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Bernhard Burgeth and Joachim Weickert

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Bernhard Burgeth
Mathematical Image Analysis Group
Saarland University
Department of Mathematics and Computer Science
Postfach 15 11 50
D–66041 Saarbrücken
Germany
burgeth@dmia.uni-saarland.de

Joachim Weickert
Mathematical Image Analysis Group
Saarland University
Department of Mathematics and Computer Science
Postfach 15 11 50
D–66041 Saarbrücken
Germany
weickert@mia.uni-saarland.de

Edited by
FR 6.1 – Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-Mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

Abstract

Dorst/van den Boomgaard and Maragos introduced the slope transform as the morphological equivalent of the Fourier transform. It formed the basis of a morphological system theory that bears an almost logarithmic relation to linear system theory. This surprising logarithmic connection, however, has not been understood so far.

Our article provides an explanation by revealing that morphology in essence is linear system theory in specific algebras. While linear system theory uses the standard plus-prod algebra, morphological system theory is based on the max-plus algebra and the min-plus algebra. We identify the nonlinear operations of erosion and dilation as linear convolutions in the latter algebras. For the subsequent theoretical analysis, it is advantageous to focus on two concepts from convex analysis: We consider the conjugacy operation and the multivariate Laplace transform instead of the closely related slope and Fourier transforms. While the Laplace transform maps convolution into multiplication, the conjugacy operation turns erosion into addition. This logarithmic connection triggers us to consider the logarithmic Laplace transform. The logarithmic Laplace transform in the plus-prod algebra corresponds to the conjugacy operation in the max-plus algebra. Its conjugate is given by the so-called Cramer transform. Originating from the theory of large deviations in stochastics, the Cramer transform maps Gaussians to quadratic functions and relates standard convolution to erosion. This fundamental transform constitutes the direct link between linear and morphological system theory. Many numerical examples are presented that illustrate the convexifying and smoothing properties of the Cramer transform.

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Key Words: linear system theory, morphology, convex analysis, max-plus algebra, min-plus algebra, slope transform, Cramer transform.

Contents

1 Introduction

Linear system theory is a successful and well established field in signal and image processing [8, 15, 16, 30]. In the n-dimensional case, shift invariant linear filters can be described as convolutions of some signal $f : \mathbb{R}^n \to \mathbb{R}$ with a kernel function $b : \mathbb{R}^n \to \mathbb{R}$:

$$(f * b)(x) := \int_{\mathbb{R}^n} f(x-y)\, b(y)\, dy.$$

Let $\langle \cdot, \cdot \rangle$ denote the Euclidean scalar product. By means of the *Fourier transform*

$$\hat{f}(u) := \mathcal{F}[f](u) := \int_{\mathbb{R}^n} f(x)\, e^{-i2\pi\langle u,x\rangle}\, dx$$

and its backtransformation

$$\mathcal{F}^{-1}[g](x) := \int_{\mathbb{R}^n} g(u)\, e^{i2\pi\langle u,x\rangle}\, du$$

one may conveniently compute a convolution in the spatial domain via a simple product in the Fourier domain:

$$\mathcal{F}[f * b] \;=\; \mathcal{F}[f] \,\cdot\, \mathcal{F}[b]\,.$$

In this context, *Gaussians*

$$K_\sigma(x) \;:=\; \frac{1}{(2\pi\sigma^2)^{n/2}}\, e^{-\frac{\langle u,x\rangle}{2\sigma^2}}$$

play an important role as convolution kernels: They are the only separable and rotationally invariant function that preserve their shape under the Fourier transform. Convolutions of a signal f with the family $\{K_\sigma \mid \sigma > 0\}$ of Gaussians create the *Gaussian scale-space* [21, 38, 40], a multiscale representation that is useful in pattern recognition, image processing and computer vision [12, 22, 25, 36]. Figure 1(a) shows an example.

Mathematical morphology is an interesting nonlinear alternative to linear systems theory [17, 28, 33, 34, 35]. It has been applied successfully to a large number of fields including cell biology, computer-aided quality control, mineralogy, remote sensing and medical imaging. Morphology is based on two fundamental processes: dilation and erosion. In the case of nonflat morphology, the *dilation* resp. *erosion* of some function $f : \mathbb{R}^n \to \mathbb{R}$ with a structuring function $b : \mathbb{R}^n \to \mathbb{R}$ can be defined as follows (see e.g. [26, 37]):

$$\begin{aligned}(f \oplus b)(x) &:= \sup \{f(y) + b(x-y) \mid y \in \mathbb{R}^n\}, \\ (f \ominus b)(x) &:= \inf \{f(y) - b(y-x) \mid y \in \mathbb{R}^n\}.\end{aligned}$$

Dorst and van den Boomgaard [10] and Maragos [26] developed independently and simultaneously a morphological system theory that closely resembles linear system theory. Following [10], one may generalise the dilation to the *tangential dilation* via

$$(f \check{\oplus} b)(x) := \operatorname*{stat}_y \Big(f(y) + b(x-y) \Big)$$

where $\operatorname*{stat}_y f(y) := \{f(z) \mid \nabla f(z) = 0\}$ denotes the stationary values of f. Then the morphological equivalent to the Fourier transform is given by the *slope transform*

$$\mathcal{S}[f](u) := \operatorname*{stat}_x (f(x) - \langle u, x \rangle),$$

a transformation that is closely related to the conjugacy operation in convex analysis [19]. Its backtransformation is given by

$$\mathcal{S}^{-1}[g](x) = \operatorname*{stat}_u (g(u) + \langle u, x \rangle).$$

