
&\leq \alpha \int_{B_{r'}(x_0)} \eta^2 \Gamma_\delta^{\frac{p-2}{2}} |\nabla \varepsilon(u_\delta)|^2 \, \mathrm{d}x + c(\alpha) \int_{B_{r'}(x_0)} \Gamma_\delta^{\frac{2-p}{2}} \, \mathrm{d}x \, . \qquad (3.18)
\end{aligned}
$$

Since in addition the integral $\int_{B_{r'}(x_0)} \Gamma_\delta^{(4-p)/2} \, \mathrm{d}x$ is also bounded by some power $(r')^\gamma$, the lemma is proved by combining (3.16)–(3.18). ■

COROLLARY 3.1 *Fix* $\alpha < 3p/(p+1)$ *if* $n = 3$, *in the two-dimensional case we assume* $\alpha < 2$. *Then, for any* $\Omega' \Subset \Omega$ *there is a constant* c, *independent of* δ, *such that*

$$
\|u_\delta\|_{W^2_\alpha(\Omega;\mathbb{R}^2)} \leq c \, .
$$

Proof. By Theorem 1.2 we have $u_\delta \in W^1_{\tilde{q},loc}$. This, together with Lemma 3.3, implies for any $\Omega' \Subset \Omega$

$$
\int_{\Omega'} \Gamma_\delta^{\frac{p-2}{2}} |\nabla \varepsilon(u_\delta)|^2 \, \mathrm{d}x \leq c(\Omega') \, .
$$

As a consequence, we obtain for any α as above (using Young's inequality)

$$
\begin{aligned}
\int_{\Omega'} |\nabla \varepsilon(u_\delta)|^\alpha \, \mathrm{d}x &= \int_{\Omega'} \Gamma_\delta^{\frac{p-2}{2}\frac{\alpha}{2}} |\nabla \varepsilon(u_\delta)|^\alpha \Gamma_\delta^{\frac{2-p}{2}\frac{\alpha}{2}} \, \mathrm{d}x \\
&\leq c \left\{ \int_{\Omega'} \Gamma_\delta^{\frac{p-2}{2}} |\nabla \varepsilon(u_\delta)|^2 \, \mathrm{d}x + \int_{\Omega'} \Gamma_\delta^{\frac{2-p}{2}\frac{\alpha}{2-\alpha}} \right\} .
\end{aligned}
$$

Hence, again by Theorem 1.2, we have the corollary. ■

Up to now we used the Caccioppoli-type inequality Lemma 3.1 to prove a first higher integrability result Lemma 3.2 which in turn gives an improved Caccioppoli-type inequality Lemma 3.3 with Theorem 1.2 as a consequence. Now we use Theorem 1.2 for a final improvement of our Caccioppoli-type inequality which is needed for the blow-up arguments of Section 5. Let

$$
h_\delta := (1 + |\varepsilon(u_\delta)|^2)^{\frac{p}{4}} \, .
$$

LEMMA 3.4 *Consider the case* $n = 3$ *together with (1.4). Then, for any matrix* $Q \in \mathbb{R}^{3\times 3}$ *and for any* $B_r(x_0) \Subset B_{r'}(x_0) \Subset \Omega$ *we have*

$$
\begin{aligned}
\int_{B_r(x_0)} |\nabla h_\delta|^2 \, \mathrm{d}x &\leq c \int_{B_r(x_0)} \Gamma_\delta^{\frac{p-2}{2}} |\nabla \varepsilon(u_\delta)|^2 \, \mathrm{d}x \\
&\leq c \left[(r'-r)^{-2} \int_{B_{r'}(x_0)} \Gamma_\delta^{\frac{q_0-2}{2}} |\nabla u_\delta - Q|^2 \, \mathrm{d}x + (r')^{4-\frac{4}{p}} + \delta c(r,r',Q) \right] ,
\end{aligned}
$$

where the constant $c(r,r',Q)$ *is just depending on* r, r' *and the matrix* Q.

Proof. We return to (3.5) where on the right-hand side u_δ is replaced by $u_\delta - Qx$. In order to reach the power $\Gamma_\delta^{\frac{q_0-2}{2}}$ on the right-hand side of Lemma 3.4 we split

$$\sigma_\delta = \sigma_\delta^1 + \sigma_\delta^2 = Df(\varepsilon(u_\delta)) + \delta q \Gamma_\delta^{\frac{q-2}{2}} \varepsilon(u_\delta)$$

and observe that the modified right-hand side of (3.5) is an upper bound for the quantity

$$\int_{B_{r'}(x_0)} \eta^2 \partial_k \sigma_\delta^1 : \partial_k \varepsilon(u_\delta)\,\mathrm{d}x$$

which in return by ellipticity of $D^2 f$ is bounded from below by

$$c\int_{B_{r'}(x_0)} \eta^2 \Gamma_\delta^{\frac{p-2}{2}} |\nabla \varepsilon(u_\delta)|^2\,\mathrm{d}x\,.$$

Observing

$$\int_{B_{r'}(x_0)} \eta^2 |\nabla h_\delta|^2\,\mathrm{d}x \le c\int_{B_{r'}(x_0)} \eta^2 \Gamma_\delta^{\frac{p-2}{2}} |\nabla \varepsilon(u_\delta)|^2\,\mathrm{d}x$$

it remains to discuss the four terms on the right-hand side of (3.5). We have

$$\begin{aligned}
&\left| 2\int_{B_{r'}(x_0)} \eta \partial_k \sigma_\delta : (\nabla \eta \odot \partial_k[u_\delta - Qx])\,\mathrm{d}x \right| \\
&\quad \le\ c\left[\int_{B_{r'}(x_0)} \eta |\nabla \sigma_\delta^1||\nabla \eta||\nabla u_\delta - Q|\,\mathrm{d}x + \int_{B_{r'}(x_0)} \eta |\nabla \sigma_\delta^2||\nabla \eta||\nabla u_\delta - Q|\,\mathrm{d}x\right] \\
&\quad \le\ \alpha \int_{B_{r'}(x_0)} \Gamma_\delta^{\frac{2-q_0}{2}} |\nabla \sigma_\delta^1|^2 \eta^2\,\mathrm{d}x + c(\alpha)(r'-r)^{-2} \int_{B_{r'}(x_0)} \Gamma_\delta^{\frac{q_0-2}{2}} |\nabla u_\delta - Q|^2\,\mathrm{d}x \\
&\quad\quad +c\int_{B_{r'}(x_0)} \eta |\nabla \sigma_\delta^2||\nabla \eta||\nabla u_\delta - Q|\,\mathrm{d}x\,. \qquad (3.19)
\end{aligned}$$

Since $\Gamma_\delta^{(2-q_0)/2}|\nabla \sigma_\delta^1|^2 \le c\partial_k \sigma_\delta^1 : \varepsilon(u_\delta)$ we may choose α in an appropriate way such that the first integral on the right-hand side of (3.19) can be absorbed, the second one occurs on the right-hand side of the inequality stated in Lemma 3.4, and the third integral is estimated as follows:

$$\begin{aligned}
&\int_{B_{r'}(x_0)} |\nabla \sigma_\delta^2| \eta |\nabla \eta||\nabla u_\delta - Q|\,\mathrm{d}x \\
&\quad \le\ c\delta \int_{B_{r'}(x_0)} \Gamma_\delta^{\frac{q-2}{2}} |\nabla \varepsilon(u_\delta)| \eta |\nabla \eta||\nabla u_\delta - Q|\,\mathrm{d}x \qquad (3.20) \\
&\quad \le\ c\delta\left[\alpha \int_{B_{r'}(x_0)} \eta^2 |\nabla \varepsilon(u_\delta)|^2 \Gamma_\delta^{\frac{p-2}{2}}\,\mathrm{d}x + c(\alpha)\int_{B_{r'}(x_0)} |\nabla \eta|^2 |\nabla u_\delta - Q|^2 \tilde{\Gamma}_\delta^{\,q-2+\frac{2-p}{2}}\,\mathrm{d}x\right]
\end{aligned}$$

(where $\tilde{\Gamma}_\delta := 1 + |\nabla u_\delta|^2$). Since $2(q-1)+2-p < 3p$ ($\Leftrightarrow q < 2p$) on account of (1.7) we see that (recall Theorem 1.2) the last integral on the right-hand side of (3.20) is bounded from above by a suitable constant $c(r, r', Q)$ independent of δ (the first one again is absorbed).

