Chernoff approximation of evolution semigroups generated by Markov processes. Feynman formulae and path integrals

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Abstract

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This work is devoted to approximation of evolution semigroups generated by Markov processes. The presented method of approximation is based on the Chernoff theorem. In this work, the so-called *Chernoff approximations* are constructed for Feller processes (in particular, for Feller diffusions) in $\mathbb{R}^d$. Moreover, this work presents the techniques to construct Chernoff approximations for semigroups corresponding to Markov processes which are obtained from other Markov processes by different operations (or, equivalently, for semigroups whose generators are obtained from other generators by different procedures): a random time-change of processes which is equivalent to a multiplicative perturbation of generators, subordination of processes (or semigroups), killing of a process upon leaving a domain, additive perturbations of generators (what allows, in particular, to add a drift and a potential term). The developed techniques can be combined to approximate semigroups generated by processes obtained via several iterative procedures listed above. The constructed Chernoff approximations lead to representations of solutions of corresponding evolution equations in the form of limits of n-fold iterated integrals of elementary functions when n tends to infinity. Such representations are called *Feynman formulae*. They can be used for direct computations, modelling of the related dynamics, simulation of stochastic processes. Furthermore, the limits in Feynman formulae sometimes coincide with path integrals with respect to probability measures (such path integrals are usually called *Feynman-Kac formulae*) or with respect to Feynman type pseudomeasures (such integrals are *Feynman path integrals*). Therefore, the constructed Feynman formulae can be used to approximate (or even sometimes to define) the corresponding path integrals; different Feynman formulae for the same semigroup allow to establish connections between different path integrals. In this work, some Feynman formulae, arising from the obtained Chernoff approximations, are presented, and their connections to path integrals are discussed. Further, the developed technique of Chernoff approximation is applied to particular problems: approximation of semigroups generated by some Markov processes on a star graph and in a Riemannian manifold, approximation of solutions of some time-fractional evolution equations.

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Contents

List of Symbols

General

$\mathbb{N}_0^n$	set of multiindices; $\alpha \in \mathbb{N}_0^n$ if $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_k \in \mathbb{N} \cup \{0\}$
$\lvert\alpha\rvert$, $\alpha \in \mathbb{N}_0^n$,	$= \alpha_1 + \dots + \alpha_n$
∂^α, $\alpha \in \mathbb{N}_0^n$,	$= \partial^{\lvert\alpha\rvert} / \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$
$x \cdot y$	scalar product of vectors $x, y \in \mathbb{R}^d$ or vectors x, y in a Hilbert space
$A \circ B$	composition (product) of operators A and B
1_G	indicator of a set G
$\overline{G}$	the closure of a set G
$\overline{L}$	the closure of a linear operator L in a Banach space X
$\bar{\lambda}$	complex conjugate to $\lambda \in \mathbb{C}$
$\widehat{H}$	operator obtained by some quantization from a given function H
$\widehat{H}_\tau$	operator obtained by the τ-quantization from a given function H, see formula (3.0.1) for the definition
$\widehat{L}$	multiplicative perturbation of an operator L, see formula (2.2.1)
$(\widehat{T}_t)_{t\geq 0}$	semigroup whose generator is $\widehat{L}$, see Section 2.2
$(\widehat{F}(t))_{t\geq 0}$	family constructed in Thm 2.2.2; Chernoff equivalent to $(\widehat{T}_t)_{t\geq 0}$
$\widetilde{\varphi}$ or $\mathcal{F}[\varphi]$	the Fourier transform of a function φ; $\widetilde{\varphi}(x) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot y} \varphi(y)\, dy$ for $\varphi \in L^1(\mathbb{R}^d)$
$\widetilde{\eta}$ or $\mathcal{F}[\eta]$	the Fourier transform of a measure η
$\mathcal{F}^{-1}[\varphi]$	the inverse Fourier transform of a function φ
$\operatorname{supp}\varphi$, $\operatorname{supp}\eta$	support of a function φ, of a measure η
∇	$= (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d})$, $\nabla\varphi = \operatorname{grad}\varphi$ for a function $\varphi : \mathbb{R}^d \to \mathbb{C}$
$\operatorname{div} b$	$= \nabla \cdot b = \sum_{k=1}^d \frac{\partial b_k}{\partial x_k}$, divergence of a vector field $b : \mathbb{R}^d \to \mathbb{R}^d$
$\operatorname{Hess}\varphi$	Hessian matrix of a function $\varphi : \mathbb{R}^d \to \mathbb{C}$
tr	trace of a matrix
$\mathcal{B}(Q)$	Borel σ-algebra of a topological space Q
$B_R(x)$	an open ball with raduis R and center x
δ_x	Dirac delta-measure concentrated at $x \in \mathbb{R}^d$
Id	Identity operator in a Banach space X
$\mathrm{Range}(A)$	Range of an operator A
$R_A(\lambda)$	Resolvent of an operator A at the point λ

Spaces

Q	a metric space (with a metric ρ); in particular, $Q = \mathbb{R}^d$
$B_b(Q)$	space of bounded Borel functions on Q
$C(Q)$	space of continuous functions on Q
$C_b(Q)$	space of bounded continuous functions on Q

$C_0(Q)$	$= \{\varphi \in C_b(Q) : \forall\, \varepsilon > 0\ \exists$ a compact $K_\varphi^\varepsilon \subset Q$ with $\lvert\varphi(q)\rvert < \varepsilon\ \forall\, q \notin K_\varphi^\varepsilon\}$, where Q is supposed to be locally compact
$C_\infty(Q)$	$= \{\varphi \in C_b(Q) : \lim\limits_{\rho(q,q_0)\to\infty} \varphi(q) = 0\}$, where q_0 is a fixed point of Q, and Q is unbounded with respect to its metric ρ
$C^m(\mathbb{R}^d)$	space of m times continuously differentiable functions on $\mathbb{R}^d$
$C^{0,\lambda}(\mathbb{R}^d)$	Hölder continuous functions on $\mathbb{R}^d$ with exponent $\lambda \in (0,1]$
$C_b^m(\mathbb{R}^d)$	$= \{\varphi \in C^m(\mathbb{R}^d) : \partial^\alpha\varphi \in C_b(\mathbb{R}^d),\ \lvert\alpha\rvert \le m\}$
$C^{m,\lambda}(\mathbb{R}^d)$	$= \{\varphi \in C^m(\mathbb{R}^d) : \partial^\alpha\varphi \in C^{0,\lambda}(\mathbb{R}^d),\ \lvert\alpha\rvert = m\}$
$C_b^{m,\lambda}(\mathbb{R}^d)$	$= C^{m,\lambda}(\mathbb{R}^d) \cap C_b^m(\mathbb{R}^d)$
$C_c(\mathbb{R}^d)$	space of continuous functions on $\mathbb{R}^d$ with compact support
$C_c^m(\mathbb{R}^d)$	$= C^m(\mathbb{R}^d) \cap C_c(\mathbb{R}^d)$
$C_c^{m,\lambda}(\mathbb{R}^d)$	$= C^{m,\lambda}(\mathbb{R}^d) \cap C_c(\mathbb{R}^d)$
$C_c^\infty(\mathbb{R}^d)$	$= \cap_{m\in\mathbb{N}} C_c^m(\mathbb{R}^d)$
$C_\infty(\mathbb{R}^d)$	$= C_0(\mathbb{R}^d)$; space of continuous functions on $\mathbb{R}^d$ vanishing at infinity
$C_\infty^m(\mathbb{R}^d)$	$= \{f \in C_\infty(\mathbb{R}^d) : \partial^\alpha f \in C_\infty(\mathbb{R}^d),\ \lvert\alpha\rvert \le m\}$
$L^p(\mathbb{R}^d)$	standard Lebesgue space over $\mathbb{R}^d$ with the Lebesgue measure
$\mathcal{L}(X)$	the space of bounded linear operators on a Banach space X
$\mathrm{Mat}(d \times d)$	the space of matrices of the size $d \times d$
$S(\mathbb{R}^d)$	Schwartz space of tempered functions on $\mathbb{R}^d$
$W^{2,p}(G)$	classical L^p-Sobolev space over G
$W_{loc}^{2,p}(G)$	the space of all Borel measurable functions φ such that $\varphi 1_K \in W^{2,p}(G)$ for all compacts $K \subset G$

The spaces of functions taking values not in $\mathbb{C}$ but in some other space X are denoted similarly; the space X is then given after the space where the functions are defined. E.g.: $C(\mathbb{R}^d; X)$.

Norms

$\lvert\cdot\rvert$	vector resp. matrix norm in $\mathbb{R}^d$
$\lVert\cdot\rVert_X$	norm in the Banach space X
$\lVert\cdot\rVert$	norm in the Banach space $\mathcal{L}(X)$ of bounded linear operators on X
$\lVert\cdot\rVert_\infty$	supremum norm; for $\varphi : Q \to \mathbb{C}$ one defines $\lVert\varphi\rVert_\infty := \sup_{q\in Q} \lvert\varphi(q)\rvert$
$\lVert\cdot\rVert_{(m)}$	norm in the Banach space $C_b^m(\mathbb{R}^d)$; $\lVert\varphi\rVert_{(m)} := \sum_{\lvert\alpha\rvert\le m} \lVert\partial^\alpha\varphi\rVert_\infty$
$\lVert\cdot\rVert_p$	norm in the Banach space $L^p(\mathbb{R}^d)$, $p \in [1,\infty)$; $\lVert\varphi\rVert_p := \left(\int_{\mathbb{R}^d} \lvert\varphi(x)\rvert^p\, dx\right)^{1/p}$

Dedicated to my father, Anatoly Butko

Chapter 1

Introduction

Let us consider the ordinary differential equation $\frac{df}{dt} = Lf$ with a constant L and an unknown function $f : [0, +\infty) \to \mathbb{R}$. For each initial data $f_0 \in \mathbb{R}$, the Cauchy problem

$$\begin{cases} \frac{df}{dt} = Lf, & t > 0, \\ f(0) = f_0 \end{cases} \tag{1.0.1}$$

has the unique solution $f(t)$ given by $f(t) := e^{tL} f_0$. And the exponential function $T_t := e^{tL}$ posesses the following properties:

(i) $T_0 = 1$,
(ii) $T_t T_s = T_{t+s}, \quad \forall t, s \geq 0$,
(iii) the function T is continuous,
(iv) $\left.\frac{dT_t}{dt}\right|_{t=0} = L$.

Let us consider now the following generalization of the above Cauchy problem. Namely, let $(X, \|\cdot\|_X)$ be a Banach space, $f_0 \in X$ and $f : [0, +\infty) \to X$. Let $L : \mathrm{Dom}(L) \subset X \to X$ be a linear operator in X. Then (1.0.1) can be read now as an abstract Cauchy problem in X for the *evolution equation* $\frac{df}{dt} = Lf$ with the operator L. If $X = \mathbb{R}^d$, the considered evolution equation is just a system of linear ordinary differential equations of the first order. If X is a Banach space of some functions on $\mathbb{R}^d$ and L is a differential operator in X, the considered evolution equation is a partial differential equation. It is natural to expect that the abstract Cauchy problem (1.0.1) in X can be solved by a generalization of the exponential function $T_t := e^{tL}$. But how to define this object? If L is a bounded operator on X, one defines $e^{tL} := \sum_{k=0}^{\infty} \frac{t^k}{k!} L^k$ and checks that $f(t) := e^{tL} f_0$ is the unique solution of (1.0.1) in X for each $f_0 \in X$. If L is a bounded from above self-adjoint operator in a Hilbert space X, one can use the spectral decompositon of L to define $T_t := e^{tL}$ as a function of L and to get the unique solution of (1.0.1) in X. In the general case, one uses the notion of *semigroup*.

Definition 1.0.1. A family $(T_t)_{t\geq 0}$ of bounded linear operators on a Banach space X is called a *(one-parameter operator) semigroup* if

(i) $T_0 = \mathrm{Id}$,
(ii) $T_t \circ T_s = T_{t+s}, \quad \forall t, s \geq 0.$

A semigroup $(T_t)_{t\geq 0}$ is called *strongly continuous* if

(iii) $\lim\limits_{t\to 0} \|T_t\varphi - \varphi\|_X = 0 \quad$ for all $\varphi \in X$.

The *generator* $(L, \mathrm{Dom}(L))$ of a strongly continuous semigroup $(T_t)_{t\geq 0}$ in X is defined by

(iv) $$L\varphi := \lim_{t\to 0} \frac{T_t\varphi - \varphi}{t}, \quad \mathrm{Dom}(L) = \left\{ \varphi \in X \,:\, \lim_{t\to 0} \frac{T_t\varphi - \varphi}{t} \text{ exists} \right\}.$$

In the sequel, a semigroup with a given generator L will be denoted both as $(T_t)_{t\geq 0}$ and as $(e^{tL})_{t\geq 0}$. Note, that condition (iii) provides actually the continuity at zero, but, together with the semigroup property (ii), it ensures the continuity at any $t \geq 0$[1]. A semigrpoup $(T_t)_{t\geq 0}$ is called a *contraction semigroup* if $\|T_t\| \leq 1$ for all $t \geq 0$.

The following fundamental result of the theory of semigroups (see, e.g., Pazy, 1983, Thm 1.2.4, Thm 4.1.3) connects strongly continuous semigroups and evolution equations.

Theorem 1.0.2. *Let $(L, \mathrm{Dom}(L))$ be a densely defined linear operator in a Banach space X with a nonempty resolvent set. The Cauchy problem (1.0.1) in X has a unique solution $f(t)$, which is continuously differentiable on $[0, +\infty)$, for every initial value $f_0 \in \mathrm{Dom}(L)$ if and only if $(L, \mathrm{Dom}(L))$ is the generator of a strongly continuous semigroup $(T_t)_{t\geq 0}$ on X. Moreover, the solution $f(t)$ is given by $f(t) := T_t f_0$.*

Let now $(\xi_t)_{t\geq 0}$ be a temporally homogeneous Markov process with some state space[2] Q and with transition kernel $P(t, x, B)$ (See Appendix B for the definitions.) Consider the Banach space $B_b(Q)$ of bounded Borel functions on Q with the supremum-norm $\|\cdot\|_\infty$. Define the family $(T_t)_{t\geq 0}$ of bounded linear operators on X via

$$T_t\varphi(x) := \int\limits_Q \varphi(y) P(t, x, dy), \quad \forall\, \varphi \in X, \quad \forall\, x \in Q.$$

Properties of transition kernels of Markov processes ensure that $(T_t)_{t\geq 0}$ is a contraction semigroup on $B_b(Q)$. The strong continuity on $B_b(Q)$ is, however, quite rare. Nevertheless, several important classes of Markov processes posess semigroups, which are strongly continuous on some smaller Banach spaces

[1] A family $(T_t)_{t\geq 0}$ of bounded linear operators on a Banach space X is called *strongly continuous*, if $\lim_{t\to t_0} \|T_t\varphi - T_{t_0}\varphi\|_X = 0$ for all $t_0 \in [0, \infty)$ and all $\varphi \in X$.

[2] In the sequel, we consider locally compact metric spaces Q (such as (subdomains of) $\mathbb{R}^d$, compact Riemannian manifolds and metric graphs).

$X \subset B_b(Q)$ (and/or on some other Banach spaces). In this case, the following three problems are essentially equivalent:

Problem 1: To construct the strongly continuous evolution semigroup $(T_t)_{t\geq 0}$ with a given generator $(L, \mathrm{Dom}(L))$ on a given Banach space X.

Problem 2: To solve a Cauchy problem for the evolution equation $\frac{df}{dt} = Lf$ in X.

Problem 3: To find a transition kernel $P(t,x,dy)$ of a corresponding Markov process $(\xi_t)_{t\geq 0}$.

Example 1.0.3. Let us present one of the basic examples. Consider the operator $(L, \mathrm{Dom}(L))$ being the closure of $\left(\frac{1}{2}\Delta, S(\mathbb{R}^d)\right)$ in the Banach space $X = C_\infty(\mathbb{R}^d)$ or $X = L^p(\mathbb{R}^d)$, $p \in [1,\infty)$. Then $(L, \mathrm{Dom}(L))$ generates[3] the strongly continuous semigroup $(T_t)_{t\geq 0}$ given for each $f_0 \in X$ by

$$T_t f_0(x) \equiv e^{\frac{t}{2}\Delta} f_0(x) := (2\pi t)^{-d/2} \int\limits_{\mathbb{R}^d} f_0(y) \exp\left\{-\frac{|x-y|^2}{2t}\right\} dy. \tag{1.0.2}$$

The function $f(t,x) := T_t f_0(x)$ solves the corresponding Cauchy problem for the heat equation

$$\begin{cases} \frac{\partial f}{\partial t}(t,x) = \frac{1}{2}\Delta f(t,x) & t>0,\ x \in \mathbb{R}^d, \\ f(0,x) = f_0(x), & x \in \mathbb{R}^d. \end{cases}$$

And the function $P^{BM}(t,x,dy) := (2\pi t)^{-d/2} \exp\left\{-\frac{|x-y|^2}{2t}\right\} dy$ is the transition kernel of a d-dimensional Brownian motion.

Usually, it is not possible to construct a strongly continuous semigroup with a given generator explicitly. Nonetheless, there exist different methods to approximate such semigroups. The method, which will be used in the sequel, is based on the Chernoff theorem (Chernoff, 1968, Chernoff, 1974):

Theorem 1.0.4. *Let $(F(t))_{t\geq 0}$ be a family of bounded linear operators on a Banach space X. Assume that*

(i) $F(0) = \mathrm{Id}$,

(ii) $\|F^k(t)\| \leq Me^{wkt}$ *for some* $M \geq 1$*, some* $w \in \mathbb{R}$*, all* $k \in \mathbb{N}$ *and all* $t \geq 0$,

(iii) the limit $L\varphi := \lim\limits_{t\to 0} \frac{F(t)\varphi - \varphi}{t}$ *exists for all* $\varphi \in D$*, where* D *and* $(\lambda_0 - L)D$ *are dense subspaces in* X *for some* $\lambda_0 > w$.

Then the closure $(L, \mathrm{Dom}(L))$ *of* (L, D) *generates a strongly continuous semigroup* $(T_t)_{t\geq 0}$ *given by*

$$T_t\varphi = \lim_{n\to\infty} [F(t/n)]^n \varphi$$

[3] For the Banach space $X = C_\infty(\mathbb{R}^d)$, it follows from Theorem 31.5 of Sato, 1999. For Banach spaces $X = L^p(\mathbb{R}^d)$, $p \in [1,\infty)$, it follows from Theorem 4.6.25 and Example 4.6.29 of Jacob, 2001, cf. Theorem 3.4.2 of Applebaum, 2009. Using the Fourier transform, one can easily show that $e^{\frac{t}{2}\Delta}(S(\mathbb{R}^d)) \subset S(\mathbb{R}^d)$ and, hence $S(\mathbb{R}^d)$ is a core for $(e^{\frac{t}{2}\Delta})_{t\geq 0}$ by the Core Criterium A.0.7, cf. Sato, 1999, Lemma 31.6.

for all $\varphi \in X$, and the convergence is locally uniform with respect to $t \geq 0$.

Remark 1.0.5. The result of Chernoff has diverse generalizations. Versions, using arbitrary partitions of the time interval $[0, t]$ instead of the equipartition $(t_k)_{k=0}^n$ with $t_k - t_{k-1} = t/n$, are presented, e.g., in Pazy, 1983, Smolyanov, Weizsäcker, and Wittich, 2003. The analogue of the Chernoff theorem for multivalued generators can be found, e.g., in Ethier and Kurtz, 1986. Analogues of Chernoff's result for semigroups, which are continuous in a weaker sense, are obtained, e.g., in Albanese and Mangino, 2004, Kúhnemund, 2001. For analogues of the Chernoff theorem in the case of nonlinear semigroups, see, e.g., Barbu, 1976, Brézis and Pazy, 1970, Brézis and Pazy, 1972. The Chernoff Theorem for two-parameter families of operators (related to evolution equations of the form (1.0.1) with time-dependent operators L in the right hand side) can be found in Obrezkov, Smolyanov, and Trumen, 2005, Plyashechnik, 2012.

Let us consider the following simplified version of Theorem 1.0.4.

Corollary 1.0.6. *Let $(F(t))_{t\geq 0}$ be a family of bounded linear operators on a Banach space X. Assume that*

(i) $F(0) = \mathrm{Id}$,

(ii) $\|F(t)\| \leq e^{wt}$ *for some $w \in \mathbb{R}$ and all $t \geq 0$,*

(iii) the limit $L\varphi := \lim_{t\to 0} \frac{F(t)\varphi - \varphi}{t}$ exists for all $\varphi \in D$, where D is a dense subspace in X such that (L, D) is closable and the closure $(L, \mathrm{Dom}(L))$ of (L, D) generates a strongly continuous semigroup $(T_t)_{t\geq 0}$.

Then the semigroup $(T_t)_{t\geq 0}$ is given by

$$T_t\varphi = \lim_{n\to\infty} [F(t/n)]^n \varphi \tag{1.0.3}$$

for all $\varphi \in X$, and the convergence is locally uniform with respect to $t \geq 0$.

The formula (1.0.3) is called *Chernoff approximation of the semigroup $(T_t)_{t\geq 0}$ by the family $(F(t))_{t\geq 0}$*. Any family $(F(t))_{t\geq 0}$, satisfying the assumptions (i)–(iii) of the Chernoff theorem in the form of Corollary 1.0.6 with respect to some semigroup $(T_t)_{t\geq 0}$, is called *Chernoff equivalent* to this semigroup.

Remark 1.0.7. Since each strongly continuous semigroup $(T_t)_{t\geq 0}$ satisfies the estimate $\|T_t\| \leq Me^{wt}$ for some $M \geq 1$ and some $w \in \mathbb{R}$ (cf. Pazy, 1983, Engel and Nagel, 2000), the family $(F(t))_{t\geq 0}$ with $F(t) := T_t$ satisfies all the assumptions of Theorem 1.0.4, i.e. each strongly continuous semigroup can be approximated by itself (via Theorem 1.0.4). This trivial case will not be considered in the sequel. The condition (ii) in Corollary 1.0.6 implies the condition (ii) in Theorem 1.0.4, is easier to check and is fullfilled for the families $(F(t))_{t\geq 0}$ constructed in this work. In the sequel, we assume the existence of the considered semigroups and use the Chernoff theorem in the form of Corollary 1.0.6

Let us consider some particular cases of families $(F(t))_{t\geq 0}$ suitable for Chernoff approximations.

Example 1.0.8. Let L be a bounded linear operator on a Banach space X. Consider a family of bounded linear operators $(F(t))_{t\geq 0}$ given by $F(t) := \mathrm{Id} + tL$. Then, obviously, the family $(F(t))_{t\geq 0}$ is Chernoff equivalent to the semigroup $(e^{tL})_{t\geq 0}$ generated by L, and hence the following Chernoff approximation holds:

$$e^{tL} = \lim_{n\to\infty}\left(\mathrm{Id} + \frac{t}{n}L\right)^n.$$

This formula is a straight generalization of the classical limit

$$e^{tx} = \lim_{n\to\infty}\left(1 \pm \frac{t}{n}x\right)^{\pm n}.$$

Moreover, even if L is unbounded, as soon as L generates a strongly continuous semigroup $(T_t)_{t\geq 0}$, the family $(F(t))_{t\geq 0}$, given by

$$F(t) := (\mathrm{Id} - tL)^{-1} \equiv \frac{1}{t}R_L(1/t),$$

where $R_L(\lambda)$ is the resolvent operator of L at the point λ, is Chernoff equivalent to the semigroup $(T_t)_{t\geq 0}$. And the identity below (known as the *Post–Widder inversion formula*) immediately follows from the Chernoff theorem 1.0.6:

$$T_t\varphi = \lim_{n\to\infty}\left(\mathrm{Id} - \frac{t}{n}L\right)^{-n}\varphi \equiv \lim_{n\to\infty}\left[\frac{n}{t}R_L(n/t)\right]^n\varphi, \quad \forall\, \varphi \in X.$$

Example 1.0.9. Let opertors $(A, \mathrm{Dom}(A))$ and $(B, \mathrm{Dom}(B))$ in a Banach space X generate strongly continuous semigroups $(e^{tA})_{t\geq 0}$ and $(e^{tB})_{t\geq 0}$ respectively. Assume that the closure of $(A + B, \mathrm{Dom}(A) \cap \mathrm{Dom}(B))$ generates a strongly continuous semigroup $(e^{t(A+B)})_{t\geq 0}$. If the operators A and B do not commute, then

$$e^{tA} \circ e^{tB} \neq e^{tB} \circ e^{tA} \neq e^{t(A+B)}.$$

Nevertheless, one can easily check that the families $(F_1(t))_{t\geq 0}$ and $(F_2(t))_{t\geq 0}$, such that $F_1(t) := e^{tA} \circ e^{tB}$ and $F_2(t) := e^{tB} \circ e^{tA}$, are Chernoff equivalent to the semigroup $(e^{t(A+B)})_{t\geq 0}$. And the identity below (known as the *Daletskii–Lie–Trotter formula*) immediately follows from the Chernoff theorem 1.0.6:

$$e^{t(A+B)}\varphi = \lim_{n\to\infty}\left[e^{\frac{t}{n}A} \circ e^{\frac{t}{n}B}\right]^n\varphi, \quad \forall\, \varphi \in X. \tag{1.0.4}$$

Chernoff approximation has the following advantage: in order to check the conditions of the Chernoff theorem with respect to a given semigroup $(T_t)_{t\geq 0}$, one has to construct a Chernoff-equivalent family $(F(t))_{t\geq 0}$ explicitly. Therefore, the expressions $[F(t/n)]^n$ can be directly used for calculations and hence for approximation of solutions of the corresponding evolution equations, for computer modelling of the considered dynamics, for approximation of transition probabilities of underlying stochastic processes and hence for simulation of these processes. If all operators $F(t)$ are integral operators with elementary kernels or pseudo-differential operators with elementary symbols, the identity (1.0.3) leads to representation of a given semigroup by n-folds iterated integrals of elementary functions when n tends to infinity. This gives rise to *Feynman formulae*.

Definition 1.0.10. A *Feynman formula* is a representation of a solution of an initial (or initial-boundary) value problem for an evolution equation (or, equivalently, a representation of the semigroup solving the problem) by a limit of n-fold iterated integrals of some functions as $n \to \infty$.

Remark 1.0.11. One should not confuse the notions of Chernoff approximation and Feynman formula. On the one hand, not all Chernoff approximations can be directly interpreted as Feynman formulae since, generally, the operators $(F(t))_{t\geq 0}$ do not have to be neither integral operators, nor pseudo-differential operators. On the other hand, representations of solutions of evolution equations in the form of Feynman formulae can be obtained by different methods, not necessarily via the Chernoff Theorem. And such Feynman formulae may have no relations to any Chernoff approximation, or their relations may be quite indirect (see, e.g., Feynman formulae (5.3.5), (6.3.9) and (6.3.13)).

Remark 1.0.12. Richard Feynman was the first who considered representations of solutions of evolution equations by limits of iterated integrals (Feynman, 1948, Feynman, 1951). He has, namely, introduced a construction of a path integral (known nowadays as *Feynman path integral*) for solving the Schrödinger equation. And this path integral was defined exactly as a limit of iterated finite dimensional integrals. Feynman path integrals can be also understood as integrals with respect to Feynman type pseudomeasures (see Section 3.5 for details). Analogously, one can sometimes obtain representations of a solution of an initial (or initial-boundary) value problem for an evolution equation (or, equivalently, a representation of an operator semigroup resolving the problem) by functional (or, path) integrals with respect to probability measures. Such representations are usually called *Feynman–Kac formulae*. It is a usual situation that limits in Feynman formulae coincide with (or in some cases define) certain path integrals with respect to probability measures or Feynman type pseudomeasures on a set of paths of a physical system. Hence the iterated integrals in Feynman formulae for some problem give approximations to path integrals representing the solution of the same problem. Therefore, representations of evolution semigroups by Feynman formulae, on the one hand, allow to establish new path-integral-representations and, on the other hand, provide an additional tool to calculate path integrals numerically. Note that different Feynman formulae for the same semigroup allow to establish relations between different path integrals (cf. Remark 3.4.5 and Remark 3.5.15 (ii)). Moreover, the method of Chernoff approximation itself can be understood in some particular cases as a construction of Markov chains approximating a given Markov process (see Smolyanov, Weizsäcker, and Wittich, 2007b; Böttcher and Schilling, 2009 for details) and as the numerical path integration method for solving the corresponding PDE/SDE (see Chen, Jakobsen, and Naess, 2016).

Let us illistrate the connection between Feynman formulae and path integrals with the following two classical examples.

Example 1.0.13. Let $V : \mathbb{R}^d \to \mathbb{R}$, $V \in C_b(\mathbb{R}^d)$, $f_0 \in X$ where $X := C_\infty(\mathbb{R}^d)$ or $X := L^p(\mathbb{R}^d)$, $p \in [1, \infty)$. Consider the Cauchy problem

$$\begin{cases} \frac{\partial f}{\partial t}(t,x) = \frac{1}{2}\Delta f(t,x) + V(x) f(t,x), & t > 0,\ x \in \mathbb{R}^d, \\ f(0,x) = f_0(x), & x \in \mathbb{R}^d. \end{cases}$$

The function V can be considered as a bounded multiplication operator $V : (V\varphi)(x) = V(x)\varphi(x)$, $\forall\, \varphi \in X$, $\forall\, x \in \mathbb{R}^d$. This operator generates the strongly continuous semigroup $(e^{tV})_{t\geq 0}$ on X; $(e^{tV}\varphi)(x) = e^{tV(x)}\varphi(x)$. And we have $\|e^{tV}\| \leq e^{t\|V\|_\infty}$. The operator $(\overline{\frac{1}{2}\Delta, S(\mathbb{R}^d)})$ generates the contraction semigroup $(e^{\frac{t}{2}\Delta})_{t\geq 0}$ described in Example 1.0.3 by formula (1.0.2). Therefore, one gets the solution of the Cauchy problem with the Daletskii–Lie–Trotter formula in the following form:

$$\begin{aligned} f(t,x_0) &= e^{t(\frac{1}{2}\Delta + V)} f_0(x_0) = \lim_{n\to\infty} \left[e^{Vt/n} \circ e^{\frac{t}{2n}\Delta} \right]^n f_0(x_0) = \\ &= \lim_{n\to\infty} \int\limits_{\mathbb{R}^d} \cdots \int\limits_{\mathbb{R}^d} e^{\frac{t}{n}\sum_{k=1}^n V(x_{k-1})} f_0(x_n) P^{BM}(t/n, x_0, dx_1) \ldots P^{BM}(t/n, x_{n-1}, dx_n) = \\ & \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad (1.0.5) \\ &= \mathbb{E}^{x_0}\left[e^{\int_0^t V(\xi_s)ds} f_0(\xi_t) \right], \qquad (1.0.6) \end{aligned}$$

where the order of integration in iterated integrals in the line (1.0.5) is from x_n to x_1, $(\xi_s)_{s\in[0,t]}$ is a d-dimensional Brownian motion starting at x_0, the expectation $\mathbb{E}^{x_0}[\ldots]$ is nothing else but a path integral with respect to the Wiener measure concentrated on paths starting at x_0. Therefore, the solution $f(t,x_0)$ is represented by a Feynman formula in the line (1.0.5) and by a Feynman–Kac formula in the line (1.0.6).

Example 1.0.14. Let now $V : \mathbb{R}^d \to \mathbb{C}$, $V \in C_b(\mathbb{R}^d)$, $f_0 \in L^2(\mathbb{R}^d)$. Consider the Cauchy problem for the Schrödinger equation

$$\begin{cases} -i\frac{\partial f}{\partial t}(t,x) = \frac{1}{2}\Delta f(t,x) + V(x) f(t,x), & t > 0,\ x \in \mathbb{R}^d, \\ f(0,x) = f_0(x), & x \in \mathbb{R}^d. \end{cases}$$

The (semi)group $(e^{\frac{it}{2}\Delta})_{t\geq 0}$ on the space $L^2(\mathbb{R}^d)$ is given by

$$e^{\frac{it}{2}\Delta}\varphi(x) := (2\pi i t)^{-d/2} \int\limits_{\mathbb{R}^d} \varphi(y) \exp\left\{ i\frac{|x-y|^2}{2t} \right\} dy$$

for each $\varphi \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. For each $\varphi \in L^2(\mathbb{R}^d)$, the semigroup $(e^{\frac{it}{2}\Delta})_{t\geq 0}$ is given by the same formula, where the integral is understood in a regularized sense, i.e. as an L^2-limit of $\frac{it}{2}\Delta\varphi_n$, $n \to \infty$, with $\varphi_n \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $\varphi_n \to \varphi$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$ (see Reed and Simon, 1975 Ch. IX, § 7, Thm. X.66). Proceeding similar to Example 1.0.13, one obtains the solution of the Cauchy problem for the Schrödinger equation in the following form (cf. Nelson, 1964, Reed and Simon, 1975 Ch. X, § 11):

$$f(t,x_0) = e^{it(\frac{1}{2}\Delta+V)}f_0(x_0) = \lim_{n\to\infty}\left[e^{iVt/n}\circ e^{\frac{it}{2n}\Delta}\right]^n f_0(x_0) =$$
$$= \lim_{n\to\infty}\int\limits_{\mathbb{R}^d}\dots\int\limits_{\mathbb{R}^d} e^{i\frac{t}{n}\sum_{k=1}^n V(x_{k-1})}f_0(x_n)e^{i\sum_{k=1}^n\frac{|x_k-x_{k-1}|^2}{t/n}}(2\pi it/n)^{-dn/2}dx_n\dots dx_1, \tag{1.0.7}$$

where the order of integration is from x_n to x_1, and iterated integrals are understood in a regularized sense: $\int_{\mathbb{R}^d} := \lim_{R\to\infty}\int_{|x|\le R}$ and limits are taken in L^2-sense. The Feynman formula in the line (1.0.7) gives rise to the formal Feynman path integral (cf. Feynman, 1948, Reed and Simon, 1975 Ch. X, § 11)

$$f(t,x_0) := \int e^{i\int_0^t[\frac{1}{2}|\dot{x}(s)|^2+V(x(s))]ds}f_0(x(t))Dx \tag{1.0.8}$$

over all paths $(x(s))_{s\in[0,t]}$ (in the configuration space of the considered system) starting at x_0. Some rigorous mathematical constructions of this object can be found, e.g., in Albeverio, Høegh-Krohn, and Mazzucchi, 2008 and in Smolyanov and Shavgulidze, 1990 (see also a discussion in Section 3.5).

In the sequel, we distinguish two particular classes of Feynman formulae. Chernoff approximation (1.0.3) is referred to as a *Lagrangian Feynman formula* if all operators $F(t)$, $t>0$, are integral operators with explicitly given kernels. If all $F(t)$, $t>0$, are pseudo-differential operators with explicitly given symbols (see Chapter 3 for the definition), we speak of a *Hamiltonian Feynman formula*. Such terminology is inspired by the fact that a Lagrangian Feynman formula usually gives approximations to a functional integral over a set of paths in the configuration space of a system (whose evolution is described by the semigroup $(T_t)_{t\ge0}$), whereas a Hamiltonian Feynman formula corresponds to a functional integral over a set of paths in the phase space of some system. Following this terminology, both Feynman formulae (1.0.5) and (1.0.7) are Lagrangian ones. And the Lagrangian Feynnman formula (1.0.5) corresponds to a functional integral with respect to a probability measure in the Feynman–Kac formula (1.0.6), whereas the Lagrangian Feynman formula (1.0.7) corresponds to the Feynman path integral (1.0.8). More Lagrangian Feynman formulae and corresponding path integrals can be found in Chapters 2, 4, 5, 6. Some Hamiltonian Feynman formulae and corresponding phase space Feynman path integrals are discussed in Chapter 3.

Remark 1.0.15. The example 1.0.14 demonstrates a rigorous mathematical justification of the heuristic representetion (obtained in Feynman, 1948) for the solution of the Cauchy problem for the Schrödinger equation with potential in terms of Feynman path integral over trajectories in the configuration space. This justification was first obtained in 1964 by E. Nelson (Nelson, 1964) exactly by means of the Daletskii–Lie–Trotter formula. A rigorous mathematical justification of the heuristic representetion (obtained in Feynman, 1951) for the solution of the

Cauchy problem for the same Schrödinger equation in terms of Feynman path integral over trajectories in the phase space has been published for the first time only in the paper Smolyanov, Tokarev, and Truman, 2002. And the authors have used the Chernoff theorem to prove their results. Relatively at the same time, the Chernoff theorem has been used in works of O.G. Smolyanov, H. v. Weizsäcker and O. Wittich (see, e.g., Smolyanov, Weizsäcker, and Wittich, 2000, Weizsäcker, Smolyanov, and Wittich, 2000, Smolyanov and Weizsäcker, 2001, Smolyanov, Weizsäcker, and Wittich, 2003, Smolyanov, Weizsäcker, and Wittich, 2005, Smolyanov, Weizsäcker, and Wittich, 2006, Smolyanov, Weizsäcker, and Wittich, 2007b) to obtain different approximations (Feynman formulae) for the heat semigroup, corresponding to the process of Brownian motion in a compact Riemannian manifold, and to construct some related surface measures on infinite dimensional manifolds of Brownian paths. Starting from these results, the Chernoff theorem has been actively used to obtain Chernoff approximations (in particular, Feynman formulae) and path integral representations for evolution semigroups related to different types of evolution equations on different geometrical objects:

- *Schrödinger type evolution equations* have been considered, e.g., in Remizov, 2016, Plyashechnik, 2012, Sakbaev and Smolyanov, 2010, Kupsch and Smolyanov, 2009, Gadèl′ya and Smolyanov, 2008, Smolyanov and Truman, 2004, Smolyanov and Shavgulidze, 2003, Smolyanov and Truman, 2000; *stochastic Schrödinger type equations* have been studied in Obrezkov and Smolyanov, 2016; Obrezkov, 2006; Obrezkov, Smolyanov, and Trumen, 2005; Gough, Obrezkov, and Smolyanov, 2005.

- *Second order parabolic equations related to diffusions in different geometrical structures* (e.g., in Euclidean spaces and their subdomains, Riemannian manifolds and their subdomains, metric graphs, Hilbert spaces) have been studied, e.g., in Butko, Grothaus, and Smolyanov, 2016, Butko, 2015, Rat′yu and Smolyanov, 2015, Butko, 2014, Plyashechnik, 2013b, Smolyanov and Tolstyga, 2013, Remizov, 2012, Böttcher et al., 2011, Butko, Schilling, and Smolyanov, 2011, Weizsäcker, Smolyanov, and Tolstyga, 2011, Butko, Grothaus, and Smolyanov, 2010, Weizsäcker and Smolyanov, 2009, Butko, Grothaus, and Smolyanov, 2008, Butko, 2008, Butko, 2007, Smolyanov, Weizsäcker, and Wittich, 2007a, Butko, 2006, Butko, 2004, Smolyanov, Weizsäcker, and Wittich, 2004, Obrezkov, 2003.

- *Evolution equations with non-local operators* generating some Markov processes in $\mathbb{R}^d$ and its subdomains have been considered in Butko, 2017a, Butko, 2017b, Butko, Grothaus, and Smolyanov, 2016, Butko, Schilling, and Smolyanov, 2012, Butko, Schilling, and Smolyanov, 2010.

- *Evolution equations with the Vladimirov operator* (this operator is a p-adic analogue of the Laplace operator) have been investigated in Smolyanov and Shamarov, 2011, Smolyanov, Shamarov, and Kpekpassi, 2011 as well as in Smolyanov and Shamarov, 2010, Smolyanov and Shamarov, 2009, Smolyanov and Shamarov, 2008.

- *Evolution equations containing Lévy Laplacians* have been considered in, e.g., Accardi and Smolyanov, 2007, Accardi and Smolyanov, 2006.
- Feynman formulae as a *method of averaging of random Hamiltonians* have been discussed in Orlov, Sakbaev, and Smolyanov, 2016; Orlov, Sakbaev, and Smolyanov, 2014.

The present work demonstrates the results of the author on Chernoff approximations of evolution semigroups corresponding to Markov processes. In Chapter 2, Chernoff approximations are obtained for semigroups generated by additively and / or multiplicatively perturbed generators of some original semigroups. It is supposed that the original semigroups are known or already Chernoff approximated. In particular, Chernoff approximations in the form of Lagrangian Feynman formulae are obtained for semigroups corresponding to Markov processes constructed through a random time-change of Markov processes with known transitional probabilities (such as, e.g., Brownian motion or Cauchy process); as well as for semigroups corresponding to second order parabolic equations with variable (position-dependent) coefficients (such equations are governing equations for diffusions with variable diffusion coefficients, variable drift and a potential term).

In Chapter 3, different types of correspondence between a pseudo-differential operator and its symbol are considered, they are parameterized by a number $\tau \in [0,1]$. They can be understood as different procedures of quantization when one considers a pseudo-differential operator as Hamiltonian of a quantum system obtained from a classical system with a given Hamilton function (symbol of this pseudo-differential operator) through a given procedure of quantization. The case $\tau = 1$ corresponds to the so-called qp-quantization which is usually used in the literature devoted to pseudo-differential operators generating Markov processes (cf. Jacob, 2001). The considered symbols are second order polynomials with variable coefficients or a more general class of continuous negative definite functions with "variable coefficients" (such symbols are related to Feller processes). Chernoff approximations are obtained for semigroups generated by such type of operators on Banach spaces $C_\infty(\mathbb{R}^d)$ and $L^1(\mathbb{R}^d)$. These Chernoff approximations are based on families of pseudo-differential operators (constructed either by the same procedure of τ-quantization as the considered semigroups, or by the qp-quantization). These Chernoff approximations give rise to Hamiltonian Feynman formulae and coincide with some phase space Feynman path integrals. In particular, Feller semigroups are considered; Chernoff approximations are obtained under very mild conditions on the symbols of the underlying Feller processes; some corresponding Hamiltonian and Lagrangian Feynman Formulae, as well as Feynman-Kac formulae and phase space Feynman path integrals are discussed.

In Chapter 4, some Chernoff approximations are constructed for semigroups obtained by the procedure of subordination. These semigroups are subordinate to some original (or "parent") semigroups which are unknown explicitly but are already Chernoff approximated. And the considered semigroups are subordinate with respect to subordinators posessing either known transitional probabilities, or known and bounded Lévy measures. In the case when transitional probabilities of subordinators are known, the obtained approximations are given as iterated integrals of elementary functions and lead to representations of the considered semigroups by Feynman formulae.

In Chapter 5, some semigroups generated by Feller processes in $\mathbb{R}^d$, killed upon leaving a given domain, are considered. Chernoff approximations for such semigroups are constructed under assumption that the semigroups generated by the same processes in the whole space (i.e. without killing) are known or already Chernoff approximated. Using the constructed Chernoff approximations, a Lagrangian Feynman formula is obtained for semigroups corresponding to a Cauchy–Dirichlet type initial exterior-value problem for evolution equations with some of operators considered in Chapter 3. As a special case, a Lagrangian Feynman formula for semigroups corresponding to killed Feller diffusions (i.e. for a Cauchy–Dirichlet initial boundary-value problem for second order parabolic equations with variable coefficients) follows.

In Chapter 6, the developed technique of Chernoff approximation is applied to several particular problems: approximation of semigroups generated by some Markov processes on a star graph and in a Riemannian manifold, approximation of solutions of some distributed order time-fractional evolution equations.

For the convenience of the reader, some prerequisites and related results are collected in the first sections of Chapters 3 — 5 as well as in Appendices A–D.

Chapter 2

Lagrangian Feynman Formulae For Evolution Semigroups

This chapter is devoted to some general results on the technique of Chernoff approximation, to Feynman formulae related to families $(F(t))_{t\geq 0}$ of integral operators with elementary kernels (i.e. to Lagrangian Feynman formulae) and to discussion of corresponding Feynman-Kac formulae. For the convenience of the reader, some basic notions of the Semigroup Theory and some results on generation of strongly continuous semigroups are collected in Appendices A and D.

2.1 Chernoff approximations for semigroups generated by a sum of operators

In this Section, we discuss a generalization of the Daletskii–Lie–Trotter formula (1.0.4) to the case when the considered semigroup is generated by a sum of operators and the semigroups, generated by each summand, are not known but are already Chernoff approximated. This generalization follows, of course, from the Chernoff theorem 1.0.6 and will be used in Chapters 2, 4, 5, 6 as an element of subsequent constructions.

Theorem 2.1.1. *Let $(T_k(t))_{t\geq 0}$, $k = 1, \dots, m$, be strongly continuous semigroups on a Banach space X with generators $(L_k, \mathrm{Dom}(L_k))$ respectively. Let $(F_k(t))_{t\geq 0}$, $k = 1, \dots, m$, be families of operators in X which are Chernoff equivalent to the semigroups $(T_k(t))_{t\geq 0}$ respectively. Assume that $L = L_1 + \dots + L_m$ defined on $\cap_{k=1}^m \mathrm{Dom}(L_k)$ is closable and that the closure $(L, \mathrm{Dom}(L))$ is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X. Let a set $D \subset \cap_{k=1}^m \mathrm{Dom}(L_k)$ be a core for $(L, \mathrm{Dom}(L))$ and assume that $\lim_{t\to 0} \left\| \frac{F_k(t)\varphi - \varphi}{t} - L_k\varphi \right\|_X = 0$ for all $\varphi \in D$ and all $k \in \{1, \dots, m\}$. Then the family $(F(t))_{t\geq 0}$, where $F(t) = F_1(t) \circ \dots \circ F_m(t)$ is Chernoff equivalent to the semigroup $(T(t))_{t\geq 0}$. And hence the Chernoff approximation*

$$T_t\varphi = \lim_{n\to\infty} \left[F(t/n)\right]^n \varphi \equiv \lim_{n\to\infty} \left[F_1(t/n) \circ \dots \circ F_m(t/n)\right]^n \varphi \tag{2.1.1}$$

holds for each $\varphi \in X$ locally uniformly with respect to $t \geq 0$.

Proof. Since the families $(F_k(t))_{t\geq 0}$, $k = 1,\dots,m$, are Chernoff equivalent to the semigroups $(T_k(t))_{t\geq 0}$ respectively, we have $F_k(0) = \mathrm{Id}$ and $\|F_k(t)\| \leq e^{a_k t}$ for some $a_k > 0$ for each $k \in \{1,\dots,m\}$. Obviously, the family $(F(t))_{t\geq 0}$ satisfies then the conditions $F(0) = \mathrm{Id}$ and

$$\|F(t)\| \leq \|F_1(t)\| \cdot \ldots \cdot \|F_m(t)\| \leq e^{(a_1+\cdots+a_m)t}.$$

Further, for each $\varphi \in D$, we have

$$\begin{aligned}
&\lim_{t\to 0}\left\|\frac{F(t)\varphi-\varphi}{t} - L\varphi\right\|_X \\
&= \lim_{t\to 0}\left\|\frac{F_1(t)\circ\cdots\circ F_m(t)\varphi-\varphi}{t} - L_1\varphi - \cdots - L_m\varphi\right\|_X \\
&= \lim_{t\to 0}\left\|F_1(t)\circ\cdots\circ F_{m-1}(t)\left(\frac{F_m(t)\varphi-\varphi}{t} - L_m\varphi\right)\right. \\
&\left. + \left(F_1(t)\circ\cdots\circ F_{m-1}(t) - \mathrm{Id}\right)L_m\varphi + \frac{F_1(t)\circ\cdots\circ F_{m-1}(t)\varphi-\varphi}{t} - L_1\varphi - \cdots - L_{m-1}\varphi\right\|_X \\
&\leq \lim_{t\to 0}\left\|\frac{F_1(t)\circ\cdots\circ F_{m-1}(t)\varphi-\varphi}{t} - L_1\varphi - \cdots - L_{m-1}\varphi\right\|_X \\
&\leq \cdots \leq \lim_{t\to 0}\left\|\frac{F_1(t)\varphi-\varphi}{t} - L_1\varphi\right\|_X \\
&= 0.
\end{aligned}$$

Therefore, all requirements of the Chernoff theorem 1.0.6 are fulfilled and hence $(F(t))_{t\geq 0}$ is Chernoff equivalent to $(T(t))_{t\geq 0}$. □

In several cases, it is too restrictive to assume that each L_k generates a strongly continuous semigroup. For example, L_1 can be a leading term (which generates a strongly continuous semigroup) and $L_2,\dots,L_m$ can be L_1-bounded additive perturbations such that $L := L_1 + L_2 + \cdots + L_m$ again generates a strongly continuous semigroup. Or even L can be a sum of operators L_k, none of which generates a strongly continuous semigroup itself. Analyzing the proof of Theorem 2.1.1, one immediately sees that the requirement on generation of a strongly continuous semigroup by each of L_k can be relaxed in the following way:

Corollary 2.1.2. *Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X with generator $(L, \mathrm{Dom}(L))$. Let D be a core for L. Let $L = L_1 + \ldots + L_m$ for some linear operators L_k, $k = 1,\dots,m$, in X such that $D \subset \mathrm{Dom}(L_k)$ for all $k = 1,\dots,m$. Let $(F_k(t))_{t\geq 0}$, $k = 1,\dots,m$, be families of bounded linear operators on X such that for all $k \in \{1,\dots,m\}$ holds: $F_k(0) = \mathrm{Id}$, $\|F_k(t)\| \leq e^{a_k t}$ for some $a_k > 0$ and all $t \geq 0$, as well as $\lim_{t\to 0}\left\|\frac{F_k(t)\varphi-\varphi}{t} - L_k\varphi\right\|_X = 0$ for all $\varphi \in D$. Then the family $(F(t))_{t\geq 0}$, where $F(t) := F_1(t)\circ\cdots\circ F_m(t)$, is Chernoff equivalent to the semigroup $(T(t))_{t\geq 0}$. And hence the Chernoff approximation*

$$T_t\varphi = \lim_{n\to\infty}\left[F(t/n)\right]^n\varphi \equiv \lim_{n\to\infty}\left[F_1(t/n)\circ\cdots\circ F_m(t/n)\right]^n\varphi$$

holds for each $\varphi \in X$ locally uniformly with respect to $t \geq 0$.

Remark 2.1.3. Obviously, the family $(F(t))_{t\geq 0}$ is strongly continuous if and only if all the families $(F_k(t))_{t\geq 0}$, $k = 1, \dots, m$, are strongly continuous.

Remark 2.1.4. Let all the assumptions of Corollary 2.1.2 be fulfilled. Consider for simplicity the case $m = 2$. Let $\theta \in [0,1]$. Similarly to the proof of Theorem 2.1.1, one shows that the following families $(F^\theta(t))_{t\geq 0}$ and $(F^{sym}(t))_{t\geq 0}$ are Chernoff equivalent to the semigroup $(T_t)_{t\geq 0}$ generated by $L = L_1 + L_2$:

$$F^\theta(t) := F_1(\theta t) \circ F_2(t) \circ F_1((1-\theta)t),$$
$$F^{sym}(t) := \frac{1}{2}\big(F_1(t) \circ F_2(t) + F_2(t) \circ F_1(t)\big).$$

Note that, for $\theta = 0$, we have $F^0(t) = F_2(t) \circ F_1(t)$, and, for $\theta = 1$, we have $F^1(t) = F_1(t) \circ F_2(t)$. Hence the parameter θ corresponds to different orderings of non-commuting terms $F_1(t)$ and $F_2(t)$. And $F^{sym}(t) = \frac{1}{2}\left(F^1(t) + F^0(t)\right)$.

Remark 2.1.5. Chernoff approximation (2.1.1) can be understood as an abstract analogue of the *operator splitting* known in numerical methods of solving PDEs (see MacNamara and Strang, 2016 and references therein). In particular, let $F_1(t) := e^{tL_1}$, $F_2(t) := e^{tL2}$ and consider the families $(F^\theta(t))_{t\geq 0}$ of Remark 2.1.4. If $\theta = 0$ and $\theta = 1$, the families $(F^\theta(t))_{t\geq 0}$ correspond to first order splitting schemes. Whereas the family $(F^\theta(t))_{t\geq 0}$ with $\theta = 1/2$ corresponds to the symmetric Strang splitting and, together with $(F^{sym}(t))_{t\geq 0}$, represents second order splitting schemes.

Let us apply Theorem 2.1.1, in order to extend the results of Examples 1.0.13 and 1.0.14 to the case of heat and Schrödinger type evolution equations containing additionally a first order term in the right hand side (i.e., the so called gradient perturbation of the Laplacian). To this aim consider first the following result (cf. Example A.0.18 in Appendix A).

Lemma 2.1.6. *Let $X = C_\infty(\mathbb{R}^d)$. Consider a non-zero vector field $B : \mathbb{R}^d \to \mathbb{R}^d$ such that $B \in C_b^2(\mathbb{R}^d; \mathbb{R}^d)$. Let $D \subseteq C_c^2(\mathbb{R}^d)$ be a dense linear subspace of X. Assume that the operator $-B\nabla$ defined on D via $-B\nabla\varphi(x) := -B(x)\cdot\nabla\varphi(x)$ is closable and its closure generates a strongly continuous semigroup $(T_t^{-B\nabla})_{t\geq 0}$ on X. Consider a family $(S(t))_{t\geq 0}$ of linear opertors on X defined by*

$$S(t)\varphi(x) := \varphi(x - tB(x)). \tag{2.1.2}$$

Then the family $(S(t))_{t\geq 0}$ is strongly continuous and is Chernoff equivalent to the semigroup $(T_t^{-B\nabla})_{t\geq 0}$.

Proof. Note, that if B is a constant vector field, the family $(S(t))_{t\geq 0}$ is just the translation semigroup $(T_t^{-B\nabla})_{t\geq 0}$ (cf. Example A.0.18 in Appendix A). If B is nonconstant, the family $(S(t))_{t\geq 0}$ doesn't posess the semigroup property. Nevertheless, it is Chernoff equivalent to $(T_t^{-B\nabla})_{t\geq 0}$. Indeed, it is obvious that $S(0) =$

Id, $\|S(t)\| = 1$. And for all $\varphi \in C_b^2(\mathbb{R}^d)$ (in particular, for all φ belonging to the core D)

$$\begin{aligned}\lim_{t\to 0}\left\|\frac{S(t)\varphi-\varphi}{t}+B\nabla\varphi\right\|_X &= \lim_{t\to 0}\sup_{x\in\mathbb{R}^d}\left|\frac{\varphi(x-tB(x))-\varphi(x)}{t}+B(x)\cdot\nabla\varphi(x)\right|\\ &\le \lim_{t\to 0}\ t\times\sup_{x\in\mathbb{R}^d,\, s\in[0,t]}\left|B(x)\cdot\operatorname{Hess}\varphi(x-sB(x))B(x)\right|\\ &= 0.\end{aligned}$$

Hence $\frac{d}{dt}S(t)\big|_{t=0}\varphi = -B\nabla\varphi$ for each $\varphi \in D$ and all requirements of the Chernoff theorem 1.0.6 are fullfilled. Moreover, the family $(S(t))_{t\ge 0}$ is strongly continuous since for all $\varphi \in C^1_\infty(\mathbb{R}^d)$ and all $t \ge 0$

$$\begin{aligned}\lim_{h\to 0}\|S(t+h)\varphi-S(t)\varphi\|_X &= \lim_{h\to 0}\sup_{x\in\mathbb{R}^d}|\varphi(x+tB(x)+hB(x))-\varphi(x+tB(x))|\\ &\le \lim_{h\to 0}|h|\|\nabla\varphi\|_\infty\|B\|_\infty = 0,\end{aligned}$$

where the supremum norm of a vector field $V:\mathbb{R}^d\to\mathbb{R}^d$ can be defined, e.g., by $\|V\|_\infty := \max_{1\le k\le d}\sup_{x\in\mathbb{R}^d}|V_k(x)|$. Since $\|S(t)\| = 1$, the 3ε-argument provides that $\lim_{h\to 0}\|S(t+h)\varphi - S(t)\varphi\|_X = 0$ for all $\varphi \in X$. □

Remark 2.1.7. An analogue of Lemma 2.1.6 holds true also in L^p-spaces (cf. Plyashechnik, 2013a): Let $X = L^p(\mathbb{R}^d)$, $p \in [1,\infty)$. Consider a vector field $B \in C_b^1(\mathbb{R}^d;\mathbb{R}^d)$. Let $D \subseteq C_c^2(\mathbb{R}^d)$ be a dense linear subspace of X. Assume that the operator $-B\nabla$ defined on D via $-B\nabla\varphi(x) := -B(x)\cdot\nabla\varphi(x)$ is closable and its closure generates a strongly continuous semigroup $(T_t^{-B\nabla})_{t\ge 0}$ on X. Since $B \in C_b^1(\mathbb{R}^d;\mathbb{R}^d)$, there exists $\delta_B > 0$ such that for all $t \in [0,\delta_B]$ the mapping $\Phi:\mathbb{R}^d\to\mathbb{R}^d$, $\Phi(x) := x - tB(x)$, is invertible. Define a function $\tau:[0,\infty)\to[0,\delta_B]$ as

$$\tau(t) := \begin{cases} t, & t\in[0,\delta_B],\\ \delta_B, & t>\delta_B.\end{cases}$$

Consider a family $(S(t))_{t\ge 0}$ of linear opertors on X defined by

$$S(t)\varphi(x) := \varphi(x-\tau(t)B(x)).$$

Then the family $(S(t))_{t\ge 0}$ is Chernoff equivalent to $(T_t^{-B\nabla})_{t\ge 0}$.

Note, that the operators $S(t)$ are not contractions any more. But it holds with some constant $c = c(B) > 0$:

$$\|S(t)\varphi\|_X^p = \int_{\mathbb{R}^d}|\varphi(x-\tau(t)B(x))|^p dx = \left|z := \Phi(x)\right| =$$

$$= \int_{\mathbb{R}^d}|\varphi(z)|^p \det(\mathrm{Id}-\tau(t)\nabla\otimes B(x))^{-1}\Big|_{x=\Phi^{-1}(z)}dz \le ((1+ct)\|\varphi\|_X)^p \le \left(e^{ct}\|\varphi\|_X\right)^p.$$

The following result follows immediately from Theorem 2.1.1, Lemma 2.1.6, Remark 2.1.7 and Example A.0.17.

Corollary 2.1.8 (Chernoff approximation for gradient and Schrödinger perturbations)**.** *Let $X = C_\infty(\mathbb{R}^d)$ or $X = L^p(\mathbb{R}^d)$, $p \in [1,\infty)$. Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup with a generator $(L, \mathrm{Dom}(L))$ and a family $(F(t))_{t\geq 0}$ be Chernoff equivalent to $(T_t)_{t\geq 0}$. Let a function $C \in C(\mathbb{R}^d)$ be such that $\inf_{x\in\mathbb{R}^d} \mathrm{Re}\, C(x) > -\infty$. Let a vector field B be of class $C_b^2(\mathbb{R}^d;\mathbb{R}^d)$. Consider the operator $L - B\nabla - C$ such that*

$$(L - B\nabla - C)\varphi(x) := L\varphi(x) - B(x)\cdot\nabla\varphi(x) - C(x)\varphi(x)$$

for all $\varphi \in \mathrm{Dom}(L - B\nabla - C) := \mathrm{Dom}(L) \cap \mathrm{Dom}(-B\nabla) \cap \mathrm{Dom}(-C)$. Assume that (the closure of) the operator $(L - B\nabla - C, \mathrm{Dom}(L - B\nabla - C))$ generates a strongly continuous semigroup $(T_t^{L-B\nabla-C})_{t\geq 0}$ on X. Let a set $D \subset \mathrm{Dom}(L - B\nabla - C) \cap C_c^2(\mathbb{R}^d)$ be a core for $(T_t^{L-B\nabla-C})_{t\geq 0}$ and assume that $\lim_{t\to 0} \|t^{-1}(F(t)\varphi - \varphi) - L\varphi\|_X = 0$ for all $\varphi \in D$. Then the family

$$(e^{-tC} \circ S(t) \circ F(t))_{t\geq 0}$$

is Chernoff equivalent to the semigroup $(T_t^{L-B\nabla-C})_{t\geq 0}$. And the Chernoff approximation

$$T_t^{L-B\nabla-C}\varphi = \lim_{n\to\infty} \left[e^{-tC/n} \circ S(t/n) \circ F(t/n)\right]^n \varphi$$

is valid for all $\varphi \in X$ locally uniformly with respect to $t \geq 0$.

Remark 2.1.9. (i) The assumption, that $(L - B\nabla - C, \mathrm{Dom}(L - B\nabla - C))$ generates a strongly continuous semigroup $(T_t^{L-B\nabla-C})_{t\geq 0}$ on X, holds, e.g., if $C \in C_b(\mathbb{R}^d)$ and the operator $-B\nabla$ is L-bounded (see Appendix D for the definition). In particular, $-B\nabla$ is L-bounded for the case of Laplacian $L = \Delta$ and fractional Laplacian (see Appendix C for the definition) $L = -(-\Delta)^{\alpha/2}$, $\alpha \in (1, 2]$, cf. Example D.0.3. Some further sufficient conditions on the operator $L - B\nabla - C$ to generate a strongly continuous semigroup can be found, e.g., in Wang, 2013; Shigekawa, 2010.

(ii) Obviously, the family $(e^{-tC} \circ S(t) \circ F(t))_{t\geq 0}$ is strongly continuous if and only if so is the family $(F(t))_{t\geq 0}$.

Example 2.1.10 (Lagrangian Feynman formula for gradient and Schrödinger perturbations of the heat semigroup)**.** Let $X = C_\infty(\mathbb{R}^d)$ or $X = L^p(\mathbb{R}^d)$, $p \in [1,\infty)$, a vector field B be of class $C_b^2(\mathbb{R}^d;\mathbb{R}^d)$, $C \in C_b(\mathbb{R}^d)$. Consider the Laplace operator $\frac{1}{2}\Delta$ on $S(\mathbb{R}^d)$. The closure of $\left(\frac{1}{2}\Delta, S(\mathbb{R}^d)\right)$ generates a strongly continuous semigroup $(T_t)_{t\geq 0}$ solving the Cauchy problem for the heat equation $\frac{\partial f}{\partial t} = \frac{1}{2}\Delta f$ (cf. Example 1.0.3); $T_0 := \mathrm{Id}$ and for all $t \geq 0$ and $\varphi \in X$

$$T_t\varphi(x) = (2\pi t)^{(-d/2)} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2t}} \varphi(y)dy.$$

Thus, by Corollary 2.1.2, the Lagrangian Feynman formula[1] is valid for the semigroup $(T_t^L)_{t\geq 0}$ generated by the closure $(L, \mathrm{Dom}(L))$ of $(L, S(\mathbb{R}^d))$ with

[1]The order of integration in iterated integrals does not matter since the corresponding multiple integrals (of the absolute value of the integrand) exist and are finite due to Gaussian fall

$L := \frac{1}{2}\Delta - B\nabla - C$:

$$T_t^L\varphi(x_0) = \lim_{n\to\infty}(2\pi t/n)^{(-dn/2)}\int\limits_{\mathbb{R}^{nd}} e^{-\frac{t}{n}\sum\limits_{k=1}^{n}C(x_{k-1})}e^{-\sum\limits_{k=1}^{n}\frac{|x_{k-1}-B(x_{k-1})t/n-x_k|^2}{2t/n}}\times$$
$$\times\,\varphi(x_n)dx_1\cdots dx_n,$$

for all $t > 0$, $\varphi \in X$. And the convergence is in the norm of Banach space X and uniform with respect to $t \in (0, t^*]$ for all $t^* > 0$. Since $|x - B(x)t - y|^2 = |x-y|^2 - 2tB(x)\cdot(x-y) + t^2|B(x)|^2$ then

$$T_t^L\varphi(x_0) = \lim_{n\to\infty}\int\limits_{\mathbb{R}^{nd}} e^{-\frac{t}{n}\sum_{k=1}^n C(x_{k-1})}e^{\sum_{k=1}^n B(x_{k-1})\cdot(x_{k-1}-x_k)}\times$$
$$\times\, e^{-\frac{t}{2n}\sum_{k=1}^n |B(x_{k-1})|^2}p_{t/n}^{BM}(x_0-x_1)\cdots p_{t/n}^{BM}(x_{n-1}-x_n)\varphi(x_n)dx_1\cdots dx_n,$$

where $p_t^{BM}(x) = (2\pi t)^{(-d/2)}\exp\left\{-\frac{|x|^2}{2t}\right\}$ and $p(t,x,y) := p_t^{BM}(x-y)$ is the transition density of Brownian motion. Moreover, one can show (cf. Simon, 1979, Sec. V.15, and Lőrinczi, Hiroshima, and Betz, 2011, Sec. 3.5, for the case of $L^2(\mathbb{R}^d)$), that the limit in the right hand side of the last formula coincides with the path integral

$$\mathbb{E}^{x_0}\left[e^{-\int_0^t C(\xi_\tau)d\tau}e^{-\int_0^t B(\xi_\tau)d\xi_\tau}e^{-\frac{1}{2}\int_0^t |B(\xi_\tau)|^2 d\tau}f(\xi_t)\right] \tag{2.1.3}$$

with respect to the Wiener measure concentrated on the paths starting at x_0 (with $\int_0^t B(\xi_\tau)d\xi_\tau$ being the Itô stochastic integral). Hence, the machinery of Feynman formulae provides another way to prove the Feynman–Kac formula (2.1.3) and leads to the Cameron–Martin–Girsanov theorem.