The slope transform allows to replace the tangential dilation by simple addition in the slope domain:

$$\mathcal{S}[f \check{\oplus} b] = \mathcal{S}[f] + \mathcal{S}[b] .$$

Paraboloids

$$b(x,t) = -\frac{\langle x, x \rangle}{4t} \qquad (t > 0)$$

are those structuring functions in morphological system theory that play a role comparable to Gaussians in linear system theory [37]: They are the only rotationally invariant and separable structuring functions that maintain their shape under the slope transformation. The corresponding dilation and erosion scale-spaces are depicted in Figure 1(b) and (c). For a detailed analysis

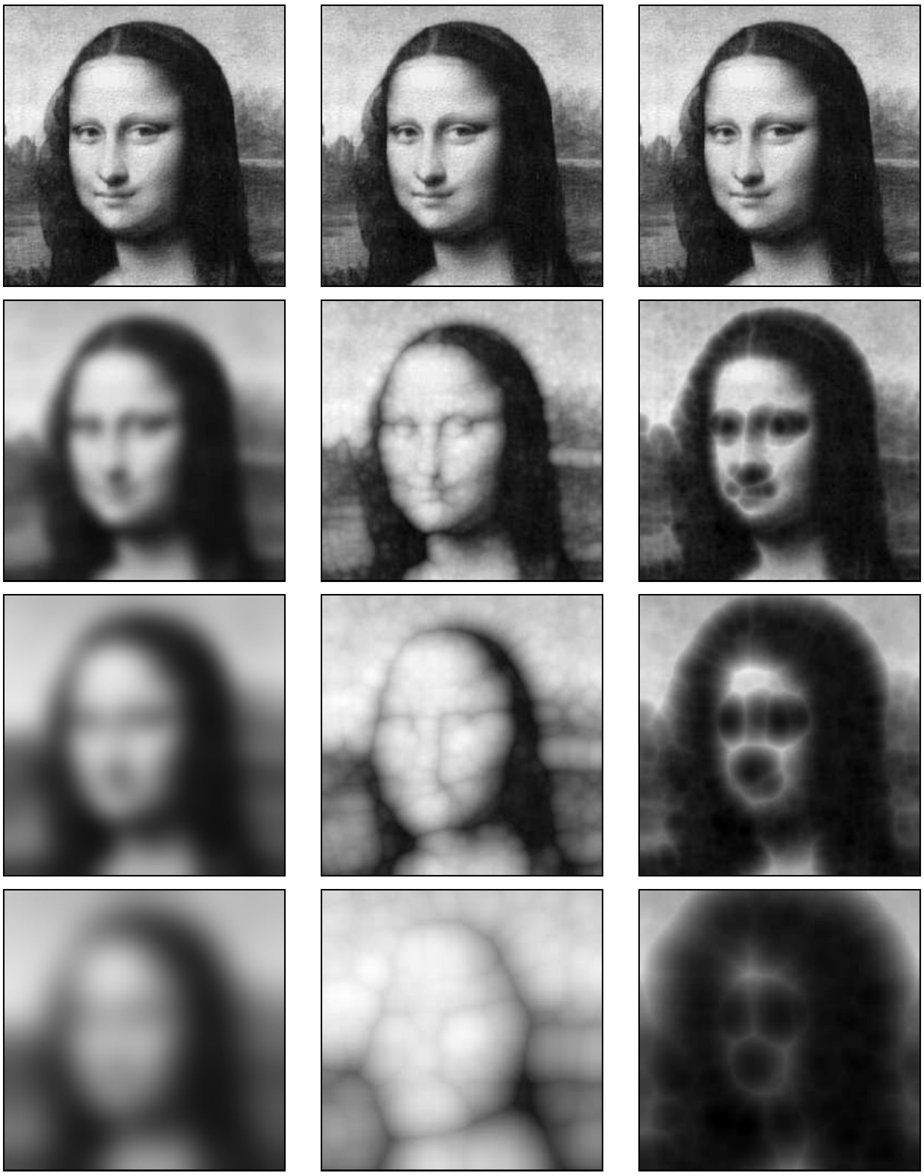

Figure 1: Linear and morphological scale-spaces. **Top:** Mona Lisa painting by Leonardo da Vinci, 256×256 pixels. **(a) Left Column:** Gaussian scale-space, top to bottom: $\sigma = 0$, 5, 10, 15. **(b) Middle Column:** Dilation scale-space with quadratic structuring function, $t = 0$, 0.25, 1, 4. **(c) Right Column:** Erosion scale-space with quadratic structuring function, $t = 0$, 0.25, 1, 4.

Table 1: Linear System Theory vs. Morphological System Theory

	Linear theory	Morphological theory
Canonical transform	$\mathcal{F}[f](u)$	$\mathcal{S}[f](u)$
"Convolution" theorem	$\mathcal{F}[f * b] \;=\; \mathcal{F}[f] \cdot \mathcal{F}[b]$	$\mathcal{S}[f \check{\oplus} b] \;=\; \mathcal{S}[f] + \mathcal{S}[b]$
Canonical kernel	$\frac{1}{(2\pi\sigma^2)^{n/2}}\, e^{-\frac{\langle x,x\rangle}{2\sigma^2}}$	$-\frac{\langle x,x\rangle}{4t}$

of their scale-space properties, we refer to Jackway and Deriche [24]. Morphological scale-spaces with paraboloids as structuring functions are useful for computing Euclidean distance transformations [37], for image enhancement [32] and for multiscale segmentation [23].
The connections mentioned above are displayed in Table 1.
This table suggests that there an almost logarithmic connection between linear and morphological system theory. The structural similarities between linear and morphological processes have triggered Florack *et al.* [13, 14] and Welk [39] to construct parameterised processes that incorporate Gaussian scale-space and both types of morphological scale-spaces as limiting cases. Heijmans and van den Boomgaard [18, 19] have investigated unifying algebraic definitions of scale-space concepts that include a number of linear and morphological approaches (cf. also [2]).
However, in spite of these very interesting contributions, the reason for the almost logarithmic connection between linear and morphological systems has not been discovered so far. To address this problem is the topic of the present paper.