The second term on the right-hand side of (3.5) can be handled as done before the inequality (3.16), i.e. an upper bound is given by

$$c\Bigg[\alpha\int_{B_{r'}(x_0)}\eta^2\Gamma_\delta^{\frac{p-2}{2}}|\nabla\varepsilon(u_\delta)|^2\,\mathrm{d}x+c(\alpha)\int_{B_{r'}(x_0)}\tilde{\Gamma}_\delta^{\frac{4-p}{2}}\,\mathrm{d}x+(r'-r)^{-2}\int_{B_{r'}(x_0)}|\nabla u_\delta-Q|^2\,\mathrm{d}x$$

where we already used Young's inequality. By Hölder's inequality we have

$$\int_{B_{r'}(x_0)}\tilde{\Gamma}_\delta^{\frac{4-p}{2}}\,\mathrm{d}x\le\Bigg[\int_{B_{r'}(x_0)}\tilde{\Gamma}_\delta^{\frac{3p}{2}}\,\mathrm{d}x\Bigg]^{\frac{4-p}{3p}}(r')^{4-\frac{4}{p}}\,, \tag{3.21}$$

and the a priori bound of Theorem 1.2 applies again.

The third term on the right-hand side of (3.5) is treated as outlined after (3.16) where for estimating T_1 we clearly can replace q by q_0 in the second line of (3.17), and T_2 is discussed in (3.18).

The last term on the right-hand side of (3.5) is estimated as follows (compare the discussion of the proof of Lemma 3.1

$$\begin{aligned}
&\Bigg|\int_{B_{r'}(x_0)}\eta\partial_k p_\delta\mathbf{1}:(\nabla\eta\odot\partial_k[u_\delta-Qx])\,\mathrm{d}x\Bigg|\\
&\quad\le\; c\Bigg[\int_{B_{r'}(x_0)}\eta|\nabla\sigma_\delta^1||\nabla\eta||\nabla u_\delta-Q|\,\mathrm{d}x\\
&\qquad+\int_{B_{r'}(x_0)}\eta|\nabla\sigma_\delta^2||\nabla\eta||\nabla u_\delta-Q|\,\mathrm{d}x\\
&\qquad+\int_{B_{r'}(x_0)}\eta|\nabla(u_\delta\otimes u_\delta)||\nabla\eta||\nabla u_\delta-Q|\,\mathrm{d}x\\
&\qquad+\int_{B_{r'}(x_0)}\eta|g||\nabla\eta||\nabla u_\delta-Q|\,\mathrm{d}x\Bigg]\\
&\quad=:\; c[F_1^1+F_1^2+F_2+F_3]\,,
\end{aligned} \tag{3.22}$$

where F_1^1 and F_1^2 have already been discussed in the beginning. Using the uniform local boundedness of u_δ, we see

$$\begin{aligned}
F_2 &\le c\Bigg[\int_{B_{r'}(x_0)}\eta^2|\nabla u_\delta|^2\,\mathrm{d}x+(r'-r)^{-2}\int_{B_{r'}(x_0)}|\nabla u_\delta-Q|^2\,\mathrm{d}x\Bigg]\\
&\le c\Bigg[\int_{B_{r'}(x_0)}\tilde{\Gamma}_\delta^{\frac{4-p}{2}}\,\mathrm{d}x+(r'-r)^{-2}\int_{B_{r'}(x_0)}|\nabla u_\delta-Q|^2\,\mathrm{d}x\Bigg]\,.
\end{aligned} \tag{3.23}$$

Finally we have

$$F_3\le c\Bigg[(r'-r)^{-2}\int_{B_{r'}(x_0)}|\nabla u_\delta-Q|^2\,\mathrm{d}x+(r')^3\Bigg]\,, \tag{3.24}$$

and by collecting our results (3.19)–(3.24) Lemma 3.4 is established. ■

4 The limit equation

In this section we pass to the limit $\delta \to 0$ and prove Theorem 1.3. Thus, consider a weakly convergent sequence

$$u_\delta \overset{\delta\to 0}{\rightharpoondown} \bar{u} \quad \text{in } W^1_p(\Omega;\mathbb{R}^n)\,,$$

where it is known from Theorem 1.2 that $\bar{u}$ is locally of class $W^1_{\tilde{q}}$. Moreover, it is established in Lemma 2.2 that

$$u_\delta \in W^2_{2,loc}(\Omega;\mathbb{R}^n)\,, \qquad Df_\delta(\varepsilon(u_\delta)) \in W^1_{q/(q-1),loc}(\Omega;\mathbb{S}^n)\,.$$

Let us finally recall that (see the proof of Lemma 2.2, [BF2])

$$|\Delta_h\{Df_\delta(\varepsilon(u_\delta))\}| \le \sqrt{\mathcal{B}_x}\,\mathcal{B}_x(\varepsilon(\Delta_h u_\delta),\varepsilon(\Delta_h u_\delta))^{\frac{1}{2}}\,.$$

Passing to the limit $h \to 0$ (which is admissible) we obtain

$$|\nabla\{Df_\delta(\varepsilon(u_\delta))\}| \le |D^2 f_\delta(\varepsilon(u_\delta))|^{\frac{1}{2}} D^2 f_\delta(\varepsilon(u_\delta))(\partial_k\varepsilon(u_\delta),\partial_k\varepsilon(u_\delta))^{\frac{1}{2}}$$

and, as a consequence, for $B_r(x_0) \Subset \Omega$

$$\begin{aligned}
&\int_{B_r(x_0)} |\nabla\{Df_\delta(\varepsilon(u_\delta))\}|^{\frac{q}{q-1}}\,\mathrm{d}x \\
&\quad\le\; c\int_{B_r(x_0)} |D^2 f_\delta(\varepsilon(u_\delta))|^{\frac{1}{2}\frac{q}{q-1}}\{D^2 f_\delta(\varepsilon(u_\delta))(\partial_k\varepsilon(u_\delta),\partial_k\varepsilon(u_\delta))\}^{\frac{1}{2}\frac{q}{q-1}}\,\mathrm{d}x \\
&\quad\le\; c\left[\int_{B_r(x_0)} D^2 f_\delta(\varepsilon(u_\delta))(\partial_k\varepsilon(u_\delta),\partial_k\varepsilon(u_\delta))\,\mathrm{d}x\right]^{\frac{q}{2(q-1)}}\left[\int_{B_r(x_0)} |D^2 f_\delta(\varepsilon(u_\delta))|^{\frac{q}{q-2}}\,\mathrm{d}x\right]^{\frac{q-2}{2q-2}} \\
&\quad=:\; I_1\cdot I_2\,.
\end{aligned}$$

Note that I_2 is bounded on account of Theorem 1.2, a bound for I_1 is found with the same arguments as given in Section 3. (We may take $\int_{B_r(x_0)} D^2 f_\delta(\varepsilon(u_\delta))(\partial_k\varepsilon(u_\delta),\partial_k\varepsilon(u_\delta))\,\mathrm{d}x$ as the left-hand side of (3.15). Recalling Theorem 1.2 we then we proceed as before with uniformly bounded right-hand sides.) Hence, it is shown that uniformly w.r.t. δ

$$Df_\delta(\varepsilon(u_\delta)) \in W^1_{q/(q-1),loc}(\Omega;\mathbb{S}^n)\,. \tag{4.1}$$