Example 2.1.11 (Lagrangian Feynman formula for dynamics of a quantum particle in potential and magnetic fields)**.** Let $X = L^2(\mathbb{R}^d)$. Consider the operator $L = \frac{i}{2}\Delta$ on $C_c^\infty(\mathbb{R}^d)$. According to Example 1.0.14 (cf. Thm. IX.27 in Reed and Simon, 1975), the closure $(L, \mathrm{Dom}(L))$ of $(L, C_c^\infty(\mathbb{R}^d))$ generates the strongly continuous (semi)group $(T_t \equiv e^{\frac{it}{2}\Delta})_{t\geq 0}$ solving the Cauchy problem for the Schrödinger equation $i\frac{\partial f}{\partial t} = -\frac{1}{2}\Delta f$:

$$T_t\varphi(x) = (2\pi i t)^{(-d/2)}\int\limits_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{2t}}\varphi(y)dy,$$

where the right hand side is understood in a regularized sense. Consider a vector field B of class $C_b^1(\mathbb{R}^d;\mathbb{R}^d)$ and a complex-valued function $C \in C_b(\mathbb{R}^d)$. Then the closure of the operator $L = \frac{i}{2}\Delta - B\nabla - iC$, defined on $C_c^\infty(\mathbb{R}^d)$, generates (by Thm. X.22 in Reed and Simon, 1975, cf. Lőrinczi, Hiroshima, and Betz, 2011, Lemma 3.64, and the Stone Theorem A.0.16) a strongly continuous (semi)group $(T_t^L)_{t\geq 0}$. Thus, the following Feynman formula is valid for the

off of the integrand with respect to each of the variable. Therefore, here and in the sequel, in such case, we use multiple integrals in Feynman formulae.

semigroup $(T_t^L)_{t\geq 0}$ by Corollary 2.1.2:

$$T_t^L\varphi(x_0) = \lim_{n\to\infty}\left[e^{-i\frac{t}{n}C}\circ e^{-\frac{t}{n}B\nabla}\circ T_{t/n}\right]^n\varphi(x_0) =$$
$$= \lim_{n\to\infty}(2\pi i t)^{(-dn/2)}\int_{\mathbb{R}^d}\dots\int_{\mathbb{R}^d} e^{-i\frac{t}{n}\sum_{k=1}^{n}C(x_{k-1})}e^{i\sum_{k=1}^{n}\frac{|x_k-x_{k-1}+\frac{t}{n}B(x_{k-1})|^2}{2t/n}}\varphi(x_n)dx_n\dots dx_1,$$

where the order of integration is from x_n to x_1, the integrals again must be understood in a regularized sense, the convergence is in the norm of the space $L^2(\mathbb{R}^d)$ uniform with respect to $t\in(0,t^*]$ for all $t^*>0$. Note that the expression in the last line can be interpreted as a Feynman path integral over the set of paths in the configuration space of the system whose quantization is described by the Hamiltonian $\widehat{H} = -\frac{1}{2}\Delta - iB\nabla + C$,

$$T_t^L\varphi(x_0)\equiv e^{-it\widehat{H}}\varphi(x_0) = \int e^{-i\int_0^t[C(\xi(s))-\frac{1}{2}|B(\xi(s))|^2]ds}e^{i\int_0^t B(\xi(s))\cdot d\xi(s)}\varphi(\xi(t))\Phi^{x_0}(d\xi),$$

where the integral is taken with respect to the Feynman pseudomeasure Φ^{q_0}, concentrated on paths starting at the point q_0, and the heuristic "stochastic integral" $\int_0^t B(\xi(s))\cdot d\xi(s)$ requires a separate justification (compare Albeverio and Brzeźniak, 1995, Thm.5.4, Obrezkov, 2005). Some other Feynman formulae (in particular, for more general classes of potential and magnetic fields $B(\cdot)$ and $C(\cdot)$) can be found in Plyashechnik, 2012, Remizov, 2016.

2.2 Chernoff approximations for semigroups generated by multiplicatively perturbed operators

Let Q be a metric space[2]. Consider the Banach space $X = C_b(Q)$ of bounded continuous functions on Q with supremum-norm $\|f\|_\infty = \sup_{q\in Q}|f(q)|$. Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup on X with generator $(L,\mathrm{Dom}(L))$. Consider a function $a\in C_b(Q)$ such that $a(q)>0$ for all $q\in Q$. Then the space X is invariant under the multiplication operator a, i.e. $a(X)\subset X$. Consider the operator $\widehat{L}$, defined for all $\varphi\in\mathrm{Dom}(\widehat{L})$ and all $q\in Q$ by

$$\widehat{L}\varphi(q) := a(q)(L\varphi)(q),\quad \text{where}\quad \mathrm{Dom}(\widehat{L}) := \mathrm{Dom}(L). \tag{2.2.1}$$

Assumption 2.2.1. We assume that $(\widehat{L},\mathrm{Dom}(\widehat{L}))$ generates a strongly continuous semigroup (which is denoted by $(\widehat{T}_t)_{t\geq 0}$) on the Banach space X.

[2]The metric space Q is not assumed to be neither linear, nor locally compact. So, Q can be, e.g., a Hilbert space, a Euclidean space, a Riemannian manifold, a metric graph, or a subdomain of any of them.

The operator $\widehat{L}$ is called a *multiplicative perturbation* of the generator L and the semigroup $(\widehat{T}_t)_{t\geq 0}$, generated by $\widehat{L}$, is called a *semigroup with the multiplicatively perturbed* with the function a *generator*. Some conditions assuring the existence and strong continuity of the semigroup $(\widehat{T}_t)_{t\geq 0}$ are discussed in Appendix D.

Theorem 2.2.2. *Let Assumption 2.2.1 hold. Let $(F(t))_{t\geq 0}$ be a strongly continuous family of bounded linear operators on the Banach space X, which is Chernoff equivalent to the semigroup $(T_t)_{t\geq 0}$. Consider the family of operators $(\widehat{F}(t))_{t\geq 0}$ defined on X by*

$$\widehat{F}(t)\varphi(q) := (F(a(q)t)\varphi)(q) \quad \textit{for all } q \in Q. \tag{2.2.2}$$

The operators $\widehat{F}(t)$ act on the space X and the family $(\widehat{F}(t))_{t\geq 0}$ is Chernoff equivalent to the semigroup $(\widehat{T}_t)_{t\geq 0}$ with multiplicatively perturbed with the function a generator, i.e. the Chernoff approximation

$$\widehat{T}_t\varphi = \lim_{n\to\infty}\left[\widehat{F}(t/n)\right]^n\varphi$$

is valid for all $\varphi \in X$ locally uniformly with respect to $t \geq 0$.

Proof. Since the strongly continuous family $(F(t))_{t\geq 0}$ of bounded linear operators on the Banach space X is Chernoff equivalent to the semigroup $(T_t)_{t\geq 0}$, we have $\|F(t)\| \leq e^{kt}$ for some $k > 0$ and all $t \geq 0$ and, moreover, there exists a core D for the generator L such that $F'(0)\varphi = L\varphi$ for all $\varphi \in D$. Let us show first that the operators $\widehat{F}(t)$ act from X into X. Let $\varphi \in X = C_b(Q)$. Let us check that, for each $t \geq 0$, the function $\widehat{F}(t)\varphi$ is bounded and continuous on Q. Denote the distance in Q between points q and q_0 as $\rho(q, q_0)$. Fix $t \geq 0$ and $q_0 \in Q$.

$$\begin{aligned}
&\lim_{\rho(q,q_0)\to 0}|\widehat{F}(t)\varphi(q) - \widehat{F}(t)\varphi(q_0)|\\
&= \lim_{\rho(q,q_0)\to 0}|F(a(q)t)\varphi(q) - F(a(q_0)t)\varphi(q_0)|\\
&\leq \lim_{\rho(q,q_0)\to 0}\Big(|F(a(q)t)\varphi(q) - F(a(q_0)t)\varphi(q)| + |F(a(q_0)t)\varphi(q) - F(a(q_0)t)\varphi(q_0)|\Big)\\
&\leq \lim_{\rho(q,q_0)\to 0}\big(\|[F(a(q)t) - F(a(q_0)t)]\varphi\|_\infty + |F(a(q_0)t)\varphi(q) - F(a(q_0)t)\varphi(q_0)|\big).
\end{aligned}$$

The limit of the second term in the last line equals zero since, for each $q_0 \in Q$, the operator $F(a(q_0)t)$ acts from X into X, i.e. $F(a(q_0)t)\varphi$ is a continuous function on Q (and hence at the point q_0). Moreover,

$$\lim_{\rho(q,q_0)\to 0}\|[F(a(q)t) - F(a(q_0)t)]\varphi\|_\infty = \lim_{\tau\to\tau_0 := a(q_0)t}\|F(\tau)\varphi - F(\tau_0)\varphi\|_\infty = 0$$

since the family $(F(t))_{t\geq 0}$ is strongly continuous. Therefore, the function $\widehat{F}(t)\varphi$ is continuous. Moreover, we have with $\overline{a} := \sup_{q\in Q} a(q)$:

$$\begin{aligned}
\|\widehat{F}(t)\varphi\|_\infty &= \sup_{q\in Q}|\widehat{F}(t)\varphi(q)| = \sup_{q\in Q}|F(a(q)t)\varphi(q)| \leq \sup_{q,q_0\in Q}|F(a(q_0)t)\varphi(q)|\\
&\leq \sup_{q_0\in Q}\|F(a(q_0)t)\varphi\|_\infty \leq \sup_{q_0\in Q}\|F(a(q_0)t)\|\cdot\|\varphi\|_\infty = e^{\overline{a}kt}\|\varphi\|_\infty < \infty.
\end{aligned}$$

Hence the function $\widehat{F}(t)\varphi$ is also bounded, i.e. belongs to the space X. Moreover, it is shown that for each $t \geq 0$ the estimate $\|\widehat{F}(t)\| \leq e^{\overline{a}kt}$ is true. Note that the condition $\widehat{F}(0) = \mathrm{Id}$ is also fulfilled.

Let us find the derivative at zero of the family $(\widehat{F}(t))_{t\geq 0}$ on the set D. For each $\varphi \in D$, we have

$$\begin{aligned}
&\lim_{t\to 0}\left\|\frac{\widehat{F}(t)\varphi-\varphi}{t}-\widehat{L}\varphi\right\|_\infty = \lim_{t\to 0}\sup_{q\in Q}\left|\frac{F(a(q)t)\varphi(q)-\varphi(q)}{t}-a(q)L\varphi(q)\right| \\
&\leq \lim_{t\to 0}\sup_{q_0,q\in Q}\left|\frac{F(a(q_0)t)\varphi(q)-\varphi(q)}{t}-a(q_0)L\varphi(q)\right| \\
&= \lim_{t\to 0}\sup_{q_0\in Q}\left\|\frac{F(a(q_0)t)\varphi-\varphi}{t}-a(q_0)L\varphi\right\|_\infty \\
&= \lim_{t\to 0}\sup_{c\in[0,\overline{a}]}\left\|\frac{F(ct)\varphi-\varphi}{t}-cL\varphi\right\|_\infty \\
&\leq \overline{a}\cdot\lim_{z\to 0}\left\|\frac{F(z)\varphi-\varphi}{z}-L\varphi\right\|_\infty \\
&= 0.
\end{aligned}$$

Therefore, all conditions of the Chernoff Theorem 1.0.6 are fullfilled and hence the family $(\widehat{F}(t))_{t\geq 0}$ is Chernoff equivalent to the semigroup $(\widehat{T}_t)_{t\geq 0}$. □

Remark 2.2.3. The construction of the family $(\widehat{F}(t))_{t\geq 0}$ (which is Chernoff equivalent to the semigroup $(\widehat{T}_t)_{t\geq 0}$ with the multiplicatively perturbed generator) in Theorem 2.2.2 requires an additional assumption that the original family $(F(t))_{t\geq 0}$ (Chernoff equivalent to the original semigroup with non-perturbed generator) is strongly continuous. Therefore, in order to combine the technique of Chernoff approximation of semigroups generated by multiplicative perturbations of generators with some other techniques, one has to deal with strongly continuous families $(F(t))_{t\geq 0}$. Due to this reason, all the families $(F(t))_{t\geq 0}$ constructed in this work will be checked additionally on strong continuity.

Lemma 2.2.4. *The family $(\widehat{F}(t))_{t\geq 0}$ constructed in Theorem 2.2.2 by formula (2.2.2) is strongly continuous.*

Proof. Since the original family $(F(t))_{t\geq 0}$ is strongly continuous, the function $t \mapsto F(t)\varphi$ is uniformly continuous for each $\varphi \in X$ on each segment $[0,T]$. Therefore, we have for each $t_0 \geq 0$ and each $\varphi \in X$

$$\begin{aligned}
&\lim_{t\to t_0}\|\widehat{F}(t)\varphi-\widehat{F}(t_0)\varphi\|_\infty = \lim_{t\to t_0}\sup_{q\in Q}|F(a(q)t)\varphi(q)-F(a(q)t_0)\varphi(q)| \\
&\leq \lim_{t\to t_0}\sup_{q\in Q}\|F(a(q)t)\varphi-F(a(q)t_0)\varphi\|_\infty = \lim_{t\to t_0}\sup_{c\in[0,\overline{a}]}\|F(ct)\varphi-F(ct_0)\varphi\|_\infty = 0.
\end{aligned}$$

□

Remark 2.2.5. The statements of Theorem 2.2.2 and Lemma 2.2.4 remain true if, instead of the Banach space $C_b(Q)$ itself, one considers any Banach subspace X of $C_b(Q)$, invariant under multiplication operator a, and requires / proves that $(\widehat{F}(t))_{t\geq 0}$ acts from X into X.

Proposition 2.2.6. *The statements of Theorem 2.2.2 and Lemma 2.2.4 remain true for the following Banach spaces:*

a) $X = C_\infty(Q) := \left\{\varphi \in C_b(Q) \,:\, \lim\limits_{\rho(q,q_0)\to\infty} \varphi(q) = 0\right\}$, *where q_0 is an arbitrary fixed point of Q and the metric space Q is unbounded with respect to its metric ρ;*

b) $X = C_0(Q) := \{\varphi \in C_b(Q) \,:\, \forall\, \varepsilon > 0 \;\exists$ *a compact* $K^\varepsilon_\varphi \subset Q$ *such that* $|\varphi(q)| < \varepsilon$ *for all* $q \notin K^\varepsilon_\varphi\}$, *where the metric space Q is assumed to be locally compact*[3].

Proof. Let $(F(t))_{t\geq 0}$ be a strongly continuous family of bounded linear operators on X. Let us check that $(\widehat{F}(t))_{t\geq 0}$ defined by (2.2.2) acts from X into X.

Case (a): It is sufficient to check that $\lim\limits_{\rho(q,q_0)\to\infty} (\widehat{F}(t)\varphi)(q) = 0$ for each $\varphi \in X$ and each $t > 0$. Let us fix $\varphi \in X$ and $t > 0$. Let as before $\overline{a} := \sup_{q\in Q} a(q)$.

$$\begin{aligned}
\lim_{\rho(q,q_0)\to\infty} |\widehat{F}(t)\varphi(q)| &= \lim_{\rho(q,q_0)\to\infty} |(F(a(q)t)\varphi)(q)| \\
&\leq \lim_{\rho(q,q_0)\to\infty} \sup_{q'\in Q} |F(a(q')t)\varphi(q)| \\
&= \lim_{\rho(q,q_0)\to\infty} \sup_{\tau\in[0,\overline{a}t]} |F(\tau)\varphi(q)|.
\end{aligned}$$

Since the family $(F(t))_{t\geq 0}$ acts from X into X, for any $\tau \in [0, \overline{a}t]$ and any $\varepsilon > 0$ there exists $R_{\varepsilon,\tau} > 0$ such that, for each $q \in Q$ with $\rho(q_0, q) > R_{\varepsilon,\tau}$, the inequality $|F(\tau)\varphi(q)| < \varepsilon/2$ holds. Since the family $(F(t))_{t\geq 0}$ is strongly continuous, the mapping $\tau \to F(\tau)\varphi$ is uniformly continuous on the segment $[0, \overline{a}t]$. Therefore, for each $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that, for all $\tau, \tau' \in [0, at]$ satisfying the condition $|\tau - \tau'| < \delta_\varepsilon$, the inequality $\|F(\tau)\varphi - F(\tau')\varphi\|_\infty < \varepsilon/2$ holds.

Fix $\varepsilon > 0$. Consider a partition $\tau_0 = 0 < \tau_1 < \ldots < \tau_N = \overline{a}t$ of the segment $[0, \overline{a}t]$ such that $\max\limits_{1\leq k\leq N} |\tau_k - \tau_{k-1}| < \delta_\varepsilon$. Then for each $\tau \in [0, \overline{a}t]$ there exists an element τ_k of the partition $(\tau_i)_{i=0}^N$ such that $|\tau - \tau_k| < \delta_\varepsilon$. Let now $R_\varepsilon = \max\limits_{0\leq k\leq N} R_{\varepsilon,\tau_k}$. Then, for any $\tau \in [0, \overline{a}t]$ and any $q \in Q$ with $\rho(q, q_0) > R_\varepsilon$, we have:

$$\begin{aligned}
|F(\tau)\varphi(q)| &\leq |F(\tau)\varphi(q) - F(\tau_k)\varphi(q)| + |F(\tau_k)\varphi(q)| \\
&\leq \|F(\tau)\varphi - F(\tau_k)\varphi\|_\infty + |F(\tau_k)\varphi(q)| \\
&\leq \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.
\end{aligned}$$

[3] If $Q = \mathbb{R}^d$, we have $C_\infty(Q) = C_0(Q)$, and we use the notation $C_\infty(\mathbb{R}^d)$ for this space. If $Q = (0, \infty)$, we have $C_0(Q) = C_\infty(Q) \cap \{\varphi : \lim\limits_{x\searrow 0} \varphi(x) = 0\}$ and $C_\infty((0,\infty)) = C_0([0,\infty))$. In general, it is not assumed in the definition of $C_\infty(Q)$ that Q is locally compact.

Therefore, $\lim\limits_{\rho(q,q_0)\to\infty} \sup\limits_{\tau\in[0,\overline{a}t]} |F(\tau)\varphi(q)| = 0$. And Case (a) is proved.

Case (b): Fix $\varphi \in X$, $t > 0$ and $\varepsilon > 0$. It is sufficient to find a compact $K^\varepsilon_\varphi \subset Q$ such that $|\widehat{F}(t)\varphi(q)| < \varepsilon$ for all $q \notin K^\varepsilon_\varphi$.

$$|\widehat{F}(t)\varphi(q)| = |F(a(q)t)\varphi(q)| \le \sup_{q_0\in Q} |F(a(q_0)t)\varphi(q)| = \sup_{\tau\in[0,\overline{a}t]} |F(\tau)\varphi(q)|.$$

Since the family $(F(t))_{t\ge0}$ acts from X into X, for each $\tau \in [0,\overline{a}t]$ there exists a compact $K^\varepsilon_\tau \subset Q$ such that $|F(\tau)\varphi(q)| < \varepsilon/2$ for all $q \notin K^\varepsilon_\tau$. Since the mapping $\tau \to F(\tau)\varphi$ is uniformly continuous on the segment $[0,\overline{a}t]$, for each $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that, for all $\tau, \tau' \in [0,\overline{a}t]$ with $|\tau - \tau'| < \delta_\varepsilon$, the inequality $\|F(\tau)\varphi - F(\tau')\varphi\|_\infty < \varepsilon/2$ holds. Consider a partition $\tau_0 = 0 < \tau_1 < \ldots < \tau_N = \overline{a}t$ of the segment $[0,\overline{a}t]$ such that $\max\limits_{1\le k\le N} |\tau_k - \tau_{k-1}| < \delta_\varepsilon$. Define $K^\varepsilon_\varphi := \cup_{i=0}^N K^\varepsilon_{\tau_i}$. Then K^ε_φ is a compact such that $|F(\tau_i)\varphi(q)| < \varepsilon/2$ for all $q \notin K^\varepsilon_\varphi$ and all $i = 0,1,\ldots,N$. Since for each $\tau \in [0,\overline{a}t]$ there exists an element τ_k of the partition $(\tau_i)_{i=0}^N$ such that $|\tau - \tau_k| < \delta_\varepsilon$, we have for all $q \notin K^\varepsilon_\varphi$:

$$|F(\tau)\varphi(q)| \le \|F(\tau)\varphi - F(\tau_k)\varphi\|_\infty + |F(\tau_k)\varphi(q)| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

Therefore, $|\widehat{F}(t)\varphi(q)| \le \sup\limits_{\tau\in[0,\overline{a}t]} |F(\tau)\varphi(q)| < \varepsilon$ for all $q \notin K^\varepsilon_\varphi$. And Case (b) is proved. □

Remark 2.2.7. Let all the operators T_t, $t \ge 0$, satisfy the estimate $\|T_t\| \le e^{tk}$ with some $k \ge 0$. Then one may take the semigroup $(T_t)_{t\ge0}$ itself (as the strongly continuous family $(F(t))_{t\ge0}$), in order to construct the family $(\widehat{F}(t))_{t\ge0}$:

$$\widehat{F}(t)\varphi(q) = (T_{a(q)t}\varphi)(q), \quad \forall\, \varphi \in X, \quad \forall\, q \in Q.$$

Note, that such family $(\widehat{F}(t))_{t\ge0}$ is not a semigroup any more.

Corollary 2.2.8. *Let $(X_t)_{t\ge0}$ be a Markov process with the state space Q and transition probability $P(t,q,dy)$. Let the corresponding semigroup $(T_t)_{t\ge0}$,*

$$T_t\varphi(q) = \mathbb{E}^q\left[\varphi(X_t)\right] \equiv \int_Q \varphi(y)P(t,q,dy),$$

be strongly continuous on the Banach space X, where $X = C_b(Q)$, $X = C_\infty(Q)$ or $X = C_0(Q)$, and Assumption 2.2.1 hold. Then by Theorem 2.2.7 and Proposition 2.2.6 the family $(\widehat{F}(t))_{t\ge0}$ defined by

$$\widehat{F}_t\varphi(q) := \int_Q \varphi(y)P(a(q)t,q,dy),$$

is strongly continuous and is Chernoff equivalent to the semigroup $(\widehat{T}_t)_{t\geq 0}$ *with multiplicatively perturbed (with the function* a*) generator. Therefore, the following Lagrangian Feynman formula is true for all* $t > 0$ *and all* $q_0 \in Q$*:*

$$\widehat{T}_t\varphi(q_0) = \lim_{n\to\infty} \int\limits_Q \cdots \int\limits_Q \varphi(q_n)P(a(q_0)t/n, q_0, dq_1)P(a(q_1)t/n, q_1, dq_2) \times \cdots \times P(a(q_{n-1})t/n, q_{n-1}, dq_n), \tag{2.2.3}$$

where the order of integration is from q_n *to* q_1 *and the convergence is uniform with respect to* $q_0 \in Q$ *and locally uniform with respect to* $t \geq 0$.

Remark 2.2.9. Corollary 2.2.8 holds true if, e.g., $(X_t)_{t\geq 0}$ is a Feller process (see Chapter 3 for definition). In this case $Q = \mathbb{R}^d$ and $(T_t)_{t\geq 0}$ is a strongly continuous contraction semigroup on $C_\infty(\mathbb{R}^d)$.

Remark 2.2.10. A multiplicative perturbation of the generator of a Markov process is equivalent to some randome time change of the process (see Volkonskiĭ, 1958, Volkonskiĭ, 1960, Ethier and Kurtz, 1986). Note that $\widehat{P}(t,q,dy) := P(a(q)t, q, dy)$ is not a transition probability any more. Nevertheless, if the transition probability $P(t,q,dy)$ of the original process is known, formula 2.2.3 allows to approximate the unknown transition probability of the modified process. Some explicilty known transitional densities can be found, e.g., in Borodin and Salminen, 2002; Cont and Tankov, 2004.

Example 2.2.11 (Lagrangian Feynman formula for a diffusion with variable diffusion coefficient)**.** Consider the Banach space $X = C_\infty(\mathbb{R}^d)$. Consider a Brownian motion in $\mathbb{R}^d$. Its generator is the Laplace operator $L = \frac{1}{2}\Delta$ and the corresponding semigroup $(T_t)_{t\geq 0}$ is given by (1.0.2). Its transition density $p(t,x,y)$ is given as $p(t,x,y) := p_t^{BM}(x-y)$ via the Gaussian exponent

$$p_t^{BM}(x) = (2\pi t)^{-d/2} \exp\left\{-\frac{|x|^2}{2t}\right\}.$$

Consider a function $a \in C_b(\mathbb{R}^d)$, $a > 0$. Let $(\widehat{T}_t)_{t\geq 0}$ be the semigroup on X with multiplicatively perturbed with the function a generator L. This semigroup corresponds to a diffusion process with the variable diffusion matrix $\sqrt{a}\,\mathrm{Id}$. The following Lagrangian Feynman formula is valid by Proposition 2.2.6 for any $\varphi \in X$ and any $t > 0$:

$$\widehat{T}_t\varphi(q_0) = \lim_{n\to\infty} \int\limits_{\mathbb{R}^d} \cdots \int\limits_{\mathbb{R}^d} (2\pi a(q_0)t/n)^{-d/2} \exp\left\{-\frac{|q_0-q_1|^2}{2a(q_0)t/n}\right\} \times \cdots \times (2\pi a(q_{n-1})t/n)^{-d/2} \exp\left\{-\frac{|q_{n-1}-q_n|^2}{2a(q_{n-1})t/n}\right\}\varphi(q_n)dq_n\cdots dq_1,$$

where the order of integration is from q_n to q_1 and the convergence is uniform with respect to $q_0 \in Q$ and $t \in (0, t^*]$, $t^* > 0$.

Example 2.2.12 (Lagrangian Feynman formula for a Cauchy type Feller process)**.** Consider the Banach space $X = C_\infty(\mathbb{R}^d)$. Consider a Cauchy process in $\mathbb{R}^d$. It is a Lévy (and hence Feller) process (see Appendix C for all the details). Its generator is given by the fractional Laplacian $L = -\sqrt{-\Delta}$ and its transition density $p(t,x,y)$ is given as $p(t,x,y) := p_t(x-y)$ via

$$p_t(x) = \Gamma\left(\frac{d}{2}+\frac{1}{2}\right)\frac{t}{[\pi|x|^2+t^2]^{(d+1)/2}},$$

where $\Gamma(\cdot)$ is Euler's Gamma function. Consider a function $a \in C_b(\mathbb{R}^d)$, $a > 0$. Let $(\widehat{T}_t)_{t\geq 0}$ be the semigroup on X with multiplicatively perturbed with the function a generator L. This semigroup corresponds to a Cauchy type Feller process. The following Lagrangian Feynman formula is valid by Proposition 2.2.6 for any $\varphi \in X$ and any $t > 0$:

$$\begin{aligned}\widehat{T}_t\varphi(q_0) = \lim_{n\to\infty}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\left[\Gamma\left(\frac{d}{2}+\frac{1}{2}\right)\right]^n \frac{a(q_0)t/n}{[(a(q_0)t/n)^2+(\pi|q_0-q_1|)^2]^{(d+1)/2}}\times\cdots\\ \times\frac{a(q_{n-1})t/n}{[(a(q_{n-1})t/n)^2+(\pi|q_{n-1}-q_n|)^2]^{(d+1)/2}}\varphi(q_n)dq_n\cdots dq_1,\end{aligned} \tag{2.2.4}$$

where the order of integration is from q_n to q_1 and the convergence is uniform with respect to $q_0 \in Q$ and locally uniform with respect to $t \geq 0$.

Some further examples are discussed in Chapter 6.

2.3 Lagrangian Feynman formulae for semigroups generated by second order elliptic operators

In this Section, we generalize the results of Section 2.1 and Section 2.2 on Chernoff approximation of semigroups generated by additive and multiplicative perturbations of the Laplace operator to the case of semigroups generated by second order elliptic operators with variable coefficients. We consider the Banach space $X = C_\infty(\mathbb{R}^d)$ in this Section.

Assumption 2.3.1. Let $A : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^d)$ be a continuous mapping such that the operator $A(x)$ is symmetric for all $x \in \mathbb{R}^d$, and let there exist $a_0, A_0 \in \mathbb{R}$ such that $0 < a_0 \leq A_0 < \infty$ and for all $x, z \in \mathbb{R}^d$

$$a_0|z|^2 \leq z\cdot A(x)z \leq A_0|z|^2. \tag{2.3.1}$$

Consider the second order elliptic operator Δ_A defined for each $\varphi \in C^2(\mathbb{R}^d)$ by

$$\Delta_A\varphi(x) := \operatorname{tr}(A(x)\operatorname{Hess}\varphi(x)) \tag{2.3.2}$$

Assumption 2.3.2. We assume that the mapping $A:\mathbb{R}^d\to\mathcal{L}(\mathbb{R}^d)$ is such that for some $\alpha\in(0,1]$ the operator $(\Delta_A, C_c^{2,\alpha}(\mathbb{R}^d))$ is closable in X and the closure $(\Delta_A, \mathrm{Dom}(\Delta_A))$ generates a strongly continuous semigroup $(T_t^{\Delta_A})_{t\geq 0}$ on X.

Remark 2.3.3. If the mapping A is uniformly continuous (or it is of the class $C_b^3(\mathbb{R}^d;\mathcal{L}(\mathbb{R}^d))$), Assumption 2.3.1 guarantees the existence of the strongly continuous semigroup $(T_t^{\Delta_A})_{t\geq 0}$ on X due to Theorem D.0.9, part (i) (or due to Theorem D.0.10 respectively). Obviously, $C_c^{2,\alpha}(\mathbb{R}^d)\subset\mathrm{Dom}(\Delta_A)$. If A is a constant matrix then the set $C_c^\infty(\mathbb{R}^d)$ (and hence $C_c^{2,\alpha}(\mathbb{R}^d)$) is a core for Δ_A due to Theorem C.0.7.

Consider the family $(F^A(t))_{t\geq 0}$ of operators on X defined by $F^A(0):=\mathrm{Id}$ and for all $t>0$ with $\varphi\in X$ and $x\in\mathbb{R}^d$

$$F^A(t)\varphi(x):=\frac{1}{\sqrt{(4\pi t)^d\det A(x)}}\int\limits_{\mathbb{R}^d}\exp\left(-\frac{A^{-1}(x)(x-y)\cdot(x-y)}{4t}\right)\varphi(y)dy. \tag{2.3.3}$$

Lemma 2.3.4. *Under Assumption 2.3.1, the family $(F^A(t))_{t\geq 0}$ is a strongly continuous family of contractions on X.*

Proof. Consider $t>0$. The operators $F^A(t)$ are obviously linear. Let us show that $F^A(t):C_c(\mathbb{R}^d)\to C_\infty(\mathbb{R}^d)$. Consider $\varphi\in C_c(\mathbb{R}^d)$. Then

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^d}|F^A(t)\varphi(x)| \qquad (2.3.4)\\
&=\sup_{x\in\mathbb{R}^d}\left|\frac{1}{\sqrt{(4\pi t)^d\det A(x)}}\int\limits_{\mathbb{R}^d}\exp\left(\frac{-A^{-1}(x)(x-y)\cdot(x-y)}{4t}\right)\varphi(y)dy\right|\\
&\leq\|\varphi\|_\infty\sup_{x\in\mathbb{R}^d}\left[\frac{1}{\sqrt{(4\pi t)^d\det A(x)}}\int\limits_{\mathbb{R}^d}\exp\left(\frac{-A^{-1}(x)(x-y)\cdot(x-y)}{4t}\right)dy\right]\\
&=\|\varphi\|_\infty,
\end{aligned}$$

i.e., the function $F^A(t)\varphi$ is well defined and bounded; $\|F^A(t)\varphi\|_\infty\leq\|\varphi\|_\infty$. Define $p_A(t,x,y)$ for all $x,y\in\mathbb{R}^d$ and all $t>0$ via

$$p_A(t,x,y):=\frac{1}{\sqrt{(4\pi t)^d\det A(x)}}\exp\left(-\frac{A^{-1}(x)(x-y)\cdot(x-y)}{4t}\right). \tag{2.3.5}$$

This function p_A is continuous on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ and hence uniformly continuous on each compact in $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$. Therefore, for each fixed $t>0$, for x, $x_0\in\mathbb{R}^d$, we have with $K_\varphi:=\operatorname{supp}\varphi$

$$|F^A(t)\varphi(x)-F^A(t)\varphi(x_0)|\leq\|\varphi\|_\infty\sup_{y\in K_\varphi}|p_A(t,x,y)-p_A(t,x_0,y)|\int\limits_{K_\varphi}dy\to 0,\ x\to x_0,$$

i.e., the function $F^A(t)\varphi$ is continuous on $\mathbb{R}^d$. Let us check that $F^A(t)\varphi(x) \to 0$ as $|x| \to \infty$. Under Assumption 2.3.1 the inequality

$$\exp\left(-\frac{A^{-1}(x)(x-y)\cdot(x-y)}{4t}\right) \leq \exp\left(-\frac{|x-y|^2}{4tA_0}\right)$$

holds for all x, $y \in \mathbb{R}^d$, $t \geq 0$. Since $\det A(x) \geq a_0^d$ due to Assumption 2.3.1, we have

$$|F^A(t)\varphi(x)| \leq \frac{\|\varphi\|_\infty}{\sqrt{(4\pi t a_0)^d}} \int_{K_\varphi} \exp\left(-\frac{|x-y|^2}{4tA_0}\right) dy \to 0 \text{ as } |x| \to \infty.$$

Therefore, $F^A(t) : C_c(\mathbb{R}^d) \to C_\infty(\mathbb{R}^d)$ with $\|F^A(t)\varphi\|_\infty \leq \|\varphi\|_\infty$. Hence operators $F^A(t)$ can be extended to contractions on $C_\infty(\mathbb{R}^d)$ by the B.L.T. Theorem A.0.19. Besides, we have shown that $\int_K p_A(t,x,y)dy \to 0$ as $|x| \to \infty$ for each $t > 0$ and each compact $K \subset \mathbb{R}^d$. Clearly, this convergence is uniform with respect to t on each compact interval $[t_1, t_2] \subset (0, \infty)$.

Let us show now that the family $(F^A(t))_{t\geq 0}$ is strongly continuous. It is sufficient to show that $\lim_{t\to t_0} \|F^A(t)\varphi - F^A(t_0)\varphi\|_\infty = 0$ for all $\varphi \in C_c(\mathbb{R}^d)$ and all $t_0 \geq 0$ and to apply the 3ε-argument. So, fix $\varphi \in C_c(\mathbb{R}^d)$. Consider first $t_0 > 0$. Let t, t_0 belong to a compact interval $[t_1, t_2] \subset (0, \infty)$. Then for each $\varepsilon > 0$ there exists a compact $K^\varepsilon_{t_1,t_2} \subset \mathbb{R}^d$ such that, for all $x \notin K^\varepsilon_{t_1,t_2}$ and all $t \in [t_1, t_2]$, we have $\int_{K_\varphi} p_A(t,x,y)dy < \varepsilon$. Therefore, we have for each $\varepsilon > 0$

$$\begin{aligned}
\|F^A(t)\varphi - &F^A(t_0)\varphi\|_\infty \\
&\leq \|\varphi\|_\infty \sup_{x\in\mathbb{R}^d} \left[\int_{K_\varphi} (p_A(t,x,y) - p_A(t_0,x,y))dy\right] \\
&\leq \|\varphi\|_\infty \left(\sup_{x\in K^\varepsilon_{t_1,t_2}, y\in K_\varphi} |p_A(t,x,y) - p_A(t_0,x,y)| \int_{K_\varphi} dy + 2\varepsilon \right) \\
&\to 2\varepsilon\|\varphi\|_\infty \text{ as } t \to t_0,
\end{aligned}$$

since the function p_A is uniformly continuous on each compact in $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$. Hence the mapping $t \to F^A(t)$ is strongly continuous on $(0, \infty)$. Consider now $t_0 = 0$. Let $\varphi \in C_\infty(\mathbb{R}^d)$. Then φ is uniformly continuous, i.e., for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in \mathbb{R}^d$ and all $z \in \mathbb{R}^d$ with $|z| \leq \delta$ the inequality

$|\varphi(x-z)-\varphi(x)|<\varepsilon$ holds. Then for each fixed $\varepsilon>0$

$$\begin{aligned}
\|F^A(t)\varphi-\varphi\|_\infty &\le \sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d} p_A(t,x,x-z)|\varphi(x-z)-\varphi(x)|dz\\
&\le \sup_{x\in\mathbb{R}^d}\left(\varepsilon\int_{|z|\le\delta} p_A(t,x,x-z)dz\right)+\sup_{x\in\mathbb{R}^d}\left(2\|\varphi\|_\infty\int_{|z|\ge\delta} p_A(t,x,x-z)dz\right)\\
&\le \varepsilon+2\|\varphi\|_\infty(A_0/a_0)^{d/2}\int_{|z|\ge\delta}(4\pi tA_0)^{-d/2}e^{-\frac{|z|^2}{4tA_0}}dz\\
&\to\varepsilon,\qquad \text{as } t\to 0.
\end{aligned}$$

Hence the mapping $t\to F^A(t)$ is strongly continuous on $[0,\infty)$ and Lemma is proved. □

Theorem 2.3.5. *Under Assumptions 2.3.1 and 2.3.2, the family $(F^A(t))_{t\ge0}$ is Chernoff equivalent to the semigroup $(T_t^{\Delta_A})_{t\ge0}$ on X. Hence*

$$T_t^{\Delta_A}\varphi=\lim_{n\to\infty}\left[F^A(t/n)\right]^n\varphi$$

for all $\varphi\in X$ locally uniformly with respect to $t\ge0$. Therefore, the following Lagrangian Feynman Formula holds for each $t>0$, $\varphi\in X$ and each $x_0\in\mathbb{R}^d$:

$$T_t^{\Delta_A}\varphi(x_0)=\lim_{n\to\infty}\int_{\mathbb{R}^{nd}} p_A(t/n,x_0,x_1)\cdots p_A(t/n,x_{n-1},x_n)\varphi(x_n)dx_1\dots dx_n, \tag{2.3.6}$$

where $p_A(t,x,y)$ is given by (2.3.5). And the convergence is uniform with respect to $x_0\in\mathbb{R}^d$ and $t\in(0,t^*]$ for all $t^*>0$.

Proof. Due to Lemma 2.3.4 and Assumption 2.3.2, it is sufficient to show for each $\varphi\in C_c^{2,\alpha}(\mathbb{R}^d)$ that $\lim_{t\to0}\|t^{-1}(F(t)\varphi-\varphi)-\Delta_A\varphi\|_\infty=0$. So, fix $\varphi\in C_c^{2,\alpha}(\mathbb{R}^d)$. Using the Taylor expansion for φ at a fixed point x, we have with some $\theta=\theta(z)\in[0,1]$:

$$\begin{aligned}
F^A(t)\varphi(x)&=\int_{\mathbb{R}^d} p_A(t,x,x-z)\varphi(x-z)dz\\
&=\int_{\mathbb{R}^d} p_A(t,x,x-z)\left[\varphi(x)+\nabla\varphi(x)\cdot z+\frac12\operatorname{Hess}\varphi(x-\theta z)z\cdot z\right]dz\\
&=\varphi(x)+t\operatorname{tr}(A(x)\operatorname{Hess}\varphi(x))+\\
&\qquad+\int_{\mathbb{R}^d} p_A(t,x,x-z)(\operatorname{Hess}\varphi(x-\theta z)-\operatorname{Hess}\varphi(x))z\cdot z\,dz.
\end{aligned}$$

Here we have used the fact that for each positive definite matrix A and each matrix M

$$\int_{\mathbb{R}^d}\left((4\pi)^d\det A\right)^{-1/2}e^{-\frac{A^{-1}z\cdot z}{4}}Mz\cdot z\,dz=2\operatorname{tr}(AM),$$

and that the integral of an odd function over the whole space is zero. Therefore, using the Hölder continuity of $\operatorname{Hess}\varphi$ with index α, we obtain with some constants $K_1 > 0$ and $K_2 > 0$

$$\begin{aligned}
&\left\| \frac{F^A(t)\varphi - \varphi}{t} - \Delta_A \varphi \right\|_\infty \\
&\leq \sup_{x\in\mathbb{R}^d} \frac{1}{t} \int_{\mathbb{R}^d} p_A(t,x,x-z)|(\operatorname{Hess}\varphi(x-\theta z) - \operatorname{Hess}\varphi(x))z\cdot z|dz \\
&\leq \frac{K_1}{t} \int_{\mathbb{R}^d} (4\pi t A_0)^{-d/2} e^{-\frac{|z|^2}{4tA_0}} |z|^{2+\alpha} dz \\
&= K_2 t^{\alpha/2} \to 0, \qquad \text{as } t \to 0.
\end{aligned}$$

Therefore, all the assumptions of the Chernoff Theorem 1.0.6 are fulfilled and the family $(F^A(t))_{t\geq 0}$ is Chernoff equivalent to the semigroup $(T_t^{\Delta_A})_{t\geq 0}$. □

The requirement $a_0|z|^2 \leq z\cdot A(x)z$ of Assumption 2.3.1 can be weaken using the results of Section 2.2. Namely, the following is true due to Theorem 2.3.5, Proposition 2.2.6 and Lemma 2.2.4.

Proposition 2.3.6. *Let A be as in Assumption 2.3.1, let $a \in C_b(\mathbb{R}^d)$ be a scalar function with $a(x) > 0$ for all $x \in \mathbb{R}^d$. Consider $\widehat{A} : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^d)$ such that $\widehat{A}(x) = a(x)A(x)$. Consider also the operator $\Delta_{\widehat{A}}$ defined (using notations of Section 2.2) for each $\varphi \in C^2(\mathbb{R}^d)$ by*

$$\Delta_{\widehat{A}}\varphi(x) := \operatorname{tr}(\widehat{A}(x)\operatorname{Hess}\varphi(x)) = a(x)\Delta_A\varphi(x) \equiv \widehat{\Delta_A}\varphi(x). \tag{2.3.7}$$

Let Assumption 2.3.2 and Assumption 2.2.1 with $L := \Delta_A$ be fullfilled. Consider the family $(\widehat{F}^A(t))_{t\geq 0}$ of linear operators on X defined by $\widehat{F}^A(0) := \mathrm{Id}$ and

$$\widehat{F}^A(t)\varphi(x) := \frac{1}{\sqrt{(4\pi t)^d \det \widehat{A}(x)}} \int_{\mathbb{R}^d} \exp\left(-\frac{\widehat{A}^{-1}(x)(x-y)\cdot(x-y)}{4t} \right) \varphi(y)dy \tag{2.3.8}$$

for all $t > 0$, $\varphi \in X$ and $x \in \mathbb{R}^d$. Then the family $(\widehat{F}^A(t))_{t\geq 0}$ is strongly continuous and Chernoff equivalent to the semigroup $(T_t^{\Delta_{\widehat{A}}})_{t\geq 0}$ on $C_\infty(\mathbb{R}^d)$ generated by the closure of $(\Delta_{\widehat{A}}, C_c^{2,\alpha}(\mathbb{R}^d))$. Therefore, the following Lagrangian Feynman Formula holds true for all $t > 0$, $\varphi \in X$ and $x_0 \in \mathbb{R}^d$

$$\begin{aligned}
&T_t^{\Delta_A}\varphi(x_0) \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} p_A(a(x_0)t/n, x_0, x_1) \cdots p_A(a(x_{n-1})t/n, x_{n-1}, x_n)\varphi(x_n)dx_n \ldots dx_1,
\end{aligned}$$

where the order of integration is from x_n to x_1. And the convergence is uniform with respect to $x_0 \in \mathbb{R}^d$ and $t \in (0, t^*]$ for all $t^* > 0$.

Combining Proposition 2.3.6, Theorem 2.1.1 and Lemma 2.1.6, we immediately obtain the following result (cf. Corollary 2.1.2).

Proposition 2.3.7. *Let A be as in Assumption 2.3.1, let $a \in C_b(\mathbb{R}^d)$ be a scalar function with $a(x) > 0$ for all $x \in \mathbb{R}^d$. Consider $\widehat{A} : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^d)$ such that $\widehat{A}(x) = a(x)A(x)$. Let a real-valued function $C \in C(\mathbb{R}^d)$ be such that $\inf_{x\in\mathbb{R}^d} C(x) > -\infty$. Let a vector field B be of class $C_b^2(\mathbb{R}^d;\mathbb{R}^d)$. Consider the operator $\Delta_{\widehat{A}} - B\nabla - C$ defined for all $\varphi \in C_c^{2,\alpha}(\mathbb{R}^d)$ by*

$$(\Delta_{\widehat{A}} - B\nabla - C)\varphi(x) := \operatorname{tr}(\widehat{A}(x)\operatorname{Hess}\varphi(x)) - B(x)\cdot\nabla\varphi(x) - C(x)\varphi(x). \quad (2.3.9)$$

Assume that the operator $(\Delta_{\widehat{A}} - B\nabla - C, C_c^{2,\alpha}(\mathbb{R}^d))$ is closable and the closure generates a strongly continuous semigroup $(T_t^{\Delta_{\widehat{A}}-B\nabla-C})_{t\geq 0}$ on $C_\infty(\mathbb{R}^d)$. Consider the families $(\widehat{F}^A(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ given by formula (2.3.8) and formula (2.1.2) respectively. Consider the family $(F(t))_{t\geq 0}$ with $F(0) := \mathrm{Id}$ and $F(t) := e^{-tC}\circ S(t)\circ \widehat{F}^A(t)$ for $t > 0$. Hence for all $t > 0$, $\varphi \in C_\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$

$$F(t)\varphi(x) := \frac{e^{-tC(x)}}{\sqrt{(4\pi t)^d \det \widehat{A}(x - tB(x))}} \times$$
$$\times \int_{\mathbb{R}^d} \exp\left(-\frac{\widehat{A}^{-1}(x-tB(x))(x-tB(x)-y)\cdot(x-tB(x)-y)}{4t}\right)\varphi(y)dy. \quad (2.3.10)$$

Then the family $(F(t))_{t\geq 0}$ is strongly continuous and Chernoff equivalent to the semigroup $(T_t^{\Delta_{\widehat{A}}-B\nabla-C})_{t\geq 0}$. And the Chernoff approximation

$$T_t^{\Delta_{\widehat{A}}-B\nabla-C}\varphi(x) = \lim_{n\to\infty}\left[F(t/n)\right]^n\varphi(x)$$

(and hence the corresponding Lagrangian Feynman formula) is valid for all $t > 0$, $\varphi \in C_\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ uniformly with respect to $x \in \mathbb{R}^d$ and $t \in (0, t^*]$ for all $t^* > 0$.

Remark 2.3.8. Some sufficient conditions on the coefficients $\widehat{A}$, B and C to ensure the existence of the strongly continuous semigroup $(T_t^{\Delta_{\widehat{A}}-B\nabla-C})_{t\geq 0}$ on X can be found, e.g., in Appendix D, Thm. D.0.9, Thm. D.0.10.

Remark 2.3.9. (i) Let $\widehat{A}$, B and C be as before. Consider the family $(F^{\widehat{A},B,C}(t))_{t\geq 0}$ on $C_\infty(\mathbb{R}^d)$ such that $F^{\widehat{A},B,C}(0) = \mathrm{Id}$ and for all $t > 0$ with $\varphi \in C_\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$

$$F^{\widehat{A},B,C}(t)\varphi(x) :=$$
$$= \frac{e^{-tC(x)}}{\sqrt{(4\pi t)^d \det \widehat{A}(x)}} \int_{\mathbb{R}^d} \exp\left(-\frac{\widehat{A}^{-1}(x)(x-tB(x)-y)\cdot(x-tB(x)-y)}{4t}\right)\varphi(y)dy. \quad (2.3.11)$$

These operators $F^{\widehat{A},B,C}$ differ from those in formula (2.3.10) by the argument of $\widehat{A}$ and $\widehat{A}^{-1}$. Slightly modifying the proofs of Theorem 2.3.5 and Lemma 2.3.4, one can easily show that, under assumptions of Proposition 2.3.7, the family $(F^{\widehat{A},B,C}(t))_{t\geq 0}$ is also Chernoff equivalent to the semigroup $(T_t^{\Delta_{\widehat{A}}-B\nabla-C})_{t\geq 0}$. This result follows also from Theorem 3.2.6 (see formula (3.2.9) in Remark 3.2.7) of Chapter 3.

(ii) The family $(F^{\widehat{A},B,C}(t))_{t\geq 0}$ has been used in Butko, Grothaus, and Smolyanov, 2008, Butko, Grothaus, and Smolyanov, 2010 (to construct Feynman formulae for solutions of Cauchy–Dirichlet problems for second order parabolic equations with variable coefficients in bounded and unbounded domains, cf. Chapter 5). Later on, A. S. Plyashechnik has generalized this result (in Plyashechnik, 2013b, Plyashechnik, 2013a) by proving the Chernoff equivalence of the family $(F^{\widehat{A},B,C}(t))_{t\geq 0}$ to the corresponding semigroup $(T_t^{\Delta_{\widehat{A}}-B\nabla-C})_{t\geq 0}$ in Banach spaces $L^p(\mathbb{R}^d)$, $p\in[1,\infty)$, and by including the case when the coefficients A, B and C depend also on time (this requires a generalization of the Chernoff Theorem to the case of two-parameter families of operators, see Obrezkov, Smolyanov, and Trumen, 2005, Plyashechnik, 2012).

(iii) Let us investigate the family $(F^{\widehat{A},B,C}(t))_{t\geq 0}$ more carefully in the case $a\equiv 1$, i.e. $\widehat{A}\equiv A$. Following the strategy of Example 2.1.10, we have

$$F^{A,B,C}(t)\varphi(x)=e^{-tC(x)}\int\limits_{\mathbb{R}^d} e^{\frac{A^{-1}(x)B(x)\cdot(x-y)}{2}}e^{-t\frac{|A^{-1/2}(x)B(x)|^2}{4}}\varphi(y)p_A(t,x,y)dy, \quad (2.3.12)$$

where the function p_A is given by (2.3.5). Therefore, under assumptions of Proposition 2.3.7, the following Lagrangian Feynman formula holds for all $t>0$, $\varphi\in X$ and $x_0\in\mathbb{R}^d$:

$$T_t^{\Delta_A-B\nabla-C}\varphi(x_0)=\lim_{n\to\infty}\int\limits_{\mathbb{R}^{dn}} e^{-\frac{t}{n}\sum\limits_{k=1}^{n}C(x_{k-1})}e^{\frac{1}{2}\sum\limits_{k=1}^{n}A^{-1}(x_{k-1})B(x_{k-1})\cdot(x_{k-1}-x_k)}\times \quad (2.3.13)$$
$$\times\, e^{-\frac{t}{4n}\sum\limits_{k=1}^{n}|A^{-1/2}(x_{k-1})B(x_{k-1})|^2}\varphi(x_n)p_A(t/n,x_0,x_1)\dots p_A(t/n,x_{n-1},x_n)\,dx_1\cdots dx_n.$$

And the convergence is uniform with respect to $x_0\in\mathbb{R}^d$ and $t\in(0,t^*]$ for all $t^*>0$. The limit in the right hand side of formula (2.3.13) coincides with the following path integral (see the discussion in Remark 3.4.5):

$$T_t^{\Delta_A-B\nabla-C}\varphi(x_0)=\mathbb{E}^{x_0}\Bigg[\exp\Bigg(-\int\limits_0^t C(X_s)ds\Bigg)\exp\Bigg(-\frac{1}{2}\int\limits_0^t A^{-1}(X_s)B(X_s)\cdot dX_s\Bigg)\times$$
$$\times\exp\Bigg(-\frac{1}{4}\int\limits_0^t A^{-1}(X_s)B(X_s)\cdot B(X_s)ds\Bigg)\varphi(X_t)\Bigg].$$

Here the stochastic integral $\int_0^t A^{-1}(X_s)B(X_s)\cdot dX_s$ is an Itô integral. And $\mathbb{E}^{x_0}$ is the expectation of a (starting at x_0) diffusion process $(X_t)_{t\geq 0}$ with the variable diffusion matrix A and without any drift, i.e the generator of $(X_t)_{t\geq 0}$ is $L=\Delta_A$ and $(X_t)_{t\geq 0}$ solves the stochastic differential equation

$$dX_t=\sqrt{2A(X_t)}dB_t$$

with a d-dimensional Brownian motion $(B_t)_{t\geq 0}$.

Chapter 3

Hamiltonian and Lagrangian Feynman formulae for semigroups generated by pseudo-differential operators related to Feller processes

Evolution of a classical physical system can be described with the help of its Hamilton function (energy) H defined on the phase space $Q \times P$ of the system. In quantum mechanical formalism, one should replace the phase space variables $(q,p) \in Q \times P$ by a couple of operators $(\widehat{q}, \widehat{p})$ acting in a Hilbert space of some functions defined on Q by the formulae $\widehat{q}\varphi(q) := q\varphi(q)$, $\widehat{p}\varphi(q) := -i\nabla\varphi(q)$. To construct a quantum analogue of a given classical system one should consider a Hamilton (energy) operator $\widehat{H}$ which somehow corresponds to the given Hamilton function H, e.g., $\widehat{H}$ is formally given by $H(\widehat{q}, \widehat{p})$. Since $\widehat{q}$ and $\widehat{p}$ do not commute, the formal expression $H(\widehat{q}, \widehat{p})$ can be interpreted in many different ways. This gives rise to different matching procedures (quantizations) $H \mapsto \widehat{H}$. So, let us consider $Q = P = \mathbb{R}^d$, a measurable function $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ and $\tau \in [0,1]$. We define a *pseudo-differential operator* $\widehat{H}_\tau$ *with* τ*-symbol* H on a Banach space $(X, \|\cdot\|_X)$ of some functions on $\mathbb{R}^d$ by

$$\widehat{H}_\tau\varphi(q) := (2\pi)^{-d} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} e^{ip\cdot(q-q')} H(\tau q + (1-\tau)q', p)\varphi(q')\, dq'\, dp, \quad q \in \mathbb{R}^d \tag{3.0.1}$$

where the domain $\mathrm{Dom}(\widehat{H}_\tau)$ is the set of all $\varphi \in X$ such that the right hand side of the formula (3.0.1) is well defined as an element of $(X, \|\cdot\|_X)$.

The mapping $H \mapsto \widehat{H}_\tau$ from the space of functions on $\mathbb{R}^d \times \mathbb{R}^d$ into the space of linear operators in $(X, \|\cdot\|_X)$ is called the τ*-quantization*, the operator $\widehat{H}_\tau$ itself is called the τ*-quantization of the function* H. Note that if the symbol H is a sum of functions depending only on one of the variables q or p then the pseudo-differential operators $\widehat{H}_\tau$ coincide for all $\tau \in [0,1]$. If $H(q,p) = qp = pq$, $q, p \in \mathbb{R}$ then

$$\widehat{H}_\tau\varphi(q) = -i\tau q \frac{\partial}{\partial q}\varphi(q) - i(1-\tau)\frac{\partial}{\partial q}(q\varphi(q)).$$

Therefore, different τ correspond to different orderings of non-commuting operators such that we have the "qp"-quantization for $\tau = 1$, the "pq"-quantization for $\tau = 0$ and the Weyl quantization for $\tau = 1/2$. The operator $\widehat{H}_\tau$ is called the *Hamiltonian* of the quantum system obtained by τ-quantization of a classical system with the Hamilton function H[1].

Motivated by the above mentioned quantum mechanical background, we consider in this chapter evolution semigroups [2] generated by pseudo-differential operators obtained by different τ-quantizations from a certain class of functions (or symbols) defined on the phase space $\mathbb{R}^d \times \mathbb{R}^d$. This class contains Hamilton functions of classical particles with variable mass in magnetic and potential fields [3] and more general symbols[4] given by the Lévy-Khintchine formula (see formula (3.1.2) below) and hence related to Feller processes (see Section 3.1).

The aim of this chapter is to restore (for a given procedure of quantization) the semigroup $e^{-t\widehat{H}}$ if the symbol $-H$ of its generator is known. Our approach is to approximate the semigroup $e^{-t\widehat{H}}$ by the family of pseudo-differential operators $\widehat{e^{-tH}}$ obtained by the same procedure of quantization from the symbol e^{-tH}. It is worth to emphasize that, if the function H depends on both variables q and p, then $e^{-t\widehat{H}} \neq \widehat{e^{-tH}}$. Nevertheless, under certain conditions, one succeeds to prove

[1] The question, which ordering of non-commuting operators $\widehat{q}$ and $\widehat{p}$ (and hence which Hamiltonian) is more appropriate can be solved differently for different problems. For example, pq-quantization ($\tau = 0$) arises on physical grounds in the discussion of the Fokker-Planck equation in Sect. 18.9 of Kleinert, 2006. For some other problems the Weyl quantization ($\tau = 1/2$) is used as it leads to symmetric operators $\widehat{H}$. However, in some cases, one could obtain formally symmetric operators also by the rule $\widehat{H} := \frac{1}{2}(\widehat{H}_\tau + \widehat{H}_{1-\tau})$, $\tau \in [0,1]$. In the case of a free one-dimensional particle with position-dependent mass, i.e., with $H(q,p) = \frac{1}{2m(q)}p^2$, this rule produces a continuous analogue of the Hamiltonian $\widehat{H} = \frac{1}{4}\left(m^\alpha(q)\widehat{p}m^\beta(q)\widehat{p}m^\gamma(q) + m^\gamma(q)\widehat{p}m^\beta(q)\widehat{p}m^\alpha(q)\right)$ which is often used in the literature (see, e.g., Ganguly et al., 2006, Bouchemla and Chetouani, 2009 and references therein). The parameters α, β, γ with $\alpha+\beta+\gamma = -1$ can be chosen differently for specific models. If the dependence of mass on position is smooth, one can consider dynamics of a particle with position-dependent mass in magnetic and potential fields in a flat space as dynamics of a particle with constant mass (also in some magnetic and potential fields) in a space with curvature. Then the operator-ordering problem solves naturally by means of geometrical principles (see Kleinert, 2006, Sect. 10.2.3). If the mass is piecewise constant and has jumps, the corresponding geometry would be rather singular. Moreover, the operators $\widehat{H}$ would have different self-adjoint extensions depending on matching conditions imposed at jumps, i.e. depending on the domain of definition (cf. Gadella, Kuru, and Negro, 2007). In this context the operator-ordering problem appears again and interplays with matching conditions at jumps. One way to solve this problem would be to consider appropriate smooth approximations of discontinuous mass, which approximate also the imposed matching conditions.

[2] For some questions of Quantum Mechanics the object $e^{-t\widehat{H}}$ is as basic as the object $e^{-it\widehat{H}}$ which describes the evolution of a quantum system with Hamiltonian $\widehat{H}$ (cf. Sect. 1.17.1 of Kleinert, 2006 and Simon, 1979, p. 7).

[3] Such Hamilton functions are second order polynomials with respect to p with variable q-dependent coefficients.

[4] This class includes, in particular, symbols corresponding to relativistic Hamiltonians and fractional Laplacians.

via the Chernoff Theorem that

$$e^{-t\widehat{H}} = \lim_{n\to\infty} \left[\widehat{e^{-\frac{t}{n}H}}\right]^n. \tag{3.0.2}$$

The limit in the right hand side is the limit of n-fold iterated integrals over the phase space when n tends to infinity. This leads to a representation of the considered semigroup $e^{-t\widehat{H}}$ in the form of a so-called *Hamiltonian Feynman formula*. This terminology is motivated by the fact, that such representation is further interpreted as a phase space (or, Hamiltonian) Feynman path integral with $\exp\left(-\int_0^t H(q(s),p(s))ds\right)$ in the integrand (see Section 3.5 for all details and definitions). Moreover, using connections between different procedures of τ-quantization, one succeeds to obtain different Hamiltinoan Feynman formulae and hence different phase space Feynman path integrals representing the same semigroup. Further, in several cases, it is possible to convert the pseudo-differential operators $\widehat{e^{-tH}}$ into integral operators with "nice" kernels by interchanging the order of integration and by proceeding the integration with respect to the variable p in the definition of pseudo-differential operators $\widehat{e^{-tH}}$. This gives rise to a representation of the considered semigroup $e^{-t\widehat{H}}$ by a so-called *Lagrangian Feynman formula*. The pre-limit expressions in the obtained Lagrangian Feynman formula approximate a path integral with respect to the probability measure corresponding to a Feller (or Feller type) process with generator $-\widehat{H}$. Hence a Feynman–Kac type representation of the semigroup $e^{-t\widehat{H}}$ arises. Finally, all these results establish a connection between some phase space Feynman path integrals and some path integrals with respect to probability measures.

3.1 Feller processes, Feller semigroups and their generators

There is no standard usage of the term *Feller semigroup* in the literature and each author exploits his or her own definition. In this section, we follow the terminology and exposition of Böttcher, Schilling, and Wang, 2013, see also Jacob, 2001.

Definition 3.1.1. Let Q be a locally compact separable metric space. Let $(T_t)_{t\geq 0}$ be a semigroup on the space $B_b(Q)$ of bounded Borel measurable functions on Q.

(i) The semigroup $(T_t)_{t\geq 0}$ is called *positivity preserving* if

$$T_t\varphi \geq 0 \quad \text{for each } \varphi \in B_b(Q) \text{ with } \varphi \geq 0.$$

(ii) The semigroup $(T_t)_{t\geq 0}$ is called *sub-Markov semigroup* if it is positivity preserving and has the *sub-Markov property*

$$T_t\varphi \leq 1 \quad \text{for each } \varphi \in B_b(Q) \text{ with } \varphi \leq 1.$$

(iii) The semigroup $(T_t)_{t\geq 0}$ is called *Markov semigroup* if it is sub-Markov and *conservative*, i.e. $T_t 1 = 1$.

(iv) The semigroup $(T_t)_{t\geq 0}$ is called *Feller semigroup* if it is sub-Markov, satisfies the *Feller property*

$$T_t\varphi \in C_0(Q) \quad \text{for each } \varphi \in C_0(Q) \text{ and each } t > 0,$$

and is strongly continuous in the Banach space $C_0(Q)$, i.e., for each $\varphi \in C_0(Q)$, holds $\lim_{t\to 0} \|T_t\varphi - \varphi\|_\infty = 0$.

Note that a sub-Markov semigroup is automatically contractive, i.e. $\|T_t\| \leq 1$, and monotone, i. e. $T_t\varphi_1 \leq T_t\varphi_2$ for all $\varphi_1 \leq \varphi_2$, $\varphi_1, \varphi_2 \in B_b(Q)$. Sometimes a strongly continuous, positivity preserving, contractive semigroup on the Banach space $C_0(Q)$ is called a Feller semigroup, although it is only defined on $C_0(Q)$. Using a version of the Riesz representation theorem (cf. Thm. 1.5 in Böttcher, Schilling, and Wang, 2013), one can extend this semigroup onto $B_b(Q)$ to a sub-Markov semigroup in the sense of Definition 3.1.1.

Considering a Feller semigroup as a strongly continuous semigroup on $C_0(Q)$, one defines its generator $(L, \mathrm{Dom}(L))$ (which will be called *Feller generator* in the sequel) in the usual way (cf. Definition 1.0.1 (iv)):

$$\mathrm{Dom}(L) := \left\{ \varphi \in C_0(Q) \,\middle|\, \lim_{t\to 0} \frac{T_t\varphi - \varphi}{t} \quad \text{exists in } C_0(Q) \right\},$$
$$L\varphi := \lim_{t\to 0} \frac{T_t\varphi - \varphi}{t} \quad \forall \varphi \in \mathrm{Dom}(L).$$

There exist also some other notions describing sub-Markovian semigroups having image in the set of continuous functions. In particular, the following notion will be used in Chapters 4 and 5.

Definition 3.1.2. A sub-Markov semigroup $(T_t)_{t\geq 0}$ is called *strong Feller semigroup* if $T_t : B_b(Q) \to C_b(Q)$ for all $t > 0$.

Note that a strong Feller semigroup need not be Feller and vice versa. Some conditions on semigroups to be strong Feller can be found in Lemma 1.12, Theorem 1.14 and Theorem 1.15 of Böttcher, Schilling, and Wang, 2013. Other related notions and connections between them can be found in Böttcher, Schilling, and Wang, 2013, p.7-11. Below, we describe the connection of Feller semigroups with Feller processes.