We provide an explanation for the structural analogies between linear and morphological systems by revealing that morphology in essence is linear system theory in a specific algebra. While classical linear system theory uses the standard plus-prod algebra, the morphological system theory is based on the max-plus algebra and the min-plus algebra. This allows us to identify the nonlinear operations of erosion and dilation as linear convolutions $*_e$ and $*_d$ induced by these algebras. In this sense, morphology may be regarded as linear system theory in disguise.
These algebraic structures have already numerous interesting applications in other fields [4]: so-called *discrete event dynamic systems (DEDS)* can be modeled as *linear* systems with respect to these algebras. Discrete event dynamic systems in this algebraic formulation are used to find shortest paths in networks or to solve scheduling and communication problems in abstract project management, for instance. They are also employed to analyse queuing systems, traffic flow and the performance of special array processors. To the best of our knowledge, however, no attempt has been made so far to tackle

problems from image analysis with this special algebraic approach.

Our paper is organised as follows. In Section 2 we introduce the max-plus and min-plus algebras that will play a fundamental role for the analysis of morphological systems. In Section 3 we show that dilation and erosion are convolutions in these algebras. Important concepts from convex analysis, in particular the conjugacy operation, are explained in Section 4. The re-interpretation of the conjugacy operation, viewed as a morphological operation, leads us to its counterpart in linear systems theory: the logarithmic Laplace transform which is introduced in Section 5. It provides an explanation for the logarithmic connection between linear and morphological systems. Another logarithmic connection is established in Section 6, where we investigate a continuous transition between convolution and morphological operations. Section 7 is devoted to the Cramer transform as the conjugate of the logarithmic Laplace transform. It constitutes a homeomorphism between linear and morphological systems. We discuss its basic properties and present a number of examples. Finally we conclude our paper with a summary in Section 8.

A preliminary version of the present paper has been presented in [7].

2 The Max-Plus and Min-Plus Algebras

The theory of discrete event dynamic systems (DEDS) is to a great extend based on two algebraic structures: the max-plus algebra $\mathbb{R}_{max}$ and, isomorphic to it, the min-plus algebra $\mathbb{R}_{min}$ [4]. Mathematical models of these systems are in general nonlinear if considered in the standard ubiquitous plus-prod algebra $(\mathbb{R}, +, \cdot)$. However, reformulating such models in terms of these new algebras transforms them into linear models. This idea of "linearisation" we are going to carry over to morphology. Formally the new algebras emerge from the standard plus-prod algebra first by an extension of the real line with either the element $-\infty$ or $+\infty$, second by replacing the addition by a max- or min-operation, and the multiplication by addition. This is captured in Table 2:

It should be noted that the max-plus and min-plus algebras are no algebras in the classical sense since they lack an inverse with respect to the max- or min-operations. For a rather exhaustive amount of details, the reader is referred to [4] and the literature cited there. The importance of these algebraic structures for our purpose will become clear in the next section, where we consider convolution operations based on these algebras.

Table 2: The Definition of the Max-Plus and the Min-Plus Algebra

	set	addition	multiplication
plus-prod algebra $\mathbb{R}$	$\mathbb{R}$	$+$	$\times$
max-plus algebra $\mathbb{R}_{max}$	$\mathbb{R} \cup \{-\infty\}$	max	$+$
min-plus algebra $\mathbb{R}_{min}$	$\mathbb{R} \cup \{+\infty\}$	min	$+$

3 Morphology as Linear System Theory in Another Algebra

The goal of the current section is to represent the basic morphological operations dilation and erosion as convolutions with respect to max-plus and the min-plus algebras. To do so, we first have to clarify the algebraic background of convolutions.

Roughly speaking, convolving two scalar-valued functions is averaging one with the translated version of the other. More precisely, the usual convolution $*$ is determined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y) \cdot g(y) \; dy,$$

for all $x \in \mathbb{R}^n$. With its use of multiplication and integration, it is based on the the standard algebra $(\mathbb{R}, +, \times)$. We give a new meaning to the term "averaging" by equipping the range of the functions with the algebraic structure introduced above, that is, we consider functions

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}_{max} \quad \text{or} \quad f : \mathbb{R}^n \longrightarrow \mathbb{R}_{min}.$$

Now the transition, indicated by $\rightleftharpoons$, from the standard algebra to the other algebras

$$(\mathbb{R}, +, \times) \quad \rightleftharpoons \quad (\mathbb{R} \cup \{+\infty\}, \min, +) \quad \text{or} \quad (\mathbb{R} \cup \{-\infty\}, \max, +)$$

amounts to the replacement of integration (=summation) by taking the infimum or the supremum, and the replacement of multiplication by addition. This gives rise to two analogs to the convolution $*$:

$$\begin{aligned}(f *_d g)(x) &:= \sup_{y \in \mathbb{R}^n} \big(f(x-y) + g(y) \big) = \sup_{y \in \mathbb{R}^n} \big(f(y) + g(x-y) \big), \\ (f *_e g)(x) &:= \inf_{y \in \mathbb{R}^n} \big(f(x-y) + g(y) \big) = \inf_{y \in \mathbb{R}^n} \big(f(y) + g(x-y) \big).\end{aligned}$$

Hence the morphological operations of dilation $\oplus$ and erosion $\ominus$ as given in [10] or [27] appear as convolutions w.r.t. these algebras:

$$
\begin{aligned}
(f \oplus g)(x) &= \sup_{y\in\mathbb{R}^n} (f(y) + g(x-y)) = f *_d g(x)\,, && (1)\\
(f \ominus g)(x) &= \inf_{y\in\mathbb{R}^n} (f(y) - g(y-x)) = f *_e \overline{g}(x)\,, && (2)
\end{aligned}
$$

with $\overline{g}(x) := -g(-x)$. This also explains the chosen notations $*_e$ and $*_d$. It should be noted that the operation $*_e$ is not only known in morphology: It coincides exactly with the so-called *inf-convolution* or *epigraphic addition* in convex analysis [20, 31] .
For our further considerations it is worthwhile to pursue an excursion into convex analysis. This shall be done next.