Now let

$$W_\delta := Df_\delta(\varepsilon(u_\delta)) = \delta q(1+|\varepsilon(u_\delta)|^2)^{\frac{q-2}{2}}\varepsilon(u_\delta) + Df(\varepsilon(u_\delta))\,.$$

The uniform estimate (4.1) yields as $\delta \to 0$

$$\left.\begin{array}{lcl} W_\delta & \rightharpoondown & W \ \text{ in } \ W^1_{q/(q-1),loc}(\Omega;\mathbb{S}^n) \\ W_\delta & \to & W \ \text{ in } \ L^{q/(q-1)}_{loc}(\Omega;\mathbb{S}^n) \ \text{ and a.e.} \end{array}\right\} \tag{4.2}$$

Moreover we have a.e. as $\delta \to 0$ (recall Theorem 1.2: $q\delta(1+|\varepsilon(u_\delta)|^2)^{\frac{q-2}{2}}\varepsilon(u_\delta) \to 0$, hence

$$\begin{aligned} Df(\varepsilon(u_\delta)) &\overset{\delta\to 0}{\to} W \ \ a.e.\,, \\ \varepsilon(u_\delta) &\overset{\delta\to 0}{\to} (Df)^{-1}(W) \ \ a.e. \end{aligned}$$

This fact, together with the weak L^p-convergence of $\varepsilon(u_\delta)$, gives

$$(Df)^{-1}(W) = \varepsilon(\bar{u}) \quad \text{or} \quad Df(\varepsilon(\bar{u})) = W\,,$$

and (4.2) reads as ($\delta \to 0$)

$$\left.\begin{array}{lll} W_\delta & \rightharpoondown & Df(\varepsilon(\bar{u})) \ \text{ in } \ W^1_{q/(q-1),loc}(\Omega;\mathbb{S}^n) \\ W_\delta & \to & Df(\varepsilon(\bar{u})) \ \text{ in } \ L^{q/(q-1)}_{loc}(\Omega;\mathbb{S}^n) \ \text{ and a.e.} \end{array}\right\} \tag{4.3}$$

Now fix a solenoidal test function $\varphi \in C_0^\infty(\Omega;\mathbb{R}^n)$. By the definition of u_δ we have

$$\int_\Omega Df_\delta(\varepsilon(u_\delta)) : \varepsilon(\varphi)\,\mathrm{d}x = \int_\Omega u_\delta \otimes u_\delta : \varepsilon(\varphi)\,\mathrm{d}x + \int_\Omega g\cdot\varphi\,\mathrm{d}x\,.$$

Then (4.3) shows that the left-hand side of the latter equation converges to $\int_\Omega Df(\varepsilon(\bar{u})) : \varepsilon(\varphi)\,\mathrm{d}x$. The a priori estimate stated in Theorem 1.2 also gives

$$\int_\Omega u_\delta \otimes u_\delta : \varepsilon(\varphi)\,\mathrm{d}x \overset{\delta\to 0}{\to} \int_\Omega \bar{u}\otimes\bar{u} : \varepsilon(\varphi)\,\mathrm{d}x$$

and Theorem 1.3 is established. ■

Next consider $\Omega' \subset \Omega$ and $w \in W^1_q(\Omega';\mathbb{R}^n)$, $\operatorname{div} w = 0$, and let

$$J[w,\Omega'] := \int_{\Omega'} f(\varepsilon(w))\,\mathrm{d}x - \int_{\Omega'} \bar{u}\otimes\bar{u} : \varepsilon(w)\,\mathrm{d}x - \int_{\Omega'} g\cdot w\,\mathrm{d}x\,.$$

We claim the local J-minimality of $\bar{u}$, to be more precise:

COROLLARY 4.1 *For any Ω' and w as above,* $\operatorname{spt}(\bar{u}-w) \Subset \Omega'$ *we have*

$$J[\bar{u},\Omega'] \le J[w,\Omega']\,.$$

REMARK 4.1 *The results of [BF3] even show that (under some suitable conditions) $\bar{u}$ is a global J-minimizer in the natural energy class.*

Proof of Corollary 4.1. We have

$$J[w,\Omega'] - J[\bar{u},\Omega'] = \int_{\Omega'} \Big[f(\varepsilon(w)) - f(\varepsilon(\bar{u}))\Big]\,\mathrm{d}x - \int_{\Omega'} \bar{u}\otimes\bar{u} : \varepsilon(w-\bar{u})\,\mathrm{d}x - \int_{\Omega'} g\cdot(w-\bar{u})\,\mathrm{d}x\,.$$

The convexity of f yields

$$f(\varepsilon(w)) - f(\varepsilon(\bar{u})) \ge Df(\varepsilon(\bar{u})) : \varepsilon(w-\bar{u})\,,$$

hence we arrive at

$$J[w,\Omega'] - J[\bar{u},\Omega'] \ge \int_{\Omega'} Df(\varepsilon(\bar{u})) : \varepsilon(w-\bar{u})\,\mathrm{d}x - \int_{\Omega'} \bar{u}\otimes\bar{u} : \varepsilon(w-\bar{u})\,\mathrm{d}x - \int_{\Omega'} g\cdot(w-\bar{u})\,\mathrm{d}x\,.$$

If we observe that (by approximation arguments and by higher integrability of $\varepsilon(\bar{u})$) the differential equation (1.6) for $\bar{u}$ remains valid whenever $\varphi \in W^1_q(\Omega;\mathbb{R}^n)$, $\operatorname{div}\varphi = 0$, φ compactly supported in Ω, then we have proved that

$$J[w,\Omega'] - J[\bar{u},\Omega'] \ge 0\,.$$

■

We finally need a limit version of Lemma 3.4 for the blow-up arguments of the next section. To this purpose we let

$$h := (1 + |\varepsilon(\bar{u})|^2)^{\frac{p}{4}}\,, \qquad \bar{\Gamma} := 1 + |\varepsilon(\bar{u})|^2\,.$$

LEMMA 4.1 *With the assumptions and the notation of Lemma 3.4 we have*

$$\int_{B_r(x_0)} |\nabla h|^2 \,\mathrm{d}x \le c\left[(r'-r)^{-2}\int_{B_{r'}(x_0)} \bar{\Gamma}^{\frac{q_0-2}{2}}|\nabla \bar{u} - Q|^2 \,\mathrm{d}x + (r')^{4-\frac{4}{p}}\right].$$

Proof. From Lemma 3.4 it is known that h_δ is uniformly bounded in $W^1_{2,loc}$, this, together with the pointwise a.e. convergence of $\varepsilon(u_\delta)$ established after (4.2) shows that passing to the limit $\delta \to 0$

$$h_\delta \rightharpoondown h \quad \text{in } W^1_{2,loc} \text{ and a.e.}$$

Lower semicontinuity implies that

$$\int_{B_r(x_0)} |\nabla h|^2 \,\mathrm{d}x \le c \liminf_{\delta\to 0}\left[(r'-r)^{-2}\int_{B_{r'}(x_0)} \Gamma_\delta^{\frac{q_0-2}{2}}|\nabla u_\delta - Q|^2 \,\mathrm{d}x + (r')^{4-\frac{4}{p}}\right]. \tag{4.4}$$

From Korn's inequality we infer that

$$\|\nabla u_\delta - \nabla \bar{u}\|_{L^{q_0}(B_{r'})} \le c\Big\{\|u_\delta - \bar{u}\|_{L^{q_0}(B_{r'})} + \|\varepsilon(u_\delta) - \varepsilon(\bar{u})\|_{L^{q_0}(B_{r'})}\Big\}\,, \tag{4.5}$$

where we already know that $\|u_\delta - \bar{u}\|_{q_0} \to 0$ as $\delta \to 0$. Moreover, $|\varepsilon(u_\delta) - \varepsilon(\bar{u})|^{q_0}$ is uniformly bounded in L^{3p/q_0}_{loc}, hence

$$|\varepsilon(u_\delta) - \varepsilon(\bar{u})|^{q_0} \overset{\delta\to 0}{\rightharpoondown:} \vartheta \quad \text{in } L^{\frac{3p}{q_0}}_{loc}\,.$$