Definition 3.1.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$, and let $(\xi_t)_{t\geq 0}$ be a temporally homogeneous Markov process with state space

$(Q, \mathcal{B}(Q))$. The process $(\xi_t)_{t\geq 0}$ is called *Feller process* if its transition semigroup $(T_t)_{t\geq 0}$,

$$T_t\varphi(x) := \int_Q \varphi(y)\mathbb{P}^x(\xi_t \in dy) = \mathbb{E}^x[\varphi(\xi_t)],$$

is a Feller semigroup.

In this setting, a Feller process $(\xi_t)_{t\geq 0}$ has *infinite life-time*, i.e. $\mathbb{P}^x(\xi_t \in Q) = 1$ for all $t > 0$ and all $x \in Q$. This means that the corresponding Feller semigroup is conservative. On the other hand, consider a conservative Feller semigroup $(T_t)_{t\geq 0}$. Using the Riesz representation theorem, one can write T_t as an integral operator $T_t\varphi(x) := \int_Q \varphi(y)P(t,x,dy)$ where a family $(P(t,x,\cdot))_{t\geq 0, x\in Q}$ is a uniquely defined transition kernel. By Kolmogorov's standard procedure, one constructs a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Markov process $(\xi_t)_{t\geq 0}$ with state space Q such that

$$\mathbb{P}^x(\xi_t \in B) \equiv \mathbb{P}(\xi_t \in B \,|\, \xi_0 = x) = P(t,x,B) \quad \text{and} \quad \mathbb{E}^x[\varphi(\xi_t)] = T_t\varphi(x).$$

Therefore, conservative Feller semigroups and Feller processes with infinite life-time are essentially in one-to-one correspondence. Let now $(T_t)_{t\geq 0}$ be a Feller semigroup which is not necessary conservative. Consider the one-point compactification $\overline{Q} := Q \cup \{\partial\}$ of Q and a family of operators $(\overline{T}_t)_{t\geq 0}$ defined on $C_0(\overline{Q}) = C_b(\overline{Q})$ by $\overline{T}_t\varphi := \varphi(\partial) + T_t(\varphi - \varphi(\partial))$. Then $(\overline{T}_t)_{t\geq 0}$ is a conservative Feller semigroup on $C_0(\overline{Q})$ and hence corresponds to a Feller process $(\xi_t)_{t\geq 0}$ with infinite life-time and with the enriched state space $\overline{Q}$ (cf. Corollary 3.2.13 in Jacob, 2005 and discussion on pp.12-13 of Böttcher, Schilling, and Wang, 2013). Such Feller process can be interpreted as a process with the state space Q and random life-time, whereas the state ∂ can be interpreted as a "cemetery" where the process ξ_t stays beyond its life-time.

Proposition 3.1.4. *Let $(\xi_t)_{t\geq 0}$ be a Feller process on $Q = \mathbb{R}^d$ (or $Q = [0,\infty)$). Let the corresponding Feller semigroup $(T_t)_{t\geq 0}$ be invariant under translations, i.e.*

$$\Theta_a(T_t\varphi) = T_t(\Theta_a\varphi) \quad \forall\, t \geq 0,\, a \in Q, \quad \textit{where} \quad \Theta_a\varphi(x) := \varphi(x+a),\ \forall\, x \in Q.$$

Then $(\xi_t)_{t\geq 0}$ is a Lévy process.

Some basic facts about Lévy processes, their generators and other related objects are exposed in Appendix C. In the general situation, Feller processes on $Q = \mathbb{R}^d$ can be considered as Lévy-type processes whose generators generalize generators of Lévy processes to the case of "variable coefficients" (i.e. variable Lévy characteristics). Namely, the following characterization (cf. Theorem C.0.6 in Appendix C) is true due to P. Courrège (Courrège, 1965/1966, Bony, Courrège, and Priouret, 1968) and W. von Waldenfels (Waldenfels, 1961, Waldenfels, 1965, Waldenfels, 1964).

Theorem 3.1.5 (Courrège, von Waldenfels)**.** *Let $(L, \mathrm{Dom}(L))$ be a Feller generator such that $C_c^\infty(\mathbb{R}^d) \subset \mathrm{Dom}(L)$. Then $L\big|_{C_c^\infty(\mathbb{R}^d)}$ is a* pseudo-differential operator with the 1-symbol[5] $-H:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{C}$, *i.e. the operator* $-\widehat{H}_1$ *defined by*

$$L\varphi(q)\equiv-\widehat{H}_1\varphi(q):=-(2\pi)^{-d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}e^{ip\cdot(q-q')}\,H(q,p)\,\varphi(q')\,dq'dp,\qquad \varphi\in C_c^\infty(\mathbb{R}^d);\tag{3.1.1}$$

the function $H:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{C}$ *is measurable, locally bounded in both variables* (q,p), *and satisfies for each fixed* $q\in\mathbb{R}^d$ *the following Lévy-Khintchine representation*

$$H(q,p)=C(q)+iB(q)\cdot p+p\cdot A(q)p+\int\limits_{y\neq0}\left(1-e^{iy\cdot p}+\frac{iy\cdot p}{1+|y|^2}\right)N(q,dy),\tag{3.1.2}$$

where $(C(q),B(q),A(q),N(q,.))$ *for each fixed* $q\in\mathbb{R}^d$ *are the Lévy characteristics of the continuous negative definite function* $H(q,.)$.

Observe that formula (3.1.2) automatically implies the continuity of the mapping $p\mapsto H(q,p)$ for each $q\in\mathbb{R}^d$. Since no more smoothness of H is assumed, such symbols H do not belong to any of the classical symbol classes of pseudo-differential operators; consequently, we do not have a Hörmander or Maslov symbolic calculus at our disposal.

Let $-\widehat{H}_1$ be a pseudo-differential operator with the 1-symbol $-H(q,p)$ as in Theorem 3.1.5. Since $H(q,p)$ is represented by the Lévy-Khintchine type formula (3.1.2), we can use Fourier inversion in (3.1.1) and find that the integro-differential operator

$$\begin{aligned}L\varphi(q)&=-C(q)\varphi(q)-B(q)\cdot\nabla\varphi(q)+\mathrm{tr}(A(q)\,\mathrm{Hess}\,\varphi(q))\\&\quad+\int_{y\neq0}\left(\varphi(q+y)-\varphi(q)-\frac{y\cdot\nabla\varphi(q)}{1+|y|^2}\right)N(q,dy)\end{aligned}\tag{3.1.3}$$

extends $\left(-\widehat{H}_1,C_c^\infty(\mathbb{R}^d)\right)$ to the set $C^2_\infty(\mathbb{R}^d)$ (cf. Theorem C.0.7 in Appendix C). Note that Lemma 3.1.8 at the end of this Section together with the integration properties of $N(q,dy)$,

$$\int\limits_{y\neq0}\frac{|y|^2}{1+|y|^2}\,N(q,dy)<\infty,\qquad\forall\,q\in\mathbb{R}^d,$$

ensure that the integral in (3.1.3) converges. From now on we will use the pseudo-differential representation (3.1.1) and the integro-differential representation (3.1.3) simultaneously. Note that, in the case $N\equiv0$, the operator L is just a second order differential operator with variable coefficients. Also in the general case, when the symbol $-H$ depends on the state space variable q, we call $-\widehat{H}_1$ an *operator with "variable coefficients"*. The role of coefficients is played by the Lévy characteristics $(C(q),B(q),A(q),N(q,.))$. In abuse of notation we also call $-H$ the 1-*symbol of a Feller process*.

[5]Hence we consider the τ-quantization with τ equal to one.

The generators with "constant coefficients" correspond to Lévy processes, i.e. to translation invariant Feller semigroups. In this case, their 1-symbols H do not depend on q and $H(q,p) \equiv \psi(p)$ for some continuous negative definite function ψ. And conversely, each continuous negative definite symbol ψ corresponds to the strongly continuous semigroup (which is a Feller semigroup, invariant under translations) generated by the closure of the pseudo-differential operator with the 1-symbol $-\psi$. Therefore, there is a one-to-one correspondence between continuous negative definite functions ψ, Lévy characteristics (C,B,A,N), translation invariant Feller semigroups and Lévy processes (with killing). This is no longer the case for general Feller semigroups and processes. For every Feller semigroup (and Feller process), whose generator $(L, \mathrm{Dom}(L))$ satisfies $C_c^\infty(\mathbb{R}^d) \subset \mathrm{Dom}(L)$, holds that $L\big|_{C_c^\infty(\mathbb{R}^d)}$ is a pseudo-differential operator $-\widehat{H}_1$ whose 1-symbol $H(q,\cdot)$ is a continuous negative definite function for each $q \in \mathbb{R}^d$. The converse statement is a difficult problem: *When does a given q-dependent function H, such that $H(q,\cdot)$ is a continuous negative definite function for each $q \in \mathbb{R}^d$, give rise to a Feller semigroup (Feller process)*? This question was first asked by N. Jacob in Jacob, 1992. This problem has been discussed at length in a series of papers see Jacob, 2002; Jacob and Schilling, 2001 and the literature given there. An overview of recent developments can be founded in Böttcher, Schilling, and Wang, 2013, where different strategies to obtain sufficient conditions on the symbol $-H$, such that $\big(-\widehat{H}_1, C_c^\infty(\mathbb{R}^d)\big)$ extends to the generator of a Feller semigroup, are presented. The problem to find optimal (necessary and) sufficient conditions remains open.

For a Lévy process $(\xi_t)_{t\geq 0}$, its symbol $-H$ coincides with $-\psi$, where ψ is the characteristic exponent of the process $(\xi_t)_{t\geq 0}$. Hence the symbol of the corresponding Feller semigroup is given by $e^{-t\psi}$ and the following holds:

$$-\psi(p) = \frac{d}{dt}e^{-t\psi(p)}\Big|_{t=0} = \lim_{t\to 0}\frac{\mathbb{E}\big[e^{ip\cdot\xi_t}\big]-1}{t} = \lim_{t\to 0}\frac{\mathbb{E}^q\big[e^{ip\cdot(\xi_t-q)}\big]-1}{t}, \qquad \forall\, q \in \mathbb{R}^d.$$

In the general case of a Feller process $(\xi_t)_{t\geq 0}$, it is still true that the corresponding Feller semigroup $(T_t)_{t\geq 0}$ consists of pseudo-differential operators T_t whose 1-symbols $\lambda_t(q,p)$ are given via

$$\lambda_t(q,p) := \mathbb{E}^q\big[e^{ip\cdot(\xi_t-q)}\big].$$

Nevertheless, $\lambda_t(q,p) \neq e^{-tH(q,p)}$ in general. The only connection between λ_t and $-H$ is given below.

Proposition 3.1.6. *Let $(\xi_t)_{t\geq 0}$ be a Feller process in $\mathbb{R}^d$ with generator $(L, \mathrm{Dom}(L))$ such that $C_c^\infty(\mathbb{R}^d) \subset \mathrm{Dom}(L)$. If the mapping $q \mapsto -H(q,p)$ is continuous for all $p \in \mathbb{R}^d$ and the operator $-\widehat{H}_1$ has "bounded coefficients" then*

$$-H(q,p) = \frac{d}{dt}\lambda_t(q,p)\Big|_{t=0} = \lim_{t\to 0}\frac{\mathbb{E}^q e^{ip\cdot(\xi_t-q)}-1}{t}.$$

Hence a natural problem arises: *How to restore the symbol λ_t of the semigroup by* *the known symbol $-H$ of the generator/process*? The next sections of Chapter 3 are devoted to the solution of this and some related problems by means of the Chernoff theorem. Before we proceed with the solution, let us present some properties of Feller generators which will be important in the sequel (cf. Thm 2.37, Thm 2.30, Lemma 2.32, Thm 2.33 in Böttcher, Schilling, and Wang, 2013).

Proposition 3.1.7. *Let $(T_t)_{t\geq 0}$ be a Feller semigroup with generator $(L, \mathrm{Dom}(L))$. Assume that $C_c^\infty(\mathbb{R}^d) \subset \mathrm{Dom}(L)$, i.e. $L\big|_{C_c^\infty(\mathbb{R}^d)}$ is a pseudo-differential operator with the 1-symbol $-H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, such that H is given by the formula (3.1.2) with some Lévy characteristics $(C(q), B(q), A(q), N(q,\cdot))$ for each fixed $q \in \mathbb{R}^d$. Then the following holds:*

(a) L has a unique extension to $C_b^2(\mathbb{R}^d)$, again denoted by L, satisfying

$$L\varphi(q) = \lim_{n\to\infty} L(\varphi\phi_n)(q), \quad \forall\, q \in \mathbb{R}^d,$$

for any sequence $(\phi_n)_{n\geq 1} \subset C_c^\infty(\mathbb{R}^d)$ with $1_{B_n(0)} \leq \phi_n \leq 1$. This extension has again the representation (3.1.2).

(b) There exists an absolute constant $c < \infty$ such that

$$|L\varphi(q)| \leq c\gamma(q)\|\varphi\|_{(2)}, \quad \forall\, q \in \mathbb{R}^d,\ \forall\, \varphi \in C_b^2(\mathbb{R}^d),\ \text{where}$$

$$\gamma(q) := C(q) + |B(q)| + |A(q)| + \int\limits_{y\neq 0} |y|^2/(1+|y|^2)\, N(q,dy).$$

(c) If H has "bounded coefficients", i.e. if the defined above γ is a bounded fucntion, then $C_\infty^2 \subset \mathrm{Dom}(L)$, i.e. $L : C_\infty^2(\mathbb{R}^d) \to C_\infty(\mathbb{R}^d)$, and for some $c > 0$ holds the estimate

$$\sup_{q\in\mathbb{R}^d} |H(q,p)| \leq c(1+|p|^2), \quad \forall\, p \in \mathbb{R}^d. \tag{3.1.4}$$

(d) If $N(q,dy)$ satisfies the condition

$$\sup_{q\in\mathbb{R}^d} N(q, \mathbb{R}^d \setminus B_r(0)) = 0$$

for some $r > 0$ (hence the corresponding Feller process has uniformly bounded jumps) then the mapping $q \mapsto H(q,p)$ is continuous for all $p \in \mathbb{R}^d$.

(e) The following two assertions are equivalent:

- *$q \mapsto H(q,0)$ is continuous;*
- *$q \mapsto H(q,p)$ is continuous for all $p \in \mathbb{R}^d$.*

(f) Let the mapping $q \mapsto H(q,p)$ be continuous for all $p \in \mathbb{R}^d$. If the Feller semigroup $(T_t)_{t\geq 0}$ is conservative, then $H(q,0) = c(q) \equiv 0$. On the other hand, if the symbol $H(q,p)$ has bounded coefficients and $H(q,0) = 0$ then $(T_t)_{t\geq 0}$ is conservative and $q \mapsto H(q,p)$ is continuous for all $p \in \mathbb{R}^d$.

Coming back to the justification of the integro-differential representation (3.1.3) of a Feller generator L, we prove

Lemma 3.1.8. *For all $\varphi \in C_b^2(\mathbb{R}^d)$ we have*

$$\left|\varphi(q+y) - \varphi(q) - \frac{y \cdot \nabla\varphi(q)}{1+|y|^2}\right| \le 2\, \frac{|y|^2}{1+|y|^2}\, \|\varphi\|_{(2)}. \tag{3.1.5}$$

Proof. By Taylor's formula, we get for all $q, y \in \mathbb{R}^d$ and some $\theta_{q,y}$ inbetween q and y

$$\begin{aligned}
&\left|(1+|y|^2)\left(\varphi(q+y) - \varphi(q) - \frac{y \cdot \nabla\varphi(q)}{1+|y|^2}\right)\right| \\
&\quad \le |\varphi(q+y) - \varphi(q) - y \cdot \nabla\varphi(q)| + |y|^2\, |\varphi(q+y) - \varphi(q)| \\
&\quad \le \frac{1}{2}\left|\sum_{j,k=1}^{d} y_j y_k \partial_j \partial_k \varphi(\theta_{q,y})\right| + 2|y|^2 \|\varphi\|_\infty \\
&\quad \le 2|y|^2 \left(\|\varphi\|_\infty + \sqrt{\sum_{j,k=1}^{d} \|\partial_j \partial_k \varphi\|_\infty^2}\right) \\
&\quad \le 2|y|^2\, \|\varphi\|_{(2)}.
\end{aligned}$$

□

In the sequel, we need also the following Lemma (cf. the proof of Lemma 3.7.2 in Jacob, 2001 and Lemma 2.15 of Hoh, 1998).

Lemma 3.1.9. *We have*

$$\frac{|y|^2}{1+|y|^2} = \int\limits_{\mathbb{R}^d} (1 - \cos(y \cdot p))\, g(p)\, dp, \qquad y \in \mathbb{R}^d,$$

where $g(p) = \frac{1}{2}\int_0^\infty (2\pi\lambda)^{-d/2}\, e^{-|p|^2/2\lambda}\, e^{-\lambda/2}\, d\lambda$ is integrable and has absolute moments of arbitrary order.

3.2 Feynman formulae for Feller semigroups

In this Section, we consider the case of τ-quantization with $\tau = 1$. Therefore, we omit the reference to τ, call the 1-symbol of a pseudo-differential operator just "the symbol" and denote the pseudo-differential operator with 1-symbol H by $\widehat{H}$.

So, let $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be a function which is measurable, locally bounded in both variables (q, p), and satisfies for each fixed $q \in \mathbb{R}^d$ the Lévy-Khintchine representation (3.1.2), i.e. $H(q, \cdot)$ is a continuous negative definite function for all $q \in \mathbb{R}^d$.

Assumption 3.2.1. Let the function H satisfy the following assertions:

$$\sup_{q\in\mathbb{R}^d}|H(q,p)|\leq\kappa(1+|p|^2)\quad\text{for all}\quad p\in\mathbb{R}^d\quad\text{and some}\quad\kappa>0,\tag{3.2.1}$$

$$p\mapsto H(q,p)\quad\text{is uniformly (w.r.t.}\quad q\in\mathbb{R}^d\text{) continuous at}\quad p=0,\tag{3.2.2}$$

$$q\mapsto H(q,p)\quad\text{is continuous for all}\quad p\in\mathbb{R}^d.\tag{3.2.3}$$

Consider the pseudo-differential operator $\widehat{H}$ with the symbol $H(q,p)$, i.e., for each $\varphi\in C_c^\infty(\mathbb{R}^d)$, we have

$$\widehat{H}\varphi(q)=(2\pi)^{-d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}e^{ip\cdot(q-q')}H(q,p)\varphi(q')dq'dp=(2\pi)^{-d/2}\int\limits_{\mathbb{R}^d}e^{ip\cdot q}H(q,p)\widetilde{\varphi}(p)dp,\tag{3.2.4}$$

where $\widetilde{\varphi}$ is the Fourier transform of φ. Note that (due to the estimate (3.1.4)) the condition (3.2.1) actually means that the pseudo-differential operator $\widehat{H}$ is an operator with "bounded coefficients" $(C(q),B(q),A(q),N(q,\cdot))$.

Assumption 3.2.2. We assume that the function $H(q,p)$ is such that the corresponding pseudo-differential operator $(-\widehat{H},C_c^\infty(\mathbb{R}^d))$ is closable and the closure (denoted by $(L,\mathrm{Dom}(L))$) generates a strongly continuous semigroup on $C_\infty(\mathbb{R}^d)$. This means, in particular, that the set $C_c^\infty(\mathbb{R}^d)$ is assumed to be an operator core for the generator $(L,\mathrm{Dom}(L))$.

Remark 3.2.3. Conditions on the function $H(q,p)$ to fulfill Assumption 3.2.2 can be found, for example, in Vol. 2 of Jacob, 2002 (Thms. 2.6.4, 2.6.9, 2.7.9, 2.7.16, 2.7.19, 2.8.1) or in Jacob and Schilling, 2001. For all these constructions $C_c^\infty(\mathbb{R}^d)$ is always an operator core. Note that Assumption 3.2.2 holds also for generators of Lévy processes (cf. Theorem C.0.7 in Appendix C).

Consider now for each $t\geq0$ the pseudo-differential operator $F(t)$ with the symbol $e^{-tH(q,p)}$, i.e. for $\varphi\in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned}F(t)\varphi(q)&=(2\pi)^{-d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}e^{ip\cdot(q-q')}e^{-tH(q,p)}\varphi(q')dq'dp\\&=(2\pi)^{-d/2}\int\limits_{\mathbb{R}^d}e^{ip\cdot q}e^{-tH(q,p)}\widetilde{\varphi}(p)dp.\end{aligned}\tag{3.2.5}$$

Lemma 3.2.4. *For each $\varphi\in C_c^\infty(\mathbb{R}^d)$, the function $F(t)\varphi$ belongs to $C_\infty(\mathbb{R}^d)$.*

Proof. The Fourier transform $\widetilde{\varphi}$ of a test function $\varphi\in C_c^\infty(\mathbb{R}^d)$ belongs to the Schwartz space $S(\mathbb{R}^d)$ of rapidly decreasing functions. Since $q\mapsto e^{-tH(q,p)}$ is continuous (by assumption (3.2.3)) and bounded ($\mathrm{Re}\ H\geq0$ due to properties of continuous negative definite functions, cf. Appendix C), Lebesgue's dominated convergence theorem shows that $F(t)$ maps $C_c^\infty(\mathbb{R}^d)$ into $C(\mathbb{R}^d)$.

Let us prove, that $F(t)\varphi(q) \to 0$ when $|q| \to \infty$. Since $H(q,\cdot)$ is continuous negative definite for all $q \in \mathbb{R}^d$ then $e^{-tH(q,\cdot)}$ is continuous positive definite for all $q \in \mathbb{R}^d$ and all $t > 0$ due to Theorem C.0.6 and Bochner's theorem. Then the function

$$\left[p \mapsto h_t(q,p) := e^{-tH(q,0)} - e^{-tH(q,p)}\right]$$

is also continuous negative definite for all $q \in \mathbb{R}^d$ by Proposition C.0.5. Hence, $h_t(q,\cdot)$ satisfies a Lévy-Khintchine representation

$$h_t(q,p) = C_t(q) + iB_t(q)\cdot p + p\cdot A_t(q)p + \int\limits_{y\neq 0}\left(1 - e^{iy\cdot p} + \frac{iy\cdot p}{1+|y|^2}\right) N_t(q,dy), \tag{3.2.6}$$

where $(C_t(q), B_t(q), A_t(q), N_t(q,\cdot))$ for each $q \in \mathbb{R}^d$ are the Lévy characteristics of $h_t(q,\cdot)$. Again, we can consider a pseudo-differential operator $\widehat{h_t}$ with the symbol $h_t(q,p)$, i.e. for each $\varphi \in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned}\widehat{h_t}\varphi(q) &= (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{ip\cdot q} h_t(q,p)\widetilde{\varphi}(p)dp \\ &= C_t(q)\varphi(q) + B_t(q)\cdot\nabla\varphi(q) - \sum_{j,k=1}^{d} A_t^{jk}(q)\partial_j\partial_k\varphi(q) \\ &\quad - \int\limits_{y\neq 0}\left(\varphi(q+y) - \varphi(q) - \frac{y\cdot\nabla\varphi(q)}{1+|y|^2}\right) N_t(q,dy)\end{aligned} \tag{3.2.7}$$

Note, that

$$F(t)\varphi(q) = (2\pi)^{-d/2}e^{-tH(q,0)}\int\limits_{\mathbb{R}^d} e^{ip\cdot q}\widetilde{\varphi}(p)dp \;-\; (2\pi)^{-d/2}\int\limits_{\mathbb{R}^d} e^{ip\cdot q}h_t(q,p)\widetilde{\varphi}(p)dp.$$

Since Re $H \geq 0$ then $\sup\limits_{q\in\mathbb{R}^d}\left|e^{-tH(q,0)}\right| \leq 1$, and the first integral in the above formula tends to zero as $|q| \to \infty$ by the Riemann–Lebesgue Theorem. Thus, we only need to show that

$$\left[q \mapsto (2\pi)^{-d/2}\int\limits_{\mathbb{R}^d} e^{ip\cdot q}h_t(q,p)\widetilde{\varphi}(p)dp\right] \in C_\infty(\mathbb{R}^d).$$

As φ has compact support, there is some $R > 0$ such that $\operatorname{supp}\varphi \subset B_R(0)$. For all $|q| > 2R$ formula 3.2.7 becomes

$$\begin{aligned}|\widehat{h_t}\varphi(q)| &= \left|\int\limits_{y\neq 0}\varphi(q+y)\,N_t(q,dy)\right| = \left|\int\limits_{|y|>R}\varphi(q+y)\,N_t(q,dy)\right| \\ &\leq 2\int\limits_{y\neq 0}\frac{|y/R|^2}{1+|y/R|^2}N_t(q,dy)\cdot\|\varphi\|_\infty.\end{aligned}$$

The last line follows from the elementary inequality $\frac{1}{2} \leq \frac{t^2}{1+t^2}$ for $|t| > 1$ which applies if $|y| > R$, and from $\varphi(q+y) = 0$ if $|q| > 2R$ and $|y| \le R$. We can now use Lemma 3.1.9, the Lévy-Khintchine representation of $h_t(q,\cdot)$ and the estimate (3.1.4) for a continuous negative definite function $h_t(q,\frac{\cdot}{R})$ to get

$$\begin{aligned}
|\widehat{h_t}\varphi(q)| &\le 2\|\varphi\|_\infty \int\limits_{y\neq 0}\int\limits_{\mathbb{R}^d}\left(1-\cos\frac{y\cdot p}{R}\right) g(p)\,dp\,N_t(q,dy)\\
&\le 2\|\varphi\|_\infty \int\limits_{\mathbb{R}^d} \operatorname{Re} h_t\left(q,\frac{p}{R}\right) g(p)\,dp\\
&\le 2\|\varphi\|_\infty \int\limits_{\mathbb{R}^d} \left|h_t\left(q,\frac{p}{R}\right)\right| g(p)\,dp\\
&\le 2\|\varphi\|_\infty \sup_{|y|\le 1/R}|h_t(q,y)| \int_{\mathbb{R}^d}\left(1+|p|^2\right) g(p)\,dp.
\end{aligned}$$

Since $g(p)$ has absolute moments of any order, we see that with some constant $c_g > 0$ holds

$$|\widehat{h_t}\varphi(q)| \le c_g\|\varphi\|_\infty \sup_{q\in\mathbb{R}^d}\sup_{|y|\le 1/R}|h_t(q,y)| \qquad \text{for all} \quad |q| > 2R.$$

As $h_t(q,0) = 0$, the condition (3.2.2) tells us that $\lim_{|q|\to\infty}\widehat{h_t}\varphi(q) = 0$. Therefore, the function $\widehat{h_t}\varphi \in C_\infty(\mathbb{R}^d)$. □

Lemma 3.2.5. *The mapping $F(t)$ can be extended to a contraction $F(t) : C_\infty(\mathbb{R}^d) \to C_\infty(\mathbb{R}^d)$ for each $t \ge 0$.*

Proof. Let us freeze the coefficients (see, e.g., Jacob and Potrykus, 2005). For each $t \ge 0$ and each $q_0 \in \mathbb{R}^d$, let us consider the pseudo-differential operator $F^{q_0}(t)$ with the symbol $e^{-tH(q_0,p)}$, i.e., for any $\varphi \in C_c^\infty(\mathbb{R}^d)$, we have

$$F^{q_0}(t)\varphi(q) = (2\pi)^{-d/2}\int\limits_{\mathbb{R}^d} e^{ip\cdot q}e^{-tH(q_0,p)}\widetilde{\varphi}(p)dp.$$

Then $F(t)\varphi(q) = F^q(t)\varphi(q)$ for any $\varphi \in C_c^\infty(\mathbb{R}^d)$ and any $q \in \mathbb{R}^d$. Since for each $q_0 \in \mathbb{R}^d$ the function $H(q_0,\cdot)$ is continuous negative definite then there exists a convolution semigroup $(\eta_t^{q_0})_{t\ge 0}$, such that $\mathcal{F}[\eta_t^{q_0}] = (2\pi)^{-d/2}e^{-tH(q_0,\cdot)}$ and $F^{q_0}(t)\varphi(q) = \int_{\mathbb{R}^d}\varphi(q-y)\eta_t^{q_0}(dy)$ (cf. Theorem C.0.6). Hence, for each $q_0 \in \mathbb{R}^d$, the family $(F^{q_0}(t))_{t\ge 0}$ is a Feller semigroup, and for each q, $q_0 \in \mathbb{R}^d$ we have

$$|F^{q_0}(t)\varphi(q)| = \left|\int\limits_{\mathbb{R}^d}\varphi(q-y)\eta_t^{q_0}(dy)\right| \le \|\varphi\|_\infty.$$

Then $\|F(t)\varphi\|_\infty = \sup_{q\in\mathbb{R}^d}|F(t)\varphi(q)| = \sup_{q\in\mathbb{R}^d}|F^q(t)\varphi(q)| \le \|\varphi\|_\infty$ for any $\varphi \in C_c^\infty(\mathbb{R}^d)$. Hence the operators $F(t)$ for all $t \ge 0$ can be extended to a contraction from $C_\infty(\mathbb{R}^d)$ into itself by the B.L.T. theorem A.0.19. □

Theorem 3.2.6. *Let the function $H:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{C}$ be measurable and locally bounded in both variables (q,p). Assume that $H(q,\cdot)$ is continuous and negative definite for all $q\in\mathbb{R}^d$ and that Assumptions 3.2.1 and 3.2.2 hold. Then the family $(F(t))_{t\geq 0}$, defined in (3.2.5), is a strongly continuous family on X and is Chernoff equivalent to the strongly continuous semigroup $(T_t)_{t\geq 0}$, generated by the closure of the pseudo-differential operator $-\widehat{H}$ with the symbol $-H(q,p)$. Hence the Chernoff approximation*

$$T_t\varphi=\lim_{n\to\infty}\left[F(t/n)\right]^n\varphi \tag{3.2.8}$$

holds for each $\varphi\in C_\infty(\mathbb{R}^d)$ locally uniformly with respect to $t\geq 0$.

Proof. By Lemma 3.2.5, each $F(t)$ is a contraction operator on $C_\infty(\mathbb{R}^d)$. Let us check the strong continuity of the family $(F(t))_{t\geq 0}$ at the point $t_0=0$. Due to the estimate (3.2.1), we have for any $\varphi\in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned}
\lim_{t\to 0}\|F(t)\varphi-\varphi\|_\infty &= \lim_{t\to 0}\sup_{q\in\mathbb{R}^d}\left|(2\pi)^{-d/2}\int\limits_{\mathbb{R}^d} e^{ip\cdot q}\widetilde{\varphi}(p)\left[e^{-tH(q,p)}-1\right]dp\right|\\
&\leq \lim_{t\to 0}(2\pi)^{-d/2}\int\limits_{\mathbb{R}^d}|\widetilde{\varphi}(p)|\sup_{q\in\mathbb{R}^d}\left\{\left|\frac{e^{-tH(q,p)}-1}{-tH(q,p)}\right||tH(q,p)|\right\}dp\\
&\leq \lim_{t\to 0}(2\pi)^{-d/2}\int\limits_{\mathbb{R}^d} t\kappa(1+|p|^2)|\widetilde{\varphi}(p)|dp\\
&=0,
\end{aligned}$$

since $\widetilde{\varphi}\in S(\mathbb{R}^d)$. Hence, $\lim_{t\to 0}\|F(t)\varphi-\varphi\|_\infty=0$ for all $\varphi\in C_c^\infty(\mathbb{R}^d)$. Due to the fact that $\|F(t)\|\leq 1$, the last equality is true for all $\varphi\in C_\infty(\mathbb{R}^d)$ by a 3-epsilon argument. The strong continuity of the family $(F(t))_{t\geq 0}$ at any other point $t_0>0$ can be shown in a similar way.

Further, we have for any $\varphi\in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned}
&\lim_{t\to 0}\left\|\frac{F(t)\varphi-\varphi}{t}+\widehat{H}\varphi\right\|_\infty\\
&\qquad= \lim_{t\to 0}\sup_{q\in\mathbb{R}^d}\left|(2\pi)^{-d/2}\int\limits_{\mathbb{R}^d} e^{ip\cdot q}\widetilde{\varphi}(p)\left[\frac{e^{-tH(q,p)}-1}{t}+H(q,p)\right]dp\right|\\
&\qquad\leq \lim_{t\to 0}(2\pi)^{-d/2}\int_{\mathbb{R}^d}|\widetilde{\varphi}(p)|\frac{t\kappa^2(1+|p|^2)^2}{2}dp\\
&\qquad=0.
\end{aligned}$$

Thus, all assumptions of the Chernoff theorem 1.0.6 are fulfilled, and the family $(F(t))_{t\geq 0}$ is Chernoff equivalent to the semigroup $(T_t)_{t\geq 0}$ generated by $-\widehat{H}$. □

Remark 3.2.7. (i) If we require in Assumption 3.2.2 the existence of not just a strongly continuous but a Feller semigroup, we obtain by Theorem 3.2.6 an approximation for the corresponding Feller process $(\xi_t)_{t\geq 0}$: for each fixed $n\in\mathbb{N}$, the operator $\left[F(t/n)\right]^n$ in the Chernoff approximation (3.2.8) corresponds to the approximation of the process ξ_t by a Markov chain $\left\{Y^{t/n}(k)\right\}_{k=0}^n$ with Lévy increments. This Markov chain is obtained by splitting the time interval $[0,t]$ onto n equal steps and "freezing" the coefficient q in the transition probabilities of ξ_t at each step (cf. Böttcher and Schilling, 2009; Böttcher and Schnurr, 2011). Moreover, the transition kernel $P(t/n,q,\cdot)$ of this Markov chain correspond to the transition operator $W_{t/n}$, $W_{t/n}\varphi(q) := \int_{\mathbb{R}^d}\varphi(y)P(t/n,q,dy)$. Hence, $\left[F(t/n)\right]^n = \left[W_{t/n}\right]^n$. This allows us to transform the obtained Chernoff approximation for the Feller semigroup $(T_t)_{t\geq 0}$ associated with the process $(\xi_t)_{t\geq 0}$ into a Lagrangian Feynman formula $T_t\varphi(q) = \lim_{n\to\infty}\left[W_{t/n}\right]^n\varphi(q)$. Let us demonstrate it in the case when the considered symbol H satisfies the additional requirements: $N(q,dy)\equiv 0$ for all $q\in\mathbb{R}^d$ and there exist constants $0<a_0\leq A_0<\infty$ such that $a_0|p|^2\leq p\cdot A(q)p\leq A_0|p|^2$ for all $q,p\in\mathbb{R}^d$. In this case, we have for all $\varphi\in C_c^\infty(\mathbb{R}^d)$ (cf. Formula (2.3.11) in Remark 2.3.9):

$$\begin{aligned}F(t)\varphi(q) &= (2\pi)^{-d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} e^{ip\cdot(q-q')}e^{-t(p\cdot A(q)p+iB(q)\cdot p+C(q))}\varphi(q')dq'dp\\ &= (4\pi t)^{-d/2}(\det A(q))^{-1/2}e^{-tC(q)}\int_{\mathbb{R}^d} e^{-\frac{(q-q'-tB((q))\cdot A^{-1}(q)(q-q'-tB(q))}{4t}}\varphi(q')\,dq'. \end{aligned}\tag{3.2.9}$$

The last expression is well-defined for all $\varphi\in C_\infty(\mathbb{R}^d)$, $t>0$ and provides the mentioned in Lemma 3.2.5 extension of $F(t)$ to a contraction $F(t): C_\infty(\mathbb{R}^d)\to C_\infty(\mathbb{R}^d)$. Therefore, the Chernoff approximation (3.2.8) in this case is nothing else but the Lagrangian Feynman formula (2.3.13).

(ii) The results of Section 3.2 have been published in Butko, Schilling, and Smolyanov, 2010 (with stronger assumptions on the symbol H) and in Butko, Schilling, and Smolyanov, 2012 (in the present form). Parallelly, it has been shown in Böttcher and Schnurr, 2011 that the Markov chain approximation $\left\{Y^{t/n}(k)\right\}_{k=0}^n$ (which corresponds to the operator $\left[F(t/n)\right]^n$ in the Chernoff approximation (3.2.8)) can be interpreted as an Euler scheme for the Feller process $(\xi_t)_{t\geq 0}$. And this scheme converges weakly in the Skorokhod space.

Remark 3.2.8. (i) Let us assume additionally that $H:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{C}$ satisfies the following condition:

$$\exists\, C>0 \quad \text{such that } \left\|\partial_q^\alpha\partial_p^\beta e^{-tH}\right\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\leq C, \tag{3.2.10}$$

where $\alpha,\beta\in\mathbb{N}_0^d$, $\alpha=0$ or 1, $\beta=0$ or 1, $\partial_q^\alpha\partial_p^\beta$ are derivatives in the distributional sense. Note, that this condition is fulfilled, e.g. if $H:|H(q,p)|\geq c|p|^r$ for $|p|\gg 1$, some $c>0$ and some $r\in(0,2)$. Then, by Theorem 2 of Hwang, 1987, we have $F(t):L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$. In this case, operators $(F(t))_{t\geq 0}$, which are defined on $C_c^\infty(\mathbb{R}^d)$ by formula (3.2.5), are given by the same formula (3.2.5) on the whole space $L^2(\mathbb{R}^d)$. However, the integrals in (3.2.5) must be understood in a regularized sense (in the same way as the Fourier transform extends onto $L^2(\mathbb{R}^d)$). Therefore, the Chernoff approximation (3.2.8) obtained in Theorem 3.2.6 can be written for each φ such that $\varphi, T_t\varphi\in C_\infty(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ in the form of Hamiltonian Feynman formula:

$$(T_t\varphi)(q_0) \tag{3.2.11}$$
$$= \lim_{n\to\infty} \frac{1}{(2\pi)^{dn}} \int\limits_{\mathbb{R}^d} \cdots \int\limits_{\mathbb{R}^d} e^{i\sum\limits_{k=1}^{n} p_k\cdot(q_{k-1}-q_k)} e^{-\frac{t}{n}\sum\limits_{k=1}^{n} H(q_{k-1},p_k)} \varphi(q_n)dq_ndp_n\cdots dq_1dp_1,$$

where the equality holds in L^2-sense, all the integrals in the right hand side must be considered in a regularized sense, and the order of integration is from q_n to p_1. We refer to Hwang, 1987 for further conditions on $H(q,p)$ ensuring that $F(t) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

(ii) If the function H satisfies sufficient conditions for $F(t)\varphi$ to be in $S(\mathbb{R}^d)$ for each $\varphi \in S(\mathbb{R}^d)$ then, for any $\varphi \in S(\mathbb{R}^d)$, the equality in the Hamiltonian Feynman formula (3.2.11) holds in each point $q_0 \in \mathbb{R}^d$, uniformly with respect to $q_0 \in \mathbb{R}^d$ and locally uniform with respect to $t \geq 0$. Such conditions can be found in the following lemma.

Lemma 3.2.9. *Let $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be such that the mapping $p \mapsto H(q,p)$ is negative definite for each $q \in \mathbb{R}^d$ and $H(\cdot,\cdot) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$. Assume that, for each $p \in \mathbb{R}^d$ and for each α, $\beta \in \mathbb{N}_0^d$, the following estimates hold:*

$$\sup_{q\in\mathbb{R}^d} |\partial_p^\alpha \partial_q^\beta H(q,p)| \leq f_{\alpha,\beta}(p), \tag{3.2.12}$$

where all functions $f_{\alpha,\beta}$ are continuous on $\mathbb{R}^d$ and have at most polynomial growth at infinity. Then $F(t)\varphi \in S(\mathbb{R}^d)$ for each $\varphi \in S(\mathbb{R}^d)$.

Proof. By Lemma 3.2.5 we have $F(t)\varphi \in C_\infty(\mathbb{R}^d)$. Let us show that for all α, $\beta \in \mathbb{N}_0^d$ the norm

$$\|F(t)\varphi\|_{\alpha,\beta} = \sup_{q\in\mathbb{R}^d} \left| q^\alpha \partial_q^\beta [F(t)\varphi](q) \right|$$

is finite. Note that the function $\partial_q^\beta e^{-tH(q,p)+ip\cdot q}$ is continuous for any $\beta \in \mathbb{N}_0^d$. By (3.2.12) it is also majorized (uniformly for all q) by some continuous function of p which has at most polynomial growth at infinity. Hence, by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} q^\alpha \partial_q^\beta [F(t)\varphi](q) &= (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} q^\alpha \partial_q^\beta e^{-tH(q,p)+ip\cdot q} \widetilde{\varphi}(p)dp \\ &= (2\pi)^{-d/2} \sum_{0\leq\gamma\leq\beta} \int\limits_{\mathbb{R}^d} q^\alpha \partial_q^\gamma (e^{ip\cdot q}) \partial_q^{\beta-\gamma} (e^{-tH(q,p)}) \widetilde{\varphi}(p)dp. \end{aligned}$$

Since $\partial_q^\gamma(e^{ip\cdot q}) = e^{ip\cdot q}R_\gamma(p)$, where R_γ is a polynomial of p, we can use integration by parts and get

$$\begin{aligned} q^\alpha\partial_q^\beta[F(t)\varphi](q) &= (2\pi)^{-d/2}\sum_{0\le\gamma\le\beta}\int\limits_{\mathbb{R}^d} q^\alpha e^{ip\cdot q}\big[R_\gamma(p)\partial_q^{\beta-\gamma}(e^{-tH(q,p)})\widetilde{\varphi}(p)\big]dp \\ &= (2\pi)^{-d/2}\sum_{0\le\gamma\le\beta} i^{|\alpha|}\int\limits_{\mathbb{R}^d} \partial_p^\alpha e^{ip\cdot q}\big[R_\gamma(p)\partial_q^{\beta-\gamma}(e^{-tH(q,p)})\widetilde{\varphi}(p)\big]dp \\ &= (2\pi)^{-d/2}\sum_{0\le\gamma\le\beta} (-i)^{|\alpha|}\int\limits_{\mathbb{R}^d} e^{ip\cdot q}\partial_p^\alpha\big[R_\gamma(p)\partial_q^{\beta-\gamma}(e^{-tH(q,p)})\widetilde{\varphi}(p)\big]dp. \end{aligned}$$

Since $\partial_p^\alpha\big[R_\gamma(p)\partial_q^{\beta-\gamma}(e^{-tH(q,p)})\widetilde{\varphi}(p)\big]$ is bounded by an L^1-function which is independent of q, we can use (3.2.12) to see that the expression in the last line is finite. Hence, the norm $\|F(t)\varphi\|_{\alpha,\beta}$ is finite. □

Example 3.2.10. Let us consider the symbol $H_{a,\alpha}(q,p) := a(q)|p|^\alpha$, where $\alpha \in (0,2]$ and $a(\cdot) \in C^\infty(\mathbb{R}^d)$ is a strictly positive and bounded function. Then $-\widehat{H_{a,\alpha}}$ extends to the generator of a Feller semigroup $(T_t^{a,\alpha})_{t\ge0}$ (see Schilling and Schnurr, 2010). If $\alpha = 2$, this semigroup corresponds to the process of diffusion with variable diffusion coefficient. All conditions of the Theorem 3.2.6 are fulfilled, and, by the Hamiltonian Feynman formula (3.2.11) we have the following Hamiltonian Feynman formula for the semigroup $(T_t^{a,\alpha})_{t\ge0}$:

$$\begin{aligned} &(T_t^{a,\alpha}\varphi)(q_0) \\ &= \lim_{n\to\infty}\frac{1}{(2\pi)^{dn}}\int\limits_{\mathbb{R}^d}\cdots\int\limits_{\mathbb{R}^d} e^{i\sum\limits_{k=1}^n p_k\cdot(q_{k-1}-q_k)} e^{-\frac{t}{n}\sum\limits_{k=1}^n a(q_{k-1})|p_k|^\alpha}\varphi(q_n)dq_ndp_n\cdots dq_1dp_1, \end{aligned}$$

where $q_0 \in \mathbb{R}^d$, $t \ge 0$, $\varphi : \varphi, T_t\varphi \in C_\infty(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, the equality is in L^2–sense, and all integrals in the pre-limit expressions are understood in a regularized sense. In the case $\alpha = 1$, this Hamiltonian Feynman formula can be transformed into the Lagrangian Feynman formula (2.2.4) of Example 2.2.12 by proceeding the integration with respect to p–variable in formula (3.2.5) defining the family $(F(t))_{t\ge0}$.

Example 3.2.11. Let us consider the symbol $H_{\alpha,m}(q,p) := \sqrt{|p|^\alpha + m^2(q)} - m(q)$, where $m(\cdot) \in C^\infty(\mathbb{R}^d)$ is a strictly positive and bounded function on $\mathbb{R}^d$, $\alpha \in (0,2]$. If additionally the function $m(\cdot)$ is such that the Assumption 3.2.2 holds (e.g. if $m \equiv$ const), then the following Hamiltonian Feynman formula is valid for the corresponding semigroup $(T_t^{\alpha,m})_{t\ge0}$:

$$\begin{aligned} (T_t^{\alpha,m}\varphi)(q_0) = \lim_{n\to\infty}\frac{1}{(2\pi)^{dn}}\int\limits_{\mathbb{R}^d}\cdots\int\limits_{\mathbb{R}^d} e^{i\sum\limits_{k=1}^n p_k\cdot(q_{k-1}-q_k)} e^{-\frac{t}{n}\sum\limits_{k=1}^n \sqrt{|p_k|^\alpha+m^2(q_{k-1})}-m(q_{k-1})}\times \\ \times\varphi(q_n)dq_ndp_n\cdots dq_1dp_1, \end{aligned}$$

where $q_0 \in \mathbb{R}^d$, $t \ge 0$, $\varphi : \varphi, T_t\varphi \in C_\infty(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, the equality is in L^2–sense, and all integrals in the pre-limit expressions are understood in a regularized sense. In the case $\alpha = 2$ the operator $-\widehat{H_{\alpha,m}}$ can be considered as the Hamiltonian of a free relativistic (quasi-)particle with variable mass (cf. Gadèl'ya and Smolyanov, 2008, Ichinose and Tamura, 1986).

Remark 3.2.12. Some other Feynman formulae for particular classes of Feller semigroups, obtained by subordination, are constructed in Chapter 4.

3.3 Feynman formulae for semigroups on the space $C_\infty(\mathbb{R}^d)$ generated by τ-quantizations of Lévy–Khintchine type symbols

In this Section, we investigate the connection between different τ-quantizations of a quadratic Hamilton function. Using this connection and results of Section 3.2, we obtain Feynman Formulae for semigroups on $C_\infty(\mathbb{R}^d)$ generated by the τ-quantization of some Lévy–Khintchine type symbols similar to (3.1.2) in the case of arbitrary $\tau \in [0,1]$. So, let us consider a quadratic Hamilton function $h : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ with

$$h(q,p) := C(q) + iB(q) \cdot p + p \cdot A(q)p, \tag{3.3.1}$$

where, for each $q \in \mathbb{R}^d$, $A(q)$ is a symmetric positive semidefinite matrix, $B(q) \in \mathbb{R}^d$ and $C(q) \in \mathbb{R}$.

Remark 3.3.1. The function h can be considered both as a Hamilton function of a particle with position-dependent mass in magnetic and potential fields in a flat space and as a Hamilton function of a particle with constant mass in magnetic and potential fields in a space with curvature (namely, in a Riemannian manifold). We follow the first interpretation (that reflects in Feynman path integrals in Section 3.5). In the context of second interpretation, some additional potentials containing geometrical characteristics of the space may arise in the effective action. This depends on the notion of distance being used: either it is the metric of the manifold, where the particle evolves, or the metric of an ambient manifold, or the distance in an ambient Euclidean space (see Refs. Weizsäcker, Smolyanov, and Wittich, 2000; Smolyanov, Weizsäcker, and Wittich, 2003; Smolyanov, Weizsäcker, and Wittich, 2007b for details).

Lemma 3.3.2. *Let $A \in C^3(\mathbb{R}^d; \mathrm{Mat}(d \times d))$, $B \in C^2(\mathbb{R}^d; \mathbb{R}^d)$, $C \in C^1(\mathbb{R}^d)$ and $\tau \in [0,1]$. The pseudo-differential operator $\widehat{h}_\tau$ is well-defined on $C_c^\infty(\mathbb{R}^d)$ and is given for each $\varphi \in C_c^\infty(\mathbb{R}^d)$ and each $q \in \mathbb{R}^d$ by the fomula*

$$\widehat{h}_\tau \varphi(q) = -\operatorname{tr}(A(q)\operatorname{Hess}\varphi(q)) + B_\tau(q) \cdot \nabla\varphi(q) + C_\tau(q)\varphi(q), \tag{3.3.2}$$

where

$$B_\tau(q) := B(q) - 2(1-\tau)\operatorname{div} A(q), \tag{3.3.3}$$
$$C_\tau(q) := C(q) + (1-\tau)\operatorname{div} B(q) - (1-\tau)^2 \operatorname{tr}(\nabla \otimes \nabla A(q)), \tag{3.3.4}$$

$\operatorname{div} A(q)$ *is the vector in* $\mathbb{R}^d$ *with coordinates*

$$(\operatorname{div} A(q))_j := \sum_{k=1}^d \frac{\partial A_{kj}(q)}{\partial q_k}, \quad j = 1, \ldots, d$$

and

$$\operatorname{tr}(\nabla \otimes \nabla A(q)) := \sum_{k,j=1}^d \frac{\partial^2 A_{kj}(q)}{\partial q_k \partial q_j} \quad \textit{with} \quad A(q) = (A_{kj}(q))_{k,j=1,\ldots,d}.$$

Therefore, the operator $\widehat{h}_\tau$ *coincides with the operator* $\widehat{h}_1^\tau$ *which is the* 1*-quantization of the quadratic Hamilton function*

$$h^\tau(q,p) := C_\tau(q) + iB_\tau(q)\cdot p + p \cdot A(q)p \tag{3.3.5}$$

with C_τ *and* B_τ *as above.*

Proof. With $A(q) = (A_{kj}(q))_{k,j=1,\ldots,d}$, $A_{kj} = A_{jk}$, and $B(q) = (B_1(q), \ldots, B_d(q))$, we have for each $\varphi \in C_c^\infty(\mathbb{R}^d)$:

$$\begin{aligned}
\widehat{h}_\tau \varphi(q) = &\sum_{k,j=1}^d (2\pi)^{-d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{ip\cdot(q-q')} A_{kj}(\tau q + (1-\tau)q') p_k p_j \varphi(q') dq' dp \\
&+ i\sum_{k=1}^d (2\pi)^{-d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{ip\cdot(q-q')} B_k(\tau q + (1-\tau)q') p_k \varphi(q') dq' dp \\
&+ (2\pi)^{-d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{ip\cdot(q-q')} C(\tau q + (1-\tau)q') \varphi(q') dq' dp
\end{aligned}$$

Since the function $f_{C,\varphi,q,\tau}(q') := C(\tau q + (1-\tau)q')\varphi(q')$ belongs to the class $C_c^1(\mathbb{R}^d)$, we have inverting the Fourier transform

$$(2\pi)^{-d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{ip\cdot(q-q')} C(\tau q + (1-\tau)q')\varphi(q') dq' dp = C(\tau q + (1-\tau)q)\varphi(q) = C(q)\varphi(q).$$

Analogously, using integration by parts and denoting $\zeta := \tau q + (1-\tau)q'$

$$
\begin{aligned}
& i\sum_{k=1}^{d}(2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{ip\cdot(q-q')}B_k(\tau q+(1-\tau)q')p_k\varphi(q')dq'dp \\
&= -\sum_{k=1}^{d}(2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left[\frac{\partial}{\partial q'_k}e^{ip\cdot(q-q')}\right]B_k(\tau q+(1-\tau)q')\varphi(q')dq'dp \\
&= \sum_{k=1}^{d}(2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{ip\cdot(q-q')}\frac{\partial}{\partial q'_k}\left[B_k(\tau q+(1-\tau)q')\varphi(q')\right]dq'dp \\
&= \sum_{k=1}^{d}(2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{ip\cdot(q-q')}\left[B_k(\zeta)\frac{\partial\varphi}{\partial q'_k}(q')+\sum_{k=1}^{d}\frac{\partial B_k}{\partial q'_k}(\zeta)\right]\varphi(q')dq'dp \\
&= B(q)\cdot\nabla\varphi(q)+(1-\tau)\operatorname{div}B(q)\varphi(q),
\end{aligned}
$$

and

$$
\begin{aligned}
& \sum_{k,j=1}^{d}(2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{ip\cdot(q-q')}A_{kj}(\tau q+(1-\tau)q')p_kp_j\varphi(q')dq'dp \\
&= \sum_{k,j=1}^{d}(2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left[-\frac{\partial^2}{\partial q'_k\partial q'_j}e^{ip\cdot(q-q')}\right]A_{kj}(\tau q+(1-\tau)q')\varphi(q')dq'dp \\
&= -\sum_{k,j=1}^{d}(2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{ip\cdot(q-q')}\frac{\partial^2}{\partial q'_k\partial q'_j}\left[A_{kj}(\tau q+(1-\tau)q')\varphi(q')\right]dq'dp \\
&= -\operatorname{tr}(A(q)\operatorname{Hess}\varphi(q))-2(1-\tau)\operatorname{div}A(q)\cdot\nabla\varphi(q)-(1-\tau)^2\operatorname{tr}(\nabla\otimes\nabla A(q))\varphi(q).
\end{aligned}
$$

□

Let us consider also a function $r:\mathbb{R}^d\to\mathbb{C}$ given by the formula

$$
r(p)=\int_{\mathbb{R}^d\setminus\{0\}}\left(1-e^{iy\cdot p}+\frac{iy\cdot p}{1+|y|^2}\right)N(dy), \tag{3.3.6}
$$

where N is a Radon measure on $\mathbb{R}^d\setminus\{0\}$ with $\int_{\mathbb{R}^d\setminus\{0\}}\frac{|y|^2}{1+|y|^2}N(dy)<\infty$. Note that we consider the case when N does not depend on $q\in\mathbb{R}^d$. We assume that for $q,p\in\mathbb{R}^d$ we have

$$
\begin{aligned}
H(q,p) &= h(q,p)+r(p) \\
&= C(q)+iB(q)\cdot p+p\cdot A(q)p+\int_{\mathbb{R}^d\setminus\{0\}}\left(1-e^{iy\cdot p}+\frac{iy\cdot p}{1+|y|^2}\right)N(dy).
\end{aligned} \tag{3.3.7}
$$

If $C\geq 0$ then the Hamilton function H is continuous negative definite with respect to the variable $p\in\mathbb{R}^d$ and the formula (3.3.7) is just a special case of the Lévy-Khintchine formula (3.1.2). We don't assume in the sequel that $C \geq 0$, that's why we call our symbol H a *Lévy-Khintchine type function*.

Remark 3.3.3. Functions of the form (3.3.6) contain in particular Hamilton functions of a relativistic massive particle: $r(p) = c\sqrt{|p|^2 + m^2c^2}$, of a massless particle: $r(p) = c|p|$, symbols of fractional Laplacians (arising, e.g., in anomalous diffusions): $r(p) = |p|^\alpha$, $\alpha \in (0,2)$, and some other functions used, e.g., for description of systems with long-range interactions (see Bouchaud and Georges, 1990, Metzler and Klafter, 2000).

Consider a pseudo-differential operator $\widehat{H}_\tau$ with the τ-symbol H for $\tau \in [0,1]$ in $C_\infty(\mathbb{R}^d)$, i.e. for any function $\varphi \in \mathrm{Dom}(\widehat{H}_\tau) \subset C_\infty(\mathbb{R}^d)$

$$\widehat{H}_\tau \varphi(q) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} e^{ip\cdot(q-q')} H(\tau q + (1-\tau)q', p)\varphi(q')\, dq'\, dp, \quad q \in \mathbb{R}^d. \tag{3.3.8}$$

Note that, under assumptions of Lemma 3.3.2, the operator $\widehat{H}_\tau$ with τ-symbol H, given by the formula (3.3.7), for each $\tau \in [0,1]$ can be extended to $C^2_\infty(\mathbb{R}^d)$ by the formula

$$\begin{aligned}
\widehat{H}_\tau \varphi(q) =& \widehat{h}_1^\tau \varphi(q) + \widehat{r}_1 \varphi(q) \\
=& -\operatorname{tr}(A(q)\operatorname{Hess}\varphi(q)) + [B(q) - 2(1-\tau)\operatorname{div} A(q)]\cdot\nabla\varphi(q) \\
&+ [C(q) + (1-\tau)\operatorname{div} B(q) - (1-\tau)^2 \operatorname{tr}(\nabla\otimes\nabla A(q))]\varphi(q) \\
&- \int_{y\neq 0} \left(\varphi(q+y) - \varphi(q) - \frac{y\cdot\nabla\varphi(q)}{1+|y|^2}\right) N(dy),
\end{aligned} \tag{3.3.9}$$

i.e. $\widehat{H}_\tau$ is a sum of a second order differential operator with variable coefficients and an integro-differential operator generating a Lévy process. Therefore, changing the parameter τ in τ-quantization of the symbol H as in (3.3.7) leads only to corrections in the magnetic and potential fields B and C.

Theorem 3.3.4. *(i) Let $A \in C^3_b(\mathbb{R}^d; \mathrm{Mat}(d\times d))$, $B \in C^2_b(\mathbb{R}^d;\mathbb{R}^d)$, $C \in C^1_b(\mathbb{R}^d)$ and $\tau \in [0,1]$. Let the symbol H be given by formulae (3.3.8)-(3.3.9) and satisfy Assumption 3.2.1. Assume that the closure of the pseudo-differential operator $(-\widehat{H}_\tau, C^\infty_c(\mathbb{R}^d))$ generates a strongly continuous semigroup $(T^\tau_t)_{t\geq 0}$ on the space $C_\infty(\mathbb{R}^d)$. Consider the family $(F^\tau_1(t))_{t\geq 0}$ of linear operators in $C_\infty(\mathbb{R}^d)$ defined for all $\varphi \in C^\infty_c(\mathbb{R}^d)$ and all $t \geq 0$ by*

$$F^\tau_1(t)\varphi(x) = (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} e^{ip\cdot q} e^{-t(h^\tau(q,p)+r(p))} \widetilde{\varphi}(p)dp.$$

Then the operators $F^\tau_1(t)$ extend to bounded linear operators on $C_\infty(\mathbb{R}^d)$; the family $(F^\tau_1(t))_{t\geq 0}$ (of these extensions) is strongly continuous and Chernoff equivalent to the semigroup $(T^\tau_t)_{t\geq 0}$. Hence the Chernoff approximation

$$T^\tau_t \varphi = \lim_{n\to\infty} \left[F^\tau_1(t/n)\right]^n \tag{3.3.10}$$

holds for each $\varphi \in C_\infty(\mathbb{R}^d)$ locally uniformly with respect to $t \geq 0$.

(ii) If the symbol $h^\tau + r$ satisfies the condition 3.2.10 of Remark 3.2.8, then the Chernoff approximation (3.3.10) converts into the following Hamiltonian Feynman formula (where the equality holds in the L^2-sense for all $t \geq 0$ and all φ such that $\varphi, T_t\varphi \in C_\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, and all integrals are understood in the regularized sense):

$$T_t^\tau \varphi(q_0) = \lim_{n\to\infty} (2\pi)^{-dn} \int\limits_{\mathbb{R}^d} \cdots \int\limits_{\mathbb{R}^d} e^{i\sum\limits_{k=1}^n p_k\cdot(q_{k-1}-q_k)} \times$$
$$\times\, e^{-\frac{t}{n}\sum\limits_{k=1}^n (h^\tau(q_{k-1},p_k)+r(p_k))} \varphi(q_n)\, dq_n\, dp_n \ldots dq_1\, dp_1$$

$$= \lim_{n\to\infty} (2\pi)^{-dn} \int\limits_{\mathbb{R}^d} \cdots \int\limits_{\mathbb{R}^d} e^{i\sum\limits_{k=1}^n p_k\cdot(q_{k-1}-q_k)} e^{-i\frac{t}{n}\sum\limits_{k=1}^n (B(q_{k-1})-2(1-\tau)\operatorname{div} A(q_{k-1})\cdot p_k} \times$$
$$\times\, e^{-\frac{t}{n}\sum\limits_{k=1}^n (C(q_{k-1})+(1-\tau)\operatorname{div} B(q_{k-1})-(1-\tau)^2 \operatorname{tr}(\nabla\otimes\nabla A(q_{k-1})))} \times$$
$$\times\, e^{-\frac{t}{n}\sum\limits_{k=1}^n p_k\cdot A(q_{k-1})p_k} e^{-\frac{t}{n}\sum\limits_{k=1}^n r(p_k)} \varphi(q_n)\, dq_n\, dp_n \ldots dq_1\, dp_1.$$

(iii) If there exist constants $0 < a_0 \leq A_0 < \infty$ such that inequalities $a_0|p|^2 \leq p\cdot A(q)p \leq A_0|p|^2$ hold for all $q, p \in \mathbb{R}^d$, then the Chernoff approximation (3.3.10) converts for all $t > 0$, all $\varphi \in C_\infty(\mathbb{R}^d)$ and all $q_0 \in \mathbb{R}^d$ into the following Lagrangian Feynman formula

$$(T_t^\tau)\varphi(q_0) = \lim_{n\to\infty} \int\limits_{\mathbb{R}^d} \cdots \int\limits_{\mathbb{R}^d} \prod_{k=1}^n \left((4\pi t/n)^d \det A(q_{k-1})\right)^{-1/2}$$
$$\times\, e^{-\sum\limits_{k=1}^n \frac{A^{-1}(q_{k-1})(q_k-q_{k-1}+z_k+B_\tau(q_{k-1})t/n)\cdot(q_k-q_{k-1}+z_k+B_\tau(q_{k-1})t/n)}{4t/n}}$$
$$\times\, e^{-\frac{t}{n}\sum\limits_{k=1}^n C_\tau(q_{k-1})} \varphi(q_n) dq_n\, \eta_{t/n}(dz_n) \ldots dq_1\, \eta_{t/n}(dz_1),$$

where $(\eta_t)_{t\geq 0}$ is the convolution semigroup on $\mathbb{R}^d$ such that $\widetilde{\eta_t}(p) = (2\pi)^{-d/2} e^{-tr(p)}$, B_τ and C_τ are given by (3.3.3)-(3.3.4). The convergence in this Lagrangian Feynman formula is uniform with respect to $q_0 \in \mathbb{R}^d$ and with respect to $t \in (0, t^*]$ for all $t^* > 0$.

Proof. The statement (i) of the theorem follows immediately from Theorem 3.2.6 in the case $C_\tau(q) \geq 0$ for all $q \in \mathbb{R}^d$. The assumption of non-negativity of C_τ can be removed applying the technique of additive perturbations, see Theorem 2.1.1, Corollary D.0.5 and Example A.0.17. The statement (ii) of the theorem follows immediately from Remark 3.2.8. Let us justify the statement (iii). Under assumption $a_0|p|^2 \leq p\cdot A(q)p \leq A_0|p|^2$ for all $q, p \in \mathbb{R}^d$ and some $0 < a_0 \leq A_0 < \infty$, we have by the Fubini–Tonelli theorem and by the properties of the Fourier transform for all $\varphi \in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned} F_1^\tau(t)\varphi(q) &= (2\pi)^{-d} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} e^{ip\cdot(q-q')} e^{-t(h^\tau(q,p)+r(p))} \varphi(q')\, dq' dp \\ &= (2\pi)^{-d} \int\limits_{\mathbb{R}^d} \varphi(q') \left[\int\limits_{\mathbb{R}^d} e^{ip\cdot(q-q')} e^{-t(h^\tau(q,p)+r(p))}\, dp \right] dq' \\ &= \int\limits_{\mathbb{R}^d} \varphi(q') \left[\mathcal{F}^{-1}\left[(2\pi)^{-d/2} e^{-t(h^\tau(q,\cdot))}\right] * \eta_t \right] (q-q')\, dq' \\ &= \int\limits_{\mathbb{R}^{2d}} (4\pi t)^{-d/2} (\det A(q))^{-1/2} e^{-tC_\tau(q)} e^{-\frac{(z-q+q'+tB_\tau((q))\cdot A^{-1}(q)(z-q+q'+tB_\tau(q))}{4t}} \varphi(q') dq' \eta_t(dz). \end{aligned} \tag{3.3.11}$$

Moreover, the latter expression is well defined for all $\varphi \in C_\infty(\mathbb{R}^d)$ and provides the extension of the operators $F_1^\tau(t)$ to bounded linear operators on $C_\infty(\mathbb{R}^d)$. $\square$

3.4 Feynman formulae for semigroups on the space $L^1(\mathbb{R}^d)$ generated by τ-quantizations of Lévy–Khintchine type symbols

A Feller semigroup $(T_t)_{t\geq 0}$ is, a priori, defined on the space $B_b(Q)$. In some cases (see, e.g., Proposition D.0.11), the operators $T_t\big|_{C_c(Q)}$ can be extended onto the spaces of integrable functions. Sub-Markovian semigroups on $L^2(\mathbb{R}^d, \mathbb{R})$ play an important role in the theory of Dirichlet forms, see, e.g., Fukushima, 1980, Fukushima, Ōshima, and Takeda, 1994, Bouleau and Hirsch, 1991, Ma and Röckner, 1992. In general, L^p-theories lead to better regularity and embedding results than the corresponding L^2-theory. Therefore, semigroups in an L^p-setting have been investigated, e.g., in Malliavin, 1997, Fukushima, 1977/78, Farkas, Jacob, and Schilling, 2001, Arendt, 2004.