4 Tools from Convex Analysis

In this section we will introduce two useful transformations: the conjugacy operation from convex analysis and the multivariate Laplace transform. They may serve as alternatives to the slope transform and the Fourier transform.
Let $\overline{\mathrm{Conv}}\,\mathbb{R}^n$ be the set of extended functions $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ which are closed convex, that is, convex, finite in at least one point and lower semicontinuous. *Lower semicontinuous* functions are precisely those that can be approximated from below by a sequence of continuous functions. This function class is still to large for our purpose, the proper definition of the conjugacy operation. We say that f has an *affine minorant* if and only if $f \geq \langle t', \cdot\rangle - c$ for some $(t', c) \in \mathbb{R}^n \times \mathbb{R}$, where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^n$. Then the convolution $f *_e g$ of two convex functions f, g that have a common affine minorant is again convex.
The *conjugacy operation (Legendre-Fenchel transform)* associates with each f with an affine minorant the function f^* defined by

$$f^*(x) := \sup_{t\in\mathbb{R}^n} [\langle t, x\rangle - f(t)]\,.$$

Remarkably, $f^* \in \overline{\mathrm{Conv}}\,\mathbb{R}^n$ as soon as f is affinely minorised, regardless of its convexity or closedness [20]. In morphology this transform is a variant of the slope transform [10, 26]. However, it offers the advantage that it generates (extended) real-valued functions, while the slope transform in general produces set-valued functions as output.
In order to clarify the algebraic properties of the conjugacy operation we mention the basic properties of the $*_e$-convolution: The $*_e$-convolution is an

associative, commutative, order-preserving binary operation. Defining for any subset $A \in \mathbb{R}^n$

$$i_A(x) := \begin{cases} 0 & x \in A\,, \\ +\infty & \text{otherwise}\,, \end{cases} \tag{3}$$

the indicator function $i_{\{0\}}$ is recognised as the neutral element with respect to this operation. This function corresponds to the structural element in [10, 24].

The following proposition sheds some light on the invertibility and algebraic properties of the conjugacy operation with respect to $*_e$.

Proposition 1. (Properties of the Conjugacy Operation I)
Let $f, g \in \overline{Conv}\,\mathbb{R}^n$. Then the following properties hold:

1. *$\overline{Conv}\,\mathbb{R}^n$ is invariant under conjugation:* $\quad f^* \in \overline{Conv}\,\mathbb{R}^n$.
2. *The conjugacy operation is its own inverse:* $\quad (f^*)^* = f$.
3. *It maps erosions into sums:* $\quad (f *_e g)^* = f^* + g^*$.

For proofs of these assertions and more detailed results on the properties of conjugation, the reader is referred to [20].

The fact that the conjugacy operation maps erosions into sums resembles the well-known property of the Fourier transform which turns convolution into multiplication. Since the Fourier transform also maps multiplication into convolution it would be interesting to know whether the conjugacy operation transforms sums into erosions. To this end we have to introduce some additional technical definitions first.

In general, semicontinuity is not preserved under $*_e$-convolution, that is, $f *_e g$ is not lower semicontinuous even if f and g are. However, if a very mild so-called qualification condition is fulfilled, semicontinuity is guaranteed. For its formulation we denote by the domain of f the set of argument values where f is finite:

$$\operatorname{dom} f := \{x \in \mathbb{R}^n \,:\, f(x) < +\infty\}.$$

Furthermore, we need the notion of *relative interior* of a set $C \subset \mathbb{R}^n$. Without going into details we characterise the relative interior $\operatorname{ri} C$ as the interior of C for the topology relative to the smallest affine manifold containing C. The aforementioned *qualification condition* requires that the relative interiors of the domains of the two functions do intersect:

$$\operatorname{ri}(\operatorname{dom} f) \cap \operatorname{ri}(\operatorname{dom} g) \neq \emptyset\,.$$

Now we are in the position to make a statement on the behaviour of the conjugacy operation with respect to standard addition.

Proposition 2. (Properties of the Conjugacy Operation II)
Let $f, g \in \overline{Conv}\, \mathbb{R}^n$ *and let* f, g *satisfy the qualification assumption*

$$\mathrm{ri}(\mathrm{dom}\, f) \cap \mathrm{ri}(\mathrm{dom}\, g) \neq \emptyset\,.$$

Then the conjugacy operation maps sums into erosions:

$$(f+g)^* = f^* *_e g^*.$$

The reader interested in the proof and in more details on this subject may consult [6] or [20].