The pointwise convergence of $\varepsilon(u_\delta)$ implies $\vartheta = 0$, thus it is shown that

$$\int_{B_{r'}(x_0)} |\varepsilon(u_\delta) - \varepsilon(\bar{u})|^{q_0} \,\mathrm{d}x \overset{\delta\to 0}{\to} 0\,,$$

and, as a consequence of (4.5)

$$\nabla u_\delta \overset{\delta\to 0}{\to} \nabla \bar{u} \ \ a.e. \tag{4.6}$$

In the same way we observe that $\Gamma_\delta^{\frac{q_0-2}{2}}|\nabla u_\delta - Q|^2$ is bounded in L^{3p/q_0}_{loc}, thus

$$\Gamma_\delta^{\frac{q_0-2}{2}}|\nabla u_\delta - Q|^2 \overset{\delta\to 0}{\rightharpoondown:} \tilde{\vartheta} \quad \text{in } L^{\frac{3p}{q_0}}_{loc}\,.$$

Now (4.6) implies $\tilde{\vartheta} = \bar{\Gamma}^{\frac{q_0-2}{2}}|\nabla \bar{u} - Q|^2$, hence

$$\int_{B_{r'}(x_0)} \Gamma_\delta^{\frac{q_0-2}{2}}|\nabla u_\delta - Q|^2 \,\mathrm{d}x \overset{\delta\to 0}{\to} \int_{B_{r'}(x_0)} \bar{\Gamma}^{\frac{q_0-2}{2}}|\nabla \bar{u} - Q|^2 \,\mathrm{d}x$$

which, together with (4.4), proves the lemma. ■

5 Partial regularity in case $n = 3$, $q_0 = 2$

In this section we are going to prove Theorem 1.4. So let us assume that all the hypotheses of this theorem are valid and recall that we already know $\nabla \bar{u} \in L^{3p}_{loc}(\Omega; \mathbb{R}^{3\times 3})$ which means $\bar{u} \in C^{0,1-1/p}(\Omega; \mathbb{R}^3)$. W.l.o.g. we may asume that $\bar{u}$ is Hölder continuous on $\bar{\Omega}$ since otherwise we just restrict the following considerations to balls compactly contained in some subdomain $\Omega' \Subset \Omega$. The proof of Theorem 1.4 is based on a blow-up argument (see Lemma 5.2), and as usual we define the excess of $\bar{u}$ w.r.t. a ball $B_r(x) \Subset \Omega$ as

$$E(x,r) := E(\bar{u}, B_r(x)) := \fint_{B_r(x)} |\varepsilon(\bar{u}) - (\varepsilon(\bar{u}))_{x,r}|^2 \,\mathrm{d}y\,,$$

$(\dots)_{x,r}$ and $\fint_{B_r(x)}$ denoting mean values. We will also make use of a Campanato-type estimate, which can be traced in [GM], a proof is also given in [FS], Lemma 3.0.5, v).

LEMMA 5.1 *Consider a matrix $A \in \mathbb{S}^3$ such that $|A| \le L$. Let $w \in W^1_2(B_1; \mathbb{R}^3)$,* $\operatorname{div} w = 0$, *satisfy*

$$\int_{B_1} D^2 f(A)(\varepsilon(w), \varepsilon(\varphi))\,\mathrm{d}z = 0$$

for all $\varphi \in \overset{\circ}{W}{}^1_2(B_1; \mathbb{R}^3)$, $\operatorname{div} \varphi = 0$. *Then there is a constant $C^* = C(p, L)$ such that*

$$\fint_{B_\tau} |\varepsilon(w) - (\varepsilon(w))_\tau|^2 \,\mathrm{d}z \le C^* \tau^2 \fint_{B_1} |\varepsilon(w) - (\varepsilon(w))_1|^2 \,\mathrm{d}z$$

for any $\tau \in (0,1)$.

REMARK 5.1 *The constant C^* – according to [FS], Lemma 3.0.5, v) – depends on the ellipticity constants of the form $D^2 f(A)$. Since*

$$\lambda (1 + |A|^2)^{\frac{p-2}{2}} |\varepsilon|^2 \le D^2 f(A)(\varepsilon, \varepsilon) \le \Lambda |\varepsilon|^2$$

we get ($p \le 2$)

$$\lambda (1 + L^2)^{\frac{p-2}{2}} |\varepsilon|^2 \le D^2 f(A)(\varepsilon, \varepsilon) \le \Lambda |\varepsilon|^2\,,$$

thus C^* is independent of the particular matrix A.

Now the main lemma of this section reads as

LEMMA 5.2 *Fix a real number $L > 0$, $\mu \in (0, \mu_0)$, $\mu_0 := \frac{3}{2} - \frac{2}{p}$, choose $C^* = C^*(p, L)$ according to Lemma 5.1 and let $C_* := 2C^*$. Then, for any $\tau \in (0, 1/4)$ there exists $\lambda = \lambda(L, \tau)$ such that: if we have for some ball $B_r(x) \Subset \Omega$*

$$|\varepsilon(\bar{u})_{x,r}| \le L \quad \text{and} \quad E(x,r) + r^{2\mu} < \lambda^2\,,$$

then it follows that

$$E(x, \tau r) \le C_* \tau^2 \Big[E(x,r) + r^{2\mu} \Big]\,.$$

From our main lemma we obtain in a standard way the following corollary which of course implies Theorem 1.4 since the complement of Ω_0 is of Lebesgue measure zero.

COROLLARY 5.1 *Denote by* Ω_0 *the set*

$$\Omega_0 := \Big\{ x \in \Omega : \sup_{r>0} |(\varepsilon(\bar u))_{x,r}| < \infty \ \text{ and } \ \liminf_{r\to 0} E(x,r) = 0 \Big\}\,.$$

Then Ω_0 *is open and we have*

$$\bar u \in C^{1,\beta}(\Omega_0;\mathbb{R}^3) \quad \textit{for any } \ 0<\beta<1\,.$$

Let us start with the **Proof of Corollary 5.1.** Following the remarks given after formula (5.11) in [FL] we get from Lemma 5.2 by a well known iteration process

$$\varepsilon(\bar u) \in C^{0,\alpha}(\Omega_0;\mathbb{S}^3) \quad \text{for some exponent } \ 0<\alpha<1\,.$$

Next fix an open set $\omega \Subset \Omega_0$, let $v = \partial_k \bar u$ and observe that analogous to (5.3) of [BF2]

$$\int_\omega D^2 f(\varepsilon(\bar u))(\varepsilon(v),\varepsilon(\varphi))\,\mathrm{d}x = \int_\Omega \partial_k(\bar u \otimes \bar u) : \varepsilon(\varphi)\,\mathrm{d}x + \int_\Omega g\cdot \partial_k \varphi\,\mathrm{d}x \tag{5.1}$$

for any solenoidal $\varphi \in C_0^\infty(\omega;\mathbb{R}^3)$. For $x_0 \in \omega$, $\varepsilon_0 := \varepsilon(\bar u)(x_0)$, $B_R(x_0) \Subset \omega$ we now choose $v_0 \in W_2^1(B_R(x_0;\mathbb{R}^3)$ as the solution w.r.t. the Dirichlet data $v_{|\partial B_R(x_0)}$ of $\operatorname{div} v_0 = 0$ together with

$$\int_{B_R(x_0)} D^2 f(\varepsilon_0)(\varepsilon(v_0),\varepsilon(\varphi))\,\mathrm{d}x = 0 \tag{5.2}$$

for any solenoidal $\varphi \in C_0^\infty(B_R(x_0);\mathbb{R}^3)$. Then it is well known (see, e.g. [GM] or [FS], Lemma 3.0.5, iii)) that for $0<r\le R$