In this Section, we consider the space $L^1(\mathbb{R}^d)$. We continue to deal with strongly continuous semigroups $\left(e^{-t\widehat{H}_\tau}\right)_{t\geq 0}$ whose generators are pseudo-differential operators $-\widehat{H}_\tau$ obtained by τ-quantization of symbols $-H$ with H given by (3.3.1) or (3.3.7). We approximate such semigroups by families $(F_\tau(t))_{t\geq 0}$ of pseudo-differential operators $F_\tau(t) = \left(\widehat{e^{-tH}}\right)_\tau$ obtained by the same procedure of quantization from the symbol e^{-tH}. To handle the proofs we need to assume (sometimes different) boundedness and smoothness conditions on the symbol $-H$. All the assumptions, we use in the sequel, are collected below.

Assumption 3.4.1. Let H be given by (3.3.7). We assume that:

(i) There exist constants $0 < a_0 \leq A_0 < +\infty$ such that for all $p \in \mathbb{R}^d$ and all $q \in \mathbb{R}^d$ the following inequalities hold

$$a_0|p|^2 \leq p \cdot A(q)p \leq A_0|p|^2.$$

(ii) The coefficients A, B, C with all their derivatives up to the 4th order are continuous and bounded.

(iii) The coefficients A, B, C are infinite differentiable and bounded with all their derivatives.

(iv) The function $H(q,\cdot)$ is of class $C^\infty(\mathbb{R}^d)$ for each $q \in \mathbb{R}^d$.

Assumption 3.4.2. Let $\tau \in [0,1]$. Let H be given by (3.3.7). We assume that the coefficients A, B, C, N are such that $\widehat{H}_\tau$ is well defined on $C_c^\infty(\mathbb{R}^d)$, $\widehat{H}_\tau : C_c^\infty(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ and the closure of $(-\widehat{H}_\tau, C_c^\infty(\mathbb{R}^d))$ generates a strongly continuous semigroup $(T_t^\tau)_{t\geq 0}$ on the space $L^1(\mathbb{R}^d)$.

Remark 3.4.3. Let Assumption 3.4.1 (ii) be fullfilled. Then, due to the representation (3.3.9) of $\widehat{H}_\tau$, it is clear that $\widehat{H}_\tau : C_c^\infty(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ for measures N having compact supports (fast enough decay at infinity is also sufficient). In the case $N \equiv 0$, the explicit conditions on A, B, C to fulfill the Assumption 3.4.2 are given, e.g., in Fornaro and Lorenzi, 2007; Cannarsa and Vespri, 1988, Stannat, 1999. Then the case with nonzero coefficient N can be proceeded, e.g., by the technique of relatively bounded perturbations of generators (see Theorem D.0.4 and Theorem C.0.7). Moreover, a sufficient condition for the generator of a pure jump Lévy process (in particular, for the fractional Laplacian) with gradient perturbation to generate a sub-Markovian strongly continuous semigroup on $L^1(\mathbb{R}^d)$ is given in Wang, 2013.

Consider now the family $(F_\tau(t))_{t\geq 0}$ of pseudo-differential operators with the τ-symbol e^{-tH} in the space $L^1(\mathbb{R}^d)$, i.e. for $\varphi \in \mathrm{Dom}(F_\tau(t))$

$$F_\tau(t)\varphi(q) = (2\pi)^{-d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{ip\cdot(q-q')}e^{-tH(\tau q+(1-\tau)q',p)}\varphi(q')dq'dp. \tag{3.4.1}$$

Theorem 3.4.4. *Consider the Banach space $L^1(\mathbb{R}^d)$. Let $\tau \in [0,1]$ and $H \equiv h$, where h is given by the formula (3.3.1) (i.e. $N \equiv 0$). Under Assumption 3.4.1 (i),(ii) and Assumption 3.4.2, the family $(F_\tau(t))_{t\geq 0}$, given by the formula (3.4.1), extends to the family of bounded linear operators on $L^1(\mathbb{R}^d)$ which is strongly continuous and Chernoff equivalent to the strongly continuous semigroup $(T_t^\tau)_{t\geq 0}$, generated by the closure of the pseudo-differential operator $(-\widehat{H}_\tau, C_c^\infty(\mathbb{R}^d))$ with the τ-symbol $-H$. Therefore, the Chernoff approximation*

$$(T_t^\tau)\varphi = \lim_{n\to\infty} (F_\tau(t/n))^n\varphi \tag{3.4.2}$$

holds for all $\varphi \in L^1(\mathbb{R}^d)$ in the norm of $L^1(\mathbb{R}^d)$ locally uniformly with respect to $t \geq 0$. Moreover, this Chernoff approximation (3.4.2) converts for all $t > 0$ into the Lagrangian

Feynman formula:

$$(T_t^\tau)\varphi(q_0) = \lim_{n\to\infty} \int\limits_{\mathbb{R}^{nd}} \varphi(q_n) \prod_{k=1}^{n} g_{t/n}^{\tau q_{k-1}+(1-\tau)q_k}(q_{k-1}-q_k)dq_1\dots dq_n, \tag{3.4.3}$$

where the Gaussian type density $g_t^x(z)$ is given by

$$g_t^x(z) = (4\pi t)^{-d/2}(\det A(x))^{-1/2}e^{-tC(x)}e^{-\frac{(z-tB(x))\cdot A^{-1}(x)(z-tB(x))}{4t}}, \tag{3.4.4}$$

and the convergence is uniform with respect to $t \in (0, t^*]$ for all $t^* > 0$.

If, additionally, Assumption 3.4.1 (iii) holds, we have $F_\tau(t) : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$. And the Chernoff approximation (3.4.2) converts for all $\varphi \in S(\mathbb{R}^d)$ and all $t \geq 0$ into the Hamiltonian Feynman formula:

$$(T_t^\tau)\varphi(q_0) = \lim_{n\to\infty}(2\pi)^{-nd}\int\limits_{\mathbb{R}^d}\cdots\int\limits_{\mathbb{R}^d} \exp\left(i\sum_{k=1}^{n} p_k\cdot(q_{k-1}-q_k)\right) \tag{3.4.5}$$
$$\times \exp\left(-\frac{t}{n}\sum_{k=1}^{n} H(\tau q_{k-1}+(1-\tau)q_k, p_k)\right)\varphi(q_n)dq_n dp_n\dots dq_1 dp_1.$$

The convergence in the Hamiltonian Feynman formula (3.4.5) is locally uniform with respect to $t \geq 0$.

Proof. Theorem 3.4.4 follows from the Chernoff Theorem 1.0.6 with the help of Lemma 3.4.7, Remark 3.4.9 and Lemma 3.4.10 below. □

Remark 3.4.5. Let $\tau = 1$. One can show (e.g., combining the results of Freidlin, 1985 §2.1, Thm.1.1, and Lunardi, 1995, Thm. 5.1.3) that, under Assumptions 3.4.1 (i), (ii) and Assumption 3.4.2 with $N \equiv 0$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$, the function $T_t^\tau\varphi$ can be represented also by the following Feynman–Kac formula:

$$T_t^\tau\varphi(q_0) = \mathbb{E}_B^{q_0}\left[\exp\left(-\int\limits_0^t C(\xi_s)ds\right)\varphi(\xi_t)\right], \tag{3.4.6}$$

where $\mathbb{E}_B^{q_0}$ is the expectation of a (starting at q_0) diffusion process $(\xi_t)_{t\geq 0}$ with the variable diffusion matrix A and the drift $-B$ (i.e. satisfying the stochastic differential equation $d\xi_t = -B(\xi_t)dt + \sqrt{2A(\xi_t)}dB_t$). Therefore, the Lagrangian Feynman formula (3.4.3) gives approximations of a functional integral in the Feynman–Kac formula (3.4.6) which are suitable for direct calculations.

Consider now the Lagrangian Feynman formula (3.4.3) for the case $B = 0$. Then the limit in the right hand side of (3.4.3) coincide with the same path integral as in (3.4.6) but with respect to the measure, generated by the diffusion process $(X_t)_{t\geq 0}$ with the variable diffusion matrix A and without any drift, and with X instead of ξ in the integrand. Moreover, one can easily distinguish, which parts of the integrands in the pre-limit expressions of the Lagrangian Feynman formula (3.4.3) are responsible for approximation of the integrand in (3.4.6) and which parts serve to produce the path integral with respect to this measure, generated by the diffusion process $(X_t)_{t\geq 0}$, in the limit.

Further, let us analyze the pre-limit expressions in the right hand side of (3.4.3) in the general case $B \neq 0$. First, note that the formula (3.4.4) for the function g_t^x can be also rearranged in the following way:

$$g_t^x(z) = (4\pi t)^{-d/2}(\det A(x))^{-1/2} e^{-tC(x)} e^{-\frac{z\cdot A^{-1}(x)z}{4t}} e^{\frac{A^{-1}(x)B(x)\cdot z}{2}} e^{-t\frac{A^{-1}(x)B(x)\cdot B(x)}{4}}.$$

Taking in mind this formula for the function g_t^x one can see that the parts of the pre-limit expressions in the Lagrangian Feynman formula (3.4.3), which do not contain B, are the same as in the case when $B = 0$, i.e. once again they approximate an integrand as in (3.4.6) and a path integral with respect to the measure, generated by the diffusion process $(X_t)_{t\geq 0}$. The other parts (i.e. exponents containing B) can be interpreted as approximations for exponents of some functionals of the diffusion process $(X_t)_{t\geq 0}$. This heuristic suggests that, in the general case $B \neq 0$, the limit in the Lagrangian Feynman formula (3.4.3) coincides with the following path integral, having a product of exponents as the integrand; and this product of exponents is integrated with respect to the law of the diffusion process $(X_t)_{t\geq 0}$ without a drift (compare with the formula (34) in Lunt, Lyons, and Zhang, 1998 and formula (3) in Lejay, 2004):

$$\begin{aligned} T_t^\tau\varphi(q_0) = &\mathbb{E}^{q_0}\Bigg[\exp\Bigg(-\int_0^t C(X_s)ds\Bigg)\exp\Bigg(-\frac{1}{2}\int_0^t A^{-1}(X_s)B(X_s)\cdot dX_s\Bigg)\times \qquad (3.4.7)\\ &\times\exp\Bigg(-\frac{1}{4}\int_0^t A^{-1}(X_s)B(X_s)\cdot B(X_s)ds\Bigg)\varphi(X_t)\Bigg], \end{aligned}$$

where $\mathbb{E}^{q_0}$ is the expectation of a diffusion process $(X_t)_{t\geq 0}$ with the variable diffusion matrix A and without any drift, the stochastic integral $\int_0^t A^{-1}(X_s)B(X_s)\cdot dX_s$ is an Itô integral. Since the functional integrals in formula (3.4.6) and formula (3.4.7) coincide, one obtains the analogue of the Girsanov–Cameron–Martin–Reimer–Maruyama formula for the case of diffusion processes with variable diffusion matrices. Due to formula (3.3.9), the similar results are then valid for all $\tau \in [0, 1]$.

Theorem 3.4.6. *Consider the Banach space $L^1(\mathbb{R}^d)$. Let $\tau = 1$ and H be given by (3.3.7). Under Assumptions 3.4.1 (i),(iii),(iv) and Assumption 3.4.2, the family $(F_\tau(t))_{t\geq 0}$, given by the formula (3.4.1), extends to the family of bounded linear operators on $L^1(\mathbb{R}^d)$. This family is strongly continuous and Chernoff equivalent to the semigroup $(T_t^\tau)_{t\geq 0}$, generated by the closure of the pseudo-differential operator $(-\widehat{H}_\tau, C_c^\infty(\mathbb{R}^d))$ with the τ-symbol $-H$. Therefore, the Chernoff approximation*

$$(T_t^\tau)\varphi = \lim_{n\to\infty}(F_\tau(t/n))^n\varphi$$

holds for all $\varphi \in L^1(\mathbb{R}^d)$ in the norm of $L^1(\mathbb{R}^d)$ locally uniformly with respect to $t \geq 0$. The obtained Chernoff approximation converts for all $t > 0$ also into the Lagrangian Feynman formula

$$(T_t^\tau)\varphi(q_0) = \lim_{n\to\infty} \int\limits_{\mathbb{R}^d} \cdots \int\limits_{\mathbb{R}^d} \prod_{k=1}^{n} \left((2\pi t/n)^d \det A(q_{k-1})\right)^{-1/2} \tag{3.4.8}$$
$$\times e^{-\sum\limits_{k=1}^{n} \frac{A^{-1}(q_{k-1})(q_k - q_{k-1} + z_k + B(q_{k-1})t/n)\cdot(q_k - q_{k-1} + z_k + B(q_{k-1})t/n)}{4t/n}}$$
$$\times e^{-\frac{t}{n}\sum\limits_{k=1}^{n} C(q_{k-1})} \varphi(q_n)dq_n\, \eta_{t/n}(dz_n)\ldots dq_1\, \eta_{t/n}(dz_1),$$

where $(\eta_t)_{t\geq 0}$ is the convolution semigroup on $\mathbb{R}^d$ such that $\widetilde{\eta_t}(p) = (2\pi)^{-d/2}e^{-tr(p)}$. And the convergence is uniform with respect to $t \in (0, t^*]$ for all $t^* > 0$.

With $\varphi \in S(\mathbb{R}^d)$, the obtained Chernoff approximation converts for all $t \geq 0$ also into the Hamiltonian Feynman formula

$$(T_t^\tau)\varphi(q_0) = \lim_{n\to\infty} \frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d} \cdots \int\limits_{\mathbb{R}^d} \exp\left(i\sum_{k=1}^{n} p_k\cdot(q_{k-1} - q_k)\right) \tag{3.4.9}$$
$$\times \exp\left(-\frac{t}{n}\sum_{k=1}^{n} H(q_{k-1}, p_k)\right)\varphi(q_n)dq_n dp_n \ldots dq_1 dp_1.$$

And the convergence is locally uniform with respect to $t \geq 0$.

Proof. The statement of Theorem 3.4.6 is a straightforward consequence of the Chernoff Theorem 1.0.6 and Lemma 3.4.7, Lemma 3.4.11, Lemma 3.4.12 below. Note, that the Lagrangian Feynman formula (3.4.8) is obtained with the help of the representation (3.4.15) below. Moreover, under Assumptions 3.4.1 (i), (iii), (iv), due to Lemma (3.2.9), we have $F_\tau(t) : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$. And, therefore, all expressions in the right hand side of the Hamiltonian Feynman formula (3.4.9) are well defined. □

Lemma 3.4.7. *Under Assumption 3.4.1 (i),(ii), we have $F_\tau(t)\varphi \in L^1(\mathbb{R}^d)$ for any $\varphi \in C_c^\infty(\mathbb{R}^d)$ and any $\tau \in [0,1]$. For all $t \geq 0$, the operators $F_\tau(t)$ can be extended to bounded mappings on the space $L^1(\mathbb{R}^d)$, and there exists a constant $k \geq 0$ such that for all $t \geq 0$ holds:*

$$\|F_\tau(t)\| \leq e^{tk}. \tag{3.4.10}$$

Proof. Using the inequalities of Assumption 3.4.1 (i) and the fact, that the real part of each continuous negative definite function is non-negative (see Appendix C or inequalities (3.123) and (3.117) in Jacob, 2001), we obtain the estimate

$$\sup_{q\in\mathbb{R}^d} |e^{-tH(q,p)}| \leq e^{-ta_0 p^2} \exp\left(-t \min_{q\in\mathbb{R}^d} C(q)\right). \tag{3.4.11}$$

Hence, the function $f_{t,q} = (2\pi)^{-d/2}e^{-tH(q,\cdot)} \in L^1(\mathbb{R}^d)$ for each $q \in \mathbb{R}^d$ and $t > 0$. Moreover, $f_{t,q}(0) = (2\pi)^{-d/2}e^{-tC(q)}$. Therefore, the inverse Fourier transform of $f_{t,q}$ has the view $e^{-tC(q)}P_t^q$, where, for each $q \in \mathbb{R}^d$ and $t > 0$, the function $P_t^q \in C_\infty(\mathbb{R}^d)$ is a density of a probability measure. This follows from the Bochner Theorem and the fact that the inverse Fourier transform maps $L^1(\mathbb{R}^d)$ into $C_\infty(\mathbb{R}^d)$.

Consider first the case $\tau = 0$. Then we have for each $\varphi \in C_c^\infty(\mathbb{R}^d)$ by the Fubini–Tonelli Theorem

$$\begin{aligned} F_0(t)\varphi(q) &= \frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} e^{ip\cdot(q-q')}e^{-tH(q',p)}\varphi(q')dq'dp \\ &= \int\limits_{\mathbb{R}^d} \varphi(q')e^{-tC(q')}P_t^{q'}(q-q')dq'. \end{aligned}$$

Again by the Fubini–Tonelli Theorem, for each $\varphi \in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned} &\|F_0(t)\varphi\|_1 \\ &= \left\| \int\limits_{\mathbb{R}^d} \varphi(q')e^{-tC(q')}P_t^{q'}(q-q')dq' \right\|_1 \\ &\le \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} |\varphi(q')|e^{-tC(q')}P_t^{q'}(q-q')dq'dq \\ &= \int\limits_{\mathbb{R}^d} |\varphi(q')|e^{-tC(q')}\left[\int\limits_{\mathbb{R}^d} P_t^{q'}(q-q')dq\right]dq' \\ &\le \exp\left(-t\min_{x\in\mathbb{R}^d} C(x)\right)\|\varphi\|_1. \end{aligned}$$

Therefore, for any $\varphi \in C_c^\infty(\mathbb{R}^d)$ we have $F_0(t)\varphi \in L^1(\mathbb{R}^d)$ and $F_0(t)$ is a bounded operator from $C_c^\infty(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$. Then, due to the B.L.T. theorem A.0.19, the operator $F_0(t)$ can be extended to a bounded operator on $L^1(\mathbb{R}^d)$ with the same norm. Hence, the statement of Lemma 3.4.7 is true for $\tau = 0$.

Let us now prove the statement of Lemma 3.4.7 for the case $\tau \in (0,1]$. Let us now consider the function H given by (3.3.7) as a sum of functions h and r given by formulae (3.3.1) and (3.3.6)). Under Assumptions 3.4.1 (i),(ii), consider the family $(G^\theta_{A,B,C}(t))_{t\ge0}$ of operators on $L^1(\mathbb{R}^d)$ defined for each fixed $\theta \in (0,1]$ by the formula

$$\begin{aligned} &G^\theta_{A,B,C}(t)\varphi(q) \\ &= \frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} e^{ip\cdot(q-q')}e^{-th(\theta q,p)}\varphi(q')dq'dp \\ &= \frac{e^{-tC(\theta q)}}{(4\pi t)^{d/2}(\det A(\theta q))^{1/2}}\times \\ &\quad\times\int\limits_{\mathbb{R}^d} \exp\left(-\frac{(q-q'-tB(\theta q))\cdot A^{-1}(\theta q)(q-q'-tB(\theta q))}{4t}\right)\varphi(q')dq', \quad q\in\mathbb{R}^d. \end{aligned} \tag{3.4.12}$$

Each $G^\theta_{A,B,C}(t)$ is a integral operator with the kernel $g^x_t(z)$ given by the formula (3.4.4) with $x = \theta q$ and $z = q - q'$. Note, that g^x_t is the inverse Fourier transform of the function $(2\pi)^{-d/2}e^{-th(x,\cdot)}$. Due to Plyashechnik, 2013a (cf. Plyashechnik, 2013b), there is a constant $k < \infty$ such that the estimate $\|G^1_{A,B,C}(t)\| \le e^{kt}$ holds for $\theta = 1$. For each fixed $\theta \in (0,1]$ the operator $G^\theta_{A,B,C}(t)$ equals the operator $G^1_{A_\theta,B_\theta,C_\theta}(t)$ with new coefficients $A_\theta(q) := A(\theta q)$, $B_\theta(q) := B(\theta q)$, $C_\theta(q) := C(\theta q)$ which remain as smooth and bounded as the original A, B and C are. Therefore, the estimate $\|G^\theta_{A,B,C}(t)\| \le e^{kt}$ still holds for each $\theta \in (0,1]$ and the same k. Hence, by the Fubini–Tonelli Theorem for $\varphi \in C^\infty_c(\mathbb{R}^d)$ we have

$$F_\tau(t)\varphi(q) \tag{3.4.13}$$
$$= (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{ip\cdot(q-q')}e^{-tH(\tau q+(1-\tau)q',p)}\varphi(q')dq'dp$$
$$= (2\pi)^{-d}\int_{\mathbb{R}^d}\varphi(q')\left[\int_{\mathbb{R}^d} e^{ip\cdot(q-q')}e^{-th(\tau q+(1-\tau)q',p)}e^{-tr(p)}dp\right]dq'$$
$$= \int_{\mathbb{R}^d}\varphi(q')\left[g_t^{\tau q+(1-\tau)q'} * \eta_t\right](q-q')dq'. \tag{3.4.14}$$

Here the function in the squared brackets is for each fixed $q, q' \in \mathbb{R}^d$ and $\tau \in (0,1]$ the inverse Fourier transform of the product

$$\left((2\pi)^{-d/2}e^{-th(\tau q+(1-\tau)q',\cdot)}\right)\cdot\left((2\pi)^{-d/2}e^{-tr(\cdot)}\right),$$

i.e. a convolution of a function $g_t^{\tau q+(1-\tau)q'}$ given by the formula (3.4.4) and the probability measure η_t such that $\widetilde{\eta_t}(p) = (2\pi)^{-d/2}e^{-tr(p)}$ (cf. formula (3.3.11)). And this function is taken at the point $(q - q')$. Hence with $y := q + \frac{1-\tau}{\tau}q'$ and $x := q'/\tau$

$$\|F_\tau(t)\varphi\|_1 = \int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\varphi(q')\left[g_t^{\tau q+(1-\tau)q'} * \eta_t\right](q-q')dq'\right|dq$$
$$\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|\varphi(q')|\left[g_t^{\tau q+(1-\tau)q'} * \eta_t\right](q-q')dq'dq$$
$$= \int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|\varphi(q')|\, g_t^{\tau q+(1-\tau)q'}(q-q'-z)dq'dq\right]\eta_t(dz)$$
$$\le \int_{\mathbb{R}^d}\eta_t(dz)\cdot\sup_{z\in\mathbb{R}^d}\left[\tau^d\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_t^{\tau y}(y-z-x)|\varphi(\tau x)|dxdy\right]$$
$$= \tau^d\sup_{z\in\mathbb{R}^d}\int_{\mathbb{R}^d} G^\tau_{A,B,C}(t)|\varphi_\tau|(y-z)dy,$$

where $\varphi_\tau(q) := \varphi(\tau q)$ and the operator $G^\tau_{A,B,C}$ is given by the formula (3.4.12) for each $\tau \in (0,1]$. Therefore, due to the estimate $\|G^\tau_{A,B,C}(t)\| \le e^{kt}$ for each

$\varphi \in C_c^\infty(\mathbb{R}^d)$, we have

$$\begin{aligned}\left\|F_\tau(t)\varphi\right\|_1 &\le \tau^d \sup_{z\in\mathbb{R}^d} \int_{\mathbb{R}^d} G^\tau_{A,B,C}(t)|\varphi_\tau|(y-z)dy \\ &\le \tau^d e^{kt}\|\varphi_\tau\|_1 = \tau^d e^{kt} \int_{\mathbb{R}^d} |\varphi(\tau q)|dq = e^{kt}\|\varphi\|_1.\end{aligned}$$

Once again by the B.L.T. theorem A.0.19, the estimate $\left\|F_\tau(t)\varphi\right\|_1 \le e^{kt}\|\varphi\|_1$ is true for all $\varphi \in L^1(\mathbb{R}^d)$. □

Remark 3.4.8. Due to Lemma 3.2.5, the statement of Lemma (3.4.7) in the case $\tau = 1$ is also valid in the space $X = C_\infty(\mathbb{R}^d)$ with $k = 0$. Therefore, in the case $\tau = 1$, by the Riesz–Thorin theorem, the estimate (3.4.10) holds also in all spaces $L^p(\mathbb{R}^d)$, p ≥ 1 (with some other constants k).

Remark 3.4.9. As it follows from the representation (3.4.13), the operators $F_\tau(t)$ can be considered as integral operators acting on $\varphi \in C_c^\infty(\mathbb{R}^d)$ as

$$F_\tau(t)\varphi(q) = \int_{\mathbb{R}^d} \varphi(q_1)\left[g_t^{\tau q+(1-\tau)q_1} * \eta_t\right](q-q_1)dq_1, \quad q \in \mathbb{R}^d. \tag{3.4.15}$$

The right hand side is, however, well defined for all $\varphi \in L^1(\mathbb{R}^d)$ and provides the extension of $F_\tau(t)$ to a bounded linear oprator on $L^1(\mathbb{R}^d)$. Hence, this representation can be used to construct a Lagrangian Feynman formula.

Lemma 3.4.10. *Let $N \equiv 0$ in the formula (3.3.6), i.e. $H \equiv h$. Under Assumption 3.4.1 (i), (ii) and Assumption 3.4.2, for any $\varphi \in C_c^\infty(\mathbb{R}^d)$, any $\tau \in [0,1]$ and any $t_0 \ge 0$, we have*

$$\lim_{t\searrow 0}\left\|\frac{F_\tau(t)\varphi-\varphi}{t} + \widehat{H}_\tau\varphi\right\|_1 = 0 \quad \textit{and} \quad \lim_{t\to t_0}\|F_\tau(t)\varphi - F_\tau(t_0)\varphi\|_1 = 0.$$

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^d) \subset \mathrm{Dom}(\widehat{H}_\tau)$, $t > 0$. By Taylor's formula with $\theta \in (0,1)$ we have

$$\begin{aligned}&\left\|\frac{F_\tau(t)\varphi-\varphi}{t} + \widehat{H}_\tau\varphi\right\|_1 \\ &= \frac{t}{(2\pi)^d}\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^{2d}} e^{ip\cdot(q-q')}h^2(\tau q+(1-\tau)q',p)e^{-\theta t h(\tau q+(1-\tau)q',p)}\varphi(q')dq'dp\right|dq.\end{aligned}$$

Here $p \mapsto h^2(\tau q+(1-\tau)q',p)$ is a 4th order polynomial with bounded coefficients continuously depending on q and q'. Let us present the calculations for the case

$d = 1$ and $b = 0$, $c = 0$ for simplicity. The general case can be handled similarly.

$$\begin{aligned}
&\left\| \frac{F_\tau(t)\varphi - \varphi}{t} + \widehat{H}_\tau \varphi \right\|_1 \\
&= \frac{t}{2\pi} \int\limits_{\mathbb{R}} \left| \int\limits_{\mathbb{R}^2} e^{ip\cdot(q-q')} A^2(\tau q + (1-\tau)q') p^4 e^{-\theta t A(\tau q + (1-\tau)q')p^2} \varphi(q') dq' dp \right| dq \\
&= \frac{t}{2\pi} \int\limits_{\mathbb{R}} \left| \int\limits_{\mathbb{R}^2} \partial^4_{q'} \left[e^{ip\cdot(q-q')} \right] \left[A^2(\tau q + (1-\tau)q') e^{-\theta t A(\tau q + (1-\tau)q')p^2} \varphi(q') \right] dq' dp \right| dq \\
&= \frac{t}{2\pi} \int\limits_{\mathbb{R}} \left| \int\limits_{\mathbb{R}^2} \left[e^{ip\cdot(q-q')} \right] \partial^4_{q'} \left[A^2(\tau q + (1-\tau)q') e^{-\theta t A(\tau q + (1-\tau)q')p^2} \varphi(q') \right] dq' dp \right| dq.
\end{aligned} \tag{3.4.16}$$

Consider first the case $\tau = 1$. Then

$$\partial^4_{q'} \left[A^2(\tau q + (1-\tau)q') e^{-\theta t A(\tau q + (1-\tau)q')p^2} \varphi(q') \right] = A^2(q) e^{-\theta t A(q)p^2} \varphi^{(4)}(q')$$

and by the Fubini–Tonelli theorem

$$\begin{aligned}
&\left\| \frac{F_1(t)\varphi - \varphi}{t} + \widehat{H}_1 \varphi \right\|_1 \\
&= t \int\limits_{\mathbb{R}} \left| (2\pi)^{-1} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \left[e^{ip\cdot(q-q')} \right] \cdot \left[A^2(q) e^{-\theta t A(q)p^2} \varphi^{(4)}(q') \right] dq' dp \right| dq \\
&= t \int\limits_{\mathbb{R}} \left| A^2(q) \int\limits_{\mathbb{R}} (4\pi\theta t A(q))^{-1/2} e^{-\frac{(q-q')^2}{4\theta t A(q)}} \varphi^{(4)}(q') dq' \right| dq \\
&\le t A_0^2 \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} (4\pi\theta t a_0)^{-1/2} e^{-\frac{(q-q')^2}{4\theta t A_0}} |\varphi^{(4)}(q')| dq' dq = t(A_0^{5/2} a_0^{-1/2}) \|\varphi^{(4)}\|_1.
\end{aligned}$$

Consider now the case when $\tau \in [0,1)$. Then

$$\begin{aligned}
&\partial^4_{q'} \left[A^2(\tau q + (1-\tau)q') e^{-\theta t A(\tau q + (1-\tau)q')p^2} \varphi(q') \right] \\
&= e^{-\theta t A(\tau q + (1-\tau)q')p^2} \sum_{k=0}^{4} (\theta t p^2)^k \psi_k(\tau q + (1-\tau)q', q'),
\end{aligned}$$

where the functions $(x, y) \mapsto \psi_k(x, y)$ are just linear combinations of the products $A^k(x)(A^2)^{(m)}(x)\varphi^{(n)}(y)$ with $m, n = 0, \ldots, 4$. Hence, $\psi_k(x, \cdot) \in C_c(\mathbb{R})$ and $\psi_k(\cdot, y) \in C_b(\mathbb{R})$ for all $x, y \in \mathbb{R}$. Therefore, with the change of variables $\sqrt{\theta t} p = \rho$,

$\frac{q-q'}{\sqrt{\theta t}} = y$ we have

$$
\begin{aligned}
&\left\| \frac{F_\tau(t)\varphi - \varphi}{t} + \widehat{H}_\tau \varphi \right\|_1 \\
&= \frac{t}{2\pi} \int\limits_{\mathbb{R}} \left| \int\limits_{\mathbb{R}^2} e^{ip\cdot(q-q')} e^{-\theta t A(\tau q + (1-\tau)q')p^2} \sum_{k=0}^{4} (\theta t p^2)^k \psi_k(\tau q + (1-\tau)q', q') dq' dp \right| dq \\
&= \frac{t}{2\pi} \int\limits_{\mathbb{R}} \left| \int\limits_{\mathbb{R}^2} e^{i\rho\cdot y} e^{-A(q-\sqrt{\theta t}(1-\tau)y)\rho^2} \sum_{k=0}^{4} (\rho^2)^k \psi_k(q - \sqrt{\theta t}(1-\tau)y, q - \sqrt{\theta t} y) d\rho dy \right| dq \\
&= t \int\limits_{\mathbb{R}} \left| \int\limits_{\mathbb{R}} \sum_{k=0}^{4} (-1)^k \, \partial_\xi^{2k} \left[\frac{\exp\left\{ -\frac{\xi^2}{4A(q-\sqrt{\theta t}(1-\tau)y)} \right\}}{(4\pi A(q-\sqrt{\theta t}(1-\tau)y))^{1/2}} \right] \right|_{\xi=y} \\
&\qquad\qquad\qquad \times \psi_k(q - \sqrt{\theta t}(1-\tau)y, q - \sqrt{\theta t} y) dy \Bigg| dq \\
&\le t \int\limits_{\mathbb{R}} (4\pi a_0)^{-1/2} e^{-\frac{y^2}{4A_0}} C_5 (1 + y^8) \sum_{k=0}^{4} C_k \int\limits_{\mathbb{R}} |\varphi^{(k)}(q - \sqrt{\theta t} y)| dq dy \\
&\le t \sum_{k=0}^{4} C_k' \|\varphi^{(k)}\|_1
\end{aligned}
$$

with some positive constants $C_k < \infty$ and $C_k' < \infty$. Analogously, for $\varphi \in C_c^\infty(\mathbb{R}^d)$ by Taylor's formula with $\theta \in (0,1)$ and $t, t_0 \ge 0$, $t \to t_0$ we have

$$
\begin{aligned}
&\|F_\tau(t)\varphi - F_\tau(t_0)\varphi\|_1 = |t - t_0| (2\pi)^{-d} \\
&\times \int\limits_{\mathbb{R}^d} \left| \int\limits_{\mathbb{R}^{2d}} e^{ip\cdot(q-q')} h(\tau q + (1-\tau)q', p) e^{-[t_0 + \theta(t-t_0)] h(\tau q + (1-\tau)q', p)} \varphi(q') dq' dp \right| dq.
\end{aligned}
$$

Once again let us present the calculations for the case $d = 1$ and $b = 0$, $c = 0$ for simplicity. For any fixed $t_0 > 0$ take $t \in (t_0/2, 2t_0)$. Hence, $\alpha(t) := t_0 + \theta(t - t_0) \in (t_0/2, 2t_0)$ and

$$
\begin{aligned}
&\|F_\tau(t)\varphi - F_\tau(t_0)\varphi\|_1 = |t - t_0| (2\pi)^{-1} \\
&\times \int\limits_{\mathbb{R}} \left| \int\limits_{\mathbb{R}^2} e^{ip\cdot(q-q')} A(\tau q + (1-\tau)q') p^2 e^{-\alpha(t) A(\tau q + (1-\tau)q')p^2} \varphi(q') dq' dp \right| dq \\
&= |t - t_0| \int\limits_{\mathbb{R}} \left| \int\limits_{\mathbb{R}} (4\pi \alpha(t) A(\tau q + (1-\tau)q'))^{-1/2} \right. \\
&\qquad\qquad \left. \times \, \partial_\xi^2 \left[e^{-\frac{\xi^2}{4\alpha(t) A(\tau q + (1-\tau)q')}} \right] \Bigg|_{\xi = q - q'} \varphi(q') dq' \right| dq \\
&\le |t - t_0| \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} (2\pi t_0 a_0)^{-1/2} e^{-\frac{(q-q')^2}{8 t_0 A_0}} C(t_0) (1 + (q - q')^2) |\varphi(q')| dq' dq \\
&\le |t - t_0| C'(t_0) \|\varphi\|_1
\end{aligned}
$$

with some positive constants $C < \infty$ and $C' < \infty$ depending only on t_0. In the case $t_0 = 0$, we have $F(t_0) = \mathrm{Id}$. And we proceed as before

$$\begin{aligned}
&\|F_\tau(t)\varphi - \varphi\|_1 \\
&= \frac{t}{2\pi} \int\limits_{\mathbb{R}} \left| \int\limits_{\mathbb{R}^2} e^{ip\cdot(q-q')} A(\tau q + (1-\tau)q')p^2 e^{-\theta t A(\tau q+(1-\tau)q')p^2} \varphi(q') dq' dp \right| dq \\
&= \frac{t}{2\pi} \int\limits_{\mathbb{R}} \left| \int\limits_{\mathbb{R}^2} \partial^2_{q'} \left[e^{ip\cdot(q-q')}\right] A(\tau q + (1-\tau)q') e^{-\theta t A(\tau q+(1-\tau)q')p^2} \varphi(q') dq' dp \right| dq \\
&= \frac{t}{2\pi} \int\limits_{\mathbb{R}} \left| \int\limits_{\mathbb{R}^2} e^{ip\cdot(q-q')} \partial^2_{q'} \left[A(\tau q + (1-\tau)q') e^{-\theta t A(\tau q+(1-\tau)q')p^2} \varphi(q')\right] dq' dp \right| dq \\
&\le t \sum_{k=0}^{2} C_k \|\varphi^{(k)}\|_1,
\end{aligned}$$

where the integrals in the penultimate line can be handled as in (3.4.16). A 3-ε–argument concludes the proof of $\lim\limits_{t\to t_0} \|F_\tau(t)\varphi - F_\tau(t_0)\varphi\|_1 = 0$ for all $\varphi \in L^1(\mathbb{R}^d)$. □

Lemma 3.4.11. *Let $\tau = 1$. Let $f, g : \mathbb{R}^d \to \mathbb{C}$ be bounded continuous functions and $\lambda : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be the 1-symbol of the pseudo-differential operator $\widehat{\lambda}_1$. Let $H(q,p) = f(q)g(p)\lambda(q,p)$, $q, p \in \mathbb{R}^d$. Then*

$$\widehat{H}_1\varphi = \left(\widehat{f} \circ \widehat{\lambda}_1 \circ \widehat{g}\right)\varphi$$

for all $\varphi \in S(\mathbb{R}^d) \cap \mathrm{Dom}(\widehat{H}_1) \cap \mathrm{Dom}(\widehat{f} \circ \widehat{\lambda}_1 \circ \widehat{g})$.

Proof. Let $\varphi \in S(\mathbb{R}^d) \cap \mathrm{Dom}(\widehat{H}_1) \cap \mathrm{Dom}(\widehat{f} \circ \widehat{\lambda}_1 \circ \widehat{g})$. Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ stand for Fourier transform and its inverse respectively. Then

$$\begin{aligned}
\widehat{H}_1\varphi(q) &= (2\pi)^{-d} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} e^{ip\cdot(q-q')} H(q,p)\varphi(q')\, dq'\, dp \\
&= (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} e^{ip\cdot q} f(q)g(p)\lambda(q,p)\mathcal{F}[\varphi](p)dp,
\end{aligned}$$

and

$$\begin{aligned}
\left(\widehat{f} \circ \widehat{\lambda}_1 \circ \widehat{g}\right)\varphi(q) &= \left(\widehat{f} \circ \widehat{\lambda}_1\right) \mathcal{F}^{-1}[g\mathcal{F}[\varphi]](q) \\
&= f(q)(2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} e^{ip\cdot q} \lambda(q,p)\mathcal{F}[\mathcal{F}^{-1}[g\mathcal{F}[\varphi]]](p)dp \\
&= (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} e^{ip\cdot q} f(q)g(p)\lambda(q,p)\mathcal{F}[\varphi](p)dp.
\end{aligned}$$

□

Lemma 3.4.12. *Let the symbol $H = h + r$ be given by the formula (3.3.7) and $\tau = 1$. Under Assumption 3.4.1 (i), (iii), (iv) and Assumption 3.4.2 for any $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $t_0 \geq 0$, we have*

$$\lim_{t\searrow 0}\left\|\frac{F_\tau(t)\varphi - \varphi}{t} + \widehat{H}_\tau\varphi\right\|_1 = 0$$

and

$$\lim_{t\to t_0}\|F_\tau(t)\varphi - F_\tau(t_0)\varphi\|_1 = 0.$$

Proof. Fix $t_0 \geq 0$ and let $t \in [0, t_0+1]$. By Taylor's formula with θ in between t and t_0, by the Fubini–Tonelli theorem, by Lemma 3.4.11 and Lemma 3.4.7 with the probability measure $\eta_\theta = (2\pi)^{-d/2}\mathcal{F}^{-1}[e^{-\theta r(p)}]$, for each $\varphi \in C_c^\infty(\mathbb{R}^d)$, we have

$$\begin{aligned}
&\|F_1(t)\varphi - F_1(t_0)\varphi\|_1 = \left\|\frac{t-t_0}{(2\pi)^d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} e^{ip\cdot(q-q')}H(q,p)e^{-\theta H(q,p)}\varphi(q')dq'dp\right\|_1\\
&= |t-t_0|\left\|\left(\widehat{He^{-\theta H}}\right)_1\varphi\right\|_1 \leq |t-t_0|\left[\left\|\left(\widehat{he^{-\theta H}}\right)_1\varphi\right\|_1 + \left\|\left(\widehat{re^{-\theta H}}\right)_1\varphi\right\|_1\right]\\
&= |t-t_0|\left[\left\|\left(\widehat{he^{-\theta h}}\right)_1\circ\left(\widehat{e^{-\theta r}}\right)\varphi\right\|_1 + \left\|\left(\widehat{e^{-\theta H}}\right)_1\circ\widehat{r}\varphi\right\|_1\right]\\
&\leq |t-t_0|\left[\left\|\left(\widehat{he^{-\theta h}}\right)_1(\eta_\theta * \varphi)\right\|_1 + \|F_1(\theta)\|\,\|\widehat{r}\varphi\|_1\right]\\
&\leq |t-t_0|\left[\sum_{k=0}^{2}C_k(t_0)\|(\eta_\theta*\varphi)^{(k)}\|_1 + e^{k\theta}\|\widehat{r}\varphi\|_1\right]\\
&\leq |t-t_0|\left[\sum_{k=0}^{2}C_k(t_0)\eta_\theta(\mathbb{R}^d)\|\varphi^{(k)}\|_1 + e^{k\theta}\|\widehat{r}\varphi\|_1\right]\\
&= |t-t_0|K(t_0,\varphi)
\end{aligned}\tag{3.4.17}$$

with some constants $C_k(t_0) < \infty$, depending only on t_0, and $K(t_0,\varphi) < \infty$, depending only on t_0 and φ. These constants $C_k(t_0)$ arise from the calculations with the operator $\left(\widehat{he^{-\theta h}}\right)_1$ obtained in the Lemma 3.4.12. Note, that all calculations in Lemma 3.4.12 remain true for any $\varphi \in S(\mathbb{R}^d)$. Moreover, by Assumption 3.4.1 (iv), we have $r \in C^\infty(\mathbb{R}^d)$ and, as a negative definite function, r grows at infinity with all its derivatives not faster than a polynomial (cf. Lemma 3.6.22 and Theo.3.7.13 in Jacob, 2001). Therefore, $\widehat{r^m}\varphi$, $\left(\widehat{r^m e^{-\theta r}}\right)\varphi \in S(\mathbb{R}^d)$ for any $\varphi \in C_c^\infty(\mathbb{R}^d) \subset S(\mathbb{R}^d)$ and any $m \in \mathbb{N}\cup\{0\}$. In the same way by Lemma 3.4.11 and Lemma 3.4.12 with $[0,1] \ni t \to 0$ and $\theta \in (0,t)$, we obtain

$$\begin{aligned}
&\left\|\frac{F_1(t)\varphi-\varphi}{t} + \widehat{H}_1\varphi\right\|_1 = t\left\|\left(\widehat{H^2e^{-\theta H}}\right)_1\varphi\right\|_1 \leq t\left\|\left(\widehat{h^2e^{-\theta h}}\right)_1\circ\left(\widehat{e^{-\theta r}}\right)\varphi\right\|_1\\
&+ t\left\|2\left(\widehat{he^{-\theta h}}\right)_1\circ\left(\widehat{re^{-\theta r}}\right)\varphi + \left(\widehat{e^{-\theta h}}\right)_1\circ\left(\widehat{r^2e^{-\theta r}}\right)\varphi\right\|_1\\
&= t\left\|\left(\widehat{h^2e^{-\theta h}}\right)_1(\eta_\theta*\varphi) + 2\left(\widehat{he^{-\theta h}}\right)_1(\eta_\theta*[\widehat{r}\varphi]) + F_1(\theta)(\eta_\theta*[\widehat{r^2}\varphi])\right\|_1\\
&\leq t\eta_\theta(\mathbb{R}^d)\left[\sum_{k=0}^{4}C_k\|\varphi^{(k)}\|_1 + 2\sum_{k=0}^{2}C'_k\|(\widehat{r}\varphi)^{(k)}\|_1 + e^{k\theta}\left\|\widehat{(r^2)}\varphi\right\|_1\right] = tK'(\varphi).
\end{aligned}$$

□

3.5 Phase space Feynman path integrals related to Feller Processes

The notion of the path integral has been introduced by Richard Feynman in the middle of XX century in Feynman, 1948, Feynman, 1951. And now the apparatus of Feynman path integrals is one of important tools in Quantum Physics. There is a great deal of contributions to the development of the theory of such integrals since the time of R. Feynman till nowadays (see, e.g., Albeverio and Mazzucchi, 2016; Albeverio, Høegh-Krohn, and Mazzucchi, 2008; Berezin, 1980; Berezin, 1971; Bock and Grothaus, 2011; Butko, Grothaus, and Smolyanov, 2016; Cartier and DeWitt-Morette, 2006; Daubechies and Klauder, 1985; DeWitt-Morette, Maheshwari, and Nelson, 1979; Elworthy and Truman, 1984; Garrod, 1966; Grosche, 2013; Grosche and Steiner, 1998; Ichinose, 2010; Ichinose, 2006; Ichinose, 2000; Johnson and Lapidus, 2000; Kitada and Kumano-go, 1981; Kleinert, 2009; Kumano-go and Fujiwara, 2008; Kumano-Go, 1996; Nelson, 1964; Simon, 1979; Smolyanov and Shavgulidze, 1990; Smolyanov, Tokarev, and Truman, 2002 and references therein). However, the subject is still far from being exhausted. There exist substantially different approaches to the definition and dealing with Feynman path integrals; in particular, to the definition and dealing with Feynman path integrals over sets of paths in the phase space of a physical system (the so-called *phase space Feynman path integrals*). Some phase space Feynman path integrals are defined via the Fourier transform and via Parseval's equality (see Smolyanov and Shavgulidze, 1990, cf. Albeverio, Høegh-Krohn, and Mazzucchi, 2008; see Elworthy and Truman, 1984; Smolyanov and Shamarov, 2010; Cartier and DeWitt-Morette, 2006; DeWitt-Morette, Maheshwari, and Nelson, 1979 and references therein); some are defined via an analytic continuation of a Gaussian measure on the set of paths in a phase space (Smolyanov and Shavgulidze, 1990), some — via regularization procedures, e.g., as limits of integrals with respect to Gaussian measures with a diverging diffusion constant (Daubechies and Klauder, 1985); the integrands of some phase space Feynman path integrals are realized as Hida distributions in the setting of White Noise Analysis (Bock and Grothaus, 2011; Bock, Grothaus, and Jung, 2012; Bock and Grothaus, 2015). A variety of approaches treats Feynman path integrals as limits of integrals over some finite dimensional subspaces of paths when the dimension tends to infinity. Such path integrals are sometimes called sequential and are most convenient for direct calculations. The general definition of a sequential Feynman pseudomeasure (Feynman path integral) in an abstract space (on a set of paths in a phase space, in particular) can be found in Smolyanov and Shavgulidze, 1990. Some concrete realizations are e.g. presented in Albeverio, Guatteri, and Mazzucchi, 2002; Garrod, 1966; Ichinose, 2000; Kitada and Kumano-go, 1981; Kumano-go and Fujiwara, 2008; Kumano-Go, 1996. In the present Section we treat a sequential approach based on Feynnman formulae (cf. Butko, Grothaus, and Smolyanov, 2016; Böttcher et al., 2011; Butko, Schilling, and Smolyanov, 2011; Smolyanov, Tokarev, and Truman, 2002).

Different methods of constructing path integrals in the frame of sequential approach produce actually different approximations of such integrals. In his paper, Berezin, 1980, Berezin has remarked that the Feynman path integral is "very sensitive to the choice of approximations, and nonuniqueness appearing due to this dependence has the same character as nonuniqueness of quantization". In other words, the Feynman path integral is different for different procedures of quantization. This difference may appear both in integrands and in the set of paths over which the integration takes place. Berezin has posed the problem of distinguishing the procedure of quantization in the language of Feynman path integrals. In the paper Berezin, 1980, Berezin has considered the case of the Weyl quantization and his calculations have lead to a quite odd expression in the integrand of his Feynman path integral. The question, how to distinguish the procedure of quantization on the language of Feynman path integrals, remained open. The present Section is an attempt to answer Berezin's problem using the method of Feynman formulae. Namely, we consider the Hamiltonian Feynman formulae, obtained in Sections 3.2-3.4. These Hamiltonian Feynman formulae represent evolution semigroups $e^{-t\widehat{H}}$ generated by different, parameterized by $\tau \in [0,1]$, quantizations of a given symbol H. We show that these Hamiltonian Feynman formulae can be interpreted as sequential phase space Feynman path integrals. To this aim, a family of phase space Feynman pseudomeasures corresponding to different procedures of quantization is introduced and the considered evolution semigroups are represented as phase space Feynman path integrals with respect to these Feynman pseudomeasures. In this way, one obtains the same integrands but different domains of integration (and different pseudomeasures) for the Feynman path integrals corresponding to the semigroups $e^{-t\widehat{H}}$ generated by different quantizations of H. Moreover, the obtained in Sections 3.2-3.4 Lagrangian Feynman formulae for the same semigroups allow to connect Feynman path integrals with some functional integrals with respect to probability measures and hence provide a tool also to calculate these phase space Feynman path integrals by means of stochastic analysis.

To obtain Hamiltonian Feynman formulae of Sections 3.2, 3.4, we have approximated the semigroup $e^{-t\widehat{H}}$ (for a given procedure of quantization, for a given symbol) by the family of pseudo-differential operators $\widehat{e^{-tH}}$ obtained by the same procedure of quantization from the function e^{-tH}. Note again, that if the function H depends on both variables q and p, then $e^{-t\widehat{H}} \neq \widehat{e^{-tH}}$. Nevertheless, we have succeeded to prove that

$$e^{-t\widehat{H}} = \lim_{n\to\infty} \left[\widehat{e^{-\frac{t}{n}H}}\right]^n. \tag{3.5.1}$$

The limit in the right hand side is actually the limit of n-fold iterated integrals over the phase space when n tends to infinity, i.e. the identity (3.5.1) leads to a Hamiltonian Feynman formula. This limit can be interpreted also as a phase space Feynman path integral with $\exp\left(-\int_0^t H(q(s),p(s))ds\right)$ in the integrand. On a heuristic level, the same approach was used already in Berezin's papers Berezin, 1971, Berezin, 1980 for investigation of Schrödinger groups $e^{-it\widehat{H}}$. In his works, Berezin has just assumed the identity

$$e^{-it\widehat{H}} = \lim_{n\to\infty}\left[\widehat{e^{-i\frac{t}{n}H}}\right]^n \tag{3.5.2}$$

and then has interpreted the pre-limit expressions in the right hand side of the identity (3.5.2) as approximations to a phase space Feynman path integral. The rigorous justification of this approach was first obtained only in Smolyanov, Tokarev, and Truman, 2002. The main technical tool suggested in this paper was the Chernoff Theorem. Recall that the Chernoff theorem is a wide generalization of the classical Daletskii–Lie–Trotter formula which has been used for handling of Feynman path integrals over paths in configuration space of a system (see, e.g., Nelson, 1964).

Before introducing the explicit family of phase space Feynman pseudomeasures corresponding to different procedures of τ-quantization, let us outline some general concepts of the construction of Feynman pseudomeasure, in particular, phase space Feynman path integrals (cf., e.g., Smolyanov and Shavgulidze, 1990 and Smolyanov and Shavgulidze, 2015).

A *Feynman pseudomeasure* on a (usually infinite dimensional) vector space is a continuous linear functional on a locally convex space of some functions defined on this vector space. The value of this functional on a function belonging to its domain is called Feynman integral with respect to this Feynman pseudomeasure. If the considered vector space is itself a set of functions taking values in classical configuration or phase space then the corresponding Feynman integral is called configuration or phase space Feynman path integral.

For a locally convex space Y denote the set of all continuous linear functionals on Y by Y^*. Let E be a real vector space and for all $x \in E$ and any linear functional g on E let $\phi_g(x) = e^{ig(x)}$. Let F_E be a locally convex set of some complex valued functions on E. Elements of the set F_E^* are called *F_E^*-distributions on E* or just *distributions on E* (if we don't specify the space F_E^* exactly). Let G be a vector space of some linear functionals on E distinguishing elements of E and let $\phi_g \in F_E$ for all $g \in G$. Then *G-Fourier transform* of an element $\eta \in F_E^*$ is a function on G denoted by $\widetilde{\eta}$ or $\mathcal{F}[\eta]$ and defined by the formula

$$\widetilde{\eta}(g) \equiv \mathcal{F}[\eta](g) := \eta(\phi_g).$$

If a set $\{\phi_g : g \in G\}$ is total in F_E (i.e. its linear span is dense in F_E) then any element η is uniquely defined by its Fourier transform.

Definition 3.5.1. Let b be a quadratic functional on G, $a \in E$ and $\alpha \in \mathbb{C}$. Then *Feynman α-pseudomeasure* on E with correlation functional b and mean a is a distribution $\Phi_{b,a,\alpha}$ on E whose Fourier transform is given by the formula

$$\mathcal{F}[\Phi_{b,a,\alpha}](g) = \exp\left(\frac{\alpha b(g)}{2} + ig(a)\right).$$

If $\alpha = -1$ and $b(x) \geq 0$ for all $x \in G$ then Feynman α-pseudomeasure is a Gaussian G-cylindrical measure on E (which however can be not σ–additive). If $\alpha = i$ then we have a "standard" Feynman pseudomeasure which is usually used for solving Schrödinger type equations. In the sequel we will consider only these "standard" Feynman i–pseudomeasures with $a = 0$.

Definition 3.5.2 (Hamiltonian (or phase space) Feynman pseudomeasure)**.** Let $E = Q \times P$, where Q and P are locally convex spaces, $Q = P^*$, $P = Q^*$ (as vector spaces) with the duality $\langle \cdot, \cdot \rangle$; the space $G = P \times Q$ is identified with the space of all linear functionals on E in the following way: for any $g = (p_g, q_g) \in G$ and $x = (q, p) \in E$ let $g(x) = \langle q, p_g \rangle + \langle q_g, p \rangle$. Then *Hamiltonian* (or *symplectic*, or *phase space*) *Feynman pseudomeasure* on E is a Feynman i-pseudomeasure Φ on E whose correlation functional b is given by the formula $b(p_g, q_g) = 2\langle q_g, p_g \rangle$ and mean $a = 0$, i.e.

$$\mathcal{F}[\Phi](g) = \exp\left(i\langle q_g, p_g \rangle\right).$$

Definition 3.5.3. Assume that there exists a linear injective mapping $B : G \to E$ such that $b(g) = g(B(g))$ for all $g \in G$ (B is called *correlation operator* of Feynman pseudomeasure). A function $\mathrm{Dom}(B^{-1}) \ni x \mapsto f(x) = e^{\frac{\alpha^{-1} B^{-1}(x)(x)}{2}}$ is called the *generalized density* of Feynman α–pseudomeasure (cf. Weizsäcker, Smolyanov, and Wittich, 2000; Garsia-Narankho, Montalʹdi, and Smolyanov, 2016).

Example 3.5.4. (i) If $E = \mathbb{R}^d = G$ then the Feynman i-pseudomeasure on E with correlation operator B can be identified with a complex-valued measure (with unbounded variation) on a δ-ring of bounded Borel subsets of $\mathbb{R}^d$ whose density with respect to the Lebesgue measure is $f(x) = e^{-\frac{i}{2}(B^{-1}x, x)}$. In this case the generalized density coincides with the density in usual sense.

(ii) If we consider the Hamiltonian Feynman pseudomeasure on $E = Q \times P$ then take $B : (p, q) \in G \subset E^* \to (q, p) \in E$. Then we have $g(B(g)) = g(B(p_g, q_g)) = g(q_g, p_g) = 2\langle q_g, p_g \rangle = b(g)$. Moreover, $B^{-1} : E \to E^*$ is defined by the formula $B^{-1}(q, p) = (p, q)$ and hence the generalized density of the Hamiltonian Feynman pseudomeasure is given by the formula $f(q, p) = \exp\{i\langle q, p \rangle\}$.

The concepts given above allow to introduce the following definition of a Feynman pseudomeasure in the frame of sequential approach (in the sequel we assume any standard regularization of oscillating integrals, e.g., $\int_E f(z)dz = \lim_{\varepsilon \to 0} \int_E f(z) e^{-\varepsilon|z|^2} dz$).

Definition 3.5.5 (Sequential Feynman pseudomeasure)**.** Let $\{E_n\}_{n \in \mathbb{N}}$ be an increasing sequence of finite dimensional subsets of $\mathrm{Dom}(B^{-1})$. Then the value of a sequential Feynman α–pseudomeasure $\Phi_{B,\alpha}^{\{E_n\}}$ (with mean $a = 0$) associated with the sequence $\{E_n\}_{n\in\mathbb{N}}$ on a function $f : E \to \mathbb{C}$ (this value is called *sequential Feynman integral* of f) is defined by the formula

$$\Phi_{B,\alpha}^{\{E_n\}}(f) = \lim_{n\to\infty}\left(\int_{E_n} e^{\frac{\alpha^{-1}B^{-1}(x)(x)}{2}}dx\right)^{-1}\int_{E_n} f(x)e^{\frac{\alpha^{-1}B^{-1}(x)(x)}{2}}dx,$$

where one integrates with respect to the Lebesgue measure on E_n, if the limit in the r.h.s. exists.

The fact that the function f belongs to the domain of the functional $\Phi^{\{E_n\}}$ depends only on restrictions of this function to the subspaces E_n. In the particular case of Hamiltonian Feynman pseudomeasure, Definition 3.5.5 can be read as follows:

Definition 3.5.6. Let $\{E_n = Q_n \times P_n\}_{n\in\mathbb{N}}$ be an increasing sequence of finite dimensional vector subspaces of $E = Q \times P$, where Q_n and P_n are vector subspaces of Q and P respectively. The value $\Phi_{\{E_n\}}(G)$ of the Hamiltonian Feynman pseudomeasure $\Phi_{\{E_n\}}$, associated with the sequence $\{E_n\}_{n\in\mathbb{N}}$, on a function $f : E \to \mathbb{C}$, i.e. a Feynman path integral of f, is defined by the formula

$$\Phi_{\{E_n\}}(f) = \lim_{n\to\infty}\Big(\int_{E_n} e^{i\langle q,p\rangle}\,dq\,dp\Big)^{-1}\int_{E_n} f(q,p)e^{i\langle q,p\rangle}\,dq\,dp, \tag{3.5.3}$$

if this limit exists. In this formula (as well as before) all integrals must be considered in a suitably regularized sense.

Let us now present the construction of the Hamiltonian Feynman pseudomeasure for a particular family of spaces $E_t^{x,\tau}$ with $\tau \in [0,1]$ (cf. Butko, Grothaus, and Smolyanov, 2016; Böttcher et al., 2011; Smolyanov, Tokarev, and Truman, 2002). For any $t > 0$, let $PC([0,t],\mathbb{R}^d)$ be the vector space of all functions on $[0,t]$ taking values in $\mathbb{R}^d$ whose distributional derivatives are measures with finite support. Let $PC^l([0,t],\mathbb{R}^d)$ denote the space of all left continuous functions from $PC([0,t],\mathbb{R}^d)$. Let $PC^\tau([0,t],\mathbb{R}^d)$ be the collection of functions f from $PC([0,t],\mathbb{R}^d)$ such that for all $s \in (0,t)$

$$f(s) = \tau f(s+0) + (1-\tau)f(s-0). \tag{3.5.4}$$

We call the elements of $PC^\tau([0,t],\mathbb{R}^d)$ *τ-continuous functions*. For each $x \in \mathbb{R}^d$ let

$$Q_t^{x,\tau} = \{f \in PC^\tau([0,t],\mathbb{R}^d) : \ f(0) = \lim_{s\to+0} f(s), \ f(t) = x\},$$

$$P_t = \{f \in PC^l([0,t],\mathbb{R}^d) : f(0) = \lim_{s\to+0} f(s)\},$$

and $E_t^{x,\tau} = Q_t^{x,\tau} \times P_t$. The spaces $Q_t^{x,\tau}$ and P_t are taken in duality by the form:

$$\langle q(\cdot),p(\cdot)\rangle \mapsto \int_0^t p(s)\dot q(s)\,ds,$$

where $\dot{q}(s)\,ds$ denotes the measure which is the distributional derivative of $q(\cdot)$. We will consider the elements of $E_t^{x,\tau}$ as functions taking values in $\mathbb{R}^d \times \mathbb{R}^d$.

Let $t_0 := 0$ and for any $n \in \mathbb{N}$ and any $k \in \mathbb{N}$, $k \le n$, let $t_k := \frac{k}{n}t$. Let $F_n \subset PC([0,t],\mathbb{R}^d)$ be the space of functions, whose restrictions to each of the intervals (t_{k-1}, t_k) are constant functions. Let $Q_n^\tau = F_n \cap Q_t^{x,\tau}$, $P_n = F_n \cap P_t$. Let J_n^τ be the mapping of $E_n^\tau = Q_n^\tau \times P_n$ to $(\mathbb{R}^d \times \mathbb{R}^d)^n$, defined by

$$\begin{aligned} J_n^\tau(q,p) &:= \left(q(\tfrac{t}{n}-0), p(\tfrac{t}{n}), \dots, q(\tfrac{(n-1)t}{n}-0), p(\tfrac{(n-1)t}{n}), q(\tfrac{nt}{n}-0), p(\tfrac{nt}{n})\right) \\ &= \left(q_1, p_1, \dots\dots, q_n, p_n\right). \end{aligned}$$

The map J_n^τ is a one-to-one correspondence of E_n^τ and $(\mathbb{R}^d \times \mathbb{R}^d)^n$. Therefore, in this particular case Definition 3.5.6 can be rewritten in the following way:

Definition 3.5.7. The Hamiltonian (or phase space) Feynman path integral

$$\Phi_x^\tau(f) := \int\limits_{E_t^{x,\tau}} f(q(s),p(s)) e^{i\int_0^t p(s)\dot{q}(s)ds} \prod_{\tau=0}^{t} dq(s)\,dp(s) := \int\limits_{E_t^{x,\tau}} f(q,p)\Phi_x^\tau(dq,dp)$$

of a function $f : Q_t^{x,\tau} \times P_t \to \mathbb{R}$ is defined as a limit:

$$\begin{aligned} \Phi_x^\tau(f) = \lim_{n\to\infty} \frac{1}{(2\pi)^{dn}} &\int\limits_{(\mathbb{R}^d\times\mathbb{R}^d)^n} f((J_n^\tau)^{-1}(q_1,p_1,\dots\dots,q_n,p_n))\times \\ &\times \exp\left[i\sum_{k=1}^n p_k\cdot(q_{k+1}-q_k)\right] dq_1\,dp_1\dots\dots dq_n\,dp_n, \end{aligned} \tag{3.5.5}$$

where $q_{n+1} := x$ in each pre-limit expression. And again all integrals must be considered in a suitably regularized sense.

Remark 3.5.8. The generalized density of the pseudomeasure Φ_x^τ can be defined through the formula

$$\begin{aligned} &\int\limits_{E_t^{x,\tau}} f(q(s),p(s))\Phi_x^\tau(dq\,dp) := \\ &\lim_{n\to\infty} C_n \int\limits_{Q_n^\tau\times P_n} f(q(s),p(s)) \exp\left[i\int_0^t p(s)\dot{q}(s)\,ds\right]\nu_n(dq)\,\nu_n(dp), \end{aligned}$$

where $(C_n)^{-1} = \int_{Q_n^\tau\times P_n} \exp\left[i\int_0^t p(s)\dot{q}(s)\,ds\right]\nu_n(dq)\,\nu_n(dp)$ and ν_n is the Lebesgue measure on $\mathbb{R}^{nd}$.