The preceding propositions state the expected analogy to the convolution theorem of the Fourier transform. However, since the Fourier transform is by definition complex-valued, it would be useful to consider a real-valued transformation with similar properties. The *multivariate Laplace transform* will serve our purpose. It is defined for any function $f : \mathbb{R}^n \longrightarrow [0, +\infty]$ by

$$L[f] : x \longmapsto L[f](x) := \int_{\mathbb{R}^n} e^{\langle x,y\rangle} f(y)\, dy \qquad \text{with } x \in \mathbb{R}^n.$$

Note that the integral is always defined, but due to its exponential kernel it might not be finite even for integrable functions. Nevertheless, the $*$-convolution of functions is transformed into a multiplication of the Laplace transforms:

$$\begin{aligned}\int_{\mathbb{R}^n} e^{\langle x,y\rangle} \int_{\mathbb{R}^n} f(y-z)\, g(z)\, dz\; dy &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\langle x,y-z\rangle} f(y-z) e^{\langle x,z\rangle} g(z)\, dz\; dy \\ &= \int_{\mathbb{R}^n} e^{\langle x,t\rangle} f(t)\, dt \cdot \int_{\mathbb{R}^n} e^{\langle x,z\rangle} g(z)\, dz\,.\end{aligned}$$

This is the real-valued counterpart to the convolution theorem of the Fourier transform.

5 Laplace Transform and Conjugation: The Logarithmic Link

In this section we reveal the counterpart of the conjugacy operation in linear systems theory, and we will see that the logarithm will make its natural appearance.

We take the point of view that the conjugacy operation is a transform defined in terms of the max-plus algebra. Then the transition ($\rightleftharpoons$) from the max-plus algebra to the plus-prod algebra in the definition of the conjugacy operation entails the transition

$$\begin{aligned} f^*(x) &= \sup_{y\in\mathbb{R}^n} (\langle y,x\rangle - f(y)) = \log \sup_{y\in\mathbb{R}^n} \left(e^{\langle y,x\rangle - f(y)}\right) \\ &\rightleftharpoons \log \int_{\mathbb{R}^n} e^{\langle y,x\rangle - f(y)}\, d\,y = \log \int_{\mathbb{R}^n} e^{\langle y,x\rangle} \cdot e^{-f(y)}\, d\,y = \log\, L[e^{-f}](x)\,. \end{aligned}$$

In other words: *The conjugate of f interpreted in the context of the max-plus algebra corresponds to this logarithmic Laplace transform of e^{-f} in the standard algebra.* A logarithmic relation between the two transforms becomes obvious: Essentially it traces back to a well-known property of the logarithm:

$$\log(a \cdot b) = \log\, a + \log\, b\,.$$

This gives the theoretical explanation for the logarithmic connection between both system theories.

It should be remarked that what we encounter here is not precisely the pairing Fourier transform – slope transform. Instead the more appropriate pairing logarithmic Laplace – conjugacy operation emerges from this change of underlying algebras.

When compared to Proposition 4, item 1, and Proposition 4, the following proposition emphasises the correspondence between conjugation and logarithmic Laplace transform.

Proposition 3. (Properties of the Logarithmic Laplace Transform) *For any functions $f, g : \mathbb{R}^n \longrightarrow [0, +\infty]$ with $f, g \not\equiv 0$ one has:*

1. *The logarithmic Laplace transform is always convex and lower semi-continuous for non-negative functions:*

$$\log L[f] \in \overline{Conv}\, \mathbb{R}^n\ .$$

2. *Convolutions are mapped into sums:*

$$\log L[f * g] = \log L[f] + \log L[g]\,.$$

PROOF:

1. Suppose $0 < \alpha < 1$. Then by Hölder's inequality with exponents $p = \frac{1}{\alpha}$ and $p' = \frac{1}{1-\alpha}$ we obtain:

$$\begin{aligned} L[f](\alpha x_1 + (1-\alpha)x_2) &= \int_{\mathbb{R}^n} e^{\langle \alpha x_1 + (1-\alpha)x_2, y\rangle} f(y)\, d\,y \\ &= \int_{\mathbb{R}^n} \left(e^{\langle x_1, y\rangle}\right)^{\alpha} \cdot \left(e^{\langle x_2, y\rangle}\right)^{(1-\alpha)} f(y)\, d\,y \\ &\leq \big(L[f](x_1)\big)^{\alpha} \cdot \big(L[f](x_2)\big)^{1-\alpha} \end{aligned}$$

   Taking the logarithm proves the claimed convexity.
   The lower-semicontinuity follows directly from Fatou's lemma [5], since

$$\lim_{n \longrightarrow +\infty} x_n = x \qquad \text{implies} \qquad L[f](x) \leq \liminf_{n \longrightarrow +\infty} L[f](x_n),$$

   for non-negative f and the fact that the logarithm is increasing and continuous.

2. Property 2 follows directly from the properties of the Laplace transform and the logarithm. □

This result establishes a convexifying property of the logarithmic Laplace transform together with its algebraic property of transforming convolution into addition.

6 A Continuous Transition between Convolution and Morphological Operations

In this section we show that there is a continuous transition from the standard $*$-convolution of two positive functions f, g to their $*_e$-convolution. With reference to the Lebesgue norms

$$\|f\|_p := \begin{cases} \Big(\int\limits_{\mathbb{R}^n} |f(x)|^p\, dx\Big)^{1/p} & \text{for } 1 \leq p < +\infty, \\ \sup\{|f(x)| : x \in \mathbb{R}^n\} & \text{for } p = +\infty\,, \end{cases}$$

we define a *p-convolution* for strictly positive functions f, g:

$$(f *_p g)(x) := \|f \cdot g(x - \cdot)\|_p \quad \text{for} \quad 1 \leq p \leq +\infty.$$

On the one hand, for $p = 1$ we regain the well-known convolution: $* = *_1$. On the other hand, we infer directly from the definitions of the operations $*_d$ that

$$\begin{aligned}
(f *_p g)(x) &= \|f \cdot g(x - \cdot)\|_p \\
&\overset{p \to +\infty}{\longrightarrow} \|f \cdot g(x - \cdot)\|_\infty \\
&= \exp\big[\log[\sup_y (f(y) \cdot g(x-y))]\big] \\
&= \exp((\log f) *_d (\log g)(x)) .
\end{aligned}$$

In a similar fashion we obtain for not necessarily positive functions f, g:

$$\log((e^{-f} *_p e^{-g})(x)) \overset{p \to 1}{\longrightarrow} \log((e^{-f} * e^{-g})(x))$$

as well as

$$\begin{aligned}
\log((e^{-f} *_p e^{-g})(x)) &\overset{p \to \infty}{\longrightarrow} \log[\sup_y (e^{-f(y)} \cdot e^{-g(x-y)}] \\
&= \log e^{-\inf_y (f(y)+g(x-y))} \\
&= -(f *_e g)(x) .
\end{aligned}$$

We observe that the logarithm makes again its natural appearance. This transition may be regarded as an alternative to the results of Florack et al. [14, 13] and Welk [39].