$$\int_{B_r(x_0)} |\nabla v_0|^2\,\mathrm{d}x \le c\Big(\frac{r}{R}\Big)^3 \int_{B_R(x_0)} |\nabla v_0|^2\,\mathrm{d}x \le c\Big(\frac{r}{R}\Big)^3 \int_{B_R(x_0)} |\nabla v|^2\,\mathrm{d}x\,, \tag{5.3}$$

where the last inequality follows from (5.2) and Korn's inequality. As a consequence of (5.3) we immediately obtain

$$\begin{aligned}\int_{B_r(x_0)} |\nabla v|^2\,\mathrm{d}x &\le c\Bigg[\int_{B_r(x_0)} |\nabla v_0|^2\,\mathrm{d}x + \int_{B_R(x_0)} |\nabla v - \nabla v_0|^2 dx\Bigg]\\ &\le c\Bigg[\Big(\frac{r}{R}\Big)^3 \int_{B_R(x_0)} |\nabla v|^2\,\mathrm{d}x + \int_{B_R(x_0)} |\nabla v - \nabla v_0|^2\,\mathrm{d}x\Bigg]\,. \end{aligned}\tag{5.4}$$

The last integral on the right-hand side of (5.4) is handled with the ellipticity condition, Korn's inequality and (5.2)

$$\begin{aligned}\int_{B_R(x_0)} |\nabla v - \nabla v_0|^2\,\mathrm{d}x &\le c\int_{B_R(x_0)} D^2 f(\varepsilon_0)\big(\varepsilon(v)-\varepsilon(v_0),\varepsilon(v)-\varepsilon(v_0)\big)\,\mathrm{d}x\\ &= c\int_{B_R(x_0)} D^2 f(\varepsilon_0)\big(\varepsilon(v),\varepsilon(v)-\varepsilon(v_0)\big)\,\mathrm{d}x\\ &= c\int_{B_R(x_0)} D^2 f(\varepsilon(\bar u))\big(\varepsilon(v),\varepsilon(v)-\varepsilon(v_0)\big)\,\mathrm{d}x\\ &\quad +c\int_{B_R(x_0)} \big[D^2 f(\varepsilon_0) - D^2 f(\varepsilon(\bar u))\big]\big(\varepsilon(v),\varepsilon(v)-\varepsilon(v_0)\big)\,\mathrm{d}x\\ &= I_1 + I_2\,. \end{aligned}\tag{5.5}$$

To proceed further we use (5.1) and Young's inequality for $0 < \gamma$ sufficiently small. This gives for I_1

$$\begin{aligned} I_1 &= c\int_{B_R(x_0)} \partial_k(\bar u \otimes \bar u) : \varepsilon(v - v_0)\,\mathrm{d}x + c\int_{B_R(x_0)} g \cdot \partial_k(v - v_0)\,\mathrm{d}x \\ &\leq \gamma\int_{B_R(x_0)} |\nabla v - \nabla v_0|^2\,\mathrm{d}x + c(\gamma)\|\bar u\|_\infty^2 \int_{B_R(x_0)} |\nabla \bar u|^2\,\mathrm{d}x + c(\gamma)\|g\|_\infty^2 R^3\,. \end{aligned}$$

Moreover, we may estimate

$$\begin{aligned} I_2 &\leq c\,\mathrm{osc}_{B_R(x_0)} D^2 f(\varepsilon(\bar u)) \int_{B_R(x_0)} |\nabla v||\nabla v - \nabla v_0|\,\mathrm{d}x \\ &\leq \gamma\int_{B_R(x_0)} |\nabla v - \nabla v_0|^2\,\mathrm{d}x + c(\gamma)\Big[\mathrm{osc}_{B_R(x_0)} D^2 f(\varepsilon(\bar u))\Big]^2 \int_{B_R(x_0)} |\nabla v|^2\,\mathrm{d}x\,. \end{aligned}$$

For γ small enough we can absorb terms on the left-hand side of (5.5). Moreover, given any $\alpha > 0$, we can calculate $R_0 = R_0(\alpha)$ in such a way that (γ being fixed)

$$c(\gamma)\mathrm{osc}_{B_R(x_0)} D^2 f(\varepsilon(\bar u)) \leq \alpha$$

for all $R \leq R_0(\alpha)$. Here of course continuity of $\varepsilon(\bar u)$ is used. Since we already know that $\nabla \bar u$ is locally of class L^{3p}, we find ($\nu := 1 - 1/p$)

$$\int_{B_R(x_0)} |\nabla \bar u|^2\,\mathrm{d}x \leq cR^{3-\frac{2}{p}} = cR^{1+2\nu}\,.$$

As a result it follows from (3.4) and (3.5)

$$\int_{B_r(x_0)} |\nabla v|^2\,\mathrm{d}x \leq c_1\Big[\Big(\frac{r}{R}\Big)^3 + \alpha\Big]\int_{B_R(x_0)} |\nabla v|^2\,\mathrm{d}x + c_2 R^{1+2\nu} \tag{5.6}$$

for any $r \leq R \leq R_0(\alpha)$. Now choosing α small enough, we get (see [Gi], p. 86, Lemma 2.1) from (3.6) the growth estimate

$$\int_{B_r(x_0)} |\nabla v|^2 \leq cr^{1+2\nu}\,,$$

hence v locally is of class $C^{0,\nu}$ on ω, thus $\bar u \in C^{1,\nu}(\omega; \mathbb{R}^3)$. This gives the better estimate

$$\int_{B_R(x_0)} |\nabla \bar u|^2\,\mathrm{d}x \leq cR^3$$

and (3.6) can be replaced by

$$\int_{B_r(x_0)} |\nabla v|^2\,\mathrm{d}x \leq c_1\Big[\Big(\frac{r}{R}\Big)^3 + \alpha\Big]\int_{B_R(x_0)} |\nabla v|^2\,\mathrm{d}x + c_2 R^3$$

from which the claim follows by a further application of [Gi], p. 86, Lemma 2.1. ∎

For the **Proof of Lemma 5.2** we assume by contradiction that for $L > 0$ fixed and for some $\tau \in (0, 1/4)$ there exists a sequence of balls $B_{r_m}(x_m) \Subset \Omega$ such that

$$\begin{aligned} |(\varepsilon(\bar{u}))_{x_m,r_m}| &< L\,, &(5.7)\\ E(x_m, r_m) + r_m^{2\mu} &=: \lambda_m^2 \overset{m\to\infty}{\to} 0\,, &(5.8)\\ E(x_m, \tau r_m) &> C_* \tau^2 \lambda_m^2\,. &(5.9)\end{aligned}$$

We then consider the scaled sequences

$$\begin{aligned} u_m(z) &= \frac{1}{\lambda_m r_m}\Big[\bar{u}(x_m + r_m z) - r_m A_m z - \gamma_m(z)\Big]\,, \quad |z| < 1\,,\\ A_m &:= (\varepsilon(\bar{u}))_{x_m,r_m}\,, \qquad \gamma_m(z) := B_m z + a_m\,, \end{aligned}$$

where we choose $a_m \in \mathbb{R}^3$ and $B_m \in \mathbb{R}^{3\times 3}$ skew-symmetric such that

$$\int_{B_1} |u_m|^2 \,\mathrm{d}x \le c_1 \int_{B_1} |\varepsilon(u_m)|^2 \,\mathrm{d}x\,. \tag{5.10}$$

From (5.8) we know that

$$\int_{B_1} |\varepsilon(u_m)|^2 \,\mathrm{d}z \le c_2\,, \tag{5.11}$$

hence (5.10) and (5.11) imply together with Korn's inequality

$$u_m \overset{m\to\infty}{\rightharpoonup}: \hat{u} \ \text{ in } \ W^1_2(B_1;\mathbb{R}^3)\,, \qquad \operatorname{div} \hat{u} \equiv 0\,. \tag{5.12}$$