Remark 3.5.9. Let us discuss the actual meaning of the words "suitably regularized sense" in Definition (3.5.7). One of the standard regularizations of a multiple integral of an oscillating function g (where g is, e.g., the integrand in the right hand side of formula (3.5.5)) is given, e.g., by

$$\int_{(\mathbb{R}^d\times\mathbb{R}^d)^n} g(q_1,p_1,\ldots..,q_n,p_n)dq_1\,dp_1\ldots\ldots dq_n\,dp_n :=$$
$$:= \lim_{\varepsilon\to 0}\int_{(\mathbb{R}^d\times\mathbb{R}^d)^n} g(q_1,p_1,\ldots..,q_n,p_n)\exp\left\{-\varepsilon\sum_{k=1}^n(|q_k|^2+|p_k|^2)\right\}dq_1\,dp_1\ldots\ldots dq_n\,dp_n.$$

Such type of regularizations is quite restrictive and the limit in the right hand side of the fourmula above does not exist for many interesting cases. Let us weaken the procedure of regularization in the following way: we do not assume the existence of multiple integrals regularized as above. Instead, we assume the existence of iterated integrals

$$\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} g(q_1,p_1,\ldots..,q_n,p_n)dq_1\,dp_1\ldots\ldots dq_n\,dp_n,$$

where the order of integration is "chronological" and a standard regularization is used at each iteration. With such interpretation of regularization, Definition (3.5.7) has the following view (cf. Smolyanov and Shamarov, 2010):

Definition 3.5.10. The Hamiltonian (or phase space) Feynman path integral

$$\Phi_x^\tau(f) := \int_{E_t^{x,\tau}} f(q(s),p(s))e^{i\int_0^t p(s)\dot q(s)ds}\prod_{\tau=0}^t dq(s)\,dp(s) := \int_{E_t^{x,\tau}} f(q,p)\Phi_x^\tau(dq,dp)$$

of a function $f : Q_t^{x,\tau}\times P_t\to\mathbb{R}$ is defined as a limit:

$$\begin{aligned}\Phi_x^\tau(f) = \lim_{n\to\infty}\frac{1}{(2\pi)^{dn}}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} f((J_n^\tau)^{-1}(q_1,p_1,\ldots..,q_n,p_n))\times\\ \times\exp\left[i\sum_{k=1}^n p_k\cdot(q_{k+1}-q_k)\right]dq_1\,dp_1\ldots\ldots dq_n\,dp_n,\end{aligned}\tag{3.5.6}$$

where $q_{n+1} := x$, the order of integration is from q_1 to p_n, and each integration is regularized in a standard way.

Let us now show that the Hamiltonian Feynman formulae obtained in Sections 3.2-3.4 can be interpreted as phase space Feynman path integrals. Therefore, these phase space Feynman path integrals do exist and coincide with some functional integrals with respect to countably additive (probability) measures associated with some Feller type processes. Feynman path integrals for the dynamics governed by Hamiltonians, whose symbols are of type (3.3.7) (however not depending on position variable), have been considered by different authors in different contexts. For example, in Chapter 20 of the book Kleinert, 2009 such integrals appear for applications in financial markets; in works Laskin, 2012; Laskin, 2007; Laskin, 2000, they serve to generalize the quantum-mechanical apparatus to the so called fractional quantum mechanics.

Theorem 3.5.11. *Let $\tau \in [0,1]$ and $H = h$, where h is given by the formula (3.3.1). Let Assumption 3.4.1 (i), (iii) and Assumption 3.4.2 be fulfilled. Let $(T_t^\tau)_{t\geq 0}$ be the semigroup generated by the closure $(L^\tau, \mathrm{Dom}(L^\tau))$ of the pseudo-differential operator $(-\widehat{H}_\tau, C_c^\infty(\mathbb{R}^d))$ with the τ-symbol H. Then the Hamiltonian Feynman formula (3.4.5) can be interpreted for each $\varphi \in S(\mathbb{R}^d)$ and each $x \in \mathbb{R}^d$ as the following phase space Feynman path integral:*

$$T_t^\tau \varphi(x) = \int\limits_{E_t^{x,\tau}} e^{-\int_0^t H(q(s),p(s))ds} \varphi(q(0)) \Phi_x^\tau(dqdp). \tag{3.5.7}$$

And the equality holds in the sense of $L^1(\mathbb{R}^d)$.

Proof. The Hamiltonian Feynman formula (3.4.5) holds true under the assumptions of Theorem 3.5.11. Let us change the notations for the variables in the iterated integrals of the prelimit expression in (3.4.5) using the opposite ordering, i.e. $(q_k, p_k) \mapsto (q_{n-k+1}, p_{n-k+1})$, $k = 1, \dots, n$. Then the Hamiltonian Feynman formula (3.4.5) has the following view for each $\varphi \in S(\mathbb{R}^d)$ and each $x \in \mathbb{R}^d$:

$$\begin{aligned}(T_t^\tau \varphi)(x) = \lim_{n\to\infty} (2\pi)^{-dn} \int\limits_{\mathbb{R}^d} \cdots \int\limits_{\mathbb{R}^d} \exp\left(i \sum_{k=1}^n p_k \cdot (q_{k+1} - q_k) \right) \\ \times \exp\left(-\frac{t}{n} \sum_{k=1}^n H(\tau q_{k+1} + (1-\tau) q_k, p_k) \right) \varphi(q_1)\, dq_1\, dp_1 \cdots dq_n\, dp_n,\end{aligned} \tag{3.5.8}$$

where, in each pre-limit expression, we have $q_{n+1} := x$. Moreover, we have $q_1 = q(t/n - 0) \to q(0)$ as $n \to \infty$ due to the definition of the space $Q_t^{x,\tau}$ and

$$\begin{aligned}\frac{t}{n} \sum_{k=1}^n H(\tau q_{k+1} + (1-\tau) q_k, p_k) = \sum_{k=1}^n H(q(t_k), p(t_k))(t_k - t_{k-1}) \\ \to \int_0^t H(q(s), p(s)) ds, \quad n \to \infty,\end{aligned}$$

since any path $(q(s), p(s)) \in E_t^{x,\tau}$ is piecewise continuous and has a finite number of bounded jumps, H is a continuous function. Therefore, using Definition 3.5.10, we get

$$(T_t^\tau \varphi)(x) = \int\limits_{E_t^{x,\tau}} e^{-\int_0^t H(q(s),p(s))ds} \varphi(q(0)) \Phi_x^\tau(dqdp).$$

□

Remark 3.5.12. (i) The pre-limit expression in the Hamilton Feynman formula (3.5.8) shows that different procedures of quantization correspond to different choice of basis-points in the time-slicing procedures for path integrals (cf. Kleinert, 2009, Section 18.12). Namely, instead of midpoints $1/2(q_k + q_{k+1})$ of Stratonovich calculus or left-side points q_k of Itô calculus, the linear combinations $\tau q_{k+1} + (1 - \tau)q_k$ are used.

(ii) Note, that the integrand in the Feynman path integral (3.5.7) is the same for all $\tau \in [0, 1]$, only the space $E_t^{x,\tau}$, defining the sequential pseudomeasure Φ_x^τ is different; this space contains those paths $q(s)$ which are "τ-continuous" (consider the formula (3.5.4) as the definition). In accordance with Berezin's comment in Berezin, 1980, (phase space) Feynman path integrals over different sets of paths can be used to represent the same solution of a given evolution equation (cf. Smolyanov and Shavgulidze, 1992). And the quality (degree of smoothness) of paths in the configuration space is in inverse proportion with the quality of paths in the momentum space. This reflects Heisenberg's uncertainty principle. In many works continuous paths in the configuration space and discontinuous paths in the momentum space are considered, cf. Albeverio, Guatteri, and Mazzucchi, 2002; Bock and Grothaus, 2011; Ichinose, 2000; Kumano-Go, 1996. In our approach, the picture is symmetric: paths have the same quality in both spaces; they are piecewise constant and only their one-side continuity is different for configuration and momentum spaces.

(iii) Our Feynman path integral suits for the interpretation of the function h as a Hamilton function of a particle with position-dependent mass in a flat space. Namely, the integration is proceeded with respect to the Feynman pseudomeasure on the paths in the flat phase space. In the frame of the interpretation of the function h as a Hamilton function of a particle with constant mass in a curvilinear space, both the integrand and the pseudomeasure must be modified. One can find such Feynman path integrals, e.g., in Bouchemla and Chetouani, 2009, or in Chapter 10 of Kleinert, 2009.

Remark 3.5.13. Under Assumptions 3.4.1 (i), (ii), Assumption 3.4.2 and due to the formula (3.3.9), we know that $\widehat{H}_\tau\varphi(x) = \widehat{H}_1^\tau\varphi(x)$, where $\widehat{H}_1^\tau\varphi(x)$ is a pseudo-differential operator with 1-symbol

$$H^\tau(q,p) := C_\tau(q) + iB_\tau(q)\cdot p + p\cdot A(q)p, \quad q, p \in \mathbb{R}^d,$$

and B_τ, C_τ are given by (3.3.3) and (3.3.4) respectively. Therefore, due to Theorem 3.4.4 and Theorem 3.5.11, there is a kind of "change of variable formula" for the case of the quadratic Hamiltonian $H(q,p) = c(q) + ib(q)\cdot p + p\cdot A(q)p$:

$$\begin{aligned} T_t^\tau\varphi(x) &= \int_{E_t^{x,1}} \exp\left[-\int_0^t H^\tau(q(s),p(s))ds\right]\varphi(q(0))\,\Phi_x^1(dq\,dp) \\ &= \int_{E_t^{x,\tau}} \exp\left[-\int_0^t H(q(s),p(s))ds\right]\varphi(q(0))\Phi_x^\tau(dqdp). \end{aligned}$$

i.e.,

$$T_t^\tau\varphi(x) = \int\limits_{E_t^x} \exp\left[-\int\limits_0^t p(s)\cdot A(q(s))p(s)\,ds\right]$$
$$\times \exp\left[-\int\limits_0^t \Big[C(q(s)) + (1-\tau)\operatorname{div} B(q(s)) - (1-\tau)^2\operatorname{tr}(\operatorname{Hess} A(q(s)))\Big]ds\right]$$
$$\times \exp\left[-i\int\limits_0^t \Big[(B(q(s)) - 2(1-\tau)\operatorname{div} A(q(s)))\cdot p(s)\Big]ds\right]\varphi(q(0))\Phi_x^1(dq\,dp)$$
$$= \int\limits_{E_t^{x,\tau}} e^{-\int\limits_0^t p(s)\cdot A(q(s))p(s)\,ds - i\int\limits_0^t B(q(s))\cdot p(s)\,ds - \int\limits_0^t C(q(s))ds}\varphi(q(0))\Phi_x^\tau(dqdp).$$

Similarly to Theorem 3.5.11, one shows the following result.

Theorem 3.5.14. *Let $\tau = 1$. Let the function $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be measurable and locally bounded in both variables (q, p). Assume that $H(q,\cdot)$ is continuous and negative definite for all $q \in \mathbb{R}^d$. Let $H(q,\cdot)$ be given by (3.1.2) with some Lévy characteristics $(A(q), B(q), C(q), N(q,\cdot))$ for all $q \in \mathbb{R}^d$.*

(1) Let $X = C_\infty(\mathbb{R}^d)$. Let Assumptions 3.2.1, 3.2.2 and the estimates 3.2.12 of Lemma 3.2.9 hold. Let $(T_t^1)_{t\geq 0}$ be the strongly continuous semigroup on X generated by the closure of the pseudo-differential operator $(-\widehat{H}_1, C_c^\infty(\mathbb{R}^d))$ with the 1-symbol $-H(q,p)$. Then the Hamiltonian Feynman formula (3.2.11) for the semigroup $(T_t^1)_{t\geq 0}$ can be interpreted for each $\varphi \in S(\mathbb{R}^d)$ and each $x \in \mathbb{R}^d$ as the Feynman path integral (3.5.9) below. And the equality in formula (3.5.9) holds pointwise.

(2) Let $X = L^1(\mathbb{R}^d)$. Let H have the view (3.3.7). Let Assumptions 3.4.1 (i), (iii), (iv) and Assumption 3.4.2 hold. Let $(T_t^1)_{t\geq 0}$ be the strongly continuous semigroup on X generated by the closure of the pseudo-differential operator $(-\widehat{H}_1, C_c^\infty(\mathbb{R}^d))$ with the 1-symbol $-H(q,p)$. Then the Hamiltonian Feynman formula (3.4.9) for the semigroup $(T_t^1)_{t\geq 0}$ can be interpreted for each $\varphi \in S(\mathbb{R}^d)$ and each $x \in \mathbb{R}^d$ as the Feynman path integral (3.5.9) below. And the equality in formula (3.5.9) holds in the sense of $L^1(\mathbb{R}^d)$.

$$T_t^1\varphi(x) = \int\limits_{E_t^{x,1}} e^{-\int\limits_0^t H(q(s),p(s))ds}\varphi(q(0))\,\Phi_x^1(dq\,dp). \tag{3.5.9}$$

Remark 3.5.15. (i) If the Hamilton function H doesn't depend on the position variables q, the Feynman path integral (3.5.7) (with the proper choice of φ) is just exactly of the same art as in the formula (20.161) of Kleinert, 2009.

(ii) Lagrangian Feynman formula (3.4.3) (resp. (3.4.8)) actually provides a tool to compute Feynman path integral (3.5.7) (resp. (3.5.9)). The limits in both

Lagrangian Feynman formulae coincide with functional integrals over probability measures generated by the corresponding Feller (Feller type) stochastic processes (see, in particular, Remark 3.4.5). Therefore, the obtained Feynman path integrals (3.5.7) and (3.5.9) do actually coincide with these functional integrals over probability measures.

Remark 3.5.16. Let now $\alpha > 0$ and all the assumptions of Theorem 3.5.14 be fulfilled. Due to formulae (3.4.9), (3.5.9) and Definition 3.5.7, we have a kind of time-rescaling formula for the semigroup $(T^1_t)_{t\geq 0}$ (in both cases (1) and (2) of Theorem 3.5.14):

$$\begin{aligned}(T^1_{\alpha t}\varphi)(q_0) &= \lim_{n\to\infty}(2\pi)^{-dn}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\exp\left(i\sum_{k=1}^n p_k\cdot(q_{k+1}-q_k)\right)\times\\ &\quad\times\exp\left(-\frac{\alpha t}{n}\sum_{k=1}^n H(q_{k+1},p_k)\right)\varphi(q_1)\,dq_1\,dp_1\cdots dq_n\,dp_n = \qquad (3.5.10)\\ &= \int_{E^{q_0,1}_t} e^{-\alpha\int_0^t H(q(s),p(s))ds}\varphi(q(0))\,\Phi^1_{q_0}(dq\,dp).\end{aligned}$$

Assume now that H is such that $(T^1_t)_{t\geq 0}$ is (extendable to) a symmetric Feller semigroup (see Definitions 4.1.4 and 4.6.23 in Jacob, 2001). Then it is an analytic semigroup (by Thm. 4.2.12 and Thm. 4.6.25 of Jacob, 2001). And the function $t\to T^1_t$ has an analytic extension to the sector $S=\{z\in\mathbb{C}\setminus\{0\},\,|\arg z|<\frac{\pi}{2}\}$ which is continuous on $\overline{S}$. Then, for each $\varphi\in S(\mathbb{R}^d)$, the left hand side of (3.5.10) is well defined for $\alpha=i$: $T^1_{it}=\lim_{[0,\pi/2)\ni\theta\to\frac{\pi}{2}}T^1_{e^{i\theta}t}$. Therefore, by passing to the limit $[0,\pi/2)\ni\theta\to\frac{\pi}{2}$ with $\alpha=e^{i\theta}$ in the formula (3.5.10), the following Feynman path integral is also well defined:

$$\int_{E^{q_0,1}_t} e^{-i\int_0^t H(q(s),p(s))ds}\varphi(q(0))\,\Phi^1_{q_0}(dq\,dp) := T^1_{it}\varphi(q_0).$$

This expression can be considered as a solution of the Schrödinger type equation

$$i\frac{\partial f}{\partial t}(t,x)=\widehat{H}_1 f(t,x)$$

with the initial condition φ.

Chapter 4

Chernoff approximations for subordinate semigroups

One of the ways to construct strongly continuous semigroups is given by the procedure of subordination. From two ingredients: an original strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ and a convolution semigroup $(\eta_t)_{t\geq 0}$ supported by $[0,\infty)$ (see all definitions in Section 4.1), this procedure produces the strongly continuous contraction semigroup $(T_t^f)_{t\geq 0}$ with $T_t^f := \int_0^\infty T_s\eta_t(ds)$. If the semigroup $(T_t)_{t\geq 0}$ corresponds to a stochastic process $(X_t)_{t\geq 0}$, then subordination is a random time-change of $(X_t)_{t\geq 0}$ by an independent increasing Lévy process (subordinator) related to $(\eta_t)_{t\geq 0}$. If $(T_t)_{t\geq 0}$ and $(\eta_t)_{t\geq 0}$ both are known explicitly, so is $(T_t^f)_{t\geq 0}$. But if, e.g., $(T_t)_{t\geq 0}$ is not known, neither $(T_t^f)_{t\geq 0}$ itself, nor even the generator of $(T_t^f)_{t\geq 0}$ are known explicitly any more. This impedes the construction of a family $(F(t))_{t\geq 0}$ with a prescribed (but unknown explicitly) derivative at $t = 0$. This difficulty is overwhelmed in the present Chapter by construction of families $(\mathcal{F}(t))_{t\geq 0}$ and $(\mathcal{F}_\mu(t))_{t\geq 0}$ (they are defined in Sections 4.2, 4.3) which incorporate approximations of the generator of $(T_t^f)_{t\geq 0}$ itself. In this Chapter, we consider the semigroup $(T_t^f)_{t\geq 0}$ which is subordinate to a given semigroup $(T_t)_{t\geq 0}$ with respect to a given subordinator. It is assumed that the subordinator is known explicitly, i.e. either its transition probability is known, or its Lévy measure is known and bounded. Chernoff approximations of the subordinate semigroup $(T_t^f)_{t\geq 0}$ are constructed in the case, when the semigroup $(T_t)_{t\geq 0}$ is not known explicitly but is already Chernoff approximated by a given family $(F(t))_{t\geq 0}$. This condition is fulfilled, e.g., for evolution semigroups corresponding to Feller (and other Markov) processes in $\mathbb{R}^d$ constructed in Chapters 2, 3, as well as for evolution semigroups corresponding to killed Feller processes and to (Feller) diffusions in Riemannian manifolds and on metric graphs constructed in Chapters 5 and 6. Therefore, the Chernoff approximations obtained in Sections 4.2 and 4.3 can be applied further to construct explicit approximations for semigroups corresponding to subordination of Feller processes, and (Feller type) diffusions in Euclidean spaces, metric graphs and Riemannian manifolds. This, in turn, can allow to establish some new Feynman and Feynman–Kac formulae for the corresponding semigroups. Further, the technique of Chernoff approximation of subordinate semigroups can be combined with the technique of Chernoff approximation of semigroups generated by additive and multplicative perturbations (of generators of some original semigroups), developed in Chapter 2, and with the technique of Chernoff approximation of semigroups corresponding to killed Feller processes, developed in Chapter 5, in order to obtain Chernoff approximations for semigroups constructed by several iterative procedures of perturbation, killing and subordination.

4.1 Subordinate semigroups

We follow the exposition of the book Jacob, 2001 in this section. Let $(\eta_t)_{t\geq 0}$ be a convolution semigroup on $\mathbb{R}^d$ (cf. Definition C.0.1). Let us denote the corresponding (backward) strongly continuous contraction semigroup on the Banach space $C_\infty(\mathbb{R}^d)$ as $(S^\eta_t)_{t\geq 0}$. Recall that $(S^\eta_t)_{t\geq 0}$ is defined by the rule (see formula (C.0.5))

$$S^\eta_t\varphi(x) := \int\limits_{\mathbb{R}^d} \varphi(x+y)\eta_t(dy), \qquad \forall\, \varphi \in C_\infty(\mathbb{R}^d). \tag{4.1.1}$$

Let now $d = 1$, i.e. $(\eta_t)_{t\geq 0}$ be a convolution semigroup of measures on $\mathbb{R}$. It is said to be *supported by* $[0,\infty)$ if $\operatorname{supp}\eta_t \subset [0,\infty)$ for all $t \geq 0$. Each convolution semigroup $(\eta_t)_{t\geq 0}$ supported by $[0,\infty)$ corresponds to a *Bernstein function* f via the Laplace transform $\mathcal{L}$: $\mathcal{L}[\eta_t](x) = e^{-tf(x)}$ for all $x > 0$ and $t > 0$. Each Bernstein function f is uniquely defined by a triplet (σ,λ,μ) with constants $\sigma,\lambda \geq 0$ and a Radon measure μ on $(0,\infty)$, such that $\int_{0+}^\infty \frac{s}{1+s}\mu(ds) < \infty$, through the representation

$$f(z) = \sigma + \lambda z + \int\limits_{0+}^{\infty} (1 - e^{-sz})\mu(ds), \qquad \forall\, z : \operatorname{Re} z \geq 0. \tag{4.1.2}$$

Note that $\eta_t(\mathbb{R}) = 1$ for all $t \geq 0$ if and only if $\sigma = 0$ (i.e. there is no "killing", cf. Böttcher, Schilling, and Wang, 2013).

Consider a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ on a Banach space $(X, \|\cdot\|_X)$. Let $(\eta_t)_{t\geq 0}$ be a convolution semigroup on $\mathbb{R}$ supported by $[0,\infty)$ with the associated Bernstein function f. The family of operators $(T^f_t)_{t\geq 0}$ defined on X by the Bochner integral

$$T^f_t\varphi := \int\limits_0^\infty T_s\varphi\, \eta_t(ds), \quad \varphi \in X, \tag{4.1.3}$$

is again a strongly continuous contraction semigroup on X. The semigroup $(T^f_t)_{t\geq 0}$ is called *subordinate* (in the sense of Bochner) to $(T_t)_{t\geq 0}$ with respect to $(\eta_t)_{t\geq 0}$.

Recall that each convolution semigroup $(\eta_t)_{t\geq 0}$ corresponds to a Lévy process $(\xi_t)_{t\geq 0}$ with $\eta_t(ds) = \mathbb{P}(\xi_{t+h} - \xi_h \in ds)$, $\forall\, t,h \geq 0$. If a convolution semigroup $(\eta_t)_{t\geq 0}$ is supported by $[0,\infty)$ then the corresponding (one-dimensional) Lévy process $(\xi_t)_{t\geq 0}$ has non-decreasing paths almost surely and is called a *subordinator*. Such processes can be used for a time-change of other processes. Namely, if $(X_t)_{t\geq 0}$ is a (decent) Markov process then the *subordinate process* $(X_{\xi_t})_{t\geq 0}$ (such that $X_{\xi_t}(\omega) := X_{\xi_t(\omega)}(\omega)$ for each $\omega \in \Omega$) is again a (decent) Markov process. Moreover, if $(X_t)_{t\geq 0}$ is a Feller process then $(X_{\xi_t})_{t\geq 0}$ is again a Feller process. If $(T_t)_{t\geq 0}$ is the (backward) strongly continuous contraction semigroup corresponding to $(X_t)_{t\geq 0}$, i.e. $T_t\varphi(x) = \mathbb{E}^x[\varphi(X_t)]$, and $(\eta_t)_{t\geq 0}$ is the convolution semigroup of the subordinator $(\xi_t)_{t\geq 0}$ then the subordinated semigroup $(T_t^f)_{t\geq 0}$, defined in (4.1.3), corresponds to the subordinate process $(X_{\xi_t})_{t\geq 0}$. Many interesting processes are obtained from a Brownian motion via subordination (see §4.4 of Cont and Tankov, 2004).

Consider the operator semigroup $(S_t^\eta)_{t\geq 0}$ given by (4.1.1), corresponding to a convolution semigroup $(\eta_t)_{t\geq 0}$ supported by $[0,\infty)$. Assume that the related Bernstein function is given by a triplet $(0,0,\mu)$. The generator $(L^\eta, \mathrm{Dom}(L^\eta))$ of $(S_t^\eta)_{t\geq 0}$ has then the following properties: $C_c^\infty(\mathbb{R}) \subset \mathrm{Dom}(L^\eta)$ and for all $\varphi \in \mathrm{Dom}(L^\eta)$

$$L^\eta\varphi(x) = \int_{0+}^{\infty} (\varphi(x+s) - \varphi(x))\mu(ds). \tag{4.1.4}$$

Let now $(T_t)_{t\geq 0}$ be a strongly continuous contraction semigroup with the generator $(L, \mathrm{Dom}(L))$ and f be a Bernstein function given by the representation (4.1.2) with an associated convolution semigroup $(\eta_t)_{t\geq 0}$ supported by $[0,\infty)$. Then the set $\mathrm{Dom}(L)$ is a core for the generator L^f of the subordinate semigroup $(T_t^f)_{t\geq 0}$ and, for $\varphi \in \mathrm{Dom}(L)$, the operator L^f has the representation

$$L^f\varphi = -\sigma\varphi + \lambda L\varphi + \int_{0+}^{\infty} (T_s\varphi - \varphi)\mu(ds). \tag{4.1.5}$$

Note that if a linear subspace $D \subset X$ is a core for L, then D is also a core for L^f (see Sato, 1999, Prop. 32.5, p. 215).The representation (4.1.5) allows, in particular, to obtain the fractional Laplacian $L^f := -(-\Delta)^\alpha$, $\alpha \in (0,1)$, via subordination of a Brownian motion (with $L := \Delta$) with respect to an α-stable subordinator.

For each convolution semigroup $(\eta_t)_{t\geq 0}$ on $\mathbb{R}^d$, the corresponding operator semigroup $(S_t^\eta)_{t\geq 0}$ extends to a contraction semigroup $(\bar{S}_t^\eta)_{t\geq 0}$ on the space $B_b(\mathbb{R}^d)$ of all bounded Borel functions on $\mathbb{R}^d$. This semigroup belongs to the class of *strong Feller* semigroups (see Definition 3.1.2) if and only if all the measures η_t admit densities of the class $L^1(\mathbb{R}^d)$ with respect to the Lebesgue measure (cf. Examle 4.8.21 of Jacob, 2001). One may consider a strong Feller semigroup $(\bar{S}_t^\eta)_{t\geq 0}$ as a semigroup on the space $C_b(\mathbb{R}^d)$ of all bounded continuous functions and define its C_b-generator $(\bar{L}^\eta, \mathrm{Dom}(\bar{L}^\eta))$ for each $x \in \mathbb{R}^d$ by

$$\bar{L}^\eta\varphi(x) := \lim_{t\to 0}\frac{\bar{S}^\eta_t\varphi(x)-\varphi(x)}{t}, \qquad \mathrm{Dom}(\bar{L}^\eta) := \Big\{\varphi \in C_b(\mathbb{R}^d)\,\Big|\, \lim_{t\to 0}\frac{\bar{S}^\eta_t\varphi(x)-\varphi(x)}{t} \text{ exists uniformly on compact subsets of } \mathbb{R}^d\Big\}.$$

Moreover, the operator $(\bar{L}^\eta, \mathrm{Dom}(\bar{L}^\eta))$ in the space $C_b(\mathbb{R}^d)$ is an extension of the generator $(L^\eta, \mathrm{Dom}(L^\eta))$ of the semigroup $(S^\eta_t)_{t\geq 0}$ on $C_\infty(\mathbb{R}^d)$, and, in particular, the inclusion $C_b^2(\mathbb{R}^d) \subset \mathrm{Dom}(\bar{L}^\eta)$ holds (cf. Example 4.8.26 of Jacob, 2001).

4.2 Chernoff approximation for subordinate semigroups in the case when transitional probabilities of subordinators are known

In this section, we consider the semigroup $(T^f_t)_{t\geq 0}$ subordinate to a given semigroup $(T_t)_{t\geq 0}$ with respect to a given convolution semigroup $(\eta_t)_{t\geq 0}$ associated to a Bernstein function f which is defined by a triplet (σ, λ, μ). We assume that the corresponding convolution semigroup $(\eta^0_t)_{t\geq 0}$, associated to the Bernstein function f_0 wth the triplet $(0, 0, \mu)$, is known explicitly and corresponds to a strong Feller semigroup $(\bar{S}^{\eta^0}_t)_{t\geq 0}$. This is the case of inverse Gaussian (including $1/2$-stable) subordinator, Gamma subordinator and some others (see, e.g., Burridge et al., 2014 for examples). We are interested in approximation of the subordinate semigroup $(T^f_t)_{t\geq 0}$ when the semigroup $(T_t)_{t\geq 0}$ is not known explicitly but is Chernoff-approximated by a given family $(F(t))_{t\geq 0}$.

Theorem 4.2.1. *Let f be a Bernstein function given by a triplet (σ, λ, μ) through the representation (4.1.2) with associated convolution semigroup $(\eta_t)_{t\geq 0}$ supported by $[0, \infty)$. Let $(\eta^0_t)_{t\geq 0}$ be the convolution semigroup (supported by $[0, \infty)$) associated to the Bernstein function f_0 defined by the triplet $(0, 0, \mu)$. Assume that the corresponding operator semigroup $(\bar{S}^{\eta^0}_t)_{t\geq 0}$ is strong Feller. Let $(T_t)_{t\geq 0}$ be a strongly continuous contraction semigroup on a Banach space $(X, \|\cdot\|_X)$ with the generator $(L, \mathrm{Dom}(L))$. Let $(F(t))_{t\geq 0}$ be a family of contractions on $(X, \|\cdot\|_X)$ which is Chernoff equivalent to $(T_t)_{t\geq 0}$, i.e. $F(0) = \mathrm{Id}$, $\|F(t)\| \leq 1$ for all $t \geq 0$ and there is a set $D \subset \mathrm{Dom}(L)$, which is a core for L, such that $\lim_{t\to 0}\left\|\frac{F(t)\varphi-\varphi}{t} - L\varphi\right\|_X = 0$ for each $\varphi \in D$. Let $m : (0, \infty) \to \mathbb{N}_0$ be a monotone function such that $m(t) \to +\infty$ as $t \to 0$[1]. Let, for each $t > 0$ and each $\varphi \in X$, the mapping $[F(\cdot/m(t))]^{m(t)}\varphi : [0, \infty) \to X$ be Bochner measurable as the mapping from $([0, \infty), \mathcal{B}([0, \infty)), \eta^0_t)$ to $(X, \mathcal{B}(X))$. Let $(T^f_t)_{t\geq 0}$ be the semigroup subordinate to $(T_t)_{t\geq 0}$ with respect to $(\eta_t)_{t\geq 0}$ and L^f be its generator. Consider the family $(\mathcal{F}(t))_{t\geq 0}$ of operators on $(X, \|\cdot\|_X)$ defined by $\mathcal{F}(0) := \mathrm{Id}$ and*

$$\mathcal{F}(t)\varphi := e^{-\sigma t} \circ F(\lambda t) \circ \mathcal{F}_0(t)\varphi, \quad t > 0,\ \varphi \in X, \tag{4.2.1}$$

[1] One can take, e.g., $m(t) := \lfloor 1/t \rfloor$ = the largest integer $n \leq 1/t$. Recall that $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

with $\mathcal{F}_0(0) = \mathrm{Id}$ *and*[2]

$$\mathcal{F}_0(t)\varphi := \int\limits_{0+}^{\infty} \left[F(s/m(t))\right]^{m(t)} \varphi\, \eta_t^0(ds), \quad t > 0,\, \varphi \in X. \tag{4.2.2}$$

The family $(\mathcal{F}(t))_{t\geq 0}$ *is Chernoff equivalent to the semigroup* $(T_t^f)_{t\geq 0}$*, and hence*

$$T_t^f \varphi = \lim_{n\to\infty} \left[\mathcal{F}(t/n)\right]^n \varphi$$

for all $\varphi \in X$ *locally uniformly with respect to* $t \geq 0$.

Before we proceed with the proof, let us illustrate the result of Theorem 4.2.1 by means of the following Example 4.2.2. Some further examples can be found in Chapter 6.

Example 4.2.2. Let $X := C_\infty(\mathbb{R}^d)$. Let $a \in C(\mathbb{R}^d)$ be a bounded and strictly positive function. Consider the operator (L, D) with $D := C_c^\infty(\mathbb{R}^d)$ and $L\varphi(x) := a(x)\Delta\varphi(x)$ for all $\varphi \in D$ and all $x \in \mathbb{R}^d$. Due to Example 2.2.11 and Lemma 2.3.4, the family $(F(t))_{t\geq 0}$ of contractions on X, given for all $\varphi \in X$ and all $x \in \mathbb{R}^d$ by

$$F(t)\varphi(x) := (2\pi t a(x))^{-d/2} \int\limits_{\mathbb{R}^d} \varphi(y) e^{-\frac{|x-y|^2}{2ta(x)}} dy,$$

is a strongly continuous family. And this family is Chernoff equivalent to the strongly continuous semigroup $(T_t)_{t\geq 0}$ generated by the closure of (L, D). In particular, $\lim_{t\to 0} \left\| \frac{F(t)\varphi - \varphi}{t} - L\varphi \right\|_X = 0$ for each $\varphi \in D$. Consider now a $1/2$-stable subordinator $(\xi_t)_{t\geq 0}$. The corresponding convolution semigroup $(\eta_t)_{t\geq 0}$ is given by

$$\eta_t(ds) := 1_{(0,\infty)}(s) \frac{t}{\sqrt{4\pi s^3}} e^{-\frac{t^2}{4s}} ds$$

and hence admits densities of the class $L^1(\mathbb{R})$ with respect to the Lebesgue measure ds. The corresponding Bernstein function is $f(x) = \sqrt{x}$ with the triplet $(0, 0, \mu)$, where $\mu(ds) := \frac{s^{-3/2}}{2\Gamma(1/2)} ds$. Take a function m as in Theorem 4.2.1. Then the family $(\mathcal{F}_0(t))_{t\geq 0}$, given for all $\varphi \in C_\infty(\mathbb{R}^d)$ and all $x_0 \in \mathbb{R}^d$ by

$$\begin{aligned}\mathcal{F}_0(t)\varphi(x_0) := \int\limits_{0+}^{\infty} \int\limits_{\mathbb{R}^{dm(t)}} \left(\prod_{k=1}^{m(t)} \frac{2\pi s a(x_{k-1})}{m(t)} \right)^{-\frac{d}{2}} e^{-\sum\limits_{k=1}^{m(t)} \frac{|x_k - x_{k-1}|^2 m(t)}{2sa(x_{k-1})}} \times \\ \times \varphi\left(x_{m(t)}\right) \frac{t}{\sqrt{4\pi s^3}} e^{-\frac{t^2}{4s}} dx_1 \dots dx_{m(t)} ds,\end{aligned}$$

is Chernoff equivalent to the semigroup $(T_t^f)_{t\geq 0}$ subordinate to $(T_t)_{t\geq 0}$ with respect to $(\eta_t)_{t\geq 0}$. Since all operators $\mathcal{F}_0(t)$ are integral operators with elementary

[2]For any bounded operator B, its zero degree B^0 is considered to be the identity operator. For each $t > 0$, a non-negative integer $m(t)$ and a bounded Bochner measurable mapping $[F(\cdot/m(t))]^{m(t)}\varphi : [0, \infty) \to X$, the integral in the right hand side of formula (4.2.2) is well defined.

kernels, the corresponding Chernoff approximation $T_t^f\varphi = \lim_{n\to\infty}[\mathcal{F}_0(t/n)]^n\varphi$ is just a Feynman formula.

Proof. We return now to the proof of Theorem 4.2.1. Let us prove that the family $(\mathcal{F}_0(t))_{t\geq 0}$ is Chernoff equivalent to the semigroup $(T_t^{f_0})_{t\geq 0}$ subordinate to $(T_t)_{t\geq 0}$ with respect to $(\eta_t^0)_{t\geq 0}$. The generator L^{f_0} of this semigroup for each $\varphi \in \mathrm{Dom}(L)$ is given by

$$L^{f_0}\varphi = \int_{0+}^{\infty}(T_s\varphi - \varphi)\mu(ds), \tag{4.2.3}$$

and $D \subset \mathrm{Dom}(L)$ is a core for L^{f_0}. The statement of Theorem 4.2.1 then follows immediately from Theorem 2.1.1 (or Corollary 2.1.2) since $F(\lambda t)$ is Chernoff equivalent to $T_{\lambda t} \equiv e^{(t\lambda)L} \equiv e^{t(\lambda L)}$ for each $\lambda > 0$. The proof, that the family $(\mathcal{F}_0(t))_{t\geq 0}$ is Chernoff equivalent to the semigroup $(T_t^{f_0})_{t\geq 0}$, is the subject of the following five Lemmata.

Lemma 4.2.3. *Operators $\mathcal{F}_0(t)$ are contractions on X for all $t > 0$.*

Proof. Taking into account that all operators $F(t)$ are contractions on X and that $\eta_t^0(\mathbb{R}) = 1$, one has for each $t > 0$

$$\begin{aligned}\|\mathcal{F}(t)\varphi\|_X &= \left\| \int_{0+}^{\infty} [F(s/m(t))]^{m(t)}\,\varphi\,\eta_t^0(ds)\right\|_X \\ &\leq \int_{0+}^{\infty} \left\|[F(s/m(t))]^{m(t)}\,\varphi\right\|_X \eta_t^0(ds) \\ &\leq \int_{0+}^{\infty} \|[F(s/m(t))]\|^{m(t)}\,\|\varphi\|_X\,\eta_t^0(ds) \\ &\leq \|\varphi\|_X\,.\end{aligned}$$

□

Lemma 4.2.4. *The family $(\mathcal{F}_0(t))_{t\geq 0}$ is strongly continuous at $t = 0$.*

Proof. Let us check that the family $(\mathcal{F}_0(t))_{t\geq 0}$ is strongly continuous at zero. For each $\varphi \in X$ we have

$$\begin{aligned}\lim_{t\to 0}\|\mathcal{F}_0(t)\varphi - \varphi\|_X &= \lim_{t\to 0}\left\|\int_{0+}^{\infty}[F(s/m(t))]^{m(t)}\,\varphi\,\eta_t^0(ds) - \varphi\right\|_X \\ &\leq \lim_{t\to 0}\left\|\int_{0+}^{\infty}[F(s/m(t))]^{m(t)}\,\varphi\,\eta_t^0(ds) - T_t^{f_0}\varphi\right\|_X + \lim_{t\to 0}\left\|T_t^{f_0}\varphi - \varphi\right\|_X\end{aligned}$$

$$
\begin{aligned}
&= \lim_{t\to 0} \left\| \int_{0+}^{\infty} \left([F(s/m(t))]^{m(t)} \varphi - T_s\varphi \right) \eta_t^0(ds) \right\|_X \\
&\le \lim_{t\to 0} \int_{0+}^{\infty} \left\| [F(s/m(t))]^{m(t)} \varphi - T_s\varphi \right\|_X \eta_t^0(ds) \\
&\le \lim_{t\to 0} \int_{0+}^{1} \left\| [F(s/m(t))]^{m(t)} \varphi - T_s\varphi \right\|_X \eta_t^0(ds) + \lim_{t\to 0} 2\|\varphi\|_X \int_{1}^{\infty} \eta_t^0(ds) \\
&\le \lim_{t\to 0} \sup_{s\in[0,1]} \left\| [F(s/m(t))]^{m(t)} \varphi - T_s\varphi \right\|_X \\
&= 0,
\end{aligned}
$$

since the convergence of $\left\| [F(s/m(t))]^{m(t)} \varphi - T_s\varphi \right\|_X$ to zero as $t \to 0$ is uniform w.r.t. s on compact intervals due to the Chernoff Theorem, and since η_t^0 weakly converges to the Dirac delta-measure δ_0 as $t \to 0$.

□

Remark 4.2.5. The family $(\mathcal{F}_0(t))_{t\ge 0}$ (and hence $(\mathcal{F}(t))_{t\ge 0}$, constructed in Theorem 4.2.1) can not be strongly continuous for all $t \in [0,\infty)$ since the function m, used in the construction of this family, is not continuous. Therefore, the family $(\mathcal{F}(t))_{t\ge 0}$ can not be further used for establishing Chernoff approximations for semigroups, generated by multiplicative perturbations of the operator L^f, via Theorem 2.2.2 (or Proposition 2.2.6). However, combination of the family $(\mathcal{F}(t))_{t\ge 0}$ with other thechniques of Chernoff approximation, developed in this work, are safely possible. Moreover, under additional assumptions on the function m and the family $(F(t))_{t\ge 0}$, assuring that the mapping $\mathcal{F}(\cdot)\varphi : [0,\infty) \to X$ is Bochner measurable for each $\varphi \in X$, the family $(\mathcal{F}(t))_{t\ge 0}$, constructed in Theorem 4.2.1, can be used to approximate solutions of time-fractional evolution equations as in Theorem (6.3.3).

Lemma 4.2.6. *For a fixed $\varphi \in D$ define the function $\Psi_t : [0,\infty) \to [0,\infty)$ by $\Psi_t(s) := \|F^{m(t)}(s/m(t))\varphi - T_s\varphi\|_X$. The estimate*

$$
\begin{aligned}
\frac{\Psi_t(s)}{s} &\le \left\| \frac{T_s\varphi - \varphi}{s} - L\varphi \right\|_X + \left\| \frac{F(s/m(t))\varphi - \varphi}{s/m(t)} - L\varphi \right\|_X \\
&+ \left\| \left(\frac{1}{m(t)} \left[F^{m(t)-1}(s/m(t)) + F^{m(t)-2}(s/m(t)) + \ldots + F(s/m(t)) + \mathrm{Id} \right] - \mathrm{Id} \right) L\varphi \right\|_X
\end{aligned}
$$

holds for each $t > 0$ and each $s > 0$.

Proof. Denote $B := F^{m(t)-1}(s/m(t)) + F^{m(t)-2}(s/m(t)) + \ldots + F(s/m(t)) + \mathrm{Id}$. Then $F^{m(t)}(s/m(t))\varphi - \varphi = B(F(s/m(t)) - \mathrm{Id})\varphi$ and $\|B\| \le m(t)$. Therefore, one has

$$\begin{aligned}\frac{\Psi_t(s)}{s} &= \left\| \frac{F^{m(t)}(s/m(t))\varphi - T_s\varphi}{s} \right\|_X \\ &\le \left\| \frac{F^{m(t)}(s/m(t))\varphi - \varphi}{s} - L\varphi \right\|_X + \left\| \frac{T_s\varphi - \varphi}{s} - L\varphi \right\|_X\end{aligned}$$

and

$$\begin{aligned}&\left\| \frac{F^{m(t)}(s/m(t))\varphi - \varphi}{s} - L\varphi \right\|_X \\ &= \left\| \frac{(m^{-1}(t)B)(F(s/m(t))\varphi - \varphi)}{s/m(t)} - (m^{-1}(t)B)L\varphi + (m^{-1}(t)B)L\varphi - L\varphi \right\|_X \\ &\le \left\| \frac{F(s/m(t))\varphi - \varphi}{s/m(t)} - L\varphi \right\|_X + \left\| (m^{-1}(t)B)L\varphi - L\varphi \right\|_X.\end{aligned}$$

□

Lemma 4.2.7. *Let Ψ_t be as in Lemma 4.2.6. For each $\varepsilon > 0$ there exist $t_\varepsilon > 0$ and $s_\varepsilon > 0$ such that the estimate*

$$\frac{\Psi_t(s)}{s} < \varepsilon$$

holds for all $t \in (0, t_\varepsilon]$ and all $s \in (0, s_\varepsilon]$.

Proof. Fix $\varepsilon > 0$. Choose $s_1 > 0$ such that $\left\| \frac{T_s\varphi - \varphi}{s} - L\varphi \right\|_X < \varepsilon/3$ for all $s \in (0, s_1]$. This can be done since $\varphi \in D \subset \mathrm{Dom}(L)$. Choose then $t_1 > 0$ such that for all $s \in (0, s_1]$ one has

$$\left\| \frac{F(s/m(t_1))\varphi - \varphi}{s/m(t_1)} - L\varphi \right\|_X < \varepsilon/3.$$

This can be done due to the assumption: $\lim_{t \searrow 0} \left\| \frac{F(t)\varphi - \varphi}{t} - L\varphi \right\|_X = 0$ for each $\varphi \in D$. Since $s/m(t) \le s_1/m(t_1)$ for all $s \in (0, s_1]$ and $t \in (0, t_1]$, one has also

$$\left\| \frac{F(s/m(t))\varphi - \varphi}{s/m(t)} - L\varphi \right\|_X < \varepsilon/3$$

for such s and t. Since the semigroup $(T_t)_{t \ge 0}$ is strongly continuous choose $s_2 \in (0, s_1]$ such that

$$\|T_\tau L\varphi - L\varphi\|_X < \varepsilon/9$$

for all $\tau \in (0, s_2]$. Due to the Chernoff Theorem it is possible to choose $K \in \mathbb{N}$ such that the inequality

$$\left\| F^{k-1}(\tau/(k-1))L\varphi - T_\tau L\varphi \right\|_X < \varepsilon/9$$

holds for all $k \geq K$ and all $\tau \in [0, s_2/m(t_1)]$. Choose $t_2 \in (0, t_1]$ such that $m(t_2) > K$. Thus, the following estimate is true for $s \in (0, s_2]$ and $t \in (0, t_2]$:

$$\begin{aligned}
&\left\| \frac{1}{m(t)} \sum_{k=1}^{m(t)} F^{k-1}(s/m(t))L\varphi - L\varphi \right\|_X \\
&\leq \frac{1}{m(t)} \sum_{k=1}^{m(t)} \left\| F^{k-1}\left(\frac{(k-1)s/m(t)}{k-1}\right) L\varphi - T_{(k-1)s/m(t)}L\varphi \right\|_X \\
&\qquad + \frac{1}{m(t)} \sum_{k=1}^{m(t)} \left\| T_{(k-1)s/m(t)}L\varphi - L\varphi \right\|_X \\
&\leq \frac{1}{m(t)} \sum_{k=K}^{m(t)} \left\| F^{k-1}\left(\frac{(k-1)s/m(t)}{k-1}\right) L\varphi - T_{(k-1)s/m(t)}L\varphi \right\|_X + \frac{2K\|L\varphi\|_X}{m(t)} + \varepsilon/9 \\
&\leq 2K\|L\varphi\|_X m^{-1}(t) + 2\varepsilon/9.
\end{aligned}$$

Due to our assumptions, the function $m : (0, \infty) \to \mathbb{N}_0$ is monotone with $m(t) \to \infty$ as $t \to 0$. Therefore, one can choose $t_3 \in (0, t_2]$ with $m(t_3) > \frac{18K\|L\varphi\|_X}{\varepsilon}$. Then we obtain, due to Lemma 4.2.6 with $t_\varepsilon := t_3$ and $s_\varepsilon := s_2$, that

$$\frac{\Psi_t(s)}{s} < \varepsilon.$$

$\square$

Lemma 4.2.8. *It holds for each $\varphi \in D$:*

$$\lim_{t\to 0} \left\| \frac{\mathcal{F}_0(t)\varphi - \varphi}{t} - L^{f_0}\varphi \right\|_X = 0.$$

Proof. With the function Ψ_t defined in Lemma 4.2.6, one has

$$\lim_{t\to 0} \left\| \frac{\mathcal{F}_0(t)\varphi - \varphi}{t} - L^{f_0}\varphi \right\|_X \leq \lim_{t\to 0} \left\| \frac{\mathcal{F}_0(t)\varphi - T_t^{f_0}\varphi}{t} \right\|_X + \lim_{t\to 0} \left\| \frac{T_t^{f_0}\varphi - \varphi}{t} - L^{f_0}\varphi \right\|_X$$

$$= \lim_{t\to 0} \frac{1}{t} \left\| \int_0^\infty (F^{m(t)}(s/m(t))\varphi - T_s\varphi)\eta_t^0(ds) \right\|_X \leq \lim_{t\to 0} \frac{1}{t} \int_0^\infty \Psi_t(s)\eta_t^0(ds).$$

Fix an arbitrary $\varepsilon > 0$. Take t_ε and s_ε as in Lemma 4.2.7. Let $r_\varepsilon := \min(s_\varepsilon, 1)$. Take $R_\varepsilon > 0$ such that $\int_{R_\varepsilon}^\infty \mu(ds) < \varepsilon$. For $k = 1, 2, 3$ choose functions $\chi_k \in C_b^2(\mathbb{R})$ with $0 \leq \chi_k \leq 1$ such that $\operatorname{supp}\chi_1 \subset (-1, r_\varepsilon)$, $\operatorname{supp}\chi_3 \subset (R_\varepsilon, \infty)$, $\operatorname{supp}\chi_2 \subset (r_\varepsilon/2, 2R_\varepsilon)$

and $\sum_{k=1}^{3} \chi_k(s) = 1$ for all $s \geq 0$. Then by Lemma 4.2.7

$$\lim_{t\to 0} \frac{1}{t} \int\limits_0^\infty \Psi_t(s)\eta_t^0(ds) \leq \lim_{t_\varepsilon > t \to 0} \frac{\varepsilon}{t} \int\limits_0^{r_\varepsilon} s\chi_1(s)\eta_t^0(ds)$$

$$+ \lim_{t\to 0} \sup_{s\in[r_\varepsilon/2, 2R_\varepsilon]} \Psi_t(s) \cdot \frac{1}{t} \int\limits_{r_\varepsilon/2}^{2R_\varepsilon} \chi_2(s)\eta_t^0(ds) + \lim_{t\to 0} \frac{2\|\varphi\|_X}{t} \int\limits_{R_\varepsilon}^\infty \chi_3(s)\eta_t^0(ds).$$

Due to the Chernoff theorem $\lim_{t\to 0} \sup\limits_{s\in[r_\varepsilon/2, 2R_\varepsilon]} \Psi_t(s) = 0$ for any fixed r_ε and R_ε. Define also χ_4 such that $\chi_4(s) := s\chi_1(s)$ for all $s \in \mathbb{R}$. Since the semigroup $(\bar{S}_t^{\eta^0})_{t\geq 0}$ is strong Feller, $\chi_k \in C_b^2(\mathbb{R}) \subset \mathrm{Dom}(\bar{L}^{\eta^0})$ and $\chi_k(0) = 0$ for $k = 2, 3, 4$, one has

$$\lim_{t\to 0} \frac{1}{t} \int\limits_{0+}^\infty \chi_k(s)\eta_t^0(ds) = \lim_{t\to 0} \frac{\bar{S}_t^{\eta^0}\chi_k - \chi_k}{t}(0) = \left(\bar{L}^\eta \chi_k\right)(0) = \int\limits_{0+}^\infty \chi_k(s)\mu(ds).$$

Therefore, $\int\limits_{0+}^\infty \chi_2(s)\mu(ds) = \int\limits_{r_\varepsilon/2}^{2R_\varepsilon} \chi_2(s)\mu(ds) \leq \mu[r_\varepsilon/2, 2R_\varepsilon] < \infty$ (cf. Lemma 2.16 of Böttcher, Schilling, and Wang, 2013). And hence

$$\lim_{t\to 0} \sup_{s\in[r_\varepsilon/2, 2R_\varepsilon]} \Psi_t(s) \cdot \frac{1}{t} \int\limits_{r_\varepsilon/2}^{2R_\varepsilon} \chi_2(s)\eta_t^0(ds) = 0.$$

Similarly,

$$\lim_{t\to 0} \frac{2\|\varphi\|_X}{t} \int\limits_{R_\varepsilon}^\infty \chi_3(s)\eta_t^0(ds) = 2\|\varphi\|_X \int\limits_{R_\varepsilon}^\infty \chi_3(s)\mu(ds) < 2\varepsilon\|\varphi\|_X.$$

And, further, with $K := \int_0^1 s\mu(ds) < \infty$

$$\lim_{t_\varepsilon > t\to 0} \frac{\varepsilon}{t} \int\limits_0^{r_\varepsilon} s\chi_1(s)\eta_t^0(ds) = \varepsilon \int\limits_0^{r_\varepsilon} s\chi_1(s)\mu(ds) \leq \varepsilon \int\limits_0^1 s\mu(ds) = K\varepsilon.$$

Thus, it is shown that for each fixed $\varepsilon > 0$

$$\lim_{t\to 0} \left\| \frac{\mathcal{F}_0(t)\varphi - \varphi}{t} - L^{f_0}\varphi \right\|_X \leq \varepsilon(K + 2\|\varphi\|_X).$$

Therefore, the statement of Lemma is true. $\square$

Hence the family $(\mathcal{F}_0(t))_{t\geq 0}$ is Chernoff equivalent to the semigroup $(T_t^{f_0})_{t\geq 0}$ subordinate to $(T_t)_{t\geq 0}$ with respect to $(\eta_t^0)_{t\geq 0}$. And Theorem 4.2.1 is proved.

$\square$

Remark 4.2.9. Let $(T_t)_{t\geq 0}$, $(\eta_t)_{t\geq 0}$ and $(F(t))_{t\geq 0}$ be as before, and let $(F(t))_{t\geq 0}$ be strongly continuous. Let $X = C_b(Q)$ or $X = C_\infty(Q)$, where Q is a metric space. Let σ, λ be not constants but continuous functions on Q such that λ is bounded and strictly positive and σ is bounded from below. Assume that the operator L^f defined as in (4.1.5) (but with variable σ and λ) with the domain D (here D is as in Theorem 4.2.1) is closable and the closure generates a strongly continuous semigroup $(T_t^f)_{t\geq 0}$ on X. Then, due to Theorem 2.1.1 (or Corollary 2.1.2), Theorem 2.2.2, Proposition 2.2.6 and Lemmas 4.2.3 – 4.2.8, the family $(\widehat{\mathcal{F}}(t))_{t\geq 0}$ of operators on $(X, \|\cdot\|_\infty)$ defined by $\widehat{\mathcal{F}}(0) := \mathrm{Id}$ and

$$\widehat{\mathcal{F}}(t)\varphi := e^{-\sigma t} \circ \widehat{F}(t) \circ \mathcal{F}_0(t)\varphi, \quad t > 0,\ \varphi \in X, \tag{4.2.4}$$

with $(\mathcal{F}_0(t))_{t\geq 0}$ as in Theorem 4.2.1 and with $(\widehat{F}(t))_{t\geq 0}$ such that

$$\widehat{F}(t)\varphi(x) := \left(F(\lambda(x)t)\varphi\right)(x), \quad \forall \varphi \in X, \quad \forall x \in Q, \tag{4.2.5}$$

is Chernoff equivalent to the semigroup $(T_t^f)_{t\geq 0}$.

Remark 4.2.10. Some other approximations for subordinate semigroups (in the case of known transitional probabilities of subordinators) are given also in Remark 6.3.5.

4.3 Chernoff approximation for subordinate semigroups in the case when Lévy measures of subordinators are known and bounded

In this section we again consider the semigroup $(T_t^f)_{t\geq 0}$ subordinate to a given semigroup $(T_t)_{t\geq 0}$ with respect to a given convolution semigroup $(\eta_t)_{t\geq 0}$, associated to a Bernstein function f which is defined by a triplet (σ, λ, μ). We assume that the corresponding convolution semigroup $(\eta_t^0)_{t\geq 0}$, associated to the Bernstein function f_0 defined by the triplet $(0, 0, \mu)$, is not known explicitly. In this case, the family $(\mathcal{F}_0(t))_{t\geq 0}$ of Theorem 4.2.1 is not known explicitly as well, and hence the formula (4.2.4) is not proper for direct computations any more. Let us assume that the Lévy measure μ of $(\eta_t^0)_{t\geq 0}$ is given explicitly and is bounded (and nonzero). In this case the generator L^{η^0} of the corresponding semigroup $S_t^{\eta^0}$ is a bounded linear operator given as in (4.1.4). The generator L^{f_0} of the semigroup $(T_t^{f_0})_{t\geq 0}$ subordinate to $(T_t)_{t\geq 0}$ with respect to $(\eta_t^0)_{t\geq 0}$ is given by (4.2.3) and is also bounded. Therefore, the semigroup $(T_t^{f_0})_{t\geq 0}$ can be constructed, e.g., via Taylor series representation (which, however, contains powers of the operator L^{f_0}, and L^{f_0} is not known explicitly). We use another approach below: we construct a Chernoff approximation for the semigroup $(T_t^{f_0})_{t\geq 0}$. This Chernoff approximation can be used further as a building block for constructing Chernoff approximations for semigroups obtained by iterative prosedures of subordination and perturbations (of generators).

Theorem 4.3.1. *Let f be a Bernstein function given by a triplet (σ, λ, μ) through the representation (4.1.2) with associated convolution semigroup $(\eta_t)_{t\geq 0}$ supported by $[0,\infty)$. Assume that the measure μ is bounded. Let $(T_t)_{t\geq 0}$ be a strongly continuous contraction semigroup on a Banach space $(X, \|\cdot\|_X)$ with the generator $(L, \mathrm{Dom}(L))$. Let $(F(t))_{t\geq 0}$ be a family of contraction operators on $(X, \|\cdot\|_X)$ which is Chernoff equivalent to $(T_t)_{t\geq 0}$, i.e. $F(0) = \mathrm{Id}$, $\|F(t)\| \leq 1$ for all $t \geq 0$ and there is a set $D \subset \mathrm{Dom}(L)$, which is a core for L, such that $\lim_{t\to 0} \left\| \frac{F(t)\varphi - \varphi}{t} - L\varphi \right\|_X = 0$ for each $\varphi \in D$. Let $m : (0,\infty) \to \mathbb{N}_0$ be a monotone function with $m(t) \to +\infty$ as $t \to 0$. Let, for each $t > 0$ and each $\varphi \in X$, the mapping $[F(\cdot/m(t))]^{m(t)}\varphi : [0,\infty) \to X$ be Bochner measurable as the mapping from $([0,\infty), \mathcal{B}([0,\infty)), \mu)$ to $(X, \mathcal{B}(X))$. Let $(T_t^f)_{t\geq 0}$ be the semigroup subordinate to $(T_t)_{t\geq 0}$ with respect to $(\eta_t)_{t\geq 0}$ and L^f be its generator. Consider a family $(\mathcal{F}_\mu(t))_{t\geq 0}$ of operators on $(X, \|\cdot\|_X)$ defined for all $\varphi \in X$ and all $t \geq 0$ by*

$$\mathcal{F}_\mu(t)\varphi := e^{-\sigma t} F(\lambda t) \left(\varphi + t \int_{0+}^{\infty} (F^{m(t)}(s/m(t))\varphi - \varphi)\mu(ds) \right). \tag{4.3.1}$$

The family $(\mathcal{F}_\mu(t))_{t\geq 0}$ is Chernoff equivalent to the semigroup $(T_t^f)_{t\geq 0}$, and hence

$$T_t^f \varphi = \lim_{n\to\infty} \left[\mathcal{F}_\mu(t/n)\right]^n \varphi$$

for all $\varphi \in X$ locally uniformly with respect to $t \geq 0$.

Proof. Let us prove that the family $(F_\mu(t))_{t\geq 0}$, defined for all $\varphi \in X$ and all $t \geq 0$ by

$$F_\mu(t)\varphi := \varphi + t \int_{0+}^{\infty} (F^{m(t)}(s/m(t))\varphi - \varphi)\mu(ds), \tag{4.3.2}$$

is Chernoff equivalent to the semigroup $(T_t^{f_0})_{t\geq 0}$ which is subordinate to $(T_t)_{t\geq 0}$ with respect to $(\eta_t^0)_{t\geq 0}$, associated to the Bernstein function f_0 which is defined by the triplet $(0, 0, \mu)$. Then the statement of Theorem 4.3.1 follows immediately from Theorem 2.1.1 (or Corollary 2.1.2). So, let $K := \mu(\mathbb{R}) < \infty$. Then, clearly, we have $F_\mu(0) = \mathrm{Id}$ and

$$\begin{aligned} &\|F_\mu(t)\varphi\|_X \\ &\leq \|\varphi\|_X + tK \sup_{s\in(0,\infty)} \left\|F^{m(t)}(s/m(t))\varphi - \varphi\right\|_X \leq \|\varphi\|_X (1 + 2tK) \leq e^{2tK} \|\varphi\|_X, \end{aligned}$$

and

$$\|F_\mu(t)\varphi - \varphi\|_X \leq tK \sup_{s\in(0,\infty)} \left\|F^{m(t)}(s/m(t))\varphi - \varphi\right\|_X \leq 2tK\|\varphi\|_X \to 0, \quad t \to 0.$$

Further, for an arbitrary $\varepsilon > 0$ choose R_ε such that $\int_{R_\varepsilon}^{\infty} \mu(ds) < \varepsilon$. Then for each $\varphi \in D$

$$\lim_{t\to 0} \left\| \frac{F_\mu(t)\varphi - \varphi}{t} - L^{f_0}\varphi \right\|_X = \lim_{t\to 0} \left\| \int_{0+}^{\infty} (F^{m(t)}(s/m(t))\varphi - T_s\varphi)\mu(ds) \right\|_X$$

$$\leq \lim_{t\to 0} \left[\int_{0+}^{R_\varepsilon} \|F^{m(t)}(s/m(t))\varphi - T_s\varphi\|_X \mu(ds) + \int_{R_\varepsilon}^{\infty} \|F^{m(t)}(s/m(t))\varphi - T_s\varphi\|_X \mu(ds) \right]$$

$$\leq 2\varepsilon\|\varphi\|_X + K \lim_{t\to 0} \sup_{s\in[0,R_\varepsilon]} \|F^{m(t)}(s/m(t))\varphi - T_s\varphi\|_X = 2\varepsilon\|\varphi\|_X$$

due to the Chernoff Theorem. Therefore,

$$\lim_{t\to 0} \left\| \frac{F_\mu(t)\varphi - \varphi}{t} - L^{f_0}\varphi \right\|_X = 0$$

which proves the Chernoff equivalence of the family $(\mathcal{F}_\mu(t))_{t\geq 0}$ to the semigroup $(T_t^f)_{t\geq 0}$.

□

Remark 4.3.2. Again, the family $(F_\mu(t))_{t\geq 0}$ (and hence $(\mathcal{F}_\mu(t))_{t\geq 0}$, constructed in Theorem 4.3.1) can not be strongly continuous for all $t \in [0,\infty)$ since the function m, used in the construction of this family, is not continuous. Therefore, the family $(\mathcal{F}_\mu(t))_{t\geq 0}$ also can not be used for establishing Chernoff approximations for semigroups, generated by multiplicative perturbations of the operator L^f, via Theorem 2.2.2 (or Proposition 2.2.6). Nevertheless, combination of the family $(\mathcal{F}(t))_{t\geq 0}$ with other thechniques of Chernoff approximation, developed in this work, are safely possible. Moreover, under additional assumptions on the function m and the family $(F(t))_{t\geq 0}$, assuring that the mapping $\mathcal{F}_\mu(\cdot)\varphi : [0,\infty) \to X$ is Bochner measurable for each $\varphi \in X$, the family $(\mathcal{F}_\mu(t))_{t\geq 0}$, constructed in Theorem 4.3.1, can be used to approximate solutions of time-fractional evolution equations as in Theorem (6.3.3).

Remark 4.3.3. The choise of the family $F_\mu(t)$ is motivated by the fact that, for each bounded linear operator A, the family $F_A(t) := \mathrm{Id} + tA$ is obviously Chernoff equivalent to the semigroup e^{tA} (cf. Example 1.0.8). We have, however, the family $F_A(t) := \mathrm{Id} + tA(t)$, where operators $A(t)$ are bounded and tend to the generator A as $t \to 0$. The natural question arises: if it is possible to find the family $F_A(t) := \mathrm{Id} + tA(t)$, where operators $A(t)$ are bounded and tend to the unbounded generator A of the semigroup e^{tA} as $t \to 0$, such that $F_A(t)$ would be Chernoff equivalent to e^{tA}? In this case it would be possible to generalize Theorem 4.3.1 to the case of unbounded Lévy measure μ, e.g., by approximating μ with bounded measures $\mu_t := 1_{[\alpha(t),\infty)}\mu$ for some proper function $\alpha(t) \to 0$ as $t \to 0$. However, the answer is NO, since the norm estimate $\|F_A(t)\| \leq e^{ct}$ for all $t \geq 0$ and some $c \in \mathbb{R}$ (or the equivalent one $\|F_A^k(t)\| \leq Me^{ckt}$ for all $k \in \mathbb{N}$, $t \geq 0$ and some $c \in \mathbb{R}$, $M \geq 1$, cf. Pazy, 1983), required in the Chernoff Theorem, fails.

Remark 4.3.4. The analogue of Remark 4.2.9 holds true also for the family $(\widehat{\mathcal{F}}_\mu(t))_{t\geq 0}$,

$$\widehat{\mathcal{F}}_\mu(t)\varphi := e^{-\sigma t}\widehat{F}(t)\left(\varphi + t\int\limits_{0+}^{\infty}(F^{m(t)}(s/m(t))\varphi - \varphi)\mu(ds)\right),$$

with $\widehat{F}(t)$ as in (4.2.5).