7 The Cramer Transform

In Section 5 we have established the importance of the logarithmic Laplace transform. Thus, it is natural to investigate its conjugate as well. This leads us to the so-called Cramer transform.

7.1 Definition and Basic Properties

The Cramer transform plays a key role in statistics, especially in the theory of large deviations [9, 11]. From a functional point of view, it allows us to make a connection between the usual convolution $*$, that appears in linear scale-space theory, and the morphological operations $\oplus$ and $\ominus$. This connection makes use of the convolutions $*_d$ and $*_e$. According to its appearance in statistics we define the Cramer transform for nonnegative functions only: For functions $f : \mathbb{R}^n \longrightarrow [0, +\infty]$, the transform

$$C[f] := (\log L[f])^*$$

is called *Cramer transform*.

The Cramer transform is the combination of the logarithmic Laplace transform with its morphological counterpart, the conjugacy operation. The reason why this transform is of importance in morphology is illuminated by the following proposition which is a direct consequence of the properties of the Laplace transform and the conjugacy operation.

Proposition 4. (Convolution Theorem for the Cramer Transform) *If f and g are non-negative functions on $\mathbb{R}^n$, then*

$$C[f * g] = C[f] *_e C[g] \, .$$

In view of equations (1) and (2) this entails for nonnegative functions $f, g \not\equiv 0$ the relations

$$-C[f * g] = (-C[f]) \oplus (-\overline{C[g]})$$

and

$$C[f * g] = C[f] \ominus \overline{C[g]} \, .$$

First we observe that, according to Proposition 1, the Cramer transform maps any nonnegative function into $\mathrm{C\overline{onv}}\,\mathbb{R}^n$. Hence it follows from Theorem 1 (2) that the conjugate of the Cramer transform is the logarithmic Laplace transform:

$$C^*[f] = \log\, L[f] \, .$$

7.2 Examples of Cramer Transforms

In order to illustrate the behaviour of the Cramer transform let us present some examples now.

1. **Dirac Measures.** Let δ_a denote the Dirac measure in $a \in \mathbb{R}^n$. Then

   $$C[\delta_a] = i_a$$

   with i_a being defined in (3).

2. **Bernoulli Distribution.** The Cramer transform is not additive:

   $$C[(1-p)\,\delta_0 + p\,\delta_1](x) = x \cdot \log\left(\frac{x}{p}\right) + (1-x) \cdot \log\left(\frac{1-x}{1-p}\right) + i_{[0,1]}(x).$$

3. **Gauß distribution.** The Gauß distributions correlate to quadratic functions with reciprocal "variance": As mentioned in [1, 3] this means for the one dimensional Gaussian with mean μ and variance σ^2 that

$$C\left[\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2}(\frac{\cdot - m}{\sigma})^2}\right](p) = \frac{1}{2}\left|\frac{p-m}{\sigma}\right|^2 .$$

This can be extended to the n-variate case of a Gaussian with diagonal covariance matrix. We give a proof of both assertions by first calculating the Laplace transform of a one-dimensional Gaussian with mean $\mu = 0$ and variance $\sigma^2 > 0$:

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{t^2}{2\sigma^2}}\, e^{p\cdot t}\, dt \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}}\, e^{-p\cdot t}\, dt \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \left(\int_0^{\infty} e^{-\frac{t^2}{2\sigma^2}}\, e^{-p\cdot t}\, dt + \int_0^{\infty} e^{-\frac{t^2}{2\sigma^2}}\, e^{-(-p)\cdot t}\, dt\right) \\
&= \frac{1}{2}\, e^{\frac{1}{2}\sigma^2 p^2} \left(\mathrm{Erfc}(\,\frac{1}{2} p\sqrt{2\sigma^2}\,) + \mathrm{Erfc}(-\,\frac{1}{2} p\sqrt{2\sigma^2}\,)\right) \\
&= e^{\frac{1}{2}\sigma^2 p^2}
\end{aligned}$$

where the complementary error function

$$\mathrm{Erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-t^2}\; dt$$

is used according to formula 5.41 in [29].
For a Gaussian with mean μ it follows immediately by a simple change of variables that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{\frac{(t-\mu)^2}{2\sigma^2}}\, e^{p\cdot t}\, dt = e^{p\,\mu}\, e^{\frac{1}{2}\sigma^2 p^2} .$$

An *n-variate Gaussian distribution* with mean vector $\mu \in \mathbb{R}^n$ and diagonal covariance matrix $D = diag(\sigma_1^2, \ldots, \sigma_n^2)$ has the separable density

$$g(y) := \frac{1}{(\sqrt{2\pi})^n \cdot \prod_{i=1}^{n} \sigma_i}\, e^{-\frac{1}{2}\sum_{i=1}^{n} \frac{(y_i-\mu_i)^2}{\sigma_i^2}} .$$