Moreover, (5.7) implies

$$A_m \overset{m\to\infty}{\to}: A \in \mathbb{S}^3\,, \qquad |A| \le L\,, \tag{5.13}$$

whereas (5.9) gives

$$\fint_{B_\tau} |\varepsilon(u_m) - (\varepsilon(u_m))_\tau|^2 \,\mathrm{d}z > C_* \tau^2\,. \tag{5.14}$$

As usual one has to prove at this point (using the convergences (5.12) and (5.13)) that $\hat{u}$ satisfies a suitable limit equation. To this purpose let $\psi \in C^\infty_0(B_1;\mathbb{R}^3)$, $\operatorname{div}\psi = 0$. The scaled version of our equation for $\bar{u}$ then reads as

$$\begin{aligned} &\int_{B_1} Df(\lambda_m \varepsilon(u_m)(z) + A_m) : \varepsilon(\psi)\,\mathrm{d}z\\ &= \ r_m \int_{B_1} \psi(z)\cdot g(x_m + r_m z)\,\mathrm{d}x\\ &\qquad + r_m^{-3} \int_{B_{r_m}(x_m)} \bar{u}(x) \otimes \bar{u}(x) : \varepsilon(\psi)\Big(\frac{x - x_m}{r_m}\Big)\,\mathrm{d}x =: I_1 + I_2\,. \end{aligned}$$

Letting $U(x) := \bar{u}(x) \otimes \bar{u}(x)$ we rewrite this in the following way

$$\begin{aligned} &\lambda_m^{-1} \int_{B_1} \Big[Df(\lambda_m \varepsilon(u_m)(z) + A_m) - Df(A_m)\Big] : \varepsilon(\psi)\,\mathrm{d}z\\ &= \ r_m \lambda_m^{-1} \int_{B_1} \psi(z)\cdot g(x_m + r_m z)\,\mathrm{d}x\\ &\qquad + \lambda_m^{-1} \fint_{B_{r_m}(x_m)} (U(x) - U(x_m)) : \varepsilon(\psi)\Big(\frac{x - x_m}{r_m}\Big)\,\mathrm{d}x =: I_1 + I_2\,. \end{aligned}$$

Here condition (5.8) ensures

$$r_m \lambda_m^{-1} \le \lambda_m^{\frac{1}{\mu}-1} \overset{m\to\infty}{\to} 0\,,$$

hence $I_1 \to 0$ as $m \to \infty$ follows since g is of class L^∞. For I_2 we observe $\bar{u} \in C^{0,\mu_0}(\bar{\Omega};\mathbb{R}^3)$ (on account of $\frac{3}{2} - \frac{2}{p} < 1 - \frac{1}{p}$) and obtain

$$|I_2| \le c r_m^{\mu_0} \lambda_m^{-1} = c\Big(\frac{r_m^\mu}{\lambda_m}\Big) r_m^{\mu_0-\mu} \overset{m\to\infty}{\to} 0\,.$$

Up to now it is proved that

$$\lim_{m\to\infty} \int_{B_1} \lambda_m^{-1}\Big[Df(\lambda_m \varepsilon(u_m) + A_m) - Df(A_m)\Big] : \varepsilon(\psi)\,\mathrm{d}z = 0$$

for any ψ as above. We then proceed exactly as in [BF2] to derive the limit equation

$$\int_{B_1} D^2 f(A)(\varepsilon(\hat{u}), \varepsilon(\psi))\,\mathrm{d}z = 0 \ \text{ for any solenoidal } \ \psi \in C_0^\infty(B_1;\mathbb{R}^3)\,. \tag{5.15}$$

In particular the Campanato inequality of Lemma 5.1 is valid and as in [BF2] we come to a contradiction (recalling (5.14)) provided that we can prove the strong convergence

$$\varepsilon(u_m) \overset{m\to\infty}{\to} \varepsilon(\hat{u}) \quad \text{in } \ L^2_{loc}(B_1;\mathbb{S}^3)\,. \tag{5.16}$$

To this purpose let us fix $\tilde{w} \in \overset{\circ}{W}{}^1_2(B_1;\mathbb{R}^3)$, $\operatorname{div}\tilde{w} = 0$, and consider

$$w(x) := r_m \lambda_m \tilde{w}\Big(\frac{x - x_m}{r_m}\Big)\,, \qquad x \in B_{r_m}(x_m)\,.$$

From Corollary 4.1 we get

$$\begin{aligned}
&\int_{B_{r_m}(x_m)} f\Big(A_m + \lambda_m \varepsilon(u_m)\Big(\frac{x-x_m}{r_m}\Big)\Big)\,\mathrm{d}x \\
&\quad \le \int_{B_{r_m}(x_m)} f\Big(A_m + \lambda_m\Big[\varepsilon(u_m)\Big(\frac{x-x_m}{r_m}\Big) + \varepsilon(\tilde{w})\Big(\frac{x-x_m}{r_m}\Big)\Big]\Big)\,\mathrm{d}x \\
&\qquad - \int_{B_{r_m}(x_m)} \bar{u}\otimes\bar{u} : \varepsilon(w)\,\mathrm{d}x - \int_{B_{r_m}(x_m)} g\cdot w\,\mathrm{d}x\,,
\end{aligned}$$

hence

$$\begin{aligned}
\int_{B_1} f(A_m + \lambda_m \varepsilon(u_m))\,\mathrm{d}z &\le \int_{B_1} f\Big(A_m + \lambda_m\big[\varepsilon(u_m) + \varepsilon(\tilde{w})\big]\Big)\,\mathrm{d}z \\
&\quad - r_m^{-3} \int_{B_{r_m}(x_m)} \bar{u}\otimes\bar{u} : \varepsilon(w)\,\mathrm{d}x - r_m^{-3}\int_{B_{r_m}(x_m)} g\cdot w\,\mathrm{d}x\,.
\end{aligned} \tag{5.17}$$

As in the proof of Proposition 5.2 from [BF2] we now show $(0 < \rho < 1)$

$$\lim_{m\to\infty} \int_{B_\rho} \int_0^1 (1 + |A_m + \lambda_m \varepsilon(\hat{u}) + t\lambda_m \varepsilon(w_m)|^2)^{\frac{p-2}{2}} |\varepsilon(w_m)|^2 (1-t)\,\mathrm{d}t\,\mathrm{d}z = 0\,, \tag{5.18}$$

where we have set $w_m = u_m - \hat{u}$. With the notation of [BF2] we let $\tilde{w} = \varphi[\hat{u} - u_m] - \varphi_m$. Using (5.17) we argue exactly as in [BF2], where we now have to consider the additional terms

$$-r_m^{-3}\lambda_m^{-2}\int_{B_{r_m}(x_m)} \bar{u}\otimes\bar{u} : \varepsilon(w)\,\mathrm{d}x - \lambda_m^{-2} r_m^{-3}\int_{B_{r_m}(x_m)} g\cdot w\,\mathrm{d}x$$

on the right-hand side of [BF2], formula (5.23). We first observe that

$$\begin{aligned} -\lambda_m^{-2} r_m^{-3}\int_{B_{r_m}(x_m)} g\cdot w\,\mathrm{d}x &= r_m\lambda_m^{-1}\int_{B_1} g(x_m + r_m z)\cdot\tilde{w}(z)\,\mathrm{d}z \\ &\to 0 \quad \text{as } m\to\infty \end{aligned}$$

since $r_m\lambda_m^{-1}\to 0$ as $m\to\infty$. Moreover, we may estimate

$$\begin{aligned} &\left| r_m^{-3}\lambda_m^{-2}\int_{B_{r_m}(x_m)} \bar{u}\otimes\bar{u} : \varepsilon(w)\,\mathrm{d}x \right| \\ &\quad\le r_m^{-3}\lambda_m^{-2}\int_{B_{r_m}(x_m)} |U(x) - U(x_m)||\varepsilon(w)|\,\mathrm{d}x \\ &\quad\le c r_m^{\mu_0}\lambda_m^{-1}\int_{B_1} |\varepsilon(\tilde{w})|\,\mathrm{d}x \overset{m\to\infty}{\to} 0\,. \end{aligned}$$