Chapter 5

Chernoff approximation of semigroups generated by killed Feller processes

5.1 Killed Feller processes and their generators

In this Section, we follow the exposition of Böttcher, Schilling, and Wang, 2013 and Baeumer, Luks, and Meerschaert, 2016. Let $(\xi_t)_{t\geq 0}$ be a Feller process on $\mathbb{R}^d$, $(T_t)_{t\geq 0}$ be the corresponding Feller semigroup and $(L, \mathrm{Dom}(L))$ be its Feller generator. The *pointwise generator* $(L_p, \mathrm{Dom}(L_p))$ of $(T_t)_{t\geq 0}$ is defined via

$$\mathrm{Dom}(L_p) := \left\{ \varphi \in C_\infty(\mathbb{R}^d) \;\middle|\; \exists\, g \in C_\infty(\mathbb{R}^d) : \lim_{t\to 0} \frac{T_t\varphi(x) - \varphi(x)}{t} = g(x)\ \forall\, x \in \mathbb{R}^d \right\},$$

$$L_p\varphi(x) := \lim_{t\to 0} \frac{T_t\varphi(x) - \varphi(x)}{t} = g(x) \quad \forall\, \varphi \in \mathrm{Dom}(L_p),\ x \in \mathbb{R}^d. \tag{5.1.1}$$

Then $(L, \mathrm{Dom}(L)) = (L_p, \mathrm{Dom}(L_p))$ by Theorem 1.33 of Böttcher, Schilling, and Wang, 2013. Recall that if $C_c^\infty(\mathbb{R}^d) \subset \mathrm{Dom}(L)$, then $L\varphi(x)$ is given by the formula (3.1.3) for each $\varphi \in C_\infty^2(\mathbb{R}^d)$ and each $x \in \mathbb{R}^d$, i.e. L is an integro-differential operator on $C_\infty^2(\mathbb{R}^d)$. The process $(\xi_t)_{t\geq 0}$ (resp., the semigroup $(T_t)_{t\geq 0}$ with $T_t\varphi(x) := \mathbb{E}^x[\varphi(\xi_t)]$) is called *doubly Feller* if it is both Feller and strong Feller (cf. Definitions 3.1.1, 3.1.2).

Let $G \subset \mathbb{R}^d$ be a bounded domain (connected open set) and let $Y := C_0(G)$ be the set of all continuous functions on G that tend to zero as $x \in G$ approaches the boundary ∂G. Then Y is a Banach space with the supremum norm $\|\cdot\|_Y$, $\|\varphi\|_Y := \sup_{x\in G} |\varphi(x)|$. For a Feller process $(\xi_t)_{t\geq 0}$ on $\mathbb{R}^d$, we define the first exit time from G by

$$\tau_G := \inf\{t > 0 : \xi_t \notin G\}.$$

Let $(\xi_t^o)_{t\geq 0}$ denote the killed process on G, i.e.,

$$\xi_t^o = \begin{cases} \xi_t, & t < \tau_G, \\ \partial, & t \geq \tau_G, \end{cases}$$

where ∂ denotes a cemetery point. We say that a boundary point $x \in \partial G$ is *regular* for G if $\mathbb{P}^x(\tau_G = 0) = 1$. We say that G is *regular* if every point $x \in \partial G$ is regular.[1] Let $(\xi_t)_{t\geq 0}$ be a doubly Feller process on $\mathbb{R}^d$ and G be regular. Then $(T_t^o)_{t\geq 0}$, such that

$$T_t^o\varphi(x) := \mathbb{E}^x\left[\varphi\left(\xi_t^o\right)\right] \equiv \mathbb{E}^x\left[\varphi\left(\xi_t\right)1_{\{t<\tau_G\}}\right], \quad x \in G,\ \varphi \in Y,\ \varphi(\partial) := 0,$$

is a Feller semigroup on Y (cf. Lemma 2.2 in Baeumer, Luks, and Meerschaert, 2016). Let $(L_o, \mathrm{Dom}(L_o))$ be the Feller generator of $(T_t^o)_{t\geq 0}$ on Y. The operator $(L_o, \mathrm{Dom}(L_o))$ is described in Proposition 5.1.1 below (cf. Thm. 2.3 and Lemma 2.6 in Baeumer, Luks, and Meerschaert, 2016). Note that each element $\varphi \in Y$ can be extended by zero outside G; this extension is again denoted by φ and belongs to the space $X := C_\infty(\mathbb{R}^d)$.

Proposition 5.1.1. *Let $(\xi_t)_{t\geq 0}$ be a doubly Feller process on $\mathbb{R}^d$, $(T_t)_{t\geq 0}$ be the corresponding Feller semigroup and $(L, \mathrm{Dom}(L))$ be its Feller generator. Let $G \subset \mathbb{R}^d$ be a bounded regular domain. The generator of the killed Feller process $(\xi_t^o)_{t\geq 0}$ is characterized as follows*

(i) $\mathrm{Dom}(L_o) = \{\varphi \in Y\ :\ L_p\varphi \in Y\}$, *where L_p is given by (5.1.1), $L_p\varphi \in Y$ means that L_p is applied to the zero extension of φ on $\mathbb{R}^d$, $L_p\varphi(x)$ exists for each $x \in G$ and the function $[x \mapsto L_p\varphi(x)]$ belongs to Y. Moreover, it holds that $L_o\varphi(x) = L_p\varphi(x)$ for all $\varphi \in \mathrm{Dom}(L_o)$ and the limit in (5.1.1) exists locally uniformly on G (i.e., uniformly with respect to $x \in K$ for each compact $K \subset G$).*

(ii) Assume that $C_c^\infty(\mathbb{R}^d) \subset \mathrm{Dom}(L)$. Then, for each $\varphi \in C_\infty(\mathbb{R}^d) \cap C^2(G)$, we have

$$L_p\varphi(x) = L\varphi(x), \quad \forall\, x \in G,$$

where L is the integro-differential operator given by the formula (3.1.3).

Remark 5.1.2. (i) The abstract Cauchy problem in Y for the evolution equation $\frac{df}{dt} = L_o f$ with an initial condition $f_0 \in \mathrm{Dom}(L_o)$ can be interpreted as the following Cauchy–Dirichlet type initial–exterior value problem[2]:

$$\begin{aligned} &\frac{\partial f}{\partial t}(t,x) = Lf(t,x), \quad t>0,\ x \in G,\\ &f(0,x) = f_0(x), \quad x \in G, \\ &f(t,x) = 0, \quad t>0, \quad x \in \mathbb{R}^d \setminus G. \end{aligned} \tag{5.1.2}$$

And the function $f(t,x) := T_t^o f_0(x)$, extended by zero outside G, solves the problem for each $f_0 \in \mathrm{Dom}(L_o)$ by Theorem 1.0.2.

(ii) Let, additionally, the operator $(L, \mathrm{Dom}(L))$ in X be a *local operator outside $\overline{G}$*, i.e. for each $x \in \mathbb{R}^d \setminus \overline{G}$ and each $\varphi_1, \varphi_2 \in \mathrm{Dom}(L)$ such that φ_1 and φ_2 coincide

[1] Due to Theorem 2.2. of Chen and Song, 1997, if a boundary point $x \in \partial G$ satisfies the external cone condition, then it is regular for the case when $(\xi_t)_{t\geq 0}$ is symmetric α-stable. In particular, any Lipschitz domain is regular in this case.

[2] Such problems are discussed, e.g., in Felsinger, Kassmann, and Voigt, 2015, see also Hoh and Jacob, 1996 for the stationary case.

on $\overline{G}$ and on some neighbourhood of x, one has $L\varphi_1(x) = L\varphi_2(x)$. For example, consider the integro-differential operator L given by (3.1.3) with $N(x,dy)$ such that

$$N(x,dy) = \begin{cases} 1_{0<|y|\le \mathrm{dist}(x,\partial G)}(y)N(x,dy), & x \in G, \\ 0, & x \in \mathbb{R}^d \setminus G. \end{cases} \tag{5.1.3}$$

The integral part of such operator L gives rise to the so-called *censored processes* in G, cf. Bogdan, Burdzy, and Chen, 2003; Rossi and Topp, 2016. If L is local outside $\overline{G}$, the abstract Cauchy problem in Y for the evolution equation $\frac{df}{dt} = L_o f$ can be interpreted as the following Cauchy–Dirichlet problem:

$$\begin{aligned} &\frac{\partial f}{\partial t}(t,x) = Lf(t,x), \quad t>0,\ x\in G, \\ &f(0,x) = f_0(x), \quad x \in G, \\ &f(t,x) = 0, \quad t>0, \quad x \in \partial G. \end{aligned} \tag{5.1.4}$$

And again the function $f(t,x) := T_t^o f_0(x)$ solves this problem for each $f_0 \in \mathrm{Dom}(L_o)$ by Theorem 1.0.2.

5.2 Chernoff approximation of semigroups generated by some killed Feller processes

Let $X = C_\infty(\mathbb{R}^d)$. Let $(\xi_t)_{t\ge0}$ be a doubly Feller process on $\mathbb{R}^d$, $(T_t)_{t\ge0}$ be the corresponding (doubly Feller) semigroup and $(L,\mathrm{Dom}(L))$ be its Feller generator. Let a family $(F(t))_{t\ge0}$ of bounded linear operators on X be Chernoff equivalent to the semigroup $(T_t)_{t\ge0}$. Therefore, $\|F(t)\| \le e^{kt}$ for some $k\in\mathbb{R}$ and all $t\ge0$, and there exists a core D for the operator $(L,\mathrm{Dom}(L))$ such that $\lim_{t\to0}\left\|\frac{F(t)\varphi-\varphi}{t} - L\varphi\right\|_X = 0$ for all $\varphi\in D$. Let us fix this core D. Let $G\subset\mathbb{R}^d$ be a bounded regular domain. Let $(T_t^o)_{t\ge0}$ be the strongly continuous semigroup on $Y := C_0(G)$ generated by the killed Feller process $(\xi_t^o)_{t\ge0}$ on G. Let $(L_o,\mathrm{Dom}(L_o))$ be the Feller generator of $(T_t^o)_{t\ge0}$. Our aim is to construct a family $(F_o(t))_{t\ge0}$ of bounded linear operators on Y which is Chernoff equivalent to $(T_t^o)_{t\ge0}$. The family $(F_o(t))_{t\ge0}$ will be constructed by a proper modification of the family $(F(t))_{t\ge0}$ which is Chernoff equivalent to the semigroup $(T_t)_{t\ge0}$ on X. To this aim, we need some preparations.

Assumption 5.2.1. We assume that there exists a set $D_o\subset\mathrm{Dom}(L_o)\cap C_b^2(G)$ and a mapping $\mathcal{E}: Y\to C_c(\mathbb{R}^d)\subset X$ such that

(i) D_o is a core for L_o;

(ii) $\mathcal{E}(\varphi)\big|_{\overline{G}} = \varphi$ for all $\varphi\in Y$;

(iii) the mapping $\mathcal{E}$ is linear;

(iv) the mapping $\mathcal{E}$ preserves the supremum norm, i.e. $\|\varphi\|_Y = \|\mathcal{E}(\varphi)\|_X$ for all $\varphi \in Y$;

(v) $\mathcal{E} : D_o \to D$, where D is a fixed core for $(L, \mathrm{Dom}(L))$;

(vi) $L(\mathcal{E}(\varphi))(x) = L_o\varphi(x)$ for each $\varphi \in D_o$ and each $x \in G$.

Remark 5.2.2. The space Y can be naturally embedded into X by assigning to each $\varphi \in Y$ zero values outside the domain G. However, such embedding produces from smooth functions in G only continuous functions in $\mathbb{R}^d$. This may violate the requirement (v) of Assumption 5.2.1. Note that $\mathrm{Dom}(L_o)$ typically contains functions whose zero extensions do not belong to $\mathrm{Dom}(L)$. Moreover, there is no reason to expect the existence of a core D_o such that the zero extensions of its elements belong to $\mathrm{Dom}(L)$. In particular, the sets of sufficiently smooth functions with compact supports in G can not serve as a core even for the Laplacian Δ in Y[3]! Indeed, assume that there exists a core $D_o \subset C_c(G)$ for $(\Delta, \mathrm{Dom}(\Delta))$ in Y. Then for each $\varphi \in \mathrm{Dom}(\Delta)$ there exists a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset D_o$ such that $\|\varphi_n - \varphi\|_Y \to 0$ and $\|\Delta\varphi_n - \Delta\varphi\|_Y \to 0$ as $n \to \infty$. By Theorem D.0.9 (ii), $\mathrm{Dom}(\Delta)$ is continuously embedded in $C^1(\overline{G})$. Hence $\left\|\frac{\partial\varphi_n}{\partial x_i} - \frac{\partial\varphi}{\partial x_i}\right\|_Y \to 0$ as $n \to \infty$ for all $i = 1, \dots, d$. Therefore, $\nabla\varphi\big|_{\partial G} = 0$ for each $\varphi \in \mathrm{Dom}(\Delta)$. This is however wrong since, e.g., the function $\varphi(x) := \sin x$ belongs to $\mathrm{Dom}(\Delta) = \mathrm{Dom}\left(\frac{d^2}{dx^2}\right)$ for $G := (0, \pi)$ and $\frac{d\varphi}{dx}(x) = \cos x$ is not equal to zero on ∂G.

Remark 5.2.3. Let the generator L of a Feller semigroup $(T_t)_{t\geq 0}$ be given by formula (2.3.9), Assumption 2.3.1 be fulfilled, coefficients A, B and C be of the class $C_b^{2,\alpha}(\mathbb{R}^d)$ for some $\alpha \in (0,1)$ and $a \equiv 1$. Let $D := C_c^{2,\alpha}(\mathbb{R}^d) \subset X$ be a core for L in X. Consider $D_o := \{\varphi \in C^{2,\alpha}(G) : \varphi, L\varphi \in Y\}$. If the boundary ∂G is of the class $C^{4,\alpha}$ then there exists a strongly continuous semigroup $(T_t^o)_{t\geq 0}$ on Y generated by the closure of (L_o, D_o) in Y, and there exists a mapping $\mathcal{E}$ such that Assumption 5.2.1 is fulfilled with respect to L, D, D_o and $\mathcal{E}$ due to Thm. 2.2 and Thm. 3.4 in Baur, Conrad, and Grothaus, 2011.

Remark 5.2.4. Let again the boundary ∂G be of the class $C^{4,\alpha}$ for some $\alpha \in (0,1)$. Let $C_c^\infty(\mathbb{R}^d) \subset \mathrm{Dom}(L)$ and L be given by formula (3.1.3). Consider L as the sum $L := L_1 + L_2$, where L_1 is the differential operator given by formula (2.3.9) and L_2 is the integral part

$$L_2\varphi(x) := \int_{y\neq 0} \left(\varphi(x+y) - \varphi(x) - \frac{y \cdot \nabla\varphi(x)}{1+|y|^2}\right) N(x,dy), \quad \forall \varphi \in C_\infty^2(\mathbb{R}^d),\ \forall\, x \in \mathbb{R}^d.$$

Let L_1 be as in Remark 5.2.3, i.e. Assumption 2.3.1 be fulfilled, coefficients A, B and C be of the class $C_b^{2,\alpha}(\mathbb{R}^d)$ for some $\alpha \in (0,1)$, $a \equiv 1$. Let $D := C_c^{2,\alpha}(\mathbb{R}^d) \subset X$ be the core for L_1. Then, due to Remark 5.2.3, there exists an extension $\mathcal{E}$ satisfying Assumption 5.2.1 (ii)-(v) with respect to L_1 and cores $D = C_c^{2,\alpha}(\mathbb{R}^d)$ and $D_o = \{\varphi \in C^{2,\alpha}(G) : \varphi, L_1\varphi \in Y\}$. Let $U \subset \mathbb{R}^d$ be another bounded domain such that

[3]This fact together with its proof have been communicated to the author by Professor Alessandra Lunardi.

$G \subset U$. One may consider, e.g., $U \equiv U_\varepsilon := \{x \in \mathbb{R}^d : \operatorname{dist}(x, G) < \varepsilon\}$ for some small constant $\varepsilon > 0$. Multiplying the extension $\mathcal{E}$ with a proper cut-off function, one obtains another extension $\mathcal{E}_U$, satisfying Assumption 5.2.1 (ii)-(v) with respect to L_1 and the condition $\mathcal{E}_U : Y \to C_c(U)$. Assume that $N(x, dy)$ is such that (L_2, D) is L_1-bounded and the closure of (L, D), $L = L_1 + L_2$, generates a doubly Feller semigroup on X. Let there exist a core $D_{oo} \subset \{\varphi \in C^{2,\alpha}(G) : \varphi,\ L_1\varphi \in Y\}$ for the corresponding "killed′" generator $(L_o, \operatorname{Dom}(L_o))$. Then the extension $\mathcal{E}_U$ satisfies Assumption 5.2.1 (ii)-(v) with respect to L, D and D_{oo}. Since L is a non-local operator, Assumption 5.2.1 (vi) is not fulfilled automatically. And, for each $\varphi \in D_{oo}$ and each $x \in G$, we have

$$\begin{aligned} L(\mathcal{E}_U(\varphi))(x) - L_o\varphi(x) &= \int\limits_{y \neq 0} (\mathcal{E}_U(\varphi)(x+y) - \varphi(x+y))\, N(x, dy) \\ &= \int\limits_{y \in \left(-x + U \setminus \overline{G}\right)} \mathcal{E}_U(\varphi)(x+y) N(x, dy). \end{aligned}$$

Let L_2 satisfy the additional condition: there exists a bounded domain $U \supset G$ such that for each $x \in G$ holds

$$\int\limits_{y \in \left(-x + U \setminus \overline{G}\right)} N(x, dy) = 0. \tag{5.2.1}$$

Then the extension $\mathcal{E}_U$ satisfies Assumption 5.2.1. The condition (5.2.1) actually means that the process $(\xi_t)_{t\geq 0}$ is allowed to leave the domain G either continuously, or by a sufficiently large jump which brings the process even out of U. Note that if $N(x, dy)$ corresponds to censored processes (i.e., $N(x, dy)$ satisfies (5.1.3)), then the condition (5.2.1) is fulfilled. The condition (5.2.1) is also fulfilled if, e.g., $\operatorname{supp} N(x, \cdot) \subset \mathbb{R}^d \setminus K$ for all $x \in G$ and some compact K such that $\cup_{x\in G}\left(-x + U \setminus \overline{G}\right) \subset K$. One can take as K, e.g., a ball $B_R(x_0)$ such that its center $x_0 \in G$ and its raduius $R > 2 \operatorname{diam} U$.

Consider now a continuous monotone function $s : (0, \infty) \to (0, \infty)$ such that

$$\lim_{t \to 0} \frac{s(t)}{t} = 0.$$

Define the set $G_{s(t)} \subset G$ by

$$G_{s(t)} := \{x \in G : \operatorname{dist}(x, \partial G) > s(t)\}.$$

Let $\left(\phi_{s(t)}\right)_{t>0}$ be a family of functions $\phi_{s(t)} : \mathbb{R}^d \to [0, 1]$ such that all $\phi_{s(t)} \in C_c^\infty(G)$, we have $\phi_{s(t)}(x) = 1$, $\forall\, x \in G_{s(t)}$, $\forall\, t > 0$, and $\lim_{t\to t^*} \|\phi_{s(t)} - \phi_{s(t^*)}\|_X = 0$ for each $t^* > 0$. Note that functions $\phi_{s(t)}$ converge pointwise to the indicator 1_G of the domain G when $t \to 0$. Consider the family $(F_o(t))_{t\geq 0}$ of operators on Y defined by $F_o(0) := \operatorname{Id}$ and for each $t > 0$, each $\varphi \in Y$ and each $x \in G$

$$F_o(t)\varphi(x) := \phi_{s(t)}(x)[F(t)\mathcal{E}(\varphi)](x) \tag{5.2.2}$$

where the given family $(F(t))_{t\geq 0}$ is Chernoff equivalent to the semigroup $(T_t)_{t\geq 0}$ on X generated by $(L, \mathrm{Dom}(L))$ and $F'(0)\varphi = L\varphi$ for all $\varphi \in D$.

Lemma 5.2.5. *The family $(F_o(t))_{t\geq 0}$ acts on Y and $\|F_o(t)\| \leq \|F(t)\|$. If the family $(F(t))_{t\geq 0}$ is strongly continuous on X then the family $(F_o(t))_{t\geq 0}$ is strongly continuous on Y.*

Proof. The family $(F_o(t))_{t\geq 0}$ acts on Y since, if $\varphi \in Y = C_0(G)$, then $\mathcal{E}(\varphi) \in C_c(\mathbb{R}^d) \subset C_\infty(\mathbb{R}^d) = X$, $F(t)\mathcal{E}(\varphi) \in X$ and $\phi_{s(t)}[F(t)\mathcal{E}(\varphi)] \in Y$. Moreover,

$$
\begin{aligned}
\|F_0(t)\varphi\|_Y &= \sup_{x\in G} |\phi_{s(t)}(x)[F(t)\mathcal{E}(\varphi)](x)| \\
&\leq \|F(t)\mathcal{E}(\varphi)\|_X \leq \|F(t)\|\|\mathcal{E}(\varphi)\|_X = \|F(t)\|\|\varphi\|_Y.
\end{aligned}
$$

Let us show the strong continuity of the family $(F_o(t))_{t\geq 0}$ under assumption that the family $(F(t))_{t\geq 0}$ is strongly continuous on X. First, for each $\varphi \in Y$

$$
\begin{aligned}
\lim_{t\to 0} \|F_o(t)\varphi - \varphi\|_Y &= \lim_{t\to 0} \sup_{x\in G} \left|\phi_{s(t)}(x)[F(t)\mathcal{E}(\varphi)](x) - \varphi(x)\right| \\
&= \lim_{t\to 0} \sup_{x\in G} \left|\phi_{s(t)}(x)\big([F(t)\mathcal{E}(\varphi)](x) - \mathcal{E}(\varphi)(x)\big) + \varphi(x)[\phi_{s(t)}(x) - 1]\right| \\
&\leq \lim_{t\to 0} \|F(t)\mathcal{E}(\varphi) - \mathcal{E}(\varphi)\|_X + \lim_{t\to 0} \sup_{x\in \overline{G\setminus G_{s(t)}}} |\varphi(x)| \\
&= 0
\end{aligned}
$$

due to strong continuity at zero of the family $(F(t))_{t\geq 0}$ on X and uniform continuity of φ on the compact $\overline{G\setminus G_{s(t)}}$. Second, for each $t^* > 0$ and each $\varphi \in Y$

$$
\begin{aligned}
&\lim_{t\to t^*} \|F_o(t)\varphi - F_o(t^*)\varphi\|_Y \\
&= \lim_{t\to t^*} \sup_{x\in G} \left|\phi_{s(t)}(x)[F(t)\mathcal{E}(\varphi)](x) - \phi_{s(t^*)}(x)[F(t^*)\mathcal{E}(\varphi)](x)\right| \\
&= \lim_{t\to t^*} \sup_{x\in G} \Big|\phi_{s(t)}(x)\big([F(t)\mathcal{E}(\varphi)](x) - [F(t^*)\mathcal{E}(\varphi)](x)\big) + \\
&\qquad\qquad + (\phi_{s(t)}(x) - \phi_{s(t^*)}(x))[F(t^*)\mathcal{E}(\varphi)](x)\Big| \\
&\leq \lim_{t\to t^*} \|\phi_{s(t)}\|_Y \cdot \|[F(t)\mathcal{E}(\varphi)] - [F(t^*)\mathcal{E}(\varphi)]\|_X + \|F(t^*)E(\varphi)\|_X \cdot \|\phi_{s(t)} - \phi_{s(t^*)}\|_Y \\
&= 0
\end{aligned}
$$

due to strong continuity of the family $(F(t))_{t\geq 0}$ on X and propersties of the family $\left(\phi_{s(t)}\right)_{t>0}$. Hence Lemma is proved. □

Theorem 5.2.6. *Under Assumption 5.2.1, the family $(F_o(t))_{t\geq 0}$ is Chernoff equivalent to the semigroup $(T_t^o)_{t\geq 0}$, i.e.*

$$T_t^o\varphi = \lim_{n\to\infty} [F_o(t/n)]^n \varphi$$

for each $\varphi \in Y$ locally uniformly with respect to $t \geq 0$.

Proof. Due to Lemma 5.2.5, we have $\|F_o(t)\| \le \|F(t)\| \le e^{kt}$ for some $k \in \mathbb{R}$ and all $t \ge 0$. Hence it is sufficient to show that $\lim_{t\to 0} \|t^{-1}(F_o(t)\varphi - \varphi) - L_o\varphi\|_Y = 0$ for all $\varphi \in D_o$. Due to Assumption 5.2.1

$$\begin{aligned}
&\left\| \frac{F_o(t)\varphi - \varphi}{t} - L_o\varphi \right\|_Y = \sup_{x\in G} \left| \frac{\phi_{s(t)}(x)[F(t)\mathcal{E}(\varphi)](x) - \varphi(x)}{t} - L_o\varphi(x) \right| \\
&\le \sup_{x\in G} \Bigg[|\phi_{s(t)}(x)| \left| \frac{F(t)\mathcal{E}(\varphi)(x) - \mathcal{E}(\varphi)(x)}{t} - L\mathcal{E}(\varphi)(x) \right| \\
&\qquad\qquad + \big(|\varphi(x)/t| + |L_o\varphi(x)|\big)|1 - \phi_{s(t)}(x)| \Bigg] \\
&\le \left\| \frac{F(t)\mathcal{E}(\varphi) - \mathcal{E}(\varphi)}{t} - L\mathcal{E}(\varphi) \right\|_X + \sup_{x\in \overline{G\setminus G_{s(t)}}} \big(|\varphi(x)/t| + |L_o\varphi(x)|\big) \\
&\to 0, \quad \text{as } t \to 0.
\end{aligned}$$

Indeed, $\lim_{t\to 0} \left\| \frac{F(t)\mathcal{E}(\varphi) - \mathcal{E}(\varphi)}{t} - L\mathcal{E}(\varphi) \right\|_X = 0$ since $\mathcal{E}(\varphi) \in D$ and $F'(0) = L$ on D by our assumptions. Further, $\varphi \in D_o \subset C_b^2(G) \cap Y$. Hence φ is Lipschitz on $\overline{G}$, i.e. there exists a constant $M > 0$ such that the inequality $|\varphi(x) - \varphi(z)| \le M|x - z|$ holds for all x, $z \in \overline{G}$. Moreover, for each $x \in G \setminus G_{s(t)}$, there exists at least one point $z_x \in \partial G$ such that $\operatorname{dist}(x, z_x) \le s(t)$. Therefore, $|\varphi(x)| = |\varphi(x) - \varphi(z_x)| \le Ms(t)$ for each $x \in G \setminus G_{s(t)}$. And

$$\lim_{t\to 0} \sup_{x\in \overline{G\setminus G_{s(t)}}} \frac{|\varphi(x)|}{t} \le \lim_{t\to 0} M \frac{s(t)}{t} = 0.$$

Besides, since $\varphi \in D_o \subset \mathrm{Dom}(L_o)$, we have $L_o\varphi \in Y = C_0(G)$. Hence

$$\lim_{t\to 0} \sup_{x\in \overline{G\setminus G_{s(t)}}} |L_o\varphi(x)| = 0$$

due to uniform continuity of the function $L_o\varphi$ on compacts $\overline{G \setminus G_{s(t)}}$. Thus, Theorem is proved. □

Remark 5.2.7. Analogues of Theorem 5.2.6 are also valid in unbounded domains $G \subset \mathbb{R}^d$, in domains G of a locally compact metric space Q and in other couples of Banach spaces X and Y (e.g., $X := L^p(\mathbb{R}^d)$ and $Y := L^p(G)$, $p \in [1, \infty)$) under corresponding modifications of Assumption 5.2.1 and properties of the family $\big(\phi_{s(t)}\big)_{t>0}$, as well as under additional assumption on the existence of the semigroup $(T_t^o)_{t\ge 0}$.

5.3 Lagrangian Feynman formulae for semigroups generated by some killed Feller processes

Let $(T_t)_{t\geq 0}$ be a doubly Feller semigroup on X whose generator $(L, \mathrm{Dom}(L))$ is such that the set $D := C_c^\infty(\mathbb{R}^d)$ is a core for L. Hence $L\varphi$ is given by formula (3.1.3) for each $\varphi \in C_\infty^2(\mathbb{R}^d)$. Assume that the coefficients A, B, C in formula (3.1.3) are bounded and continuous. Let there exist a_0, $A_0 \in \mathbb{R}$ with $0 < a_0 \leq A_0 < \infty$ such that condition (2.3.1) holds. Let the measure $N(x, dy)$ in formula (3.1.3) do not depend on x, i.e. $N(x, dy) := N(dy)$ for all $x \in \mathbb{R}^d$. Let $(\eta_t)_{t\geq 0}$ be the convolution semigroup on $\mathbb{R}^d$ corresponding[4] to $N(dy)$. Then, by Theorem 3.2.6 and Theorem 3.3.4 (iii), the following family $(F(t))_{t\geq 0}$ on X is Chernoff equivalent to $(T_t)_{t\geq 0}$: $F(0) = \mathrm{Id}$ and for all $t > 0$, all $\varphi \in X$ and all $x \in \mathbb{R}^d$

$$F(t)\varphi(x) := \frac{e^{-tC(x)}}{\sqrt{(4\pi t)^d \det A(x)}} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} e^{-\frac{A^{-1}(x)(z-x+tB(x)+y)\cdot(z-x+tB(x)+y)}{4t}} \varphi(y)dy\, \eta_t(dz). \tag{5.3.1}$$

Moreover, the family $(F(t))_{t\geq 0}$ is a strongly continuous family of contractions. Note also that, for $g(x) \equiv 1$, we have $F(t)g(x) = \exp\{-tC(x)\} \leq 1$ for all $x \in \mathbb{R}^d$.

Let $G \subset \mathbb{R}^d$ be a regular bounded domain. Consider the corresponding Feller semigroup $(T_t^o)_{t\geq 0}$ on Y. Let Assumption 5.2.1 be fulfilled for some core D_o of the generator of $(T_t^o)_{t\geq 0}$ and for some extension $\mathcal{E} : Y \to X$ with respect to D_o and $D := C_c^\infty(\mathbb{R}^d)$. Then, by Theorem 5.2.6, the family $(F_o(t))_{t\geq 0}$, constructed from the family $(F(t))_{t\geq 0}$ in (5.3.1) through the formula (5.2.2), is Chernoff equivalent to the semigroup $(T_t^o)_{t\geq 0}$. Hence $F_o(0) = \mathrm{Id}$ and for all $t > 0$ and all $\varphi \in Y$

$$\begin{aligned} F_o(t)\varphi(x) &:= \frac{\phi_{s(t)}(x)e^{-tC(x)}}{\sqrt{(4\pi t)^d \det A(x)}} \times \\ &\times \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \exp\left\{-\frac{A^{-1}(x)(z-x+tB(x)+y)\cdot(z-x+tB(x)+y)}{4t}\right\} \mathcal{E}(\varphi)(y)dy\, \eta_t(dz). \end{aligned} \tag{5.3.2}$$

Therefore, we have uniformly with respect to $x_0 \in G$ and uniformly with respect to $t \in (0, t^*]$ for all $t^* > 0$

$$\begin{aligned} T_t^o\varphi(x_0) = \lim_{n\to\infty} F_o^n(t/n)\varphi(x_0) = \lim_{n\to\infty} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \dots \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \left(\prod_{k=1}^n \phi_{s(t/n)}(x_{k-1})\right) \mathcal{E}(\varphi)(x_n) \times \\ \times \Psi_{t,n}^{x_0}(x_1, \dots, z_n)\, dx_n \eta_{t/n}(dz_n) \cdots dx_1 \eta_{t/n}(dz_1), \end{aligned} \tag{5.3.3}$$

[4] I.e. the Fourier transforms $\mathcal{F}[\eta_t]$ of sub-probability measures η_t for all $t \geq 0$ are given by $\mathcal{F}[\eta_t](x) = (2\pi)^{-d/2}e^{-tr(x)}$, where the function $r : \mathbb{R}^d \to \mathbb{C}$ is defined by $r(x) := \int_{\mathbb{R}^d\setminus\{0\}} \left(1 - e^{iy\cdot x} + \frac{iy\cdot x}{1+|y|^2}\right) N(dy)$.

where

$$\Psi^{x_0}_{t,n}(x_1,\dots,z_n) := \left(\prod_{k=1}^{n}(4\pi t/n)^{-d/2}(\det A(x_{k-1}))^{-1/2}\right)\exp\left\{-\frac{t}{n}\sum_{k=1}^{n}C(x_{k-1})\right\}\times$$
$$\times\exp\left\{-\sum_{k=1}^{n}\frac{A^{-1}(x_{k-1})(z_k-x_{k-1}+tB(x_{k-1})+x_k)\cdot(z_k-x_{k-1}+tB(x_{k-1})+x_k)}{4t/n}\right\}.$$

Since all $\phi_{s(t)}$ are smooth functions with compact supports in G and $\mathcal{E}(\varphi)$ is a continuous function with compact support $K := \operatorname{supp}\mathcal{E}(\varphi)$, the $2n$-fold iterated integrals over $\mathbb{R}^d$ in (5.3.3) coincide with the following $2n$-fold multiple integral

$$\Phi^{\varphi}_n(t,x_0) := \int\limits_{G^{n-1}\times K\times\mathbb{R}^{nd}}\left(\prod_{k=1}^{n}\phi_{s(t/n)}(x_{k-1})\right)\mathcal{E}(\varphi)(x_n)\times$$
$$\times\Psi^{x_0}_{t,n}(x_1,\dots,z_n)\,dx_1\cdots dx_n\eta_{t/n}(dz_1)\cdots\eta_{t/n}(dz_n).$$

Consider also

$$\Theta^{\varphi}_n(t,x_0) := \int\limits_{G^{n}\times\mathbb{R}^{nd}}\varphi(x_n)\Psi^{x_0}_{t,n}(x_1,\dots,z_n)\,dx_1\cdots dx_n\eta_{t/n}(dz_1)\cdots\eta_{t/n}(dz_n).$$

Let us show that for all $t>0$ and all $x_0\in G$ holds

$$T^o_t\varphi(x_0) = \lim_{n\to\infty}\Theta^{\varphi}_n(t,x_0). \tag{5.3.4}$$

And the convergence in (5.3.4) is locally uniform with respect to $x_0\in G$ and uniform with respect to $t\in(0,t^*]$ for all $t^*>0$. So, consider a number $t^*>0$ and a compact $\Upsilon\subset G$. Let $x_0\in\Upsilon$ and $t\in(0,t^*]$. Then

$$|\Phi^{\varphi}_n(t,x_0)-\Theta^{\varphi}_n(t,x_0)| \le \int\limits_{G^{n-1}\times K\times\mathbb{R}^{nd}}\left(\prod_{k=1}^{n}|\phi_{s(t/n)}(x_{k-1})-1_G(x_{k-1})|\right)|\mathcal{E}(\varphi)(x_n)|\times$$
$$\times\Psi^{x_0}_{t,n}(x_1,\dots,z_n)dx_1\cdots dx_n\eta_{t/n}(dz_1)\cdots\eta_{t/n}(dz_n)$$
$$+\int\limits_{G^{n-1}\times\mathbb{R}^{nd}}\left(\int\limits_{K\smallsetminus G}|\mathcal{E}(\varphi)(x_n)|\Psi^{x_0}_{t,n}(x_1,\dots,z_n)dx_n\right)dx_1\cdots dx_{n-1}\eta_{t/n}(dz_1)\cdots\eta_{t/n}(dz_n).$$

Let us estimate each of the summands separately. Denote the first summand by $I^{\varphi}_n(t,x_0)$ and the second by $J^{\varphi}_n(t,x_0)$. We have with $g(x)\equiv 1$

$$I^{\varphi}_n(t,x_0)$$
$$\le \|\varphi\|_Y|\phi_{s(t/n)}(x_0)-1_G(x_0)|\int\limits_{\mathbb{R}^d}\dots\int\limits_{\mathbb{R}^d}\Psi^{x_0}_{t,n}(x_1,\dots,z_n)dx_n\eta_{t/n}(dz_n)\cdots dx_1\eta_{t/n}(dz_1)$$
$$= \|\varphi\|_Y|\phi_{s(t/n)}(x_0)-1_G(x_0)|\,(F^n(t/n)g(x_0)) \le \|\varphi\|_Y|\phi_{s(t/n)}(x_0)-1_G(x_0)|.$$

By the construction of the sets $G_{s(t)}$, there exists $N \in \mathbb{N}$ such that $\Upsilon \subset G_{s(t^*/N)}$. Hence for all $n \geq N$, $x_0 \in \Upsilon$, $t \in (0,t^*]$ holds $|\phi_{s(t/n)}(x_0) - 1_G(x_0)| = 0$. Therefore,

$$\lim_{n\to\infty} I_n^\varphi(t,x_0) = 0 \quad \text{uniformly with respect to } x_0 \in \Upsilon,\ t \in (0,t^*].$$

Consider now the second summand $J_n^\varphi(t,x_0)$. Due to condition (2.3.1), we have for all $x \in \overline{G}$, $y \in K$, $z \in \mathbb{R}^d$, $t \in (0,t^*]$ and $n \in \mathbb{N}$

$$\begin{aligned}
&\frac{e^{-tC(x)/n}}{\sqrt{(4\pi t/n)^d \det A(x)}} \exp\left\{-\frac{A^{-1}(x)(z-x+tB(x)/n+y)\cdot(z-x+tB(x)/n+y)}{4t/n}\right\}\\
&\leq M(A_0/a_0)^{d/2}(4A_0\pi t/n)^{-d/2}\exp\left\{-\frac{|x-y|^2}{4A_0 t/n}\right\},
\end{aligned}$$

where

$$M := \sup_{x\in\overline{G},\, y\in K,\, z\in\mathbb{R}^d,\, t\in(0,t^*],n\in\mathbb{N}} \exp\left\{-\frac{|z+tB(x)/n|^2 + 2(z+tB(x)/n)\cdot(y-x)}{4A_0 t/n}\right\} < \infty.$$

Therefore, with $c := M(A_0/a_0)^{d/2}$ and $p_{A_0}(t,x,y) := (4A_0\pi t)^{-d/2}\exp\left\{-\frac{|x-y|^2}{4A_0 t}\right\}$

$$\begin{aligned}
J_n^\varphi(t,x_0) &\leq c\left(\sup_{x\in\overline{G}}\int_{K\setminus G} p_{A_0}(t/n,x,y)|\mathcal{E}(\varphi)(y)|\,dy\right)\left(F^{n-1}(t/n)g(x_0)\right)\\
&\leq c\left(\sup_{x\in\overline{G}}\int_{K\setminus G} p_{A_0}(t/n,x,y)|\mathcal{E}(\varphi)(y)|\,dy\right).
\end{aligned}$$

Denote by G^δ the δ-neighborhood of G in $\mathbb{R}^d$, i.e. $G^\delta := \{x \in \mathbb{R}^d : \operatorname{dist}(x,G) < \delta\}$, $\delta > 0$. Fix any $\varepsilon > 0$. Since $\mathcal{E}(\varphi)$ is a continuous function on $\mathbb{R}^d$ which equals zero on ∂G, there exists $\delta > 0$ such that $|\mathcal{E}(\varphi)| \leq \varepsilon/2$ on $G^\delta \setminus G$. Hence

$$\begin{aligned}
&\sup_{x\in\overline{G}}\int_{K\setminus G} p_{A_0}(t/n,x,y)|\mathcal{E}(\varphi)(y)|\,dy \leq \sup_{x\in\overline{G}}\int_{G^\delta\setminus G} p_{A_0}(t/n,x,y)|\mathcal{E}(\varphi)(y)|\,dy\\
&+\sup_{x\in\overline{G}}\int_{K\setminus G^\delta} p_{A_0}(t/n,x,y)|\mathcal{E}(\varphi)(y)|\,dy \leq \frac{\varepsilon}{2} + \|\varphi\|_Y \sup_{x\in\overline{G}}\int_{K\setminus G^\delta} p_{A_0}(t/n,x,y)\,dy.
\end{aligned}$$

Due to Gaussian fall off of p_{A_0} there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and all $t \in (0,t^*]$ holds

$$\|\varphi\|_Y \sup_{x\in\overline{G}}\int_{K\setminus G^\delta} p_{A_0}(t/n,x,y)\,dy \leq \frac{\varepsilon}{2}.$$

Consequently, since $\varepsilon > 0$ has been chosen arbitrary,

$$\lim_{n\to\infty} J_n^\varphi(t,x_0) = 0 \quad \text{uniformly with respect to } x_0 \in \Upsilon,\ t \in (0,t^*].$$

Therefore, the following statement is proved.

Proposition 5.3.1. *Under all assumptions of this Section, the following Feynman formula holds for the semigroup $(T_t^o)_{t\geq 0}$:*

$$T_t^o\varphi(x_0)=\lim_{n\to\infty}\Theta_n^\varphi(t,x_0)=\lim_{n\to\infty}\int\limits_{G^n\times\mathbb{R}^{nd}}\varphi(x_n)\left(\prod_{k=1}^n(4\pi t/n)^{-d/2}(\det A(x_{k-1}))^{-1/2}\right)\times$$
$$\times\exp\left\{-\sum_{k=1}^n\frac{A^{-1}(x_{k-1})(z_k-x_{k-1}+tB(x_{k-1})+x_k)\cdot(z_k-x_{k-1}+tB(x_{k-1})+x_k)}{4t/n}\right\}\times$$
$$\times\exp\left\{-\frac{t}{n}\sum_{k=1}^nC(x_{k-1})\right\}dx_1\cdots dx_n\eta_{t/n}(dz_1)\cdots\eta_{t/n}(dz_n),\quad\forall\,\varphi\in Y,\ \forall\,x_0\in G.\tag{5.3.5}$$

And the convergence in this Feynman formula is locally uniform with respect to $x_0\in G$ and uniform with respect to $t\in(0,t^*]$ for all $t^*>0$.

As a particular case, we have the following result.

Corollary 5.3.2. *Let all the assumptions of this Section be fulfilled. Let $N(dy)\equiv 0$, i.e. $\eta_t=\delta_0$ for all $t\geq 0$. Then the family $(F(t))_{t\geq 0}$ given in (5.3.1) coincides with the family in formula (2.3.12) (and in formula (3.2.9)), i.e. $(F(t))_{t\geq 0}$ has the following view: $F(0):=\mathrm{Id}$ and for all $t>0$ and all $\varphi\in X$*

$$F(t)\varphi(x):=\frac{e^{-tC(x)}}{\sqrt{(4\pi t)^d\det A(x)}}\int\limits_{\mathbb{R}^d}e^{-\frac{A^{-1}(x)(x-tB(x)-y)\cdot(x-tB(x)-y)}{4t}}\varphi(y)dy$$
$$\equiv e^{-tC(x)}\int\limits_{\mathbb{R}^d}e^{\frac{A^{-1}(x)B(x)\cdot(x-y)}{2}}e^{-t\frac{|A^{-1/2}(x)B(x)|^2}{4}}\varphi(y)p_A(t,x,y)dy,$$

where $p_A(t,x,y)$ is given for all $x,y\in\mathbb{R}^d$ by formula (2.3.5). This family $(F(t))_{t\geq 0}$ is a strongly continuous family of contractions on X which is Chernoff equivalent to the semigroup $(T_t)_{t\geq 0}$. Moreover, the corresponding semigroup $(T_t^o)_{t\geq 0}$ can be approximated via the following Feynman formula:

$$T_t^o\varphi(x_0)=\lim_{n\to\infty}\int\limits_{G^n}\exp\left(-\frac{t}{n}\sum_{j=1}^n\left(C(x_{j-1})+\frac14\left|A^{-1/2}(x_{j-1})B(x_{j-1})\right|^2\right)\right)$$
$$\times\exp\left(-\frac12\sum_{j=1}^nA^{-1}(x_{j-1})B(x_{j-1})\cdot(x_j-x_{j-1})\right)\varphi(x_n)$$
$$\times p_A(t/n,x_0,x_1)\cdots p_A(t/n,x_{n-1},x_n)dx_1\dots dx_n,\tag{5.3.6}$$

for each $\varphi\in Y$, each $x_0\in G$ and each $t>0$. The convergence in the Feynman formula (5.3.6) is locally uniform with respect to $x_0\in G$ and uniform with respect to $t\in(0,t^*]$ for all $t^*>0$.

Remark 5.3.3. It is assumed in this Subsection that $C_c^\infty(\mathbb{R}^d)$ is a core for L. If L is given by formula (2.3.9) (with $a\equiv 1$) with continuous and bounded coefficients A, B, C, then $C_c^\infty(\mathbb{R}^d)\subset C_c^{2,\alpha}(\mathbb{R}^d)\subset\mathrm{Dom}(L)$ and hence $C_c^{2,\alpha}(\mathbb{R}^d)$ is also a core for L. Therefore, one may consider $D:=C_c^{2,\alpha}(\mathbb{R}^d)$ (or any other bigger core for L)

and find corresponding D_o and $\mathcal{E}$ such that Assumption 5.2.1 holds. In principle, the bigger D is chosen, the easier is to find $\mathcal{E}$ satisfying the condition (v) of Assumption 5.2.1 that $\mathcal{E}(D_o) \subset D$. As it follows from the proof of Theorem 2.3.5, the family $(F(t))_{t\geq 0}$ given in (2.3.12) satisfies the condition

$$\lim_{t\to 0}\left\|\frac{F(t)\varphi-\varphi}{t}-L\varphi\right\|_X = 0 \quad \text{for all } \varphi \in C_c^{2,\alpha}(\mathbb{R}^d),\ \alpha\in(0,1).$$

Therefore, it is sufficient to assume that the set $C_c^{2,\alpha}(\mathbb{R}^d)$ is a core for the operator L in Corollary 5.3.2.

Remark 5.3.4. Under assumptions of Corollary 5.3.2, the function $f(t,x) := T_t^o f_0(x)$ (with $(T_t^o)_{t\geq 0}$ from Corollary 5.3.2) solves the Cauchy–Dirichlet initial-boundary value problem (5.1.4) with the operator L given by formula (2.3.9) (with $a\equiv 1$) and $f_0 \in \mathrm{Dom}(L_o)$. The Feynman formula (5.3.6) holds for each $f_0 \in Y$. Generally, the function $f(t,x) := T_t^o f_0(x)$, $f_0 \in Y$, is only a mild (but not classical) solution to the Cauchy–Dirichlet problem (5.1.4) because it might not be differentiable. However, under assumptions of Corollary 5.3.2, $(T_t^o)_{t\geq 0}$ is analytic due to Theorem D.0.9. Hence $T_t^o f_0$ is differentiable for all $f_0 \in Y$. Therefore, we can represent a classical solution to the Cauchy–Dirichlet problem (5.1.4) for the second order parabolic equation with L given by (2.3.12) by formula (5.3.6) even for all $f_0 \in Y = C_0(G)$.

Remark 5.3.5. Let $f_0 \in D(L_o)$ and $0 < T < \infty$. A (local in time) solution of the the Cauchy–Dirichlet initial-boundary value problem (5.1.4) for the second order parabolic equation with L given by (2.3.9) (with $a\equiv 1$) can be represented by the Feynman–Kac formula (see Zhang and Jiang, 2001, Lem. 3.1, Thm. 3.3, cf. Pinsky, 1995, Thm. 3.4.1.):

$$f(t,x) = \mathbb{E}^x\left[\exp\left(-\int_0^t C(\xi_s)ds\right) f_0(\xi_t)1_{\{t<\tau_G\}}\right], \quad x\in G,\ t\in(0,T), \tag{5.3.7}$$

where $\mathbb{E}^x$ is the law of the (starting in $x\in G$) diffusion process $(\xi_t)_{t\in[0,T]}$ with the variable diffusion matrix A and drift $-B$, τ_G is the first exit time of $(\xi_t)_{t\in[0,T]}$ from G. Hence, finite-dimensional integrals in the Feynman formula (5.3.6) give us an approximation to the functional integral in the Feynman–Kac formula (5.3.7) and this approximation contains only integrals of elementary functions and does not contain transitional densities of the diffusion process.

Chapter 6

Applications

This Chapter illustrates a range of applications of the results of Chapters 2 — 5 and presents some explicit formulae, suitable for numerical methods. In Sections 6.1, 6.2, we apply the results of Chapters 2, 4, namely, the technique of Chernoff approximation for semigroups generated by subordination as well as by additive and multiplicative perturbations (of generators) of some original semigroups, in order to approximate solutions of some evolution equations on different geometrical structures. Further, Section 6.3 demonstrates how the Chernoff approximations obtained in Chapters 2 — 5 and in Sections 6.1, 6.2 of the present Chapter can be used for constructing some approximations for solutions of time-fractional evolution equations.

6.1 Chernoff approximations for some subordinate diffusions in a star graph

In the recent years, there was a growing interest in metric graphs because of their wide range of important applications (see, e.g., Exner et al., 2008, Berkolaiko and Kuchment, 2013 and references therein). The simplest metric graphs are *star* or *single vertex graphs*. We consider a star graph Γ with vertex v and $d \in \mathbb{N}$ external edges $l_1, \dots, l_d$. Let ρ be the metric on Γ induced by the isomorphism $l_k \cong [0, +\infty)$. Thus (Γ, ρ) is a locally compact metric space. Consider also $\Gamma^o := \Gamma \setminus \{v\} = \sqcup_{k=1}^d l_k^o$, where $l_k^o \cong (0, +\infty)$. Each point $\chi \in \Gamma^o$ is in one-to-one correspondence with its local coordinates (k, x), where $k \in \{1, \dots, d\}$ is the index of the edge χ belongs to, $x = \rho(\chi, v) > 0$. For each function $\varphi : \Gamma \to \mathbb{C}$ denote its restriction to the edge l_k^o by $\varphi_k(x) := \varphi(\chi)\big|_{\chi \in l_k^o}$. Define $\int\limits_\Gamma \varphi(\chi)d\chi := \sum\limits_{k=1}^d \int\limits_0^\infty \varphi_k(x)dx$. Let $C_\infty(\Gamma)$ be the Banach space of continuous functions on Γ vanishing at infinity equipped with the sup-norm $\|\cdot\|_\infty$. Consider the set

$$C_\infty^2(\Gamma) := \left\{\varphi \in C_\infty(\Gamma) \quad \middle| \quad \varphi \in C_\infty^2(\Gamma^o), \varphi'' \text{ extends to } \Gamma \text{ as a function in } C_\infty(\Gamma)\right\}.$$

It holds for each $\varphi \in C_\infty^2(\Gamma)$ due to Lemma 1.3 in Kostrykin, Potthoff, and Schrader, 2012a: φ' vanishes at infinity and the limits $\varphi_k'(0) := \lim_{\chi \to v,\, \chi \in l_k^o} \varphi'(\chi)$

exist for all $k \in \{1, \dots, d\}$. However, we have in general $\varphi'_k(0) \neq \varphi'_j(0)$ for $k \neq j$. In contrast, it holds $\lim_{\chi \to v,\, \chi \in l_k^o} \varphi''(\chi) = \varphi''(v)$ for all $k \in \{1, \dots, d\}$.

Let δ_v be the Dirac delta-measure concentrated at the vertex v. Let 1_k be the indicator of the edge l_k^o for each $k \in \{1, \dots, d\}$, i.e. $1_k(\chi) = 1$ if $\chi \in l_k^o$, and $1_k(\chi) = 0$ if $\chi \notin l_k^o$. Consider also the Gaussian kernel $g(t,z) = (2\pi t)^{-1/2} \exp\left(\frac{-z^2}{2t}\right)$. Define

$$\rho_v(\chi, \zeta) := \rho(\chi, v) + \rho(v, \zeta)$$

for all $\chi, \zeta \in \Gamma$. Then $\rho_v(\chi, \zeta)$ is the distance between χ and ζ via the vertex v; and if χ and ζ do not belong to the same edge, then $\rho_v(\chi, \zeta) = \rho(\chi, \zeta)$. Define the following kernels on Γ for $t > 0$, $\chi, \zeta \in \Gamma$

$$p(t, \chi, \zeta) := \sum_{k=1}^{d} 1_k(\chi) 1_k(\zeta) g(t, \rho(\chi, \zeta)),$$
$$p_v(t, \chi, \zeta) := \sum_{k=1}^{d} 1_k(\chi) 1_k(\zeta) g(t, \rho_v(\chi, \zeta))$$

Hence in local coordinates $\chi = (k, x)$, $\zeta = (m, y)$, $x, y \geq 0$, $k, m \in \{1, \dots, d\}$, these kernels read

$$p(z, (k,x), (m,y)) = (2\pi t)^{-1/2} \exp\left(\frac{-(x-y)^2}{2t}\right) \delta_{km},$$
$$p_v(z, (k,x), (m,y)) = (2\pi t)^{-1/2} \exp\left(\frac{-(x+y)^2}{2t}\right) \delta_{km}.$$

Define also the Dirichlet kernel p^D on Γ for $t > 0$, $\chi, \zeta \in \Gamma$ by

$$p^D(t, \chi, \zeta) := p(t, \chi, \zeta) - p_v(t, \chi, \zeta).$$

It is the transition density of a strong Markov process with state space $\Gamma^o \cup \{\partial\}$ which on every edge of Γ^o is equivalent to a Brownian motion until the moment of reaching the vertex when it is killed, and ∂ denotes a cemetery state considered adjoint to Γ as an isolated point.

Let now a, c, $b_k \in [0,1]$, $k \in \{1, \dots, d\}$, $a \neq 1$ and $a + c + \sum_{k=1}^{d} b_k = 1$. We consider the (half of) Laplacian L_0 on Γ with Wentzell boundary conditions at the vertex v determined by constants a, c, b_k. Namely,

$$\mathrm{Dom}(L_0) := \left\{ \varphi \in C^2_\infty(\Gamma) \quad \middle| \quad a\varphi(v) + \frac{c}{2}\varphi''(v) = \sum_{k=1}^{d} b_k \varphi'_k(v) \right\} \subset C_\infty(\Gamma),$$
$$L_0 \varphi := \frac{1}{2}\varphi'' \quad \text{for all} \quad \varphi \in \mathrm{Dom}(L_0).$$

Due to results of Kostrykin, Potthoff, and Schrader, 2012a; Kostrykin, Potthoff, and Schrader, 2012b, the following statement is true.

Proposition 6.1.1. *The operator* $(L_0, \mathrm{Dom}(L_0))$ *is the generator of a strongly continuous semigroup* $(T_t^0)_{t\ge 0}$ *on the space* $C_\infty(\Gamma)$ *and for each* $\varphi \in C_\infty(\Gamma)$ *one has*

$$T_t^0\varphi(\chi) = \int_\Gamma \varphi(\zeta)P(t,\chi,d\zeta),$$

where the transition kernel $P(t,\chi,d\zeta)$ *is given explicitly by the following formulae:*

(i) for the case $a+c \in (0,1)$ *with* $w_k := \frac{b_k}{1-a-c}$, $\beta := \frac{a}{1-a-c}$, $\gamma := \frac{c}{1-a-c}$ *and*

$$g_{\beta,\gamma}(t,z) := \frac{1}{\gamma^2}(2\pi t)^{-1/2}\int_0^t \frac{s+\gamma z}{(t-s)^{3/2}}\exp\left(-\frac{(s+\gamma z)^2}{2\gamma^2(t-s)}\right)e^{-\beta s/\gamma}ds,$$

one has

$$P(t,\chi,d\zeta) = \tag{6.1.1}$$
$$= p^D(t,\chi,\zeta)d\zeta + \sum_{k,j=1}^d 1_k(\chi)1_j(\zeta)2w_j g_{\beta,\gamma}(t,\rho_v(\chi,\zeta))d\zeta + \gamma g_{\beta,\gamma}(t,\rho(\chi,v))\delta_v(d\zeta);$$

(ii) for the case $a+c=0$ *with* $w_k := b_k$, *one has*

$$P(t,\chi,\zeta) = p^D(t,\chi,\zeta)d\zeta + \sum_{k,j=1}^d 1_k(\chi)1_j(\zeta)2w_j g(t,\rho_v(\chi,\zeta))d\zeta; \tag{6.1.2}$$

(iii) for the case $a+c=1$ *with* $a = \frac{\beta}{1+\beta}$ *and* $c = \frac{1}{1+\beta}$, *one has*

$$P(t,\chi,\zeta) = p^D(t,\chi,\zeta)d\zeta - \left(\int_0^t e^{-\beta(t-s)}\frac{\rho(\chi,v)}{\sqrt{2\pi s^3}}\exp\left(-\frac{\rho(\chi,v)^2}{2s}\right)ds,\right)\delta_v(d\zeta). \tag{6.1.3}$$

Remark 6.1.2. The case (ii) corresponds to the so-called *Walsh process*. Starting at $\chi \in \Gamma^o$, this process moves as a Brownian motion on the edge containing χ until it hits the vertex, and then performs Brownian excursions from the vertex v into the edges l_k, $k \in \{1,\dots,d\}$, whereby the edge l_k is selected with probability w_k. The case (iii) describes the process which, starting at $\chi \in \Gamma^o$, moves as a Brownian motion on the edge containing χ until it hits the vertex; then the process is killed after an exponential holding time (independent of the Brownian motion) with the rate $\beta \ge 0$. If $\beta = 0$, then the process is simply a Brownian motion on $(0,+\infty)$ with absorption at the origin. In the case (i), the heat kernel $P(t,\chi,d\zeta)$ is the transition kernel of the process of Brownian motion on Γ constructed by killing (after an exponential holding time with the rate β at the vertex) the Walsh process with sticky vertex with stickness parameter γ (see Kostrykin, Potthoff, and Schrader, 2012a; Kostrykin, Potthoff, and Schrader, 2012b for the detailed exposition).

Consider now a function $A(\cdot) \in C(\Gamma)$ such that there exist $a_0, A_0 \in (0, +\infty)$ with $a_0 \le A(\chi) \le A_0$ for all $\chi \in \Gamma$. Then the operator $(\widehat{L}_0, \mathrm{Dom}(L_0))$, such that

$$\widehat{L}_0\varphi(\chi) = A(\chi)\varphi''(\chi)$$

for all $\varphi \in \mathrm{Dom}(L_0)$, generates a strongly continuous semigroup $(\widehat{T}_t^0)_{t\ge 0}$ on $X = C_\infty(\Gamma)$ by Theorem D.0.7. Consider the family $(\widehat{F}^0(t))_{t\ge 0}$ on X defined by

$$\widehat{F}^0(t)\varphi(\chi) := \int_\Gamma \varphi(\zeta)P(2A(\chi)t, \chi, d\zeta) \tag{6.1.4}$$

with $P(t, \chi, d\zeta)$ as in Proposition 6.1.1 (formulae (6.1.1), (6.1.2), (6.1.3)). The family $(\widehat{F}^0(t))_{t\ge 0}$ is strongly continuous and Chernoff equivalent to the semigroup $(\widehat{T}_t^0)_{t\ge 0}$ by Theorem 2.2.2, Proposition 2.2.6 and Lemma 2.2.4.

Consider a function $B : \Gamma \to \mathbb{R}$ such that $B(v) = 0$ and $B \in C_b^2(\Gamma)$, where $C_b^2(\Gamma) := \left\{\varphi \in C_b(\Gamma) \,\middle|\, \varphi \in C_b^2(\Gamma^o), \varphi'' \text{ extends to } \Gamma \text{ as a function in } C_b(\Gamma)\right\}$. Define the operator $-B\nabla$ with $\mathrm{Dom}(-B\nabla) := C_\infty^2(\Gamma)$ by

$$-B\nabla\varphi(\chi) := \begin{cases} -B(\chi)\varphi'(\chi), & \chi \in \Gamma^o, \\ 0, & \chi = v, \end{cases} \qquad \text{for all} \quad \varphi \in \mathrm{Dom}(-B\nabla).$$

Since $B \in C_b(\Gamma)$, $B(v) = 0$ and $\varphi_k'(0)$, $k \in \{1, \dots, d\}$, are bounded for each $\varphi \in C_\infty^2(\Gamma)$, it holds that $-B\nabla : \mathrm{Dom}(-B\nabla) \to C_\infty(\Gamma)$. Moreover, we have

Lemma 6.1.3. *The operator $(\widehat{L_0} - B\nabla, C_\infty^2(\Gamma))$ generates a strongly continuous contraction semigroup on X.*

Proof. First, the operator $(-B\nabla, C_\infty^2(\Gamma))$ is dissipative due to Remark A.0.14. Indeed, for each $\varphi \in C_\infty^2(\Gamma)$ take $l_{\chi_0} \in \mathcal{J}(\varphi)$ such that $l_{\chi_0} = \overline{\varphi(\chi_0)}\delta_{\chi_0}$, where $\chi_0 : |\varphi(\chi_0)| = \|\varphi\|_X$. Then

$$\begin{aligned} \mathrm{Re}\,\langle -B\nabla\varphi, l_{\chi_0}\rangle &= \mathrm{Re}\left(-\int_\Gamma B\nabla\varphi(\chi)\overline{\varphi(\chi_0)}\delta_{\chi_0}(d\chi)\right) \\ &= \begin{cases} 0, & \chi_0 = v, \\ -B(\chi_0)\,\mathrm{Re}\,(\varphi'(\chi_0)), & \chi_0 \ne v. \end{cases} \end{aligned}$$

If $\chi_0 = (k, x_0) \in l_k^o$ for some $k \in \{1, \dots, d\}$ then $\varphi'(\chi_0) = \varphi_k'(x_0) = 0$ since x_0 is a local maximum for $|\varphi_k'|$. Therefore, $\mathrm{Re}\,\langle -B\nabla\varphi, l_{\chi_0}\rangle = 0$ and hence $(-B\nabla, C_\infty^2(\Gamma))$ is dissipative. Moreover, the operator $(-B\nabla, C_\infty^2(\Gamma))$ is $\widehat{L_0}$-bounded. Indeed, using Example D.0.2, we have for each $\varphi \in \mathrm{Dom}(\widehat{L_0}) \equiv \mathrm{Dom}(L_0)$ and each $\alpha > 0$

with some $k_0 \in \{1, \dots, d\}$ and some $\beta \geq 0$

$$\begin{aligned}
\| - B\nabla\varphi\|_X &\leq \|B\|_\infty \max_{1\leq k\leq d} \sup_{x\in(0,+\infty)} |\varphi_k'(x)| \\
&= \|B\|_\infty \sup_{x\in(0,+\infty)} |\varphi_{k_0}'(x)| \leq \|B\|_\infty \left(\alpha \sup_{x\in(0,+\infty)} |\varphi_{k_0}''(x)| + \beta \sup_{x\in(0,+\infty)} |\varphi_{k_0}(x)|\right) \\
&\leq (\alpha\|B\|_\infty)\|\varphi''\|_X + (\beta\|B\|_\infty)\|\varphi\|_X.
\end{aligned}$$

Hence the operator $(\widehat{L_0} - B\nabla, C^2_\infty(\Gamma))$ generates a strongly continuous semigroup on X by Theorem D.0.4 and Corollary D.0.5 (cf., e.g., Pazy, 1983, Corollary III.3.3.). $\square$

Let us construct an analogue of the family $(S(t))_{t\geq0}$ of Lemma 2.1.6. Namely, define the family $(S_t)_{t\geq0}$ on $C_\infty(\Gamma)$ by

$$S_t\varphi(\chi) := \varphi(\chi - tB(\chi)) := \begin{cases} \varphi_k(x - tB_k(x)), & \chi = (k, x), \quad x - tB_k(x) > 0, \\ \varphi(v), & \chi = (k, x), \quad x - tB_k(x) \leq 0, \\ \varphi(v), & \chi = v. \end{cases} \tag{6.1.5}$$

Lemma 6.1.4. *Let the family $(S_t)_{t\geq0}$ be defined by (6.1.5). Then the following holds:*

(i) $S_t\varphi \in C_\infty(\Gamma)$ *for each* $\varphi \in C_\infty(\Gamma)$*;*

(ii) the family $(S_t)_{t\geq0}$ *is a strongly continuous family of linear contractions on* $X = C_\infty(\Gamma)$*;*

(iii) $\lim\limits_{t\to0} \left\|\frac{S_t\varphi-\varphi}{t} + B\nabla\varphi\right\|_X = 0$ *for all* $\varphi \in C^2_\infty(\mathbb{R}^d)$.