Making use of the results above, Fubini's theorem immediately gives

$$\begin{aligned}
\int_{\mathbb{R}^n} g(y)\, e^{\langle p,y\rangle}\, dy &= \int_{\mathbb{R}^n} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(y_i-\mu_i)^2}{\sigma_i^2}}\, dy \\
&= \prod_{i=1}^{n} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(y_i-\mu_i)^2}{\sigma_i^2}}\, dy_i \\
&= \prod_{i=1}^{n} e^{p_i\mu_i} \cdot e^{\frac{1}{2}p_i^2\sigma_i^2} \\
&= e^{\langle p,\mu\rangle + \frac{1}{2}p^\top Dp}\,.
\end{aligned}$$

Hence we have

$$\log L[g](p) = \langle p,\mu\rangle + \frac{1}{2}p^\top Dp\,.$$

Furthermore a straightforward calculation gives an optimal $p = D^{-1}(s-\mu)$ in $\sup_{p\in\mathbb{R}}\{\langle s,p\rangle - \log\, L[g](p)\}$ which results in

$$\begin{aligned}
(\log L[g])^*(s) &= \left(p \mapsto \langle p,\mu\rangle + \frac{1}{2}p^\top Dp\right)^*(s) \\
&= \frac{1}{2}\left\langle s-\mu, D^{-1}(s-\mu)\right\rangle
\end{aligned}$$

for $x \in \mathbb{R}^n$. This result is in complete accordance with the findings in [37], where parabolic structuring functions have been identified as the morphological equivalent to Gaussian convolution kernels.

4. **Piecewise Constant Functions.** We continue this set of examples with the numerical evaluation of Cramer transforms of positive, piecewise constant one-dimensional signals f sampled at equidistant points over the interval $[0,1[$. Defining the *characteristic function* $\mathbf{1}_A$ as

$$\mathbf{1}_A(x) := \begin{cases} 1 & x \in A\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

these signals are of the form

$$f(x) = \sum_{i=1}^{n} \alpha_i \mathbf{1}_{[\frac{i-1}{n},\frac{i}{n}[}\,.$$

Their logarithmic Laplace transforms read as

$$\log L[f](s) = \log\left(\frac{1}{s}\sum_{i=1}^{n} \alpha_i\left(e^{s\frac{i}{n}} - e^{s\frac{i-1}{n}}\right)\right)\,.$$

However, the corresponding conjugates, that means their Cramer transforms, cannot be computed explicitly. Therefore we depict the graphs of some signals together with their Cramer transforms in Figures 2 and 3 below.

5. **Approximated Signals.** We obtain numerically the Cramer transform for the follwing functions: A truncated and renormalised *Lorentz distribution*

$$f(x) = \mathbf{1}_{[0,1]}(x) \cdot \frac{3.306}{1 + 76.91\,(x-0.5)^2}\,,$$

the triangular impulse

$$f(x) = \mathbf{1}_{[0,1]}(x) \cdot (1 - 2\cdot|x - 0.5|)\,,$$

and a *beta distribution*

$$f(x) = \mathbf{1}_{[0,1]}(x) \cdot 56\,x^2(1-x)^5.$$

We use a piecewise constant approximation on 513 subintervals to calculate the corresponding Cramer transforms depicted in Figure 3.

These results illustrate the strong smoothing and convexifying properties of the Cramer transform.

8 Conclusions

In this paper we have given an explanation for the almost logarithmic connection between morphological systems and linear systems: The conjugacy operation considered in the max-plus algebra corresponds to the logarithmic Laplace transform in the standard algebra. In fact, morphological systems become linear when viewed in appropriate algebras. We were able to identify corresponding notions: convolution $\leftrightarrow$ erosion, logarithmic Laplace transform $\leftrightarrow$ conjugacy operation, and Gaussians $\leftrightarrow$ quadratic functions. The composition of the logarithmic Laplace transform and the conjugacy operation results in the Cramer transform. This fundamental transform gives the direct link between morphological and linear system theory.
The present article can be regarded as a step towards the unification of linear and morphological scale–space theory on the basis of a general linear system theory in an appropriate algebra. Taking full advantage of this connection may allow to translate results directly from one area to the other. This may

