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Analytical Methods for PDEs (SoSe 2018)

Hometask N 6

Ex. 20 Calculate the following convolutions:

a) $\theta * \theta$, θ is the Heaviside function,

b) $\theta(x) * (x^2\theta(x))$,

c) $\theta * \chi_{[a,b]}$, where $\chi_{[a,b]}(x) := \begin{cases} 1, & x \in [a,b], \\ 0, & x \notin [a,b] \end{cases}$ is the indicator function of the segment $[a,b]$,

d) $\chi_{[a,b]} * \chi_{[c,d]}$,

e) $e^{-|x|} * e^{-|x|}$,

f) $(x^2\theta(x)) * (\sin x\theta(x))$,

g) $(\sin x\,\theta(x)) * (\sinh x\,\theta(x))$,

h) $e^{-ax^2} * e^{-ax^2}$, $a > 0$,

i) $\theta(a - |x|) * \theta(a - |x|)$,

j) $\chi_{[a,b]} * \Lambda$, where $\Lambda(x) := \begin{cases} x-1, & x \in [-1,0], \\ 1-x, & x \in [0,1], \\ 0, & x \notin [-1,1] \end{cases}$ is the triangular impulse.

Ex. 21 With the help of the results of the previous task, find the following convolutions without calculating integrals:

a) $e^{-|x-2|} * e^{-|x|}$,

b) $e^{-ax^2} * (-2axe^{-ax^2})$, $a > 0$,

c) $(xe^{-ax^2}) * (xe^{-ax^2})$, $a > 0$,

d) $((x^2 + 4x + 4)\theta(x)) * (\sin x\theta(x))$,

e) $((x^2 - 2x + 1)\theta(x)) * (\cos x\theta(x))$,

Ex. 22 Let $f \in L_1(\mathbb{R})$. Prove:

a) f is even if and only if $\widehat{f}$ is even;

b) f is odd if and only if $\widehat{f}$ is odd;

c) if $f : \mathbb{R} \to \mathbb{R}$ then $\overline{\widehat{f}(\lambda)} = \widehat{f}(-\lambda)$;

d) if $if : \mathbb{R} \to \mathbb{R}$ then $\overline{\widehat{f}(\lambda)} = -\widehat{f}(-\lambda)$;

e) if $\overline{f(t)} = f(-t)$ then $\widehat{f} : \mathbb{R} \to \mathbb{R}$;

f) if $\overline{f(t)} = -f(-t)$ then $i\widehat{f} : \mathbb{R} \to \mathbb{R}$.

Ex. 23 Find the Fourier transform $\widehat{f}$ of the following functions f:

a) $f(t) = \chi_{[a,b]}(t)$,

b) $f(t) = \Lambda(t) \equiv \begin{cases} t-1, & t \in [-1,0], \\ 1-t, & t \in [0,1], \\ 0, & t \notin [-1,1]. \end{cases}$,

c) $f(t) = e^{-t}\theta(t)$.

Ex. 24 Prove the following properties of the Fourier transform:

a) Rescaling: $\widehat{f(at)}(\lambda) = \frac{1}{|a|}\widehat{f}\left(\frac{\lambda}{a}\right)$ for any $a \in \mathbb{R} \setminus \{0\}$.

b) Shift versus "damping": $\widehat{f(t+a)}(\lambda) = e^{ia\lambda}\widehat{f}(\lambda)$ for any $a \in \mathbb{R}$.

c) "Damping" versus shift: $\widehat{e^{-iat}f(t)}(\lambda) = \widehat{f}(\lambda + a)$ for any $a \in \mathbb{R}$.