Proof. Obviously, S_t are linear operators and $S_0 = \mathrm{Id}$. Let us check that, for each fixed $\varphi \in X$ and each fixed $t > 0$, $S_t\varphi$ is continuous at each point $\chi_0 \in \Gamma$. Consider the case $\chi_0 = (k, x_0) \in l_k^o$, $k \in \{1, \dots, d\}$. If $x_0 - tB_k(x_0) > 0$ then there exists a neigbourhood $U(\chi_0) \subset l_k^o$ such that for all $\chi = (k, x) \in U(\chi_0)$ holds $x - tB_k(x) > 0$ due to continuity of B. Hence, for $\chi \in U(\chi_0)$, we have $S_t\varphi(\chi) = \varphi_k(x - tB_k(x)) \to \varphi_k(x_0 - tB_k(x_0)) = S_t\varphi(\chi_0)$ as $\rho(\chi, \chi_0) \to 0$. If $x_0 - tB_k(x_0) < 0$ then there exists a neigbourhood $U(\chi_0) \subset l_k^o$ such that for all $\chi = (k, x) \in U(\chi_0)$ holds $x - tB_k(x) < 0$. Hence, for $\chi \in U(\chi_0)$, we have $S_t\varphi(\chi) = \varphi(v) = S_t\varphi(\chi_0)$. If $x_0 - tB_k(x_0) = 0$ then $S_t\varphi(\chi_0) = \varphi(v)$. Fix an arbitrary $\varepsilon > 0$. Due to continuity of φ, for each $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\eta \in \Gamma$ with $\rho(\eta, v) < \delta$ holds $|\varphi(\eta) - \varphi(v)| < \varepsilon$. Since $B \in C_b(\Gamma)$, for each $\delta > 0$ there exists $\gamma > 0$ such that for all $\chi = (k, x) \in l_k^o$ with $\rho(\chi, \chi_0) < \gamma$ holds $|x - tB_k(x)| < \delta$. Then, for such χ, we have $S_t\varphi(\chi) = \varphi(\eta)$ with $\rho(\eta, v) < \delta$, where $\eta = v$ or $\eta = x - tB_k(x)$ depending on the sign of $x - tB_k(x)$. Therefore, $|S_t\varphi(\chi) - S_t\varphi(\chi_0)| < \varepsilon$. The case $\chi_0 = v$ can be considered analogously. Hence $S_t\varphi$ is a continuous on Γ function. Obviously, for $\chi = (k, x) \in \Gamma$ with $\rho(\chi, v) > t\|B\|_\infty$, we have $S_t\varphi(\chi) = \varphi_k(x - tB_k(x)) \to 0$ as $x \to \infty$ for each $\varphi \in C_\infty(\Gamma)$. And $S_t\varphi \in C_\infty(\Gamma)$, i.e. $S_t : C_\infty(\Gamma) \to C_\infty(\Gamma)$.

Obviously, $\|S_t\| \le 1$. So, let us check now that the family $(S_t)_{t\ge0}$ is strongly continuous. Consider the continuity at $t_0 = 0$. We have for each $\varphi \in C_\infty(\Gamma)$ and some $k_0 \in \{1,\dots,d\}$:

$$\|S_t\varphi - \varphi\|_X = \max_{1\le k\le d} \sup_{x\in(0,\infty)} |(S_t\varphi)_k(x) - \varphi_k(x)| = \sup_{x\in(0,\infty)} |(S_t\varphi)_{k_0}(x) - \varphi_{k_0}(x)|$$

$$= \sup\left(\sup_{x>0,\, x-tB_{k_0}(x)\le 0} |\varphi_{k_0}(0) - \varphi_{k_0}(x)|; \sup_{x>0,\, x-tB_{k_0}(x)>0} |\varphi_{k_0}(x - tB_{k_0}(x)) - \varphi_{k_0}(x)| \right).$$

Further, let $\|\varphi'\|_\infty := \sup_{\chi\in\Gamma^o} |\varphi'(\chi)|$. We have with some z_x in between x and $x - tB_{k_0}(x)$

$$\sup_{x>0,\, x-tB_{k_0}(x)>0} |\varphi_{k_0}(x - tB_{k_0}(x)) - \varphi_{k_0}(x)| \le \sup_{x>0} |\varphi'(z_x)tB_{k_0}(x)| \le t\|B\|_\infty \|\varphi'\|_\infty.$$

In the case $x - tB_{k_0}(x) \le 0$, we have $|x| \le t\|B\|_\infty$. And hence it holds with some $z_x \in (0, x)$

$$\sup_{x>0,\, x-tB_{k_0}(x)\le 0} |\varphi_{k_0}(0) - \varphi_{k_0}(x)| \le \sup_{0<x\le t\|B\|_\infty} |x\varphi'(z_x)| \le t\|B\|_\infty \|\varphi'\|_\infty.$$

Therefore, $\|S_t\varphi - \varphi\|_X \le t\|B\|_\infty\|\varphi'\|_\infty \to 0$ as $t \to 0$. The continuity at $t_0 > 0$ can be shown in a similar way. Hence the family $(S_t)_{t\ge0}$ is a strongly continuous family of cantractions on X.

Let us check finally that $\lim_{t\to0} \left\| \frac{S_t\varphi - \varphi}{t} + B\nabla\varphi \right\|_X = 0$ for all $\varphi \in C^2_\infty(\mathbb{R}^d)$. First, we have for each $\varphi \in C^2_\infty(\mathbb{R}^d)$ and for each $k \in \{1,\dots,d\}$

$$\sup_{x>0,\, x-tB_k(x)>0} \left| \frac{(S_t\varphi)_k(x) - \varphi_k(x)}{t} + B_k(x)\varphi_k'(x) \right|$$

$$= \sup_{x>0,\, x-tB_k(x)>0} \left| \frac{\varphi_k(x - tB_k(x)) - \varphi_k(x) + tB_k(x)\varphi_k'(x)}{t} \right|$$

$$\le t\|\varphi''\|_\infty \|B\|^2_\infty \to 0, \quad t \to 0.$$

Second, we have for each $\varphi \in C^2_\infty(\mathbb{R}^d)$ and for each $k \in \{1,\dots,d\}$ with some $z_x \in (0,x)$

$$\sup_{x>0,\, x-tB_k(x)\le 0}\left|\frac{(S_t\varphi)_k(x)-\varphi_k(x)}{t}+B_k(x)\varphi_k'(x)\right|$$

$$=\sup_{0<x\le tB_k(x)}\left|\frac{\varphi_k(0)-\varphi_k(x)+tB_k(x)\varphi_k'(x)}{t}\right|$$

$$=\sup_{0<x\le tB_k(x)}\left|\frac{-x\varphi'(x)+x^2\varphi''(z_x)+tB_k(x)\varphi_k'(x)}{t}\right|$$

$$\le t\|B\|^2_\infty\|\varphi''\|_\infty+\sup_{0<x\le tB_k(x)}\left(\frac{|x-tB_k(x)|}{t}\right)\|\varphi'\|_\infty$$

$$\le t\|B\|^2_\infty\|\varphi''\|_\infty+\sup_{0<x\le t\|B\|_\infty}\left(2|B_k(x)|\right)\|\varphi'\|_\infty\to 0,\quad t\to 0,$$

since $B(v)=0$ and $B\in C_b(\Gamma)$. Therefore,

$$\lim_{t\to 0}\left\|\frac{S_t\varphi-\varphi}{t}+B\nabla\varphi\right\|_X=\lim_{t\to 0}\max_{1\le k\le d}\sup_{x\in(0,\infty)}\left|\frac{(S_t\varphi)_k(x)-\varphi_k(x)}{t}+B_k(x)\varphi_k'(x)\right|=0.$$

□

Consider now $C\in C_b(\Gamma)$ such that $C(\chi)\ge 0$ for all $\chi\in\Gamma$. Let as before a, c, $b_k\in[0,1]$, $k\in\{1,\dots,d\}$, $a\ne 1$ and $a+c+\sum_{k=1}^d b_k=1$. Consider the operator L in the space $X=C_\infty(\Gamma)$ defined by

$$\mathrm{Dom}(L):=\left\{\varphi\in C^2_\infty(\Gamma)\ \middle|\ a\varphi(v)+\frac{c}{2}\varphi''(v)=\sum_{k=1}^d b_k\varphi_k'(v)\right\},$$

$$L\varphi(\chi):=A(\chi)\varphi''(\chi)-B\nabla\varphi(\chi)-C(\chi)\varphi(\chi)\quad\text{for all}\quad\varphi\in\mathrm{Dom}(L).\tag{6.1.6}$$

Then the operator $(L,\mathrm{Dom}(L))$ is the generator of a strongly continuous contraction semigroup $(T_t)_{t\ge0}$ on the space X by Lemma 6.1.3 and Corollary D.0.5 as a bounded additive perturbation of $\widehat{L_0}-B\nabla$.

Let now the semigroup $(T^f_t)_{t\ge0}$ be subordinate to the semigroup $(T_t)_{t\ge0}$ with respect to a given convolution semigroup $(\eta_t)_{t\ge0}$ associated to a Bernstein function f defined by a triplet (σ,λ,μ). The statement (i) of Theorem 6.1.5 below follows immediately from Lemma 6.1.4, Proposition 6.1.1 and Corollary 2.1.2. The statements (ii) and (iii) follow from Theorem 4.2.1 and Theorem 4.3.1 respectively.

Theorem 6.1.5. *(i) Let $A(\cdot) \in C(\Gamma)$ be such that there exist $a_0, A_0 \in (0, +\infty)$ with $a_0 \leq A(\chi) \leq A_0$ for all $\chi \in \Gamma$. Let $B : \Gamma \to \mathbb{R}$ be such that $B(v) = 0$ and $B \in C_b^2(\Gamma)$. Let $C \in C_b(\Gamma)$ and $C(\chi) \geq 0$ for all $\chi \in \Gamma$. Let $(T_t)_{t\geq 0}$ be the strongly continuous contraction semigroup generated by $(L, \mathrm{Dom}(L))$ which is given by (6.1.6). Consider the family $(F(t))_{t\geq 0}$ of bounded linear contrations on $X = C_\infty(\Gamma)$ defined by*

$$F(t) := e^{-tC} \circ S_t \circ \widehat{F}^0(t) \quad \forall\, t \geq 0; \quad \textit{i.e. for all} \quad \varphi \in X$$

$$F(t)\varphi(\chi) := e^{-tC(\chi)} \int\limits_{\Gamma} \varphi(\zeta) P(2A(\chi - tB(\chi))t, \chi - tB(\chi), d\zeta), \quad \forall \chi \in \Gamma, \tag{6.1.7}$$

where the operators $\widehat{F}^0(t)$ and S_t are given by (6.1.4) and (6.1.5) respectively. Then the family $(F(t))_{t\geq 0}$ is strongly continuous and Chernoff equivalent to the semigroup $(T_t)_{t\geq 0}$.

(ii) Under assumptions and notations of Theorem 4.2.1, the family $(\mathcal{F}(t))_{t\geq 0}$, constructed with the help of $(F(t))_{t\geq 0}$ given by (6.1.7), is Chernoff equivalent to the semigroup $(T_t^f)_{t\geq 0}$. The operators $\mathcal{F}(t)$ have the following explicit view:

$$\mathcal{F}(0) := \mathrm{Id}$$

and and for all $t > 0$, $\varphi \in X$, $\chi \in \Gamma$

$$\mathcal{F}(t)\varphi(\chi) := e^{-\sigma t} \int\limits_{\Gamma} \int\limits_{0+}^{\infty} \int\limits_{\Gamma} \cdots \int\limits_{\Gamma} \exp\left(-\lambda t C(\chi) - \frac{s}{m(t)} \sum_{k=1}^{m(t)} C(\chi_{k+1}) \right) \varphi(\chi_1)$$

$$\times \prod_{k=1}^{m(t)} P\big(2A(\chi_{k+1} - (s/m(t))B(\chi_{k+1}))s/m(t), \chi_{k+1} - (s/m(t))B(\chi_{k+1}), d\chi_k\big)$$

$$\times\, \eta_t^0(ds)\, P\big(2A(\chi - t\lambda B(\chi))t\lambda, \chi - t\lambda B(\chi), d\chi_{m(t)+1}\big).$$

(iii) Under assumptions and notations of Theorem 4.3.1, the family $(\mathcal{F}_\mu(t))_{t\geq 0}$, constructed with the help of $(F(t))_{t\geq 0}$ given by (6.1.7), is Chernoff equivalent to the semigroup $(T_t^f)_{t\geq 0}$. The operators $\mathcal{F}_\mu(t)$ have the following explicit view: $\mathcal{F}_\mu(0) := \mathrm{Id}$ and

for all $t > 0$, $\varphi \in X$, $\chi \in \Gamma$

$$\mathcal{F}_\mu(t)\varphi(\chi) :=$$

$$= e^{-\sigma t - \lambda t C(\chi)} \int_\Gamma \Bigg[\varphi(\chi_{m(t)+1}) + t \int_{0+}^{\infty} \Bigg(\int_\Gamma \cdots \int_\Gamma \exp\Bigg(- \frac{s}{m(t)} \sum_{k=1}^{m(t)} C(\chi_{k+1}) \Bigg) \varphi(\chi_1)$$

$$\times \prod_{k=1}^{m(t)} P\big(2A(\chi_{k+1} - (s/m(t))B(\chi_{k+1}))s/m(t), \chi_{k+1} - (s/m(t))B(\chi_{k+1}), d\chi_k\big)$$

$$- \varphi(\chi_{m(t)+1}) \Bigg) \mu(ds) \Bigg] P\big(2A(\chi - t\lambda B(\chi))t\lambda, \chi - t\lambda B(\chi), d\chi_{m(t)+1}\big).$$

6.2 Chernoff approximation of subordinate diffusions in a Riemannian manifold

Theory of diffusions on manifolds is a classical topic with many contributions (see works of H. Airault, S. Albeverio, V. I. Bogachev, Z. Brzeźniak, Ya. I. Belopolskaya, A. B. Cruzeiro, Yu. L. Daletskii, R. W. R. Darling, B. K. Driver, K. D. Elworthy, M. Emery, S. Fang, A. A. Grigor'yan, L. Gross, E. P. Hsu, N. Ikeda, K. Itô, W. S. Kendall, W. S. H. Kunita, P. Malliavin, P. A. Meyer, J. R. Norris, M. Röckner, L. Schwartz, I. Shigekawa, D. W. Stroock, A. Thalmaier, Sh. Watanabe, K. Yosida and many many others; some expositions can be found, e.g., in Hackenbroch and Thalmaier, 1994; Driver, 2004; Émery, 1989; Elworthy, 1982; Ikeda and Watanabe, 1989).

Let Γ be a compact connected Riemannian manifold of class C^∞ without boundary, $\dim\Gamma = d$. Let ρ_Γ be the distance in Γ generated by the Riemannian metric of Γ. Let vol_Γ be the corresponding Riemannian volume measure on Γ. Assume also that Γ is isometrically embedded into a Riemannian manifold G (of some higher dimension) and into some Euclidean space $\mathbb{R}^D$, Φ is a C^∞-smooth isometric embedding of Γ into $\mathbb{R}^D$ and Φ_G is a C^∞-smooth isometric embedding of Γ into G. Let ρ_G be the distance in G generated by the Riemannian metric of G. Consider (a non-positive version of) the Laplace–Beltrami operator Δ_Γ, $\Delta_\Gamma\varphi := \mathrm{div}\,\mathrm{grad}\,\varphi$, $\varphi \in C^2(\Gamma)$. The closure of $(\Delta_\Gamma, C^3(\Gamma))$ generates the *heat semigroup*, i.e. the strongly continuous contraction semigroup $(e^{\frac{t}{2}\Delta_\Gamma})_{t\geq 0}$ on the space $C(\Gamma)$. Due to Section 5 of Smolyanov, Weizsäcker, and Wittich, 2007b, the following is true.

Proposition 6.2.1. *For all $t > 0$ and all $x, y \in \Gamma$ consider the following (pseudo-) Gaussian kernels*

$$K_1(t,x,y) := (2\pi t)^{-d/2} \exp\left(-\frac{\rho_\Gamma(x,y)^2}{2t}\right),$$
$$K_2(t,x,y) := (2\pi t)^{-d/2} \exp\left(-\frac{\rho_G(\Phi_G(x),\Phi_G(y))^2}{2t}\right),$$
$$K_3(t,x,y) := (2\pi t)^{-d/2} \exp\left(-\frac{|\Phi(x)-\Phi(y)|^2}{2t}\right).$$

For each kernel K_i, $i = 1,2,3$, define the family $(F_i^0(t))_{t\geq 0}$, $i = 1,2,3$, of contractions on $C(\Gamma)$ by

$$F_i^0(0) := \mathrm{Id} \quad \textit{and for each} \quad \varphi \in C(\Gamma) \quad \textit{and each} \quad t > 0$$
$$F_i^0(t)\varphi(x) := \frac{\int_\Gamma K_i(t,x,y)\varphi(y)\,\mathrm{vol}_\Gamma(dy)}{\int_\Gamma K_i(t,x,y)\,\mathrm{vol}_\Gamma(dy)}. \tag{6.2.1}$$

Then each family $(F_i^0(t))_{t\geq 0}$, $i = 1,2,3$, is strongly continuous and Chernoff equivalent to the heat semigroup $(e^{\frac{t}{2}\Delta_\Gamma})_{t\geq 0}$ on the space $C(\Gamma)$ with $\lim_{t\to 0}\left\|\frac{F_i^0(t)\varphi-\varphi}{t} - \frac{1}{2}\Delta_\Gamma\varphi\right\|_\infty = 0$ for all $\varphi \in C^3(\Gamma)$.

Remark 6.2.2. Instead of operators $F_i^0(t)$, one may consider just integral operators with kernels K_i composed with a proper function of the form $e^{-tf_i(x)}$ (for certain functions f_i containing geometrical characteristics of Γ and of the ambient space G or $\mathbb{R}^D$) which compensates the normalization procedure in the construction of $F_i^0(t)$ (i.e. $e^{tf_i(x)}$ is the leading term in the short-time asymptotics of $\int_\Gamma K_i(t,x,y)\,\mathrm{vol}_\Gamma(dy)$). In this case, one obtains families of integral operators which are also Chernoff equivalent to the heat semigroup $(e^{\frac{t}{2}\Delta_\Gamma})_{t\geq 0}$. Note, however, that these operators do not need to be contractions any more. For example, using the kernel K_3, one obtains in this way the following family $(F_4^0(t))_{t\geq 0}$ (see Cor. 6 in Smolyanov, Weizsäcker, and Wittich, 2007b):

$$F_4^0(t)\varphi(x) := e^{\frac{t}{4}\mathrm{scal}(x) - \frac{t}{8}|\tau_\Phi(x)|^2} \int_\Gamma K_3(t,x,y)\varphi(y)\,\mathrm{vol}_\Gamma(dy), \tag{6.2.2}$$

where scal is the scalar curvature of Γ and τ_Φ is the tension vector field of the embedding Φ. Some further examples of such families can be found in Section 5 of Smolyanov, Weizsäcker, and Wittich, 2007b.

Consider now a function $a(\cdot) \in C(\Gamma)$ such that there exist $a_0, A_0 \in (0,+\infty)$ with $a_0 \leq a(x) \leq A_0$ for all $x \in \Gamma$. Then the closure of the operator $(\widehat{L}_0, C^3(\Gamma))$, such that

$$\widehat{L}_0\varphi(x) = a(x)\Delta_\Gamma\varphi(x)$$

for all $\varphi \in C^3(\Gamma)$, generates a strongly continuous semigroup $(\widehat{T}_t^0)_{t\geq 0}$ on $X =$

$C(\Gamma)$ by Theorem D.0.7. Consider the families $(\widehat{F}_i^0(t))_{t\geq 0}$, $i=1,2,3$, of contractions on X defined by

$$\widehat{F}_i^0(t)\varphi(x) := (F_i^0(2a(x)t)\varphi)(x). \tag{6.2.3}$$

Hence the families $(\widehat{F}_i^0(t))_{t\geq 0}$, $i=1,2,3$, are strongly continuous and Chernoff equivalent to the semigroup $(\widehat{T}_t^0)_{t\geq 0}$ by Theorem 2.2.2 and Lemma 2.2.4.

Let now $B(\cdot):\Gamma\to T\Gamma$ be a bounded vector field of class $C^2(\Gamma)$. Denote the inner product (defined by the Riemannian metric) of vectors $u(x)$ and $v(x)$ in the tangent space $T_x\Gamma$ as $u(x)\cdot v(x)$. For each point $x\in\Gamma$ let $\gamma^x(\cdot):[0,\infty)\to\Gamma$ be a geodesic with the starting point x (i.e. $\gamma^x(0)=x$) and the direction vector $-B(x)$ (i.e. $\dot\gamma^x(0)=-B(x)$). This geodesic is uniquely defined and depends smoothly on x and $B(x)$ (e.g., due to Thm. 1.4.2 and Thm. 1.4.7 in Jost, 1998[1]). Define the family $(S_t)_{t\geq 0}$ on $C(\Gamma)$ by

$$S_t\varphi(x) := \varphi(\gamma^x(t)). \tag{6.2.4}$$

Since the manifold is smooth and compact, and the vector field B is smooth, the family $(S_t)_{t\geq 0}$ is well defined as a family of strongly continuous contractions on $C(\Gamma)$ and it holds, in particular,

$$\lim_{t\to 0}\left\|t^{-1}(S_t\varphi-\varphi)+B\cdot\nabla_\Gamma\varphi\right\|_\infty = 0$$

for all $\varphi\in C^3(\Gamma)$. Here $\nabla_\Gamma\varphi$ is the gradient of f.

Further, let $C(\cdot)\in C(\Gamma)$ be a nonnegative function. Consider the operator L defined on the set $C^3(\Gamma)$ by

$$L\varphi(x) := a(x)\Delta_\Gamma\varphi(x) - B(x)\cdot\nabla_\Gamma\varphi(x) - C(x)\varphi(x). \tag{6.2.5}$$

Using the similar argumentation as in Section 6.1, one can show that the closure of $(L, C^3(\Gamma))$ generates a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ on $C(\Gamma)$. Let now the semigroup $(T_t^f)_{t\geq 0}$ be subordinate to the semigroup $(T_t)_{t\geq 0}$ with respect to a given convolution semigroup $(\eta_t)_{t\geq 0}$ associated to a Bernstein function f defined by a triplet (σ,λ,μ). The statement below follows immediately from Proposition 6.2.1, Theorems 2.2.2, 4.2.1, 4.3.1, Lemma 2.2.4 and Corollary 2.1.2.

Theorem 6.2.3. *Let a function $a(\cdot)\in C(\Gamma)$ be such that there exist $a_0, A_0\in(0,+\infty)$ with $a_0\leq a(x)\leq A_0$ for all $x\in\Gamma$, let $B(\cdot):\Gamma\to T\Gamma$ be a bounded vector field of class $C^2(\Gamma)$ and let $C(\cdot)\in C(\Gamma)$ be a nonnegative function. Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup on $X=C(\Gamma)$ generated by the closure of the operator $(L, C^3(\Gamma))$ given by* (6.2.5).

[1]Note that $\gamma^x(t)=\exp_x(-tB(x))$ in the normal coordinates with center x on Γ (cf. Def. 1.4.4 in Jost, 1998).

(i) Define the families $(F_i(t))_{t\ge0}$, $i = 1,2,3$, *by*

$$F_i(t)\varphi(x) := \left(e^{-tC}\circ S_t\circ \widehat{F}_i^0(t)\right)\varphi(x) = \frac{\int_\Gamma e^{-tC(x)}K_i(2a(\gamma^x(t))t,\gamma^x(t),y)\varphi(y)\,\mathrm{vol}_\Gamma(dy)}{\int_\Gamma K_i(2a(\gamma^x(t))t,\gamma^x(t),y)\,\mathrm{vol}_\Gamma(dy)},$$

where the families $(F_i^0(t))_{t\ge0}$ *and* $(S_t)_{t\ge0}$ *are given by* (6.2.1) *and* (6.2.4) *respectively. Then the families* $(F_i(t))_{t\ge0}$, $i = 1,2,3$, *are strongly continuous and Chernoff equivalent to the semigroup* $(T_t)_{t\ge0}$ *on* X.

(ii) Under assumptions and notations of Theorem 4.2.1 *the families* $(\mathcal{F}^i(t))_{t\ge0}$, $k = 1,2,3$, *constructed as in Theorem* 4.2.1 *with the help of* $(F_i(t))_{t\ge0}$ *given above, are Chernoff equivalent to the semigroup* $(T_t^f)_{t\ge0}$ *and have the following explicit view:* $\mathcal{F}^i(0) := \mathrm{Id}$ *and for all* $t>0$, $\varphi\in X$, $x\in\Gamma$

$$\begin{aligned}\mathcal{F}^i(t)\varphi(x) &= e^{-\sigma t}\int\limits_\Gamma\int\limits_{0+}^{\infty}\int\limits_{\Gamma^{m(t)}}\exp\left(-\lambda tC(x)-\frac{s}{m(t)}\sum_{k=1}^{m(t)}C(x_{k+1})\right)\varphi(x_1)\\
&\times\prod_{k=1}^{m(t)}\frac{K_i\big(2a(\gamma^{x_{k+1}}(s/m(t)))s/m(t),\ \gamma^{x_{k+1}}(s/m(t)),\ x_k\big)}{\int_\Gamma K_i\big(2a(\gamma^{x_{k+1}}(s/m(t)))s/m(t),\ \gamma^{x_{k+1}}(s/m(t)),\ x_k\big)\,\mathrm{vol}_\Gamma(dx_k)}\prod_{k=1}^{m(t)}\mathrm{vol}_\Gamma(dx_k)\\
&\times\eta_t^0(ds)\times\frac{K_i\big(2a(\gamma^x(\lambda t))\lambda t,\ \gamma^x(\lambda t),\ x_{m(t)+1}\big)}{\int_\Gamma K_i\big(2a(\gamma^x(\lambda t))\lambda t,\ \gamma^x(\lambda t),\ x_{m(t)+1}\big)\,\mathrm{vol}_\Gamma(dx_{m(t)+1})}\mathrm{vol}_\Gamma(dx_{m(t)+1}).\end{aligned}$$

(iii) Under assumptions and notations of Theorem 4.3.1 *tha families* $(\mathcal{F}_\mu^i(t))_{t\ge0}$, $k = 1,2,3$, *constructed as in Theorem* 4.2.1 *with the help of* $(F_i(t))_{t\ge0}$ *given above, are Chernoff equivalent to the semigroup* $(T_t^f)_{t\ge0}$ *and have the following explicit view:* $\mathcal{F}_\mu^i(0) := \mathrm{Id}$ *and for all* $t>0$, $\varphi\in X$, $x\in\Gamma$

$$\begin{aligned}&\mathcal{F}_\mu^i(t)\varphi(x)\\
&= e^{-\sigma t-\lambda tC(x)}\int\limits_\Gamma\Bigg(\varphi(x_{m(t)+1})+t\int\limits_{0+}^{\infty}\Bigg[\int\limits_{\Gamma^{m(t)}}\exp\left(-\frac{s}{m(t)}\sum_{k=1}^{m(t)}C(x_{k+1})\right)\varphi(x_1)\\
&\times\prod_{k=1}^{m(t)}\frac{K_i\big(2a(\gamma^{x_{k+1}}(s/m(t)))s/m(t),\ \gamma^{x_{k+1}}(s/m(t)),\ x_k\big)}{\int_\Gamma K_i\big(2a(\gamma^{x_{k+1}}(s/m(t)))s/m(t),\ \gamma^{x_{k+1}}(s/m(t)),\ x_k\big)\,\mathrm{vol}_\Gamma(dx_k)}\prod_{k=1}^{m(t)}\mathrm{vol}_\Gamma(dx_k)\\
&-\varphi(x_{m(t)+1})\Bigg]\mu(ds)\Bigg)\frac{K_i\big(2a(\gamma^x(\lambda t))\lambda t,\ \gamma^x(\lambda t),\ x_{m(t)+1}\big)\,\mathrm{vol}_\Gamma(dx_{m(t)+1})}{\int_\Gamma K_i\big(2a(\gamma^x(\lambda t))\lambda t,\ \gamma^x(\lambda t),\ x_{m(t)+1}\big)\,\mathrm{vol}_\Gamma(dx_{m(t)+1})}.\end{aligned}$$

Remark 6.2.4. By Theorem 6.2.3 and Remark 6.2.2, the solution $f(t,x)$ of the evolution equation $\frac{\partial f}{\partial t}(t,x) = Lf(t,x)$ with L given by (6.2.5) and an initial condition $\varphi\in\mathrm{Dom}(L)$ can be found through the Chernoff approximations

$$f(t,x_0) = T_t\varphi(x_0) = \lim_{n\to\infty}\left[F_i(t/n)\right]^n\varphi(x_0),\quad i=1,2,3,4,\quad x_0\in\Gamma, \tag{6.2.6}$$

with $F_i(t) := e^{-tC} \circ S_t \circ \widehat{F}_i^0(t)$, $i = 1,2,3,4$. Consider the case $a(x) \equiv \frac{1}{2}$ for all $x \in \Gamma$. Using the Chernoff approximation (6.2.6) with $i = 4$ and the same strategy as in Example 2.1.10, one can show (cf. Butko, 2008) that the limit in the right hand side of (6.2.6) coincides with the following Feynman formula

$$\begin{aligned} f(t,x_0) = \lim_{n\to\infty} \int\limits_{\Gamma^n} & e^{-\frac{t}{n}\sum_{k=1}^n C(x_{k-1})} e^{\frac{t}{4n}\sum_{k=1}^n \mathrm{scal}(x_{k-1}) - \frac{t}{8n}\sum_{k=1}^n |\tau_\Phi(x_{k-1})|^2} \\ & \times e^{\sum_{k=1}^n B_\Phi(x_{k-1})\cdot(\Phi(x_{k-1})-\Phi(x_k))} e^{-\frac{t}{2n}\sum_{k=1}^n |B_\Phi(x_{k-1})|^2} \varphi(x_n) \qquad (6.2.7) \\ & \times K_3(t/n, x_0, x_1)\dots K_3(t/n, x_{n-1}, x_n)\,\mathrm{vol}_\Gamma(dx_1)\dots\mathrm{vol}_\Gamma(x_n), \end{aligned}$$

where, for each $x \in \Gamma$, the vector $-B_\Phi(x) \in \mathbb{R}^D$ is the direction vector of the smooth curve $\Phi(\gamma^x(\cdot)) : [0,\infty) \to \mathbb{R}^D$ at the point 0. The prelimit expressions in the right hand side of (6.2.7), in turn, can be interpreted (cf. Smolyanov, Weizsäcker, and Wittich, 2007b; Butko, 2008) as the following path integral with respect to the Wiener measure corresponding to a Brownian motion $(\xi_t)_{t\geq 0}$ in a Riemannian manifold Γ starting at the point $x_0 \in \Gamma$ [2]:

$$f(t,x_0) = \mathbb{E}^{x_0}\left[e^{-\int_0^t \left(C(\xi_s)+\frac{1}{2}|B(\xi_s)|^2\right)ds - \int_0^t B_\Phi(\xi_s)\cdot d\xi_s^\Phi} \varphi(\xi_t)\right],$$

where $\int_0^t B_\Phi(\xi_s)\cdot d\xi_s^\Phi$ is the Itô stochastic integral and $(\xi_t^\Phi)_{t\geq 0}$ is the process in $\mathbb{R}^D$ obtained by the horizontal lifting of $(\xi_t)_{t\geq 0}$ via the isometry Φ. For the stochastic derivation of this Feynman-Kac formula see Chapter IX, §11 of Elworthy, 1982.

6.3 Approximation of solutions of distributed order time-fractional evolution equations

Fractional derivatives are natural extensions of their integer-order analogues (see, e.g., Miller and Ross, 1993, Samko, Kilbas, and Marichev, 1993). Evolution equations with partial derivatives of fractional order (fractional evolution equations), in particular, time- (and, possibly, space-) fractional diffusion equations (modelling anomalous diffusion) have been applied to problems in physics, chemistry, biology, medicine, finance, hydrology and other areas (see, e.g., Meerschaert and Sikorskii, 2012, Metzler and Klafter, 2000, Metzler and Klafter, 2004, Miller and Ross, 1993, Samko, Kilbas, and Marichev, 1993 and references therein). Many time-fractional evolution equations serve as governing equations for stochastic processes. However, the processes, whose marginal density function evolves in time according to a given time-fractional evolution equation, are usually non-Markovian (and hence, there is no semigroup structure behind the equation) and are non-uniquely defined by this marginal density function (therefore, very different stochastic representations for a solution of a given fractional evolution equation are possible, see, e.g., Baeumer et al.,

[2]Note that each compact Riemannian manifold is always martingale-complete, i.e. martingales cannot explode in finite "intrinsic time" (see, e.g., Chapter 5 of Émery, 1989).

2016, Hahn and Umarov, 2011, Mura, Taqqu, and Mainardi, 2008). The absence of the semigroup property for solutions of time-fractional evolution equations does not allow to apply the method of Chernoff approximation for such equations directly. Nevertheless, several relations exist between time-fractional and "standard" (time-non-fractional) evolution equations: via a kind of subordination (see, e.g., Prüss, 1993, Saichev and Zaslavsky, 1997, Metzler and Klafter, 2000, Baeumer and Meerschaert, 2001, Mura, Taqqu, and Mainardi, 2008) and via higher order operators (see, e.g., Orsingher and Beghin, 2004, Orsingher and Beghin, 2009, Baeumer, Meerschaert, and Nane, 2009, Orsingher and D'Ovidio, 2012, Garra, Orsingher, and Polito, 2015). These relations allow to construct some approximations for solutions of such time-fractional evolution equations via Chernoff approximations for solutions of some related "standard" evolution equations. Below we present approximations for solutions of (a class of) time-fractional evolution equations using their connection to time-non-fractional equations via a kind of subordination.

Let us introduce some needed definitions and facts about time-fractional evolution equations. First, note that there exist many different notions of fractional derivatives. We discuss only two versions of them. One defines the *Caputo* (or *Caputo-Dzhrbashyan*) *fractional derivative* of order β, $\beta \in (0,1)$, for a (sufficiently good) function u by

$$\frac{\partial^\beta}{\partial t^\beta} u(t) := \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{u'(r)}{(t-r)^\beta} dr,$$

where Γ is the Euler's Gamma-function. Let U be the Laplace transform of u, i.e. $U(s) := \int_0^\infty e^{-st} u(t)dt$. Then the Laplace transform of the Caputo derivative $\frac{\partial^\beta}{\partial t^\beta} u$ of u can be calculated as follows:

$$\int_0^\infty e^{-st} \frac{\partial^\beta}{\partial t^\beta} u(t)dt = s^\beta U(s) - s^{\beta-1} u(0+).$$

The *Riemann–Liouville fractional derivative* of order β, $\beta \in (0,1)$, for a (sufficiently good) function u is defined by

$$\left(\frac{d}{dt}\right)^\beta u(t) := \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{u(r)}{(t-r)^\beta} dr.$$

Then the Laplace transform of the Riemann-Liouville derivative $\left(\frac{d}{dt}\right)^\beta u$ of u can be calculated as follows:

$$\int_0^\infty e^{-st} \left(\frac{d}{dt}\right)^\beta u(t)dt = s^\beta U(s).$$

Comparing both Laplace transforms and taking into account that the Laplace transform of $t^{-\beta}$ is $s^{\beta-1}\Gamma(1-\beta)$, one sees that if u is absolutely continuous on bounded intervals then the Riemann-Liouville and Caputo derivatives of u are related by

$$\frac{\partial^\beta}{\partial t^\beta} u(t) = \left(\frac{d}{dt}\right)^\beta u(t) - \frac{t^{-\beta} u(0+)}{\Gamma(1-\beta)}. \tag{6.3.1}$$

The Riemann-Liouville fractional derivative is more general since it does not require the first derivative of u to exist. Therefore, one may adopt the right hand side of the formula (6.3.1) to define the Caputo derivative. The further generalization is to consider the so-called *distributed order fractional derivative* $\mathcal{D}^\mu$ with the order μ determined by a finite Borel measure μ defined on the interval $(0,1)$ and such that $\mu(0,1) > 0$ (cf. Kochubei, 2008; Mainardi et al., 2008; Meerschaert and Scheffler, 2006; Umarov and Gorenflo, 2005a):

$$\mathcal{D}^\mu u(t) := \int_0^1 \frac{\partial^\beta}{\partial t^\beta} u(t) \mu(d\beta) = \int_0^1 \left[\left(\frac{d}{dt}\right)^\beta u(t) - \frac{t^{-\beta} u(0)}{\Gamma(1-\beta)}\right] \mu(d\beta).$$

If μ is the Dirac delta-measure δ_{β_0} concentrated at a point $\beta_0 \in (0,1)$, we return to Caputo fractional derivative of β_0–th order.

We are interested now in distributed order time-fractional evolution equations of the form

$$\mathcal{D}^\mu f(t,x) = Lf(t,x), \tag{6.3.2}$$

where $\mathcal{D}^\mu$ is the distributed order fractional derivative with respect to the time variable t and L is the generator of a strongly continuous semigroup $(T_t)_{t\geq 0}$ on some Banach space $(X, \|\cdot\|_X)$ of functions of the space variable x. Equations of such type are called *time-fractional Fokker–Planck–Kolmogorov equations* and arise in the framework of continuous time random walks (CTRWs) and fractional kinetic theory (Gillis and Weiss, 1970; Montroll and Shlesinger, 1984; Metzler and Klafter, 2000; Zaslavsky, 2002). As it is shown in papers Hahn and Umarov, 2011; Hahn, Kobayashi, and Umarov, 2012; Mijena and Nane, 2014 (see also papers Gorenflo and Mainardi, 1998; Baeumer and Meerschaert, 2001; Meerschaert and Scheffler, 2008; Umarov and Gorenflo, 2005b; Meerschaert and Straka, 2013 for the case $\mu = \delta_{\beta_0}$, $\beta_0 \in (0,1)$), such time-fractional Fokker–Planck–Kolmogorov equations are governing equations for stochastic processes which are weak limits of certain sequences or triangular arrays of CTRWs. These limit processes are actually time-changed Lévy processes, where the time-change arises as the first hitting time of level $t > 0$ (or, equivalently, as the inverse process) for a mixture of independent stable subordinators with some mixing measure μ.

Recall that a process $(D_t^\beta)_{t\geq 0}$ with $\beta \in (0,1)$ is *β-stable subordinator* if it is a one-dimensional Lévy proces with almost surely non-decreasing paths such that the corresponding Bernstein function is $f(s) := s^\beta$ (see, e.g., Section 3.9 of Jacob, 2001 for the definitions). For a given finite Borel measure μ with $\operatorname{supp}\mu \in (0,1)$, consider the function f^μ given by

$$f^\mu(s) := \int\limits_0^1 s^\beta \mu(d\beta), \quad s > 0.$$

It is a Bernstein function given by (cf. formula (3.246) of Jacob, 2001):

$$f^\mu(s) = \int\limits_{0+}^\infty (1 - e^{-ts})\, m(dt), \quad \text{where} \quad m(dt) := \left(\int\limits_0^1 \frac{\beta t^{-\beta-1}}{\Gamma(1-\beta)} \mu(d\beta) \right) dt.$$

Let $(D_t^\mu)_{t\geq 0}$ be a subordinator corresponding to the Bernstein function f^μ. This process represents a mixture of independent stable subordinators with a mixing measure μ. Define now the process $(E_t^\mu)_{t\geq 0}$ by

$$E_t^\mu := \inf\{\tau \geq 0 : D_\tau^\mu > t\}.$$

The process $(E_t^\mu)_{t\geq 0}$ is the first hitting time of the level t of the process $(D_\tau^\mu)_{\tau\geq 0}$ or, equivalently, the inverse to $(D_t^\mu)_{t\geq 0}$. This process $(E_t^\mu)_{t\geq 0}$ is sometimes called *inverse subordinator*. However, note that it is not a Markov process. It is known that $(E_t^\mu)_{t\geq 0}$ possesses a marginal density function $p^\mu(t,x)$ (with respect to the Lebesgue measure dx), i.e. $\mathbb{P}(E_t^\mu \in B) = \int_B p^\mu(t,x)dx$ for all $B \in \mathcal{B}(\mathbb{R})$, and $p^\mu(t,x) = 0$ for all $x < 0$. The marginal density function $p^\mu(t,x)$ has many nice properties (see Lemma 2.4 and Lemma 2.5 in Hahn, Kobayashi, and Umarov, 2011; for the case $\mu = \delta_{\beta_0}$, $\beta_0 \in (0,1)$, see also Meerschaert and Scheffler, 2004; D'Ovidio, 2010). In particular, $p^\mu \in C^\infty((0,\infty)\times(0,\infty))$. In the sequel, we need the following simple property of p^μ:

Lemma 6.3.1. *For each* $\varepsilon > 0$ *and each* $T > 0$ *there exists* $R_{T,\varepsilon} > 0$ *such that for all* $t \in [0,T]$ *holds*

$$\int\limits_{R_{T,\varepsilon}}^\infty p^\mu(t,x)dx < \varepsilon.$$

Proof. Choose arbitrary $\varepsilon > 0$ and $T > 0$. Consider $R > 0$. We have for all $t \in [0,T]$:

$$\begin{aligned} \int\limits_R^\infty p^\mu(t,x)dx &= \mathbb{P}\left(E_t^\mu \geq R\right) = \mathbb{P}\left(D_R^\mu \leq t\right) \\ &\leq \mathbb{P}\left(D_R^\mu \leq T\right) = \mathbb{P}\left(E_T^\mu \geq R\right) = \int\limits_R^\infty p^\mu(T,x)dx \\ &< \varepsilon \end{aligned}$$

for sufficiently large R since $\int_0^\infty p^\mu(T,x)dx = 1$. □

The marginal density function $p^\mu(t,x)$ allows to represent solutions of Cauchy problems for distributed order time-fractional evolution equations of the form (6.3.2) in the following way (cf. Thm. 3.2 in Hahn, Kobayashi, and Umarov, 2012 and Thm. 4.2 in Mijena and Nane, 2014):

Proposition 6.3.2. *Let $(X, \|\cdot\|_X)$ be a Banach space. Let $(L, \mathrm{Dom}(L))$ be the generator of a uniformly bounded*[3]*, strongly continuous semigroup $(T_t)_{t\geq 0}$ on X. Let $f_0 \in \mathrm{Dom}(L)$. Let μ be a finite Borel measure with $\operatorname{supp}\mu \in (0,1)$. Then the family of linear operators $(\mathcal{T}_t)_{t\geq 0}$ from X into X given by*

$$\mathcal{T}_t\varphi := \int_0^\infty T_\tau\varphi\, p^\mu(t,\tau)\,d\tau, \quad \forall\,\varphi\in X, \tag{6.3.3}$$

is uniformly bounded and strongly analytic in a sectorial region. Furthermore, the family $(\mathcal{T}_t)_{t\geq 0}$ is strongly continuous and the function $f(t) := \mathcal{T}_t f_0$ is a solution of the Cauchy problem

$$\begin{aligned} \mathcal{D}^\mu f(t) &= Lf(t), \quad t>0,\\ f(0) &= f_0. \end{aligned} \tag{6.3.4}$$

This result shows that solutions of time-fractional evolution equations are a kind of subordination of solutions of the corresponding time-non-fractional evolution equations with respect to "subordinators" $(E_t^\mu)_{t\geq 0}$. And respectively, if a time-non-fractional evolution equation is a governing equation for some Markov process then the related time-fractional evolution equation is a governing equation for a (already non-markovian) process which is a "subordination", i.e. a time-change of this Markov process by means of $(E_t^\mu)_{t\geq 0}$. The non-Markovity of the resulting process corresponds to the fact that the family $(\mathcal{T}_t)_{t\geq 0}$ is not a semigroup any more. Note also that some other types of time-fractional evolution equations have a similar "subordination-like" structure of solutions (see Prüss, 1993; Mura, Taqqu, and Mainardi, 2008).

Assume now that the semigroup $(T_t)_{t\geq 0}$ is not known explicitly but is already Chernoff approximated. We have no chances to construct Chernoff approximations for the family $(\mathcal{T}_t)_{t\geq 0}$ which is not a semigroup. Nevertheless, the following is true.

Theorem 6.3.3. *Let $(X, \|\cdot\|_X)$ be a Banach space. Let $(L, \mathrm{Dom}(L))$ be the generator of a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ on X. Let $f_0 \in \mathrm{Dom}(L)$. Let the family $(F(t))_{t\geq 0}$ of contractions on X be Chernoff equivalent to $(T_t)_{t\geq 0}$. Let, for each $f_0 \in \mathrm{Dom}(L)$, the mapping $F(\cdot)f_0 : [0,\infty) \to X$ be Bochner measurable as a mapping from $([0,\infty), \mathcal{B}([0,\infty)), dx)$ to $(X, \mathcal{B}(X))$. Let μ be a finite Borel measure with $\operatorname{supp}\mu \in (0,1)$ and the family $(\mathcal{T}_t)_{t\geq 0}$ be given by formula (6.3.3). Let $f : [0,\infty) \to X$*

[3] This means that $\|T_t\| \leq M$ for some $M>0$ and all $t\geq 0$.

be defined via $f(t) := T_t f_0$. *For each* $n \in \mathbb{N}$ *define the mappings* $f_n : [0, \infty) \to X$ *by*

$$f_n(t) := \int_0^\infty F^n(\tau/n) f_0 \, p^\mu(t,\tau)\, d\tau. \tag{6.3.5}$$

Then it holds locally uniformly with respect to $t \geq 0$ *that*

$$\|f_n(t) - f(t)\|_X \to 0, \quad n \to \infty.$$

Proof. Take any $T > 0$ and any $\varepsilon > 0$. Due to Lemma 6.3.1, there exists $R_{T,\varepsilon} > 0$ such that

$$\int_{R_{T,\varepsilon}}^\infty p^\mu(t,\tau) d\tau < \varepsilon$$

for all $t \in [0, T]$. Then it holds for $t \in [0, T]$

$$\begin{aligned}
\|f_n(t) - f(t)\|_X &= \left\| \int_0^\infty F^n(\tau/n) f_0 \, p^\mu(t,\tau)\, d\tau - \int_0^\infty T_\tau f_0 \, p^\mu(t,\tau)\, d\tau \right\|_X \\
&\leq \int_0^\infty \|T_\tau f_0 - F^n(\tau/n) f_0\|_X \, p^\mu(t,\tau)\, d\tau \\
&\leq \int_0^{R_{T,\varepsilon}} \|T_\tau f_0 - F^n(\tau/n) f_0\|_X \, p^\mu(t,\tau)\, d\tau + \int_{R_{T,\varepsilon}}^\infty \|T_\tau f_0 - F^n(\tau/n) f_0\|_X \, p^\mu(t,\tau)\, d\tau \\
&\leq \sup_{\tau \in [0, R_{T,\varepsilon}]} \|T_\tau f_0 - F^n(\tau/n) f_0\|_X \int_0^{R_{T,\varepsilon}} p^\mu(t,\tau)\, d\tau + 2\varepsilon \|f_0\|_X \\
&\to 2\varepsilon \|f_0\|_X, \quad n \to \infty,
\end{aligned}$$

due to the fact that the convergence in the Chernof theorem is locally uniform with respect to the time variable. Since $\varepsilon > 0$ was chosen arbitrary, the statement follows. $\square$

Remark 6.3.4. Consider a time-fractional Fokker–Planck–Kolmogorov equation of the form (6.3.2). Assume that the semigroup $(T_t)_{t \geq 0}$, whose generator L stands in the right hand side of the equation, corresponds to a Markov process $(\xi(t))_{t \geq 0}$. Then this time-fractional Fokker–Planck–Kolmogorov equation is a governing equation for the stochastic process $(\xi(E_t^\mu))_{t \geq 0}$ which is the time-change of $(\xi(t))_{t \geq 0}$ by means of the inverse subordinator $(E_t^\mu)_{t \geq 0}$. And the function

$$f(t,x) := \mathbb{E}\left[f_0 \left(\xi \left(E_t^\mu \right) \right) \mid \xi(E_0^\mu) = x \right] \tag{6.3.6}$$

solves the Cauchy problem (6.3.4) (cf. Theorem 3.6 in Hahn, Kobayashi, and Umarov, 2012, see also Baeumer and Meerschaert, 2001; Meerschaert and Scheffler, 2004). Since the process $(\xi(E_t^\mu))_{t\geq 0}$ is not Markov, its marginal density function (together with the initial distribution) does not determine all finite dimensional distributions of the process. And there exist many different processes with the same marginal density function. Hence there exist many other stochastic representations for the function $f(t,x)$ in formula (6.3.6) (see, e.g., Thm 3.3 in Baeumer et al., 2016, Cor. 3.4 in Baeumer, Meerschaert, and Nane, 2009 and results of Orsingher and Beghin, 2009). Furthermore, the considered time-fractional Fokker–Planck–Kolmogorov equations (with $\mu = \delta_{\beta_0}$, $\beta_0 \in (0,1)$) are related to some time-non-fractional evolution equations of hihger order (see, e.g., Baeumer, Meerschaert, and Nane, 2009; Orsingher and D'Ovidio, 2012). Therefore, the approximations f_n constructed in Theorem 6.3.3 can be used simultaneousely to approximate path integrals appearing in different stochastic representations of the same function $f(t,x)$ and to approximate solutions of corresponding time-non-fractional evolution equations of hihger order.

Remark 6.3.5. Obviously, approximations f_n similar to those of Theorem 6.3.3 can be constructed also for subordinate semigroups discussed in Section 4.2. Namely, assume that a semigroup $(T_t^f)_{t\geq 0}$ is subordinate to a given semigroup $(T_t)_{t\geq 0}$ on a Banach space $(X, \|\cdot\|_X)$ with respect to a given convolution semigroup $(\eta_t)_{t\geq 0}$ associated to a Bernstein function f, i.e. $T_t^f\varphi = \int_0^\infty T_s\varphi\,\eta_t(ds)$ for all $\varphi \in X$; the convolution semigroup $(\eta_t)_{t\geq 0}$ is known explicitly; a given family $(F(t))_{t\geq 0}$ (of contractions) is Chernoff equivalent to $(T_t)_{t\geq 0}$ and the mapping $F(\cdot)\varphi : [0,\infty) \to X$ is, for each $\varphi \in X$ and each $t \geq 0$, Bochner measurable as a mapping from $([0,\infty), \mathcal{B}([0,\infty)), \eta_t)$ to $(X, \mathcal{B}(X))$. Then, similarly to the proof of Theorem 6.3.3, one shows that the functions $f_n(t)$,

$$f_n(t) := \int_0^\infty F^n(s/n)\varphi\,\eta_t(ds), \tag{6.3.7}$$

approximate the function $T_t^f\varphi$ in the norm $\|\cdot\|_X$ locally uniformly with respect to $t \geq 0$ for all $\varphi \in X$. Note that such approximations f_n are much simpler than Chernoff approximations based on the family $(\mathcal{F}(t))_{t\geq 0}$ constructed in Theorem 4.2.1. However, the Chernoff approximations, based on the families $(\mathcal{F}(t))_{t\geq 0}$ which are presented in Theorem 4.2.1, can be used as a building block for further purposes. First, families $(\mathcal{F}(t))_{t\geq 0}$ can be used to obtain Chernoff approximations for semigroups, constructed by several iterative procedures of subordination, killing of an underlying process upon leaving a given domain, additive and multiplicative perturbations (of generators) of some original semigroups. Second, such Chernoff approximations, in turn, can be used to obtain approximations for solutions of the corresponding time-fractional evolution equations. Whereas the approximations f_n in formula (6.3.7) can be used only to approximate $T_t^f f_0$ and not for further purposes.

Example 6.3.6 (Feynman formula solving the Cauchy problem for a class of time-fractional diffusion equations). Let $(T_t)_{t\geq 0}$ be a Feller semigroup[4] whose generator $(L, \mathrm{Dom}(L))$ is given for all $\varphi \in C^2_\infty(\mathbb{R}^d)$ by formula (2.3.9). Let coefficients A, B, C be bounded and continuous and (2.3.1) hold. Let $C^{2,\alpha}_c(\mathbb{R}^d)$ be a core for $(L, \mathrm{Dom}(L))$. Consider the family $(F(t))_{t\geq 0}$ of bounded linear operators on X given by formula (2.3.12). Under the assumptions above, this family is strongly continuous and Chernoff equivalent to the semigroup $(T_t)_{t\geq 0}$. And all operators $F(t)$ are contractions since $C(x) \geq 0$ for all $x \in \mathbb{R}^d$. Let $f_0 \in \mathrm{Dom}(L)$. Due to Proposition 6.3.2, the function

$$f(t,x) := \int_0^\infty T_\tau f_0(x)\, p^\mu(t,\tau)\, d\tau$$

solves the Cauchy problem for the distributed order time-fractional diffusion equation

$$\mathcal{D}^\mu f(t,x) = -C(x) f(t,x) - B(x) \cdot \nabla f(t,x) + \mathrm{tr}(A(x)\,\mathrm{Hess}\, f(t,x)) \tag{6.3.8}$$

with the initial condition $f(0,x) = f_0(x)$. Due to Theorem 6.3.3, the following functions $f_n(t,x)$ approximate the solution $f(t,x)$ in supremum norm with respect to $x \in \mathbb{R}^d$ uniformly with respect to $t \in (0,t^*]$ for all $t^* > 0$ as $n \to \infty$:

$$\begin{aligned} f_n(t,x_0) := \int_0^\infty \int_{\mathbb{R}^{nd}} & e^{-\frac{\tau}{n}\sum\limits_{j=1}^n \left(C(x_{j-1}) + \frac{1}{4}|A^{-1/2}(x_{j-1})B(x_{j-1})|^2\right)} e^{\frac{1}{2}\sum\limits_{j=1}^n A^{-1}(x_{j-1})B(x_{j-1})\cdot(x_{j-1}-x_j)} \\ & \times p_A(\tau/n, x_0, x_1) \cdots p_A(\tau/n, x_{n-1}, x_n) f_0(x_n) p^\mu(t,\tau) dx_1 \ldots dx_n d\tau. \end{aligned}$$

Since for each $x_0 \in \mathbb{R}^d$ the solution $f(t,x_0)$ is the limit of $f_n(t,x_0)$, i.e. the limit of $(n+1)$–fold iterated integrals as $n \to \infty$, the approximations $f_n(t,x_0)$ provide us just a Feynman formula for $f(t,x_0)$. Namely, the following statement holds.

Proposition 6.3.7. *Under assumptions of Example 6.3.6, the function $f(t,x_0)$, given by the Feynman formula (6.3.9) below, solves the Cauchy problem for the distributed order time-fractional diffusion equation (6.3.8) with the initial condition f_0.*

$$\begin{aligned} & f(t,x_0) \\ & = \lim_{n\to\infty} \int_0^\infty \int_{\mathbb{R}^{nd}} e^{-\frac{\tau}{n}\sum\limits_{j=1}^n \left(C(x_{j-1}) + \frac{1}{4}|A^{-1/2}(x_{j-1})B(x_{j-1})|^2\right)} e^{\frac{1}{2}\sum\limits_{j=1}^n A^{-1}(x_{j-1})B(x_{j-1})\cdot(x_{j-1}-x_j)} \\ & \qquad \times p_A(\tau/n, x_0, x_1) \cdots p_A(\tau/n, x_{n-1}, x_n) f_0(x_n) p^\mu(t,\tau) dx_1 \ldots dx_n d\tau, \end{aligned} \tag{6.3.9}$$

where p_A is given by (2.3.5). And the convergence is uniform with respect to $x_0 \in \mathbb{R}^d$ and with respect to $t \in (0,t^*]$ for all $t^* > 0$.

[4] Hence all T_t are contractions.

In particular, consider a $1/2$-stable inverse subordinator $(E_t^{1/2})_{t\ge0}$. Its marginal probability density is known explicitly (see Cor. 3.1 in Meerschaert and Scheffler, 2004 and the discussion after Lemma 3 in D'Ovidio, 2010)

$$p^{1/2}(t,\tau) = \frac{1}{\sqrt{\pi t}} e^{-\frac{\tau^2}{4t}}.$$

Therefore, in the case when $\mathcal{D}^\mu$ is the Caputo derivative of $1/2$-th order, the function $f(t,x_0)$, represented (uniformly with respect to $x_0 \in \mathbb{R}^d$ and with respect to $t \in (0,t^*]$ for all $t^* > 0$) by the following Feynman formula (6.3.10), solves the Cauchy problem for the time-fractional diffusion equation (6.3.8) with initial condition f_0:

$$\begin{aligned} &f(t,x_0)\\ &= \lim_{n\to\infty} \int\limits_0^\infty \int\limits_{\mathbb{R}^{nd}} e^{-\frac{\tau}{n}\sum\limits_{j=1}^n \left(C(x_{j-1})+\frac14 |A^{-1/2}(x_{j-1})B(x_{j-1})|^2\right)} e^{\frac12 \sum\limits_{j=1}^n A^{-1}(x_{j-1})B(x_{j-1})\cdot(x_{j-1}-x_j)}\\ &\qquad\qquad \times p_A(\tau/n,x_0,x_1)\cdots p_A(\tau/n,x_{n-1},x_n) f_0(x_n) \frac{1}{\sqrt{\pi t}} e^{-\frac{\tau^2}{4t}}\, dx_1\dots dx_n d\tau. \end{aligned} \tag{6.3.10}$$

Analogous results hold true also for distributed order time-fractional Fokker–Planck–Kolmogorov equations with operators L considered in Chapters 2—4 and in Sections 6.1, 6.2 of the present Chapter.

Example 6.3.8 (**Feynman formula solving the Cauchy–Dirichlet problem for a class of time-fractional diffusion equations**). We keep on working in the situation of Example 6.3.6. Let additionally $(T_t)_{t\ge0}$ be doubly Feller, A, B, C be of the class $C_b^{2,\alpha}(\mathbb{R}^d)$ for some $\alpha \in (0,1)$. Let $G \subset \mathbb{R}^d$ be a bounded domain with the boundary ∂G of the class $C^{4,\alpha}$ for some $\alpha \in (0,1)$. Then we have by Theorem 5.2.6 and Remark 5.2.3 for the corresponding semigroup $(T_t^o)_{t\ge0}$ on Y

$$T_t^o \varphi(x) = \lim_{n\to\infty} \left(\left(F_o(t/n)\right)^n \varphi\right)(x) \quad \text{for all } \varphi \in C_0(G),$$

uniformly in $x \in G$ and locally uniformly in $t \in [0,\infty)$. Here the family $(F_o(t))_{t\ge0}$ has been constructed from the family $(F(t))_{t\ge0}$ given in (2.3.12) by the formula (5.2.2). Let $f_0 \in \mathrm{Dom}(L_o)$. Due to Proposition 6.3.2 and Remark 5.1.2, the function

$$f(t,x) := \int\limits_0^\infty T_\tau^o f_0(x)\, p^\mu(t,\tau)\, d\tau \tag{6.3.11}$$

solves the following Cauchy–Dirichlet problem for the distributed order time-fractional diffusion equation

$$\mathcal{D}^\mu f(t,x) = -C(x)f(t,x) - B(x)\cdot\nabla f(t,x) + \operatorname{tr}(A(x)\operatorname{Hess} f(t,x)), \quad t>0,\ x\in G, \tag{6.3.12}$$

$$f(0,x) = f_0(x), \quad x\in G,$$
$$f(t,x) = 0, \quad t>0,\ x\in\partial G.$$

Due to Theorem 6.3.3, the following functions $f_n(t,x)$ approximate the solution $f(t,x)$ in supremum norm with respect to $x\in G$ uniformly with respect to $t\in(0,t^*]$ for all $t^*>0$:

$$\begin{aligned} f_n(t,x_0) &:= \int_0^\infty F_o^n(\tau/n) f_0(x_0)\, p^\mu(t,\tau)\,d\tau \\ &= \int_0^\infty \int_{\mathbb{R}^{nd}} e^{-\frac{\tau}{n}\sum_{j=1}^n \left(C(x_{j-1})+\frac14|A^{-1/2}(x_{j-1})B(x_{j-1})|^2\right)} e^{\frac12\sum_{j=1}^n A^{-1}(x_{j-1})B(x_{j-1})\cdot(x_{j-1}-x_j)} \mathcal{E}(f_0)(x_n) \\ &\times \left(\prod_{j=1}^n \phi_{s(\tau/n)}(x_{j-1})\right) p_A(\tau/n,x_0,x_1)\cdots p_A(\tau/n,x_{n-1},x_n)p^\mu(t,\tau)\,dx_1\dots dx_n d\tau. \end{aligned}$$

For each $\tau\in[0,\infty)$, each $x_0\in G$ and each $n\in\mathbb{N}$, let us define $\Theta_n^{f_0}(\tau,x_0)$ by

$$\begin{aligned} \Theta_n^{f_0}(\tau,x_0) := \int_{G^n} & e^{-\frac{\tau}{n}\sum_{j=1}^n \left(C(x_{j-1})+\frac14|A^{-1/2}(x_{j-1})B(x_{j-1})|^2\right)} e^{\frac12\sum_{j=1}^n A^{-1}(x_{j-1})B(x_{j-1})\cdot(x_{j-1}-x_j)} \\ & \times f_0(x_n) p_A(\tau/n,x_0,x_1)\cdots p_A(\tau/n,x_{n-1},x_n)dx_1\dots dx_n. \end{aligned}$$

Then we have $\sup_{x_0\in G}|\Theta_n^{f_0}(\tau,x_0)| \le \|f_0\|_Y$ for all $\tau\in[0,\infty)$ and $n\in\mathbb{N}$. Let us show that the functions

$$g_n(t,x_0) := \int_0^\infty \Theta_n^{f_0}(\tau,x_0)p^\mu(t,\tau)d\tau$$

approximate the function $f(t,x_0)$ in formula (6.3.11) solving the Cauchy–Dirichlet problem (6.3.12) as $n\to\infty$ locally uniformly with respect to $x_0\in G$ and uniformly with respect to $t\in(0,t^*]$ for all $t^*>0$. So, fix any $\varepsilon>0$, $T>0$ and a compact $K\subset G$. Let $x_0\in K$ and $t\in[0,T]$. Due to Lemma 6.3.1, there exists $R_{T,\varepsilon}>0$ such that $\int_{R_{T,\varepsilon}}^\infty p^\mu(t,\tau)d\tau<\varepsilon$ for all $t\in[0,T]$. Then, similarly to the proof of Theorem 6.3.3,

$$\begin{aligned}
&|g_n(t,x_0)-f(t,x_0)| \le |g_n(t,x_0)-f_n(t,x_0)| + |f_n(t,x_0)-f(t,x_0)| \\
&\le \int_0^{R_{T,\varepsilon}} \|\Theta_n^{f_0}(\tau,\cdot)-F_o^n(\tau/n)f_0\|_{C(K)} p^\mu(t,\tau)\,d\tau + 2\varepsilon\|f_0\|_Y + \|f_n(t,\cdot)-f(t,\cdot)\|_Y \\
&\le \sup_{\tau\in[0,R_{T,\varepsilon}]} \|\Theta_n^{f_0}(\tau,\cdot)-F_o^n(\tau/n)f_0\|_{C(K)} + 2\varepsilon\|f_0\|_Y + \|f_n(t,\cdot)-f(t,\cdot)\|_Y \\
&\to 2\varepsilon\|f_0\|_Y \quad \text{as } n\to\infty,
\end{aligned}$$

since the convergence of $\Theta_n^{f_0}(\tau,x_0)$ to $F_o^n(\tau/n)f_0(x_0)$ is locally uniform with respect to $x_0\in G$ and uniform with respect to $t\in(0,t^*]$ for all $t^*>0$ (cf. Corollary 5.3.2) and due to Theorem 6.3.3. Therefore, since $\varepsilon>0$ is arbitrary, the following statement is proved.

Proposition 6.3.9. *Let $(T_t)_{t\ge0}$ be a doubly Feller semigroup on $X=C_\infty(\mathbb{R}^d)$ whose generator $(L,\mathrm{Dom}(L))$ is given for all $\varphi\in C^2_\infty(\mathbb{R}^d)$ by the formula (2.3.9). Let the coefficients A, B, C in (2.3.9) be of the class $C_b^{2,\alpha}(\mathbb{R}^d)$ for some $\alpha\in(0,1)$. Let there exist a_0, $A_0\in\mathbb{R}$ such that (2.3.1) holds. Assume that the coefficients A, B, C are such that the set $C_c^{2,\alpha}(\mathbb{R}^d)$ is a core for the generator L in X. Let $G\subset\mathbb{R}^d$ be a bounded domain with the boundary ∂G of the class $C^{4,\alpha}$. Let $(T_t^o)_{t\ge0}$ be the corresponding Feller semigroup on $Y=C_0(G)$ with the generator $(L_o,\mathrm{Dom}(L_o))$. Let $f_0\in\mathrm{Dom}(L_o)$. Let μ be a finite Borel measure with $\operatorname{supp}\mu\in(0,1)$. Then the function $f(t,x_0)$, which is given for all $t>0$ and all $x_0\in G$ by the Feynman formula (6.3.13) below, solves the Cauchy–Dirichlet problem (6.3.12). And the convergence in the Feynman formula (6.3.13) is locally uniform with respect to $x_0\in G$ and uniform with respect to $t\in(0,t^*]$ for all $t^*>0$:*

$$\begin{aligned}
&f(t,x_0) \\
&= \lim_{n\to\infty}\int_0^\infty\int_{G^n} e^{-\frac{\tau}{n}\sum_{j=1}^n\left(C(x_{j-1})+\frac14|A^{-1/2}(x_{j-1})B(x_{j-1})|^2\right)} e^{\frac12\sum_{j=1}^n A^{-1}(x_{j-1})B(x_{j-1})\cdot(x_{j-1}-x_j)} \\
&\qquad\times p_A(\tau/n,x_0,x_1)\cdots p_A(\tau/n,x_{n-1},x_n)f_0(x_n)p^\mu(t,\tau)\,dx_1\ldots dx_n\,d\tau,
\end{aligned} \tag{6.3.13}$$

where p_A is given by (2.3.5).