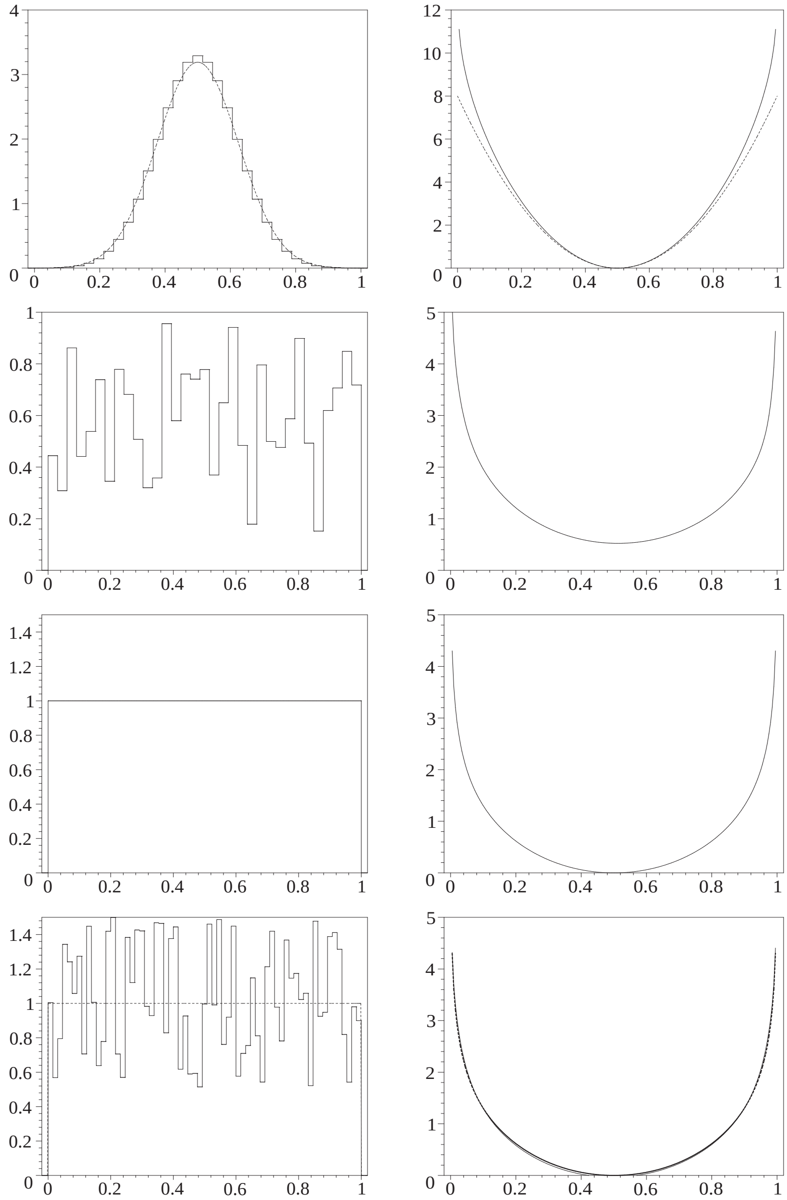

Figure 2: The smoothing property of the Cramer transform (CT). **Top Row:** A Gaussian and its piecewise constant approximation on 33 subintervals (left) and their CTs (right). **2nd Row:** A random signal, piecewise constant on 33 subintervals (left) and its CT. **3rd Row:** 0-1 signal on the interval [0,1[(left) and its CT. **4th Row:** 0-1 signal on the interval [0,1[with 100% additive uniform noise (left) and its CT vs. the results of the 3rd row.

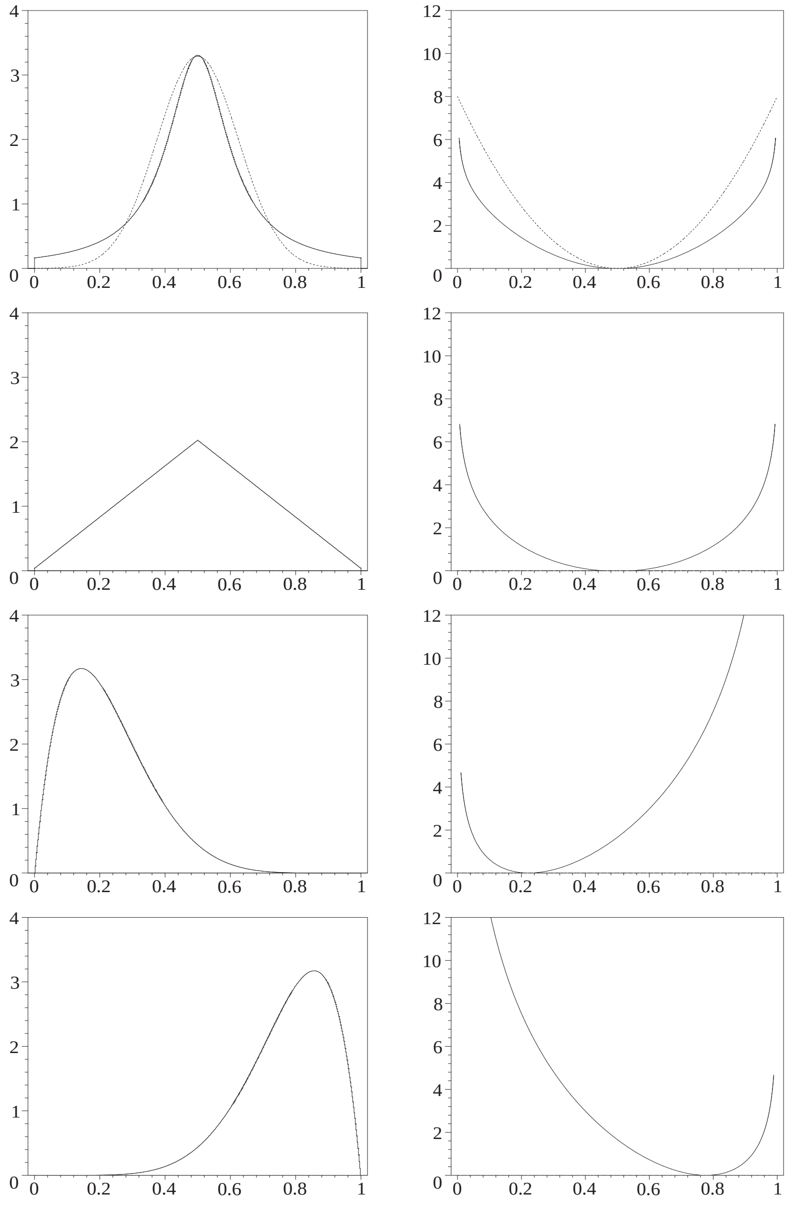

Figure 3: The smoothing property of the Cramer transform (CT). **Top Row:** A truncated Lorentzian distribution with polynomial decay together with a Gaussian of the same height (left) and their CTs (right). **2nd Row:** A triangular shaped impulse and its CT. **3rd Row:** A beta distribution and its CT. **4rd Row:** Another beta distribution and its CT.

trigger a more fruitful interaction of both paradigms that have evolved independently to powerful image processing tools. Finally, a unification within a more general algebraic framework may also help to identify novel image processing approaches that are based on other algebras. These points will be addressed in our future publications.

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