As a result, (5.18) can be established in an analogous way as outlined in [BF2]: letting

$$\psi_m := \lambda_m^{-1}\left[(1 + |A_m + \lambda_m\varepsilon(u_m)|^2)^{\frac{p}{4}} - (1 + |A_m|^2)^{\frac{p}{4}}\right]$$

it finally remains to prove

$$\int_{B_\rho} |\nabla\psi_m|^2\,\mathrm{d}z \le c(\rho)\,. \tag{5.19}$$

But from Lemma 4.1 we get

$$\begin{aligned} \int_{B_\rho} |\nabla\psi_m|^2\,\mathrm{d}z &= \lambda_m^{-2} r_m^{-1}\int_{B_{\rho r_m}(x_m)} |\nabla\{(1 + |\varepsilon(u)|^2)^{p/4}\}|^2\,\mathrm{d}x \\ &\le c(\rho)\lambda_m^{-2} r_m^{-3}\int_{B_{r_m}(x_m)} |\nabla\bar{u} - Q|^2\,\mathrm{d}x \\ &\quad + c\lambda_m^{-2} r_m^{-1} r_m^{4-\frac{4}{p}}\,, \end{aligned}$$

and if we choose $Q = A_m + r_m^{-1}B$ (recall the definition of u_m) the boundedness of

$$c(\rho)\lambda_m^{-2} r_m^{-3}\int_{B_{r_m}(x_m)} |\nabla\bar{u} - Q|^2\,\mathrm{d}x$$

is immediate. Finally, our choice of μ gives

$$\lambda_m^{-2} r_m^{3-\frac{4}{p}} = \frac{r_m^{2\mu}}{\lambda_m^2} r_m^{3-\frac{4}{p}-2\mu} \overset{m\to\infty}{\to} 0\,,$$

thus (5.19) is established. Now we can follow [BF2], proof of Proposition 5.2, Case 2, to get (5.16) which completes the proof of Lemma 5.2. ■

REMARK 5.2 *If $q_0 > 2$ (together with (1.4) and $n = 3$) then Lemma 5.2 has to be modified according to Lemma 5.2 of [BF2] which makes things more technical but the result will be the extension of Theorem 1.4 under condition (1.4). If for some reason the reader is directly interested in the general case "$n = 3$ + (1.4)", some aspects can be simplified: one starts with the observation that w.l.o.g. q_0 may be assumed to be larger than 3 (so the right-hand side of (1.3) may be enlarged still having (1.4)). Then the regularization f_δ is defined with respect to the exponent q_0 (no additional q occurs) and all results up to Section 4 remain valid. But, as mentioned before, the formulation and the proof of Lemma 5.2 are more involved since we now have to take care of the additional excess-type quantity*

$$\fint_{B_r(x)} |\varepsilon(\bar{u}) - (\varepsilon(\bar{u}))_{x,r}|^{q_0}\, \mathrm{d}x\,.$$

6 The case $n = 2$, $q_0 = 2$

In this section we consider the case $n = 2$ together with $q_0 = 2$, i.e. we are going to prove Theorem 1.5. Let us note that in contrast to Section 5 it is not evident how to generalize the result to the case $q_0 > 2$. A powerful tool for proving full regularity in two dimensions is a lemma due to Frehse and Seregin (see Lemma 4.1 of [FrS]). Our main task in this section is (compare Section 6 of [BF2]) to verify the assumptions of the Frehse-Seregin Lemma. To this purpose we recall Corollary 3.1, i.e.

$$\bar{u} \in W^2_{2-\varepsilon,loc}(\Omega;\mathbb{R}^2) \quad \text{for any } \varepsilon > 0\,. \tag{6.1}$$

Now let us fix two discs $B_{2r}(\bar{x}) \Subset B_{2R}(x_0) \Subset \Omega$ and choose $\eta \in C_0^\infty(B_{2r}(\bar{x}))$, $\eta \equiv 1$ on $B_r(\bar{x})$, $|\nabla \eta| \le c/r$. Using (6.1) it is proved in [BF4] that we may pass to the limit $\delta \to 0$ in (3.5). Moreover, it is obvious that on the right-hand side of the resulting inequality $\bar{u}$ can be replaced by $\bar{u} - Qx$ for some arbitrary matrix $Q \in \mathbb{R}^{2\times 2}$ (see again [BF4] for details). If we let $\bar{\sigma} := Df(\varepsilon(\bar{u}))$, then we end up with

$$\begin{aligned}\int_{B_{2r}(\bar{x})} \eta^2 \partial_k \bar{\sigma} : \partial_k \varepsilon(\bar{u})\, \mathrm{d}x \;\; &\le \;\; -2\int_{B_{2r}(\bar{x})} \eta \partial_k \bar{\sigma} : \big[\nabla \eta \odot \partial_k[\bar{u} - Qx]\big]\, \mathrm{d}x \\ &\quad + \int_{B_{2r}(\bar{x})} \partial_k[\bar{u} \otimes \bar{u}] : \varepsilon\big(\eta^2 \partial_k[\bar{u} - Qx]\big)\, \mathrm{d}x \\ &\quad - \int_{B_{2r}(\bar{x})} g \cdot \partial_k\big[\eta^2 \partial_k[\bar{u} - Q]\big]\, \mathrm{d}x\,.\end{aligned}$$

Letting $T_{2r}(\bar{x}) := B_{2r}(\bar{x}) - B_r(\bar{x})$ and

$$\begin{aligned} H &:= \big[D^2 f(\varepsilon(\bar{u}))(\partial_k \varepsilon(\bar{u}), \partial_k \varepsilon(\bar{u}))\big]^{\frac{1}{2}}\,, \\ h &:= (1 + |\varepsilon(\bar{u})|^2)^{\frac{p}{4}}\,, \end{aligned}$$

we obtain in the same way as outlined in [BF2], Section 6,

$$\begin{aligned}\int_{B_{2r}(\bar{x})} \eta^2 H^2\,\mathrm{d}x \;\le\; & \frac{c}{r}\left[\int_{T_{2r}(\bar{x})} H^2\,\mathrm{d}x\right]^{\frac{1}{2}} \int_{T_{2r}(\bar{x})} hH\,\mathrm{d}x \\ & + \int_{B_{2r}(\bar{x})} \partial_k[\bar{u}\otimes\bar{u}] : \varepsilon\big(\eta^2\partial_k[\bar{u}-Qx]\big)\,\mathrm{d}x \\ & - \int_{B_{2r}(\bar{x})} g\cdot\partial_k\left[\eta^2\partial_k[\bar{u}-Q]\right]\mathrm{d}x \\ & -2\int_{B_{2r}(\bar{x})} \eta\partial_k\bar{p}\mathbf{1} : [\nabla\eta\odot\partial_k(\bar{u}-Qx)]\,\mathrm{d}x\,, \end{aligned} \tag{6.2}$$

where $\bar{p}$ denotes the pressure function related to the limit problem. Since $\bar{u}$ is locally bounded, it follows for $\alpha > 0$ sufficiently small ($\bar{\Gamma} := 1 + |\varepsilon(\bar{u})|^2$, $\tilde{\Gamma} := 1 + |\nabla\bar{u}|^2$)

$$\begin{aligned}\int_{B_{2r}(\bar{x})} \partial_k[\bar{u}\otimes\bar{u}]\eta^2\varepsilon(\partial_k\bar{u})\,\mathrm{d}x \;\le\; & c\int_{B_{2r}(\bar{x})} |\partial_k\bar{u}|\eta^2\bar{\Gamma}^{\frac{2-p}{4}}\bar{\Gamma}^{\frac{p-2}{4}}|\varepsilon(\partial_k\bar{u})|\,\mathrm{d}x \\ \le\; & \alpha\int_{B_{2r}(\bar{x})} \eta^2\bar{\Gamma}^{\frac{p-2}{2}}\varepsilon(\partial_k\bar{u}) : \varepsilon(\partial_k\bar{u})\,\mathrm{d}x \\ & + c(\alpha)\int_{B_{2r}(\bar{x})} \eta^2|\nabla\bar{u}|^2\bar{\Gamma}^{\frac{2-p}{2}}\,\mathrm{d}x \\ \le\; & c\alpha\int_{B_{2r}(\bar{x})} \eta^2 H^2\,\mathrm{d}x + c(\alpha)\int_{B_{2r}(\bar{x})} \tilde{\Gamma}^{\frac{4-p}{2}}\,\mathrm{d}x\,. \end{aligned} \tag{6.3}$$