Analogous results hold true also for distributed order time-fractional Fokker–Planck–Kolmogorov equations with non-local operators L considered in Section 5.3. Other representations for solutions of some distributed order time-fractional Fokker–Planck–Kolmogorov equations in bounded domains can be found, e.g., in Meerschaert, Nane, and Vellaisamy, 2011; Chen, Meerschaert, and Nane, 2012.

Appendix A

Essentials of the Semigroup Theory

Some classical results of the Semigroup Theory are summarized in this Appendix. The standard references are Pazy, 1983, Goldstein, 1985, Engel and Nagel, 2000, Jacob, 2001. Other sources are cited directly in the text.

Proposition A.0.1. *Let X be a Banach space endowed with a norm $\|\cdot\|_X$. Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup on X with generator $(L, \mathrm{Dom}(L))$. Then the following is true.*

(a) There exist constants $\omega > 0$ and $M \geq 1$ such that $\|T_t\| \leq Me^{\omega t}$ for all $t \geq 0$.

(b) For any numbers $\alpha > 0$, $\beta \in \mathbb{C}$, the rescaled semigroup $(U_t)_{t\geq 0}$, defined by $U_t := e^{\beta t}T_{\alpha t}$, is again strongly continuous on X.

Remark A.0.2 (cf. Ethier and Kurtz, 1986, Rem.1.3)**.** Let constants $\omega > 0$ and $M \geq 1$ be such that a strongly continuous semigroup $(T_t)_{t\geq 0}$ satisfies the estimate $\|T_t\| \leq Me^{\omega t}$ for all $t \geq 0$. Consider the rescaled semigroup $(U_t)_{t\geq 0}$, $U_t := e^{-\omega t}T_t$. Then $\|U_t\| \leq M$ for all $t \geq 0$. Define a new norm $|||\cdot|||_X$ on X via $|||\varphi|||_X := \sup_{t\geq 0}\|T_t\varphi\|_X$. Hence it holds $\|\varphi\|_X \leq |||\varphi|||_X \leq M\|\varphi\|_X$, i.e., two norms $\|\cdot\|_X$ and $|||\cdot|||_X$ are equivalent. Moreover, the semigroup $(U_t)_{t\geq 0}$ is a contraction semigroup with respect to the norm $|||\cdot|||_X$.

Proposition A.0.3. *Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X with generator $(L, \mathrm{Dom}(L))$. Then the following is true.*

(a) For every $\varphi \in X$, the mapping $t \to T_t\varphi$ is a continuous function from $[0,\infty)$ into X.

(b) For each $\varphi \in X$, it holds $\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} T_s\varphi\, ds = T_t\varphi$.

(c) For each $\varphi \in X$, it holds $\int_0^t T_s\varphi\, ds \in \mathrm{Dom}(L)$ and $L\left(\int_0^t T_s\varphi\, ds\right) = T_t\varphi - \varphi$.

(d) For each $\varphi \in \mathrm{Dom}(L)$, it holds $T_t\varphi \in \mathrm{Dom}(L)$ and $\frac{d}{dt}T_t\varphi = LT_t\varphi = T_tL\varphi$.

(e) For each $\varphi \in \mathrm{Dom}(L)$, it holds $T_t\varphi - T_s\varphi = \int_s^t T_\tau L\varphi\, d\tau = \int_s^t LT_\tau\varphi\, d\tau$.

Proposition A.0.4. *Let $(T_1(t))_{t\geq 0}$ and $(T_2(t))_{t\geq 0}$ be two strongly continuous semigroups on X such that $T_1(t)\circ T_2(t) = T_2(t)\circ T_1(t)$ for all $t \geq 0$. Then the operators $U_t := T_1(t)\circ T_2(t)$ form a strongly continuous semigroup $(U_t)_{t\geq 0}$, called the* product semigroup *of $(T_1(t))_{t\geq 0}$ and $(T_2(t))_{t\geq 0}$.*

Definition A.0.5. Let X, Y be Banach spaces, $A : X \to Y$ be a linear operator.

(i) A is a *closed* operator if its graph $G(A) := \{(\varphi, A\varphi),\ \varphi \in \mathrm{Dom}(A)\} \subset X \oplus Y$ is closed in $X \oplus Y$; in other words, if the sequence $(\varphi_n)_{n\in\mathbb{N}}$ satisfies that $A\varphi_n \to \phi$ and $\varphi_n \to \varphi$ as $n \to \infty$, then $\varphi \in \mathrm{Dom}(A)$ and, moreover, $\phi = A\varphi$ holds.

(ii) A is *closable* if there exists a closed extension of A, i.e., a closed operator $B : X \to Y$ such that $A \subset B$ ($\Leftrightarrow \mathrm{Dom}(A) \subset \mathrm{Dom}(B)$ and $B\big|_{\mathrm{Dom}(A)} = A$).

(iii) The *closure* of A, denoted by $\overline{A}$, is the smallest closed extension of A, i.e. $A \subset \overline{A}$ and if B is another closed extension of A, then $\overline{A} \subset B$.

(iv) A subspace $D \in X$ is a *core* of A if $\overline{A\big|_D} = A$; in other words, if for every $\varphi \in \mathrm{Dom}(A)$ there is a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset D$ such that $\varphi_n \to \varphi$ and $A\varphi_n \to A\varphi$ as $n \to \infty$.

Proposition A.0.6. *(a) If $(L, \mathrm{Dom}(L))$ is the generator of a strongly continuous semigroup on X then $\mathrm{Dom}(L)$ is dense in X and L is a closed linear operator.*

(b) Let $(T_1(t))_{t\geq 0}$ and $(T_2(t))_{t\geq 0}$ be two strongly continuous semigroups on X with generators $(L_1, \mathrm{Dom}(L_1))$ and $(L_2, \mathrm{Dom}(L_2))$ respectively. If $L_1 = L_2$ then $T_1(t) = T_2(t)$, $t \geq 0$.

Proposition A.0.7 (A core criterium, cf. Sato, 1999, Lemma 31.6)**.** *Let $(T_t)_{t\geq 0}$ be a strongly continuous contraction semigroup on a Banach space X with generator $(L, \mathrm{Dom}(L))$. If D_0 and D are linear subspaces of X such that*

$$D_0 \subset D \subset \mathrm{Dom}(L), \quad D_0 \ \textit{is dense in}\ X$$

and

$$\varphi \in D_0 \ \textit{implies}\ T_t\varphi \in D \ \textit{for any}\ t > 0,$$

then D is a core for L.

Obviously, if D is a core for L then any subspace D' with $D \subset D' \subset \mathrm{Dom}(L)$ is also a core for L. It follows immediately from the Core Criterium A.0.7 and Proposition A.0.3, part (d), that $\mathrm{Dom}(L^k)$, $k \in \mathbb{N}$, and $D^\infty := \cap_{k\in\mathbb{N}} \mathrm{Dom}(L^k)$ are cores for L since D^∞ is dense in X by Theorem 1.2.7 in Pazy, 1983.

Definition A.0.8. Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X with generator $(L, \mathrm{Dom}(L))$. Let a Banach space Y be continuously embedded in X. The *part* of L in Y is the operator L_Y defined by

$$L_Y\varphi := L\varphi, \quad \mathrm{Dom}(L_Y) := \{\varphi \in \mathrm{Dom}(L) \cap Y \ :\ L\varphi \in Y\}.$$

Proposition A.0.9. *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X with generator $(L, \mathrm{Dom}(L))$. Let a Banach space Y be continuously embedded in X. Let Y be $(T(t))_{t\geq 0}$-invariant, i.e. $T(t)Y \subset Y$ for all $t \geq 0$.*

(a) If Y is a closed subspace of X then the restrictions $T_Y(t) := T(t)\big|_Y$ form a strongly continuous semigroup $(T_Y(t))_{t\geq 0}$, called the subspace semigroup *on the Banach space Y.*

(b) If the restricted semigroup $T_Y(t) := T(t)\big|_Y$ is strongly continuous on Y then the generator of $(T_Y(t))_{t\geq 0}$ is the part $(L_Y, \mathrm{Dom}(L_Y))$ of L in Y.

Definition A.0.10. Let $(L, \mathrm{Dom}(L))$ be a closed linear operator in X. If for some $\lambda \in \mathbb{C}$, the operator $\lambda - L$ is one-to-one, $\mathrm{Range}(\lambda - L) = X$ and $R_L(\lambda) := (\lambda - L)^{-1}$ is a bounded linear operator on X, then λ is said to belong to the *resolvent set* $\rho(L)$ of L and $R_L(\lambda)$ is called the *resolvent* of L at λ.

Theorem A.0.11 (Hille–Yosida)**.** *A linear operator $(L, \mathrm{Dom}(L))$ is the generator of a strongly continuous semigroup $(T_t)_{t\geq 0}$ on X satisfying $\|T_t\| \leq Me^{\omega t}$, if and only if*

(i) L is closed and $\mathrm{Dom}(L)$ is dense in X.

(ii) The resolvent set $\rho(L)$ of L contains the ray (ω, ∞) and

$$\|R_L(\lambda)^n\| \leq \frac{M}{(\lambda - \omega)^n} \quad \textit{for } \lambda > \omega,\, n \in \mathbb{N}.$$

Definition A.0.12. A linear operator $L : \mathrm{Dom}(L) \to X$, $\mathrm{Dom}(L) \subset X$, is called *$X$-dissipative*, if

$$\|\lambda\varphi - L\varphi\|_X \geq \lambda\|\varphi\|_X$$

holds for all $\lambda > 0$ and all $\varphi \in \mathrm{Dom}(L)$.

Theorem A.0.13 (Lumer–Phillips)**.** *A linear operator $(L, \mathrm{Dom}(L))$ is the generator of a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ on X, if and only if*

(i) $\mathrm{Dom}(L)$ is dense in X.

(ii) L is X-dissipative.

(iii) The operator $\lambda - L$ is surjective for some $\lambda > 0$.

Remark A.0.14 (cf. Prop. II.3.23, Example II.3.26 in Engel and Nagel, 2000)**.** Let X' be dual to X. For every $\varphi \in X$, the following set, called its *duality set*,

$$\mathcal{J}(\varphi) := \left\{ l \in X' : \langle \varphi, l \rangle = \|\varphi\|_X^2 = \|l\|_{X'}^2 \right\}$$

is nonempty. Then an operator $(L, \mathrm{Dom}(L))$ is dissipative if and only if for every $\varphi \in \mathrm{Dom}(L)$ there exists $l \in \mathcal{J}(\varphi)$ such that

$$\mathrm{Re}\,\langle L\varphi, l\rangle \leq 0.$$

In paticular, consider $X := C_0(Q)$, Q locally compact. Then for any $0 \neq \varphi \in X$

$$\left\{ \overline{\varphi(q_0)}\delta_{q_0} : q_0 \in Q \text{ and } |\varphi(q_0)| = \|\varphi\|_X \right\} \subset \mathcal{J}(\varphi).$$

Definition A.0.15. Let $(L, \mathrm{Dom}(L))$ be a densely defined linear operator on a Hilbert space X. The operator $(L^*, \mathrm{Dom}(L^*))$ in X defined by

$$Lx \cdot y = x \cdot L^* y, \quad \forall\, x \in \mathrm{Dom}(L),\, y \in \mathrm{Dom}(L^*)$$

is called the *adjoint* to $(L, \mathrm{Dom}(L))$. The operator $(L, \mathrm{Dom}(L))$ is called *selfadjoint* if $(L, \mathrm{Dom}(L)) = (L^*, \mathrm{Dom}(L^*))$, i.e. $L = L^*$ and $\mathrm{Dom}(L) = \mathrm{Dom}(L^*)$.

Theorem A.0.16 (Stone Theorem)**.** *A family of bounded linear operators $(T_t)_{t\in\mathbb{R}}$ on a Hilbert space X is a strongly continuous unitary group if and only if there exists a self-adjoint operator A in X such that $T_t = e^{itA}$ for all $t \in \mathbb{R}$.*

Finally, let us consider a few basic examples of strongly continuous semigroups. Recall that the heat semigroup is discussed in Example 1.0.3.

Example A.0.17 (Multiplication semigroup)**.** (i) Consider the spaces $X = L^p(\mathbb{R}^d)$ with $p \in [1, \infty)$. Let $C : \mathbb{R}^d \to \mathbb{C}$ be a Borel function such that $\operatorname{ess\,inf}_{x\in\mathbb{R}^d} \operatorname{Re} C(x) > -\infty$. Define a multiplication operator $-C : X \to X$ via

$$-C\varphi(x) := -C(x)\varphi(x), \quad x \in \mathbb{R}^d, \qquad \mathrm{Dom}(-C) := \{\varphi \in X : -C\varphi \in X\}.$$

Then the operator $(-C, \mathrm{Dom}(-C))$ is closed, densely defined and generates a strongly continuous semigroup $(T_t^{-C})_{t\geq 0}$ (on X) which is called *multiplication semigroup* and is given by

$$T_t^{-C}\varphi(x) := e^{-tC(x)}\varphi(x), \quad x \in \mathbb{R}^d.$$

If $\operatorname{ess\,sup}_{x\in\mathbb{R}^d} |C(x)| < \infty$ then the operator $-C$ is bounded and $\mathrm{Dom}(-C) = X$. If $C \in C(\mathbb{R}^d)$ then the set $C_c(\mathbb{R}^d)$ is a core for $-C$ by the Core Criterium A.0.7.

(ii) Analogously, consider the space $X = C_\infty(\mathbb{R}^d)$ and let $C : \mathbb{R}^d \to \mathbb{C}$ be a continuous function such that $\inf_{x\in\mathbb{R}^d} \operatorname{Re} C(x) > -\infty$. Then the multiplication operator $(-C, \mathrm{Dom}(-C))$ (as above) is closed, densely defined and generates a strongly continuous multiplication semigroup $(T_t^{-C})_{t\geq 0}$ (as above) on X. Once again, $C_c(\mathbb{R}^d)$ is a core for $-C$ by the Core Criterium A.0.7. Moreover, a set $C_c^\infty(\mathbb{R}^d)$ is also a core for $-C$.

Indeed, let $\varphi \in \mathrm{Dom}(-C)$. Then for each $\varepsilon > 0$ there exists a compact $K_\varepsilon \subset \mathbb{R}^d$ such that $|\varphi(x)| < \varepsilon$ and $|C(x)\varphi(x)| < \varepsilon$ for all $x \notin K_\varepsilon$. Take any sequence $(\varphi_n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^d)$ converging to φ in X such that $|\varphi_n(x)| \leq |\varphi(x)|$ for all $x \in \mathbb{R}^d$ and all $n \in \mathbb{N}$. Then one has for each $\varepsilon > 0$:

$$\begin{aligned}
&\lim_{n\to\infty} \| -C\varphi_n + C\varphi\|_\infty \leq \\
&\leq \lim_{n\to\infty}\left(\sup_{x\in K_\varepsilon} |C(x)| \|\varphi_n - \varphi\|_\infty + \sup_{x\notin K_\varepsilon} |C(x)\varphi_n(x)| + \sup_{x\notin K_\varepsilon} |C(x)\varphi(x)| \right) < 2\varepsilon,
\end{aligned}$$

i.e., the sequence $(-C\varphi_n)_{n\in\mathbb{N}}$ converges to $-C\varphi$ in X. Therefore, $C_c^\infty(\mathbb{R}^d)$ is a core for $-C$ by definition.

Example A.0.18 (Translation semigroup)**.** (i) Let $X = L^p(\mathbb{R}^d)$, $p \in [1, \infty)$, or $X = C_\infty(\mathbb{R}^d)$. Let $B \in \mathbb{R}^d$. For $t \geq 0$, consider translation operators $T_t^{-B\nabla}$ given by

$$T_t^{-B\nabla}\varphi(x) := \varphi(x - tB).$$

It is easy to check that $(T_t^{-B\nabla})_{t\geq 0}$ is a strongly continuous contraction semigroup on X and its generator L is given by $L\varphi(x) := -B \cdot \nabla\varphi(x)$ on a proper domain. This statement also follows immediately from Theorem C.0.7.

Since the action of the translation operators $T_t^{-B\nabla}$ doesn't change the rate of smoothness of $\varphi \in X$ and preserves compactness of support, it follows immediately from the Core Criterium A.0.7 that all spaces $S(\mathbb{R}^d)$, $C_c^\infty(\mathbb{R}^d)$, $C_c^k(\mathbb{R}^d)$, $C_c^{k,\lambda}(\mathbb{R}^d)$, $k \in \mathbb{N}$, $\lambda \in (0, 1)$, are cores for L.

(ii) Let now $X = C_\infty(\mathbb{R}^d)$ and B be a vector field of the class $C_b^1(\mathbb{R}^d; \mathbb{R}^d)$. Consider the operator $(-B\nabla, C_c^1(\mathbb{R}^d))$ such that

$$-B\nabla\varphi(x) = -B(x) \cdot \nabla\varphi(x) \equiv -\sum_{k=1}^{d} B_k(x)\frac{\partial\varphi}{\partial x_k}(x)$$

for all $\varphi \in C_c^1(\mathbb{R}^d)$. Then $(-B\nabla, C_c^1(\mathbb{R}^d))$ is dissipative and its closure generates a strongly continuous semigroup $(T^{-B\nabla})_{t\geq 0}$ on X due to Section II.3.28 in Engel and Nagel, 2000.

Some other examples of strongly continuous semigroups are presented in Appendices C and D. Finally, let us remind the following classical theorem about a bounded linear transformation (B.L.T.).

Theorem A.0.19 (the B.L.T. theorem; cf. Reed and Simon, 1980, Thm. I.7)**.** *Let T be a bounded linear transformation from a normed linear space $(V_1, \|\cdot\|_1)$ to a Banach space $(V_2, \|\cdot\|_2)$. Then T can be uniquely extended to a bounded linear tranformation (with the same bound) from the completion ov V_1 to $(V_2, \|\cdot\|_2)$.*

Appendix B

Markov processes

A *probability space* or a *probability triple* is a mathematical construct that models a real-world process (or "experiment") consisting of states that occur randomly. A probability space consists of three parts: a sample space Ω which is the set of all possible outcomes, a σ-algebra $\mathcal{F}$ which is a collection of all possible events[1] and a probability measure $\mathbb{P}$, i.e. a map $\mathbb{P}: \mathcal{F} \to [0,1]$ such that $\mathbb{P}(\varnothing) = 0$, $\mathbb{P}(\Omega) = 1$ and $\mathbb{P}(\cup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mathbb{P}(B_n)$ for all pairwise disjoint events $B_n \in \mathcal{F}$, $n \in \mathbb{N}$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let Q be a Borel subset of a complete separable metric space and $\mathcal{B}(Q)$ be the (induced) Borel σ-algebra of Q. A family $(\xi_t)_{t\geq 0}$ of measurable functions $\xi_t : (\Omega, \mathcal{F}) \to (Q, \mathcal{B}(Q))$ is called a *stochastic process* (with continuous time) with *state space* Q. A family $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ of σ-algebras $\mathcal{F}_t$ such that $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for all $0 \leq s \leq t$ is called a *filtration* on $(\Omega, \mathcal{F}, \mathbb{P})$. A filtration represents the information available up to and including each time t. Let a filtration $\mathbb{F}$ be given. A stochastic process $(\xi_t)_{t\geq 0}$ is called to be *adapted* to the filtration $\mathbb{F}$ if ξ_t is $\mathcal{F}_t$-measurable for each $t \geq 0$. In other words, $\mathcal{F}_t$ contains all the information about the progress of the process ξ up to time t inclusively. An important class of adapted processes is given by *Markov processes*.

Definition B.0.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$. Let Q be a Borel subset of a complete separable metric space and $(\xi_t)_{t\geq 0}$ be an adapted process with the state space Q. The process $(\xi_t)_{t\geq 0}$ is called *Markov process* if[2]

$$\mathbb{P}\{\xi_{t+s} \in B \mid \mathcal{F}_t\} = \mathbb{P}\{\xi_{t+s} \in B \mid \xi_t\} \quad \text{for all } s, t \geq 0,\ B \in \mathcal{B}(Q).$$

In other words, a Markov process is an adapted process that has no memory, i.e. one can make predictions for the future of the process based solely on its present state just as well as one could knowing the process's full history. Or, equivalently, conditional on the present state of the system, its future and past are independent. Note that all $\mathbb{R}^d$-valued processes $(\xi_t)_{t\geq 0}$ with independent increments (e.g., Brownian motion and Poisson process) are Markov processes.

[1] Each event is a subset of Ω; $\mathcal{F}$ includes the empty subset and Ω itself, is closed under complement, and is closed under countable unions and countable intersections.

[2] Conditional probability of an event must be understood as the conditional expectation of the indicator of this event, i.e., $\mathbb{P}\{\xi_{t+s} \in B \mid \mathcal{F}_t\} := \mathbb{E}\left[1_{\{\xi_{t+s}\in B\}} \mid \mathcal{F}_t\right]$ and $\mathbb{P}\{\xi_{t+s} \in B \mid \xi_t\} := \mathbb{E}\left[1_{\{\xi_{t+s}\in B\}} \mid \sigma(\xi_t)\right]$, where $\sigma(\xi_t)$ is the σ-algebra generated by ξ_t.

Markov processes are closely related to the notion of *Markovian kernel* (this notion is defined below only in the time homogeneous case).

Definition B.0.2. Let $(Q, \mathcal{B}(Q))$ be as above. A map $P : [0, +\infty) \times Q \times \mathcal{B}(Q) \to [0, 1]$ is called a *sub-Markovian kernel* if

(i) $P(\cdot, \cdot, B) : (t, x) \mapsto P(t, x, B)$ is (Borel) measurable for all $B \in \mathcal{B}(Q)$,

(ii) $P(t, x, \cdot) : B \mapsto P(t, x, B)$ is a sub-probability measure (i.e. $P(t, x, Q) \leq 1$) for all $t \in [0, +\infty)$ and all $x \in Q$,

(iii) $P(0, x, \{x\}) = 1$ for all $x \in Q$,

(iv) the *Chapman–Kolmogorov equations*

$$P(t + s, x, B) = \int_Q P(s, y, B) P(t, x, dy)$$

hold for all $s, t \geq 0$, $x \in Q$, $B \in \mathcal{B}(Q)$.

A sub-Markovian kernel P is called a *Markovian kernel* if $P(t, x, Q) = 1$ for all $t \geq 0$ and all $x \in Q$.

Definition B.0.3. Let $(\xi_t)_{t\geq 0}$ be a Markov process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$. Let $(Q, \mathcal{B}(Q))$ be the state space of $(\xi_t)_{t\geq 0}$. Let $P : [0, +\infty) \times Q \times \mathcal{B}(Q) \to [0, 1]$ be a Markovian kernel. We say that the process $(\xi_t)_{t\geq 0}$ is a time-homogeneous Markov process and P is a *transition kernel* of $(\xi_t)_{t\geq 0}$ if for all $s, t \geq 0$, $B \in \mathcal{B}(Q)$

$$\mathbb{P}\{\xi_{t+s} \in B \mid \xi_t\} = P(s, \xi_t, B) \quad \text{almost sure} \tag{B.0.1}$$

or, equivalently,

$$\mathbb{P}\{\xi_{t+s} \in B \mid \xi_t = x\} = P(s, x, B) \quad \text{for } P_{\xi_t} \text{ almost all } x \in \mathbb{R}^d, \tag{B.0.2}$$

where P_{ξ_t} is the distribution of the random variable ξ_t, i.e. $P_{\xi_t}(B) := \mathbb{P}(\xi_t^{-1}(B))$.

In general, the value $\mathbb{P}\{\xi_u \in B \mid \xi_t = x\}$, $u \geq t \geq 0$, depends on t, u, x, B and describes the probability of a transition of the Markov process $(\xi_t)_{t\geq 0}$ from the point x at the time t to the set B at the time u. In the sequel, we consider only time-homogeneous Markov processes: $\mathbb{P}\{\xi_u \in B \mid \xi_t = x\} = P(u - t, x, B)$, i.e. the probability of a transition of the process depends on u and t only through the length $u - t$ of the time interval. In principle, any Markov process can be reduced to a time-homogeneous Markov process by a proper extension of its state space (cf. Wentzell, 1979, 8.5.5.).

If a Markov process posesses a transition kernel P, the properties (i)-(iv) of Definition B.0.2 for P can be derived from (B.0.1). In particular, the Chapman–Kolmogorov equations

$$P(t + s, \xi_u, B) = \int_Q P(s, y, B) P(t, \xi_u, dy), \quad s, t, u \geq 0, \ B \in \mathcal{B}(Q),$$

hold almost surely due to the tower property of conditional expectation. Converse, let $P : [0,+\infty)\times Q\times\mathcal{B}(Q) \to [0,1]$ be a Markovian kernel. The construction of Kolmogorov assures the existence of a Markov process with the state space $(Q, \mathcal{B}(Q))$ such that P is its transition kernel. Approximating measurable functions with step functions, one can show that (B.0.1) is equivalent to the equality

$$\mathbb{E}\left[\varphi(\xi_{t+s}) \mid \mathcal{F}_t\right] = \int\limits_Q \varphi(y)P(s,\xi_t,dy) \quad \text{almost sure}$$

for all $t, s \geq 0$ and all $\varphi : Q \to \mathbb{R}$ with $\mathbb{E}[|\varphi(\xi_t)|] < \infty$, $\forall\, t \geq 0$. Moreover, it is sufficient to know the initial state of a Markov process and its transition kernel to determine all finite dimensional distributions of the process. Indeed, let $0 < t_1 < \ldots < t_n < \infty$ and $B_1, \ldots, B_n \in \mathcal{B}(Q)$. Then, using the notation $\mathbb{E}^x[...] := \mathbb{E}[...\,|\,\xi_0 = x]$, one has

$$\begin{aligned}
&\mathbb{P}\left\{\xi_{t_1} \in B_1, \ldots, \xi_{t_n} \in B_n \mid \xi_0 = x\right\}\\
&= \mathbb{E}\left[1_{\{\cap_{k=1}^{n}\{\xi_{t_k}\in B_k\}\}} \mid \xi_0 = x\right]\\
&= \mathbb{E}^x\left[1_{\{\cap_{k=1}^{n-1}\{\xi_{t_k}\in B_k\}\}} \cdot \mathbb{E}\left[1_{\{\xi_{t_n}\in B_n\}} \mid \mathcal{F}_{n-1}\right]\right]\\
&= \mathbb{E}^x\left[1_{\{\cap_{k=1}^{n-1}\{\xi_{t_k}\in B_k\}\}} \cdot P(t_n - t_{n-1}, \xi_{t_{n-1}}, B_n)\right]\\
&= \mathbb{E}^x\left[1_{\{\cap_{k=1}^{n-2}\{\xi_{t_k}\in B_k\}\}} \cdot \mathbb{E}\left[1_{\{\xi_{t_{n-1}}\in B_{n-1}\}} \cdot P(t_n - t_{n-1}, \xi_{t_{n-1}}, B_n) \mid \mathcal{F}_{n-2}\right]\right]\\
&= \mathbb{E}^x\left[1_{\{\cap_{k=1}^{n-2}\{\xi_{t_k}\in B_k\}\}} \cdot \int\limits_{B_{n-1}} P(t_n - t_{n-1}, x_{n-1}, B_n)P(t_{n-1} - t_{n-2}, \xi_{t_{n-2}}, dx_{n-1})\right]\\
&= \ldots\\
&= \mathbb{E}^x\left[\int\limits_{B_1}\cdots\int\limits_{B_{n-1}} P(t_n - t_{n-1}, x_{n-1}, B_n)P(t_{n-1} - t_{n-2}, x_{n-2}, dx_{n-1})\cdots P(t_1, \xi_0, dx_1)\right]\\
&= \int\limits_{B_1}\cdots\int\limits_{B_{n-1}} P(t_n - t_{n-1}, x_{n-1}, B_n)P(t_{n-1} - t_{n-2}, x_{n-2}, dx_{n-1})\cdots P(t_1, x, dx_1).
\end{aligned}$$

Let now $P : [0,+\infty) \times Q \times \mathcal{B}(Q) \to [0,1]$ be a sub-Markovian kernel. Consider the one-point compactification $\overline{Q} := Q \cup \{\partial\}$, where ∂ is a point not belonging to Q (and is an isolated point if Q is compact). On $\overline{Q}$ we consider the Borel σ-algebra $\mathcal{B}(\overline{Q})$. We extend the sub-Markovian kernel P to a kernel $\overline{P} : [0,+\infty) \times \overline{Q} \times \mathcal{B}(\overline{Q}) \to [0,1]$ by

$$\overline{P}(t,x,B) := \begin{cases} P(t,x,B), & \text{for } x \in Q, \quad B \in \mathcal{B}(Q), \\ 1 - P(t,x,B), & \text{for } x \in Q, \quad B = \{\partial\}, \\ 0, & \text{for } x = \partial, \quad B \in \mathcal{B}(Q), \\ 1, & \text{for } x = \partial, \quad B = \{\partial\}. \end{cases}$$

Then $\overline{P}$ is a Markovian kernel, and there exists a Markov process with the extended state space $(\overline{Q}, \mathcal{B}(\overline{Q}))$ having $\overline{P}$ as a transition kernel. An interpretation of this extension procedure is discussed in Section 3.1 of Chapter 3. Further theory of Markov processes can be found, e.g., in Dynkin, 1965; Ethier and Kurtz, 1986; Jacob, 2005; Casteren, 2011.

Appendix C

Convolution semigroups, continuous negative definite functions as well as Lévy processes and their generators

Our standard references for this Appendix are Jacob, 2001, Sato, 1999, Böttcher, Schilling, and Wang, 2013 and Applebaum, 2009.

Definition C.0.1. A family $(\eta_t)_{t\geq 0}$ of bounded Borel measures on $\mathbb{R}^d$ is called a *convolution semigroup* on $\mathbb{R}^d$ if the following conditions are fulfilled

(i) $\eta_t(\mathbb{R}^d) \leq 1$ for all $t \geq 0$, i.e. all η_t are sub-probability measures;

(ii) $\eta_t * \eta_s = \eta_{t+s}$ for all $t, s \geq 0$ and $\eta_0 = \delta_0$, where δ_0 is the Dirac delta-measure concentrated at zero and $\eta_t * \eta_s$ denotes the convolution of measures η_t and η_s;

(iii) $\eta_t \to \delta_0$ vaguely as $t \to 0$, i.e. $\lim_{t\to 0} \int_{\mathbb{R}^d} \varphi(x)\eta_t(dx) = \int_{\mathbb{R}^d} \varphi(x)\delta_0(dx) \equiv \varphi(x)$ for all $\varphi \in C_c(\mathbb{R}^d)$ and all $x \in \mathbb{R}^d$.

As it follows from Lemma 3.6.3 of Jacob, 2001, $\eta_t \to \eta_{t_0}$ weakly as $t \to t_0 \geq 0$ for any convolution semigroup $(\eta_t)_{t\geq 0}$. The semigroup property (ii) implies that η_t is an *infinitely divisible measure*, i.e., for each $n \in \mathbb{N}$, η_t is the n-th convolution power of another measure (which is $\eta_{t/n}$):

$$\eta_t = \eta_{t/n}^{*n} \quad \text{or } \widetilde{\eta_t} = (2\pi)^{d(n-1)/2}(\widetilde{\eta_{t/n}})^n,$$

where $\widetilde{\eta_s}(p) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot p}\eta_s(dx)$ is the Fourier transform of the measure η_s. Conversely, each infinitely divisible measure η, $\eta(\mathbb{R}^d) \leq 1$, generates a convolution semigroup $(\eta_t)_{t\geq 0}$ with $\eta_t\big|_{t=1} := \eta$.

Definition C.0.2. A stochastic process $(\xi_t)_{t\geq 0}$ (on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, adapted to a (right-continuous and complete) filtration $(\mathcal{F}_t)_{t\geq 0}$) with values in $\mathbb{R}^d$ is called a *Lévy process* if

(0) $\xi_0 = 0$ almost surely;

(i) $(\xi_t)_{t\geq 0}$ has stationary increments, i.e. for all $x \in \mathbb{R}^d$, $B \in \mathcal{B}(\mathbb{R}^d)$, $0 \leq s < t$

$$\mathbb{P}(\xi_t - \xi_s \in B) = \mathbb{P}(\xi_{t-s} \in B);$$

(ii) $(\xi_t)_{t\geq 0}$ has independent increments, i.e. the random variable $\xi_t - \xi_s$ is independent from $\mathcal{F}_s$ for all $t > s \geq 0$;

(iii) $(\xi_t)_{t\geq 0}$ is stochastically continuous, i.e. $\lim_{t\to s} \mathbb{P}\{|\xi_t - \xi_s| > \varepsilon\} = 0$ for all $\varepsilon > 0$ and all $s \geq 0$.

Note that each Lévy process has a modification with càdlàg paths, i.e. the mapping $t \to \xi_t(\omega)$ is almost surely right-continuous with finite left limits.

Proposition C.0.3 (cf. Sato, 1999, Thm. 7.10 and Thm. 10.5). *(a) Let $(\xi_t)_{t\geq 0}$ be a Lévy process. Then, for any $t \geq 0$, the distribution P_{ξ_t} of the random variable ξ_t is an infinitely divisible probability measure and, with $\eta := P_{\xi_1}$, we have $P_{\xi_t} = \eta_t$, where $(\eta_t)_{t\geq 0}$ is the convolution semigroup generated by η. Conversely, if η is an infinitely divisible probability measure, then there is a Lévy process $(\xi_t)_{t\geq 0}$ such that $P_{\xi_1} = \eta$.*

(b) Let $(\xi_t)_{t\geq 0}$ be a Lévy process and $\eta := P_{\xi_1}$ be the corresponding infinitely divisible probability measure. Define $P(t,x,B)$ for each $t \geq 0$, $x \in \mathbb{R}^d$, $B \in \mathcal{B}(\mathbb{R}^d)$ by

$$P(t,x,B) := \eta_t(B - x).$$

Then $P(t,x,B)$ is a temporally and spatially homogeneous transition kernel and $(\xi_t)_{t\geq 0}$ is a Markov process with this transition kernel.

Therefore, there is a one-to-one correspondence between convolution semigroups of probability measures $(\eta_t)_{t\geq 0}$, infinitely divisible probability measures $\eta := \eta_1$ and sets of Lévy processes $(\xi_t)_{t\geq 0}$ with $P_{\xi_1} = \eta$. This correspondence extends to arbitrary convolution semigroups $(\eta_t)_{t\geq 0}$. Then the corresponding stochastic processes $(\xi_t)_{t\geq 0}$ are Lévy processes with *killing*. And the condition $\eta_t(\mathbb{R}^d) \equiv \mathbb{P}(\xi_t \in \mathbb{R}^d) < 1$ means that the process $(\xi_t)_{t\geq 0}$ "dies", i.e. leaves the state space $\mathbb{R}^d$. The use of the one-point compactification $\overline{\mathbb{R}^d}$ of $\mathbb{R}^d$ (the added point is called *cemetery*; the process arrives there immediately by leaving the state space) and a proper extention of the kernel $P(t,x,B) := \eta_t(B - x)$ allow to convert $(\xi_t)_{t\geq 0}$ into a Markov process with the enriched state space $\overline{\mathbb{R}^d}$.

Consider a convolution semigroup $(\eta_t)_{t\geq 0}$. By the Bochner Theorem, Fourier tranforms $\widehat{\eta_t}$ of measures η_t are positive definite functions. Recall that a function $u : \mathbb{R}^d \to \mathbb{C}$ is called *positive definite* if for any choice of $k \in \mathbb{N}$ and vectors $p_1, \ldots, p_k \in \mathbb{R}^d$ the matrix $(u(p_i - p_j))_{i,j=1,\ldots,k}$ is positive Hermitian, i.e. for all $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ we have $\sum_{i,j=1}^k u(p_i - p_j)\lambda_i\overline{\lambda_j} \geq 0$. Positive definite functions are closely related to *negative definite* ones. Negative definite functions have been introduced by I.J. Schönberg in connection with isometric embeddings of metric spaces into a Hilbert space. His original definition is the following.

Definition C.0.4. A function $\psi : \mathbb{R}^d \to \mathbb{C}$ is called *negative definite* if for any $m \in \mathbb{N}$ and all $p_1, \ldots, p_m \in \mathbb{R}^d$ the $m \times m$ matrix $\left(\psi(p_j) + \overline{\psi(p_k)} - \psi(p_j - p_k)\right)_{j,k=1,\ldots,m}$ is positive hermitian, i.e., if for all $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$

$$\sum_{j,k=1}^{m} \left(\psi(p_j) + \overline{\psi(p_k)} - \psi(p_j - p_k)\right) \lambda_j \overline{\lambda_k} \geq 0.$$

It is not hard to see from Definition C.0.4 that a negative definite function ψ has a non-negative real part $\operatorname{Re} \psi \geq 0$, satisfies $\overline{\psi(p)} = \psi(-p)$ and that $\sqrt{|\psi|}$ is subadditive, i.e.,

$$\sqrt{|\psi(p_1 + p_2)|} \leq \sqrt{|\psi(p_1)|} + \sqrt{|\psi(p_2)|}, \quad p_1, p_2 \in \mathbb{R}^d.$$

If ψ is continuous, repeated applications of the subadditivity estimate yield the following growth bound

$$|\psi(p)| \leq 2 \sup_{|y| \leq 1} |\psi(y)| \left(1 + |p|^2\right), \quad p \in \mathbb{R}^d. \tag{C.0.1}$$

A *negative* definite function is NOT the negative of a *positive definite* function. However, we have

Proposition C.0.5. *If $u : \mathbb{R}^d \to \mathbb{C}$ is a positive definite function, then the function $[p \mapsto u(0) - u(p)]$ is negative definite.*

The deeper connection between positive definite and negative definite functions can be seen from the following Theorem (cf. Jacob, 2001, Thm. 3.6.16, Thm. 3.7.7 and Thm. 3.7.8).

Theorem C.0.6. *(i) For any convolution semigroup $(\eta_t)_{t \geq 0}$ on $\mathbb{R}^d$ there exists a uniquely determined continuous negative definite function $\psi : \mathbb{R}^d \to \mathbb{C}$ such that*

$$\widetilde{\eta}_t(p) = (2\pi)^{-d/2} e^{-t\psi(p)}, \quad \text{for all } t \geq 0,\, p \in \mathbb{R}^d. \tag{C.0.2}$$

Conversely, given a continuous negative definite function $\psi : \mathbb{R}^d \to \mathbb{C}$, there exists a unique convolution semigroup $(\eta_t)_{t \geq 0}$ on $\mathbb{R}^d$ such that (C.0.2) holds.

(ii) Let $\psi : \mathbb{R}^d \to \mathbb{C}$ be a continuous negative definite function. Then there exist a constant $C \geq 0$, a vector $B \in \mathbb{R}^d$, a symmetric positive semidefinite matrix $A \in \mathbb{R}^{d \times d}$ and a Radon measure N on $\mathbb{R}^d \setminus \{0\}$ with $\int_{y \neq 0} |y|^2 (1 + |y|^2)^{-1} N(dy) < \infty$ such that ψ is given by the Lévy-Khintchine formula

$$\psi(p) = C + iB \cdot p + p \cdot Ap + \int_{y \neq 0} \left(1 - e^{iy \cdot p} + \frac{iy \cdot p}{1 + |y|^2}\right) N(dy). \tag{C.0.3}$$

The characteristics (C, B, A, N) are uniquely determined by ψ. Moreover, for any given quadruple (C, B, A, N), the right-hand side of (C.0.3) is well-defined and is a continuous negative definite function.

The function ψ is called the *characteristic exponent*, $C = \psi(0)$ is the *killing rate* since $\eta_t(\mathbb{R}^d) = (2\pi)^{d/2}\widetilde{\eta_t}(0) = e^{-t\psi(0)}$, (B, A, N) is the Lévy triplet, B is the drift coefficient, A is the covariance matrix and N is usually called the *Lévy measure*. Note, that there are different forms of the Lévy-Khintchine formula in the literature (see Jacob, 2001, Rem. 3.7.10) since there are different ways to choose the measure N in (C.0.3). Different choises of N cause changings of the drift coefficient B in (C.0.3).

Let $(\eta_t)_{t\geq 0}$ be the convolution semigroup associated with the continuous negative definite function ψ by (C.0.2) and $(\xi_t)_{t\geq 0}$ be a corresponding Lévy process (with killing) such that $P_{\xi_t} = \eta_t$. Consider the measures $(\eta_t^\sharp)_{t\geq 0}$ obtained from measures η_t by reflection at the origin $\eta_t^\sharp(B) := \eta_t(-B)$, $B \in \mathcal{B}(\mathbb{R}^d)$. Then $(\eta_t^\sharp)_{t\geq 0}$ is again a convolution semigroup and the associated continuous negative definite function $\psi^\sharp$ is obtained by reflection of the function ψ at the origin, i.e. $\psi^\sharp(p) := \psi(-p)$, $p \in \mathbb{R}^d$. On the Schwartz space $S(\mathbb{R}^d)$, we can[1] define *pseudo-differential operators* T_t^{bw}, $t \geq 0$, with *symbols* $e^{-t\psi^\sharp}$, i.e.

$$T_t^{bw}\varphi(x) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} e^{ip\cdot(x-q)}e^{-t\psi^\sharp(p)}\varphi(q)\,dqdp = \left(\mathcal{F}^{-1} \circ e^{-t\psi^\sharp} \circ \mathcal{F}\right)\varphi(x), \quad \text{(C.0.4)}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier and the inverse Fourier transforms, $\varphi \in S(\mathbb{R}^d)$. The convolution theorem yelds

$$\begin{aligned} T_t^{bw}\varphi(x) &= \varphi * \eta_t^\sharp(x) = \int\limits_{\mathbb{R}^d} \varphi(x-y)\eta_t^\sharp(dy) \\ &= \int\limits_{\mathbb{R}^d} \varphi(x+y)\eta_t(dy) = \int\limits_{\mathbb{R}^d} \varphi(y)P(t,x,dy) = \mathbb{E}^x[\varphi(\xi_t)]. \end{aligned} \quad \text{(C.0.5)}$$

The family $(T_t^{bw})_{t\geq 0}$ extends to a semigroup on the space $B_b(\mathbb{R}^d)$. This semigroup is called *backward semigroup* associated to the Lévy process (with killing) $(\xi_t)_{t\geq 0}$. Analogousely, one can consider pseudo-differential operators $(T_t^{fw})_{t\geq 0}$ with symbols $e^{-t\psi}$ on $S(\mathbb{R}^d)$. Then

$$T_t^{fw}\varphi(x) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} e^{ip\cdot(x-q)}e^{-t\psi(p)}\varphi(q)\,dqdp = \varphi * \eta_t(x) = \mathbb{E}^x[\varphi(-\xi_t)]. \quad \text{(C.0.6)}$$

The family $(T_t^{fw})_{t\geq 0}$ has similar properties as $(T_t^{bw})_{t\geq 0}$ and is called *forward semigroup* associated to the process $(\xi_t)_{t\geq 0}$. Note, that $(T_t^{bw})_{t\geq 0}$ is simultaneously the forward semigroup for the Lévy process (with killing) $(-\xi_t)_{t\geq 0}$. It can be easy checked that (extensions of) operators T_t^{fw} and T_t^{bw} are adjoint to each other on $L^2(\mathbb{R}^d)$. The further properties of $(T_t^{fw})_{t\geq 0}$ (and, therefore, of $(T_t^{bw})_{t\geq 0}$) are summarized in the following theorem (cf. Sato, 1999, Thm. 31.5, Applebaum, 2009, Thm. 3.4.2., Jacob, 2001, Thm. 4.4.3).

[1]Indeed, since $\operatorname{Re}\psi \geq 0$, we have $e^{-t\psi^\sharp}\mathcal{F}[\varphi] \in L^1(\mathbb{R}^d)$ for each $\varphi \in S(\mathbb{R}^d)$. Hence $T_t^{bw}\varphi \in C_\infty(\mathbb{R}^d)$.

Theorem C.0.7. *(i) The family $(T_t^{fw})_{t\geq 0}$ of pseudo-differential operators given on $S(\mathbb{R}^d)$ by (C.0.6) extends to a strongly continuous contraction semigroup (denoted again by $(T_t^{fw})_{t\geq 0}$) on each of the spaces $C_\infty(\mathbb{R}^d)$, $L^p(\mathbb{R}^d)$, $p \in [1,\infty)$.*

(ii) Let $(L^{fw}, \mathrm{Dom}(L^{fw}))$ be the generator of the semigroup $(T_t^{fw})_{t\geq 0}$ on the space $C_\infty(\mathbb{R}^d)$. Then $C_c^\infty(\mathbb{R}^d)$ is a core for L^{fw}, $C_\infty^2(\mathbb{R}^d) \subset \mathrm{Dom}(L^{fw})$ and

$$\begin{aligned} L^{fw}\varphi(x) = &- C\varphi(x) - B\cdot\nabla\varphi(x) + \mathrm{tr}(A\,\mathrm{Hess}\,\varphi(x)) \\ &+ \int\limits_{y\neq 0}\left(\varphi(x+y) - \varphi(x) - \frac{y\cdot\nabla\varphi(x)}{1+|y|^2}\right)N(dy) \end{aligned} \tag{C.0.7}$$

for $\varphi \in C_\infty^2(\mathbb{R}^d)$, where (C,B,A,N) are the characteristics of the continuous negative definite function ψ as in Theorem C.0.6.

It is easy to see that the operator $(L^{fw}, C_\infty^2(\mathbb{R}^d))$ given by (C.0.7) extends the pseudo-differential operator with symbol $-\psi$ defined on $S(\mathbb{R}^d)$. So, let us note, that symbols of the (restricted to $S(\mathbb{R}^d)$) operators L^{fw} and $T_t^{fw} = e^{tL^{fw}}$ are connected in the natural manner: they are $-\psi$ and $e^{-t\psi}$ respectively.

Example C.0.8. Let us present some important continuous negative definite functions $\psi : \mathbb{R}^d \to \mathbb{C}$ and the related objects.

(a) $\psi(p) = C \geq 0$. Hence L^{fw} is multiplication with the constant $-C$.

(b) $\psi(p) = iB\cdot p$ for some $B \in \mathbb{R}^d$. Then $L^{fw} = -B\cdot\nabla$ and the corresponding semigroup is the translation semigroup $(T_t^{-B\nabla})_{t\geq 0}$ discussed in Example A.0.18. This semigroup is the forward semigroup for the process $(\xi_t)_{t\geq 0}$ which is the deterministic drift $\xi_t = Bt$ in the direction $B/|B|$ with speed $|B|$. The corresponding convolution semigroup is given by $\eta_t := \delta_{tB}$, where δ_{tB} is the Dirac delta measure concentrated at the point $tB \in \mathbb{R}^d$.

(c) $\psi(p) = \frac{1}{2}|p|^2$. Then $L^{fw} = \frac{1}{2}\Delta$, the corresponding convolution semigroup is given by $\eta_t(dx) = (2\pi t)^{-d/2}\exp\left\{-\frac{|x|^2}{2t}\right\}$, and $(\xi_t)_{t\geq 0}$ is a standard Brownian motion. In this case, we have also $T_t^{fw} = T_t^{bw}$ on $L^2(\mathbb{R}^d)$ and the generator $(L, \mathrm{Dom}(L)) := (L^{fw}, \mathrm{Dom}(L^{fw})) = (L^{bw}, \mathrm{Dom}(L^{bw})) = \overline{(\frac{1}{2}\Delta, S(\mathbb{R}^d))}^{L^2(\mathbb{R}^d)}$ is a self-adjoint operator.

(d) $\psi(p) = |p|^\alpha$ with $\alpha \in (0,2)$. Then the Lévy characteristics are $C = 0$, $B = 0$, $A = 0$ and $N(dy) = k_\alpha|y|^{-d-\alpha}dy$ with $k_\alpha = \alpha 2^{\alpha-1}\pi^{-d/2}\Gamma\left(\frac{\alpha+d}{2}\right)/\Gamma\left(1-\frac{\alpha}{2}\right)$ and Euler Gamma-function Γ. The corresponding generator L is called *fractional Laplacian* and is denoted as $-(-\Delta)^{\alpha/2}$. A corresponding Lévy process $(\xi_t)_{t\geq 0}$ is a (symmetric) α-stable process; in particular, ξ_{at} has the same law as $a^{1/\alpha}\xi_t$ for all $a > 0$, $t \geq 0$. The corresponding convolution semigroup $(\eta_t)_{t\geq 0}$ is known in closed form only for the case $\alpha = 1$. In this case, it is given by the Cauchy distribution $\eta_t(dx) := \Gamma\left(\frac{d+1}{2}\right)\frac{t}{\pi(|x|^2+t^2)^{(d+1)/2}}$.

(e) $\psi(p) = 1 - e^{ia\cdot p}$, with some fixed $a \in \mathbb{R}^d$. Then the Lévy characteristics are $C = 0$, $A = 0$, $B = -\frac{a}{1+|a|^2}$, $N = \delta_a$. A corresponding process $(\xi_t)_{t\geq 0}$

is a Poisson process with the jump "size" $-a$ (and the intensity of jumps 1): $\mathbb{P}(\xi_t = -an) = e^{-t}t^n/(n!)$. The (forward) Poisson semigroup $(T_t)_{t\geq 0}$ on $C_\infty(\mathbb{R}^d)$ is defined via $T_t\varphi(x) := e^{-t}\sum_{n=0}^{\infty}\varphi(x+na)t^n/(n!)$ and its (bounded) generator L is given by $L\varphi(x) = \varphi(x+a) - \varphi(x)$.

(f) $\psi(p) = \sqrt{|p|^2 + m^2}$ for some $m > 0$. This function is the symbol of a relativistic Hamiltonian. The corresponding convolution semigroup $(\eta_t)_{t\geq 0}$ is known in explicit form, see formula (3.251) on page 182 of Jacob, 2001.

Appendix D

Some results on generation of strongly continuous semigroups on different Banach spaces

Definition D.0.1. Let $(A, \mathrm{Dom}(A))$ and $(B, \mathrm{Dom}(B))$ be two linear operators in X such that $\mathrm{Dom}(A) \subset \mathrm{Dom}(B)$ and for some $\alpha \in [0,1)$ and $\beta \geq 0$

$$\|B\varphi\|_X \leq \alpha \|A\varphi\|_X + \beta \|\varphi\|_X$$

holds for all $\varphi \in \mathrm{Dom}(A)$. Then the operator B is called *A-bounded* and α is called an *A-bound* for B.

Example D.0.2 (Example III.2.2. of Engel and Nagel, 2000). Let $I \subseteq \mathbb{R}$ be an interval, $X_p := L^p(I)$, $p \in [1, \infty]$, and $X_0 := C_0(I)$. Consider the operators A and B in Banach spaces X_p and X_0 such that

$$\begin{aligned}
&A := \frac{d^2}{dx^2} \quad \text{with} \\
&\mathrm{Dom}(A) := W^{2,p}(I) \subset X_p \quad \text{or} \quad \mathrm{Dom}(A) := \left\{\varphi \in C_0^2(I) \,|\, \varphi', \varphi'' \in C_0(I)\right\} \subset X_0, \\
&\text{and} \\
&B := \frac{d}{dx} \quad \text{with} \\
&\mathrm{Dom}(B) := W^{1,p}(I) \subset X_p \quad \text{or} \quad \mathrm{Dom}(B) := \left\{\varphi \in C_0^1(I) \,|\, \varphi' \in C_0(I)\right\} \subset X_0.
\end{aligned}$$

Then B is A-bounded with A-bound $\alpha = 0$.

Example D.0.3. Consider the Banach space $L^2(\mathbb{R}^d)$. Let ψ be a continuous negative definite function and $(L, S(\mathbb{R}^d))$ be the corresponding pseudo-differential operator, i.e. $L\varphi = [\mathcal{F}^{-1} \circ (-\psi) \circ \mathcal{F}]\,\varphi$ for all $\varphi \in S(\mathbb{R}^d)$ (this operator is well-defined on $S(\mathbb{R}^d)$ via Berg and Forst, 1973, cf. Thm. 3.4.4 in Applebaum, 2009). Due to (C.0.1), we have for each $\varphi \in S(\mathbb{R}^d)$ with $C := 2\sup_{|y|\leq 1} |\psi(y)|$

$$\|L\varphi\|_2 \leq \|\mathcal{F}^{-1}\| \cdot \|(-\psi)\mathcal{F}[\varphi]\|_2 \leq C\|(1+|p|^2)\mathcal{F}[\varphi](p)\|_2 \leq C\left(\|\varphi\|_2 + \|\Delta\varphi\|_2\right).$$

Consider now the particular case $\psi(p) := iB \cdot p$ for a given $B \in \mathbb{R}^d$, i.e. $L\varphi = -B \cdot \nabla\varphi$, $\varphi \in S(\mathbb{R}^d)$. Since $|p| \le (1 + |p|^2)/2$, we have (with $C := |B|/2$)

$$\begin{aligned}\|L\varphi\|_2 &\le \|\,|B| \cdot |p|\,\mathcal{F}[\varphi](p)\|_2 \\ &\le \frac{|B|}{2}\left(\|\mathcal{F}[\varphi](p)\|_2 + \||p|^2\mathcal{F}[\varphi](p)\|_2\right) = \frac{|B|}{2}\left(\|\varphi\|_2 + \|\Delta\varphi\|_2\right).\end{aligned}$$

Take now any $a > 0$ and consider the function $\widetilde{\varphi}_a$ defined by the equality $\widetilde{\varphi}_a(p) = a^{1+d/2}\mathcal{F}[\varphi](ap)$ for all $p \in \mathbb{R}^d$. Then

$$\begin{aligned}\|\,|B| \cdot |p|\,\mathcal{F}[\varphi](p)\|_2 &= \|\,|B| \cdot |p|\,\widetilde{\varphi}_a(p)\|_2 \\ &\le \frac{|B|}{2}\left(\|\widetilde{\varphi}_a(p)\|_2 + \|\,|p|^2\widetilde{\varphi}_a(p)\|_2\right) \\ &= \frac{|B|}{2}\left(a\|\mathcal{F}[\varphi](p)\|_2 + a^{-1}\|\,|p|^2\mathcal{F}[\varphi](p)\|_2\right) \\ &= \frac{|B|}{2}\left(a\|\varphi\|_2 + a^{-1}\|\Delta\varphi\|_2\right).\end{aligned}$$

Therefore, the operator $L = -B\cdot\nabla$ is Δ-bounded, and its Δ-bound is any number from the interval $(0, 1)$. Similarly, using the inequality $|p| \le (1 + |p|^\alpha)$ for all $\alpha \in (1, 2]$, one obtains that the operator $L = -B \cdot \nabla$ is $-(-\Delta)^{\alpha/2}$-bounded, and its $-(-\Delta)^{\alpha/2}$-bound is any number from the interval $(0, 1)$.

The following result can be found in Ethier and Kurtz, 1986, Thm. 1.7.1, and Jacob, 2001, Thm. 4.4.3.

Theorem D.0.4. *Let $(A, \mathrm{Dom}(A))$ be closable and its closure be the generator of a strongly continuous contraction semigroup on X. If $(B, \mathrm{Dom}(B))$ is an A-bounded dissipative operator on X, then the operator $(A+B, \mathrm{Dom}(A))$ is closable and its closure generates a strongly continuous contraction semigroup on X. Moreover, $\overline{A+B} = \overline{A} + B$; and if A is closed then $\mathrm{Dom}(\overline{A+B}) = \mathrm{Dom}(A)$.*

Corollary D.0.5. *Let $(A, \mathrm{Dom}(A))$ be the generator of a strongly continuous semigroup $(T_t)_{t\ge0}$ on X, satisfying $\|T_t\| \le Me^{\omega t}$. If B is a bounded linear operator on X, then the operator $(A+B, \mathrm{Dom}(A))$ is the generator of a strongly continuous semigroup $(U_t)_{t\ge0}$, satisfying $\|U_t\| \le Me^{(\omega+M\|B\|)t}$.*

Theorem D.0.6 (Pazy, 1983, Cor. III.3.5)**.** *Let X be a reflexive Banach space and an operator $(A, \mathrm{Dom}(A))$ be the generator of a strongly continuous contraction semigroup on X. Let an operator $(B, \mathrm{Dom}(B))$ be dissipative such that $\mathrm{Dom}(B) \supset \mathrm{Dom}(A)$ and*

$$\|B\varphi\|_X \le \|A\varphi\|_X + \beta\|\varphi\|_X$$

for all $\varphi \in \mathrm{Dom}(A)$ and some $\beta \ge 0$. Then the closure of $(A + B, \mathrm{Dom}(A))$ also generates a strongly continuous contraction semigroup on X.

Some other results on additive perturbations can be found in Lumer, 1989a. Some generation results in the case of relatively bounded but not dissipative perturbations are discussed in Lasiecka and Triggiani, 1985. Perturbation theory for analytic semigroups is available, e.g., in Kato, 1966. Perturbations of positive semigroups are discussed in Arendt and Rhandi, 1991 (see also references therein). Generalizing the techniques of Dyson–Phillips perturbation series, some generation results (in particular, for gradient perturbations and perturbations by singular potentials) are obtained in the series of works Bogdan, Hansen, and Jakubowski, 2008; Bogdan and Jakubowski, 2012; Bogdan, Hansen, and Jakubowski, 2013; Bogdan and Szczypkowski, 2014; Bogdan and Sydor, 2015; Bogdan, Butko, and Szczypkowski, 2016. Perturbation theory in L^p for sub-Markovian generators can be founded, e.g., in Liskevich, Perelmuter, and Semenov, 1996.

Theorem D.0.7 (Dorroh, 1966)**.** *Let Q be a set. Let X be a complex Banach space (under the supremum norm) of bounded complex valued functions on Q, a be a bounded positive function on Q which is bounded away from zero, $aX \subset X$, and $(L, \mathrm{Dom}(L))$ be the generator of a strongly continuous contraction semigroup on X. Then the operator $(\widehat{L}, \mathrm{Dom}(L))$ with $\widehat{L}\varphi(q) := a(q)L\varphi(q)$ for all $\varphi \in \mathrm{Dom}(L)$, $q \in Q$, is also the generator of a strongly continuous contraction semigroup on X.*

Theorem D.0.8 (Lumer, 1973)**.** *Let Q be a separable locally compact metric space. Let $X = C_0(Q)$. Let $a \in C_b(Q)$ and $a > 0$ on Q. Let $(L, \mathrm{Dom}(L))$ be the generator of a strongly continuous contraction semigroup on X. Then the closure of $(\widehat{L}, \mathrm{Dom}(L))$ with $\widehat{L}\varphi(q) := a(q)L\varphi(q)$ for all $\varphi \in \mathrm{Dom}(L)$, $q \in Q$, is also the generator of a strongly continuous contraction semigroup on X. The same statement holds in the case when X is a Banach subspace of $C_0(Q)$, invariant under the function a.*

Further results on multiplicative perturbations can be found in Lumer, 1989b. For the case of Banach spaces $L^p(Q)$, see, e.g., Dorroh and Holderrieth, 1993. Multiplicative perturbations of the Laplace operator are considered also in Altomare, Milella, and Musceo, 2011.

Let now G be either $\mathbb{R}^d$ or an open bounded domain in $\mathbb{R}^d$ with uniformly C^2 boundary ∂G. Consider a second order differential operator L,

$$L\varphi(x) := \mathrm{tr}(A(x)\,\mathrm{Hess}\,\varphi(x)) - B(x)\cdot\nabla\varphi(x) - C(x)\varphi(x)$$

$$\equiv \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2\varphi}{\partial x_i \partial x_j}(x) - \sum_{i=1}^{d} B_i(x)\frac{\partial\varphi}{\partial x_i}(x) - C(x)\varphi(x).$$

We assume that the coefficients a_{ij}, B_i, C are real, bounded and continuous on $\mathbb{R}^d$, the matrix $A = (a_{ij})_{i,j=1,\dots,d}$ is symmetric and satisfies for some $a_0 > 0$ the uniform ellipticity condition

$$A(x)z\cdot z \geq a_0|z|^2, \quad \forall\, x \in \overline{G}, z \in \mathbb{R}^d.$$

The following Theorem summarizes the results of Cor. 3.1.9 (iii), Thm. 3.1.7, Cor. 3.1.21 (i) and (ii), Rem. 2.1.5. in Lunardi, 1995.

Theorem D.0.9. *Let the coefficients a_{ij}, B_i, C be uniformly continuous on G.*

(i) Let $G = \mathbb{R}^d$ and $X = C_\infty(\mathbb{R}^d)$. Define

$$\mathrm{Dom}(L) := \left\{\varphi \in \cap_{p\geq 1} W^{2,p}_{loc}(\mathbb{R}^d) \,:\, \varphi, L\varphi \in X\right\}.$$

Then $(L, \mathrm{Dom}(L))$ generates a strongly continuous semigroup on X. Moreover, this semigroup is analytic. And $\mathrm{Dom}(L)$ is continuously embedded in $C^{1,\alpha}(\mathbb{R}^d)$ for every $\alpha \in (0,1)$.

(ii) Let G be an open bounded domain in $\mathbb{R}^d$ with uniformly C^2 boundary ∂G. Let $Y = C_0(G)$. Define

$$\mathrm{Dom}(L) := \left\{\varphi \in \cap_{p\geq 1} W^{2,p}_{loc}(G) \,:\, \varphi, L\varphi \in Y\right\}.$$

Then $(L, \mathrm{Dom}(L))$ generates a strongly continuous semigroup on Y. Moreover, this semigroup is analytic. And $\mathrm{Dom}(L)$ is continuously embedded in $C^{1,\alpha}(\overline{G})$ for every $\alpha \in (0,1)$.

Theorem D.0.10 (Jacob, 2002, Thm. 2.1.43.)**.** *Let $G = \mathbb{R}^d$. Let $a_{ij} \in C^3_b(\mathbb{R}^d)$, $B_i \in C^2_b(\mathbb{R}^d)$ and $C \in C^1_b(\mathbb{R}^d)$. Then the closure of the operator $(L, W^{3,p}(\mathbb{R}^d))$, $p > d$, generates a strongly continuous semigroup on $X = C_\infty(\mathbb{R}^d)$.*

More results on generation of strongly continuous semigroups on the spaces of continuous functions (in particular, Feller semigroups generated by second order elliptic operators) can be found in Arendt and Schätzle, 2014; Arendt and Bénilan, 1999; Fornaro and Lorenzi, 2007; Da Prato and Goldys, 2001; Cerrai, 2000; Roth, 1978; Roth, 1977; Roth, 1976.

Let Q be a locally compact separable space. Let $(T_t)_{t\geq 0}$ be a Feller semigroup on $C_\infty(Q)$ (see Section 3.3 for the definition). This semigroup is, a priori, defined on (or can be extended onto) the space $B_b(Q)$. In several situations, the operators $T_t\big|_{C_c(Q)}$ can be extended onto spaces of integrable functions $L^p(Q,m)$, where m is a positive Radon measure with full topological support (i.e. $m(U) > 0$ for any open sets $U \subset Q$). We call the operators T_t *m-symmetric* if

$$\int_Q T_t\varphi(x)\cdot u(x)\, m(dx) = \int_Q \varphi(x)\cdot T_t u(x)\, m(dx), \quad \forall\ \varphi, u \in C_c(Q).$$

Denote by T_t^* the formal adjoint of T_t with respect to $L^2(Q,m)$, i.e. the linear operator defined by

$$\int_Q T_t\varphi(x)\cdot u(x)\, m(dx) = \int_Q \varphi(x)\cdot (T_t^* u)(dx), \quad \forall\ \varphi, u \in C_c(Q).$$

Note that $(T_t^* u)$ is a bounded Radon measure since the set of all bounded Radon measures is the topological dual of $C_\infty(Q)$.

Proposition D.0.11 (cf. Böttcher, Schilling, and Wang, 2013, Lemma 1.45)**.** *Let $(T_t)_{t\geq 0}$ be a Feller semigroup and assume that the operators T_t are m-symmetric or that* *the operators*

$$T_t^{\circledast} := T_t^{*}\big|_{L^1(Q,m)}$$

map $L^1(Q,m)$ *into itself and are sub-Markovian, i.e.* $0 \le T_t^{\circledast}\varphi \le 1$ *for all* $\varphi \in L^1(Q,m)$ *with* $0 \le \varphi \le 1$. *Then* $(T_t)_{t\ge 0}$ *has for every* $1 \le p < \infty$ *an extension* $(T_t^{(p)})_{t\ge 0}$ *to a strongly continuous, positivity preserving, sub-Markovian contraction semigroup on* $L^p(Q,m)$.

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