Since $|\nabla\bar{u}|$ is locally summable w.r.t. any power we get

$$\int_{B_{2r}(\bar{x})} \tilde{\Gamma}^{\frac{4-p}{2}}\,\mathrm{d}x \le c(\alpha_1)r^{\alpha_1} \quad \text{for any exponent } \alpha_1 < 2\,.$$

Next we estimate

$$\begin{aligned}& \int_{B_{2r}(\bar{x})} \partial_k[\bar{u}\otimes\bar{u}] : \nabla\eta^2\odot\partial_k[\bar{u}-Qx]\,\mathrm{d}x \\ & \le\; c\int_{T_{2r}(\bar{x})} \eta|\nabla\eta||\nabla\bar{u}||\nabla\bar{u}-Q|\,\mathrm{d}x \\ & \le\; cr^{-1}\left[\int_{T_{2r}(\bar{x})} |\nabla\bar{u}|^2\,\mathrm{d}x\right]^{\frac{1}{2}}\left[\int_{T_{2r}(\bar{x})} |\nabla\bar{u}-Q|^2\,\mathrm{d}x\right]^{\frac{1}{2}} \\ & \le\; cr^{-1}r^{\alpha_2}\int_{T_{2r}(\bar{x})} hH\,\mathrm{d}x\,, \quad \alpha_2 < 1\,, \end{aligned} \tag{6.4}$$

where the last inequality follows as in [BF2] (compare the calculations after (6.3) in [BF2] for handling $\int_{T_{2r}(\bar{x})} |\nabla\bar{u}-Q|^2\,\mathrm{d}x$). Moreover, higher integrability is used again. Let us consider the volume force term: we have

$$\begin{aligned}\left|\int_{B_{2r}(\bar{x})} g\cdot\partial_k\big(\eta^2\partial_k[\bar{u}-Qx]\big)\,\mathrm{d}x\right| \;\le\; & c\left[\int_{B_{2r}(\bar{x})} |\nabla\eta||\nabla\bar{u}-Q|\,\mathrm{d}x\right. \\ & \left. + \int_{B_{2r}(\bar{x})} \eta^2|\nabla\varepsilon(\bar{u})|\,\mathrm{d}x\right] =: c\{T_1+T_2\}\,,\end{aligned}$$

where

$$T_2 \le \gamma \int_{B_{2r}(\bar{x})} \eta^2 \bar{\Gamma}^{\frac{p-2}{2}} |\nabla \varepsilon(\bar{u})|^2 \, \mathrm{d}x + c(\gamma) \int_{B_{2r}(\bar{x})} \eta^2 \bar{\Gamma}^{\frac{2-p}{2}} \, \mathrm{d}x$$

(recall $|\nabla^2 \bar{u}| \le c|\nabla \varepsilon(\bar{u})|$). For $\gamma > 0$ sufficiently small, the first integral on the right-hand side may be absorbed on the left-hand side of (6.2) and the second one is of growth order r^{α_1} for any $\alpha_1 < 2$. Moreover,

$$\begin{aligned} T_1 &\le cr^{-1} \left[\int_{T_{2r}(\bar{x})} |\nabla \bar{u} - Q|^2 \, \mathrm{d}x \right]^{\frac{1}{2}} |B_{2r}(\bar{x})|^{\frac{1}{2}} \\ &\le cr^{-1} \int_{T_{2r}(\bar{x})} hH \, \mathrm{d}x \, r^{\alpha_2} . \end{aligned}$$

Let us finally discuss the pressure term in (6.2):

$$\begin{aligned} &\left| \int_{B_{2r}(\bar{x})} \eta \partial_k \bar{p} \mathbf{1} : [\nabla \eta \odot \partial_k (\bar{u} - Qx)] \, \mathrm{d}x \right. \\ &\le c \int_{B_{2r}(\bar{x})} \eta |\nabla \bar{\sigma}| |\nabla \eta| |\nabla \bar{u} - Q| \, \mathrm{d}x \\ &\quad + c \int_{B_{2r}(\bar{x})} \eta |\nabla (\bar{u} \otimes \bar{u})| |\nabla \eta| |\nabla \bar{u} - Q| \, \mathrm{d}x \\ &\quad + c \int_{B_{2r}(\bar{x})} \eta |g| |\nabla \eta| |\nabla \bar{u} - Q| \, \mathrm{d}x =: T_1 + T_2 + T_3 . \end{aligned}$$

Since $\bar{u}$ is uniformly bounded, T_2 (and of course T_3) is estimated in the same way as outlined in (6.4). We further have

$$\begin{aligned} T_1 &\le cr^{-1} \left[\int_{T_{2r}(\bar{x})} |H|^2 \, \mathrm{d}x \right]^{\frac{1}{2}} \left[\int_{T_{2r}(\bar{x})} |\nabla \bar{u} - Q|^2 \, \mathrm{d}x \right]^{\frac{1}{2}} \\ &\le cr^{-1} \left[\int_{T_{2r}(\bar{x})} |H|^2 \, \mathrm{d}x \right]^{\frac{1}{2}} \int_{T_{2r}(\bar{x})} hH \, \mathrm{d}x . \end{aligned}$$

Combining this inequality with (6.2)–(6.4) we end up with

$$\int_{B_r(\hat{x})} H^2 \, \mathrm{d}x \le c_1 r^{-1} \left[\int_{T_{2r}(\bar{x})} H^2 \, \mathrm{d}x + r^\alpha \right]^{\frac{1}{2}} \int_{T_{2r}(\bar{x})} hH \, \mathrm{d}x + c_2 r^\beta \tag{6.5}$$

being valid for any exponents α, $\beta < 2$. Note that the additional term $c_2 r^\beta$ on the right-hand side of (6.5) is rapidly decreasing and completely irrelevant for the proof of Lemma 4.1 of [FrS]. To be more precise, inequality (A 3.6) of [FrS] now reads (using the notation of [FrS])

$$\int_{B_r(\bar{x})} H^2 \, \mathrm{d}x \le C_3 \left[\sqrt{\log_2 \left(\frac{2R}{r} \right)} \int_{T_{2r}(\bar{x})} H^2 \, \mathrm{d}x + r^\alpha \sqrt{\log_2 \left(\frac{2R}{r} \right)} \right] + C_4 r^\beta$$

which is an immediate consequence of inequality (6.5). If we choose $\beta > \alpha$, then clearly the latter estimate reduces to the original form of (A 3.6) in [FrS]. Hence we may apply the lemma with the result

$$\int_{B_r(\bar{x})} H^2 \, \mathrm{d}x \le K(t) |\ln r|^{-t} \quad \text{for any} \quad t > 1 \,. \tag{6.6}$$

Observing $|\nabla \bar{\sigma}| \le cH$, (6.6) together with the version of the Dirichlet-Growth Theorem given in [Fre], p. 287, implies the continuity of $\bar{\sigma} = Df(\varepsilon(\bar{u}))$. Now we can proceed as in the proof of Corollary 5.1 to get the statement of Theorem 1.5. ∎

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