Calculus of Variations
Summer Term 2014

Lecture 1

28. April 2014

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  Office hours: Wednesday 09:00-10:00 or by appointment

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- More details on the web page:

  http://www.math.uni-sb.de/ag/fuchs/ag-fuchs.html

What type of Lectures is it?

- Lectures (3h) with exercises (1h) (6 ECTS points)
- Time and Location: **Wednesday 12-14 c.t.** and **Friday 10-12 c.t.**, Building E1.3, HS 001
- Exercises: Every second Wednesday instead of a lecture
- First exercise: 14 May 2014

What Prerequisites are required?

- Undergraduate knowledge in mathematics
  (i.e., Calculus I and II, Linear Algebra, basic knowledge of PDEs)
- Passive knowledge of simple English.
  (solutions in assignments or exam can be also submitted in German or in Russian)
- Basic knowledge of Maple

Assignments:

- Homework will be assigned bi-weekly.
- To qualify for the exam you need 50% of the points from these assignments.
- Working in groups of up to 2 people is permitted

Exams:

There will be a written/oral exam.

Contents:

1. The Euler-Lagrange equation,
2. Isoperimetric problems,
3. Broken extremals,
4. Direct methods in calculus of variations,
5. Variational problems in image processing

Script:

Course material would be available on the webpage in order to **support** the classroom teaching, **not to replace** it.

Additional organisational information, examples and explanations that may be relevant for your understanding and the exam are provided in the lectures.

It is solely **your** responsibility to make sure that you receive this infomation.

Literature:

- G. Aubert, P. Kornprobst,
  Mathematical Problem in Image Processing. PDEs and the Calculus of Variations
  Applied Mathematical Sciences, vol. 147
  Springer, 2006
- G. Butazzo, M. Giaquinta, S. Hildebrandt
  One-dimensional Variational Problems: An Introduction.
  Oxford University Press, 1998
- R. Weinstock
  Calculus of Variations with Applications to Physics and Engineering
  Dover Publications, Inc. New York, 1974

Purpose of Lesson:

- To introduce the idea of a functional (function of a function).
- A very common type of functionals is the integral

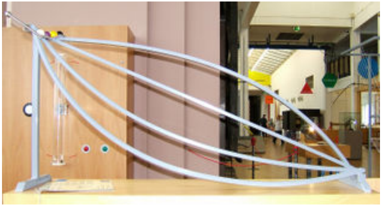
$$J[y] = \int_a^b F(x, y, y')dx$$

  where the functional J is considered a function of y (a function), and the integrand $F(x, y, y')$ is assumed known.

- We also show how to find the function $\bar{y}(x)$ that minimizes $J[y]$ by finding an equation (**Euler-Lagrange equation**) in $\bar{y}$ that must always be true when $\bar{y}$ is a minimizing function.

§1. Introduction and Examples

- Calculus of variations was originally studied about the same time as calculus and deals with maximizing and minimizing **functions of functions** (called functionals).
- Calculus $\Longleftrightarrow$ Calculus of Variations
- One of the first problems in calculus of variations was the Brachistochrone problem proposed by John Bernoulli in 1696.
- **Brachistochrone Problem:** to find the path $y(x)$ that minimizes the sliding time of a particle along a frictionless path between two points.



- **Solution of the Brahistochrone problem:**
  Bernoulli showed that the sliding time T could be written

$$T = \int_0^T dt = \int_0^L \frac{dt}{ds} ds = \int_0^L \frac{ds}{v} = \frac{1}{\sqrt{2g}} \int_0^L \frac{ds}{\sqrt{y}}$$
$$= \frac{1}{\sqrt{2g}} \int_a^b \sqrt{\frac{1 + y'^2}{y}} dx$$

- Hence, the total time $T[y]$ can be thought of as a function of a function.

- Since many functionals in nature are of the above type, it will suffice for us to study the general form

$$J[y]=\int_a^b F(x,y,y')dx \tag{1.1}$$

- **Our goal:** to find functions $y(x)$ that minimize (or maximize) functionals of the form (1.1).
- The strategy for this task is somewhat the same as for minimizing functions $f(x)$ in calculus.
- In calculus, we found the critical points of a function by setting $f'(x)=0$ and solving for x. In calculus of variations, things are much more subtle, since our argument is not a number, but, in fact, a function itself. However, the general philosophy is the same.

- We take a functional derivative (so to speak) with respect to the function $y(x)$ and set this to zero. This new equation is analogous to the equation

$$\frac{df(x)}{dx} = 0$$

from calculus, but now it is an ordinary differential equation, known as the Euler-Lagrange equation.

Problem 1-1

We consider the problem of finding the function $y(x)$ that minimizes the functional

$$J[y] = \int_a^b F(x, y, y')dx$$

among a class of smooth functions satisfying the boundary conditions

$$y(a) = A$$
$$y(b) = B$$

- To find the minimizing function, call it $\bar{y}$, we introduce a small variation from $\bar{y}(x)$, namely

$$\bar{y}(x) + \varepsilon\eta(x)$$

where ε is a small parameter and $\eta(x)$ is a smooth curve satisfying the BC

$$\eta(a) = \eta(b) = 0.$$

- it should be clear that if we evaluate the integral J at a neighboring function $\bar{y} + \varepsilon\eta$, then the functional J will be greater; that is

$$J[\bar{y}] \leqslant J[\bar{y} + \varepsilon\eta]$$

for all ε.

- Our strategy for finding $\bar{y}$ is to take the derivative of

$$\phi(\varepsilon) = J[\bar{y} + \varepsilon\eta]$$

with respect to ε, evaluate it $\varepsilon = 0$, and set this equal to zero; that is,

$$\begin{aligned} \frac{d\phi(\varepsilon)}{d\varepsilon} &= \frac{d}{d\varepsilon} J[\bar{y} + \varepsilon\eta]\Big|_{\varepsilon=0} \\ &= \int\limits_a^b \left[\frac{\partial F}{\partial \bar{y}}\eta(x) + \frac{\partial F}{\partial \bar{y}'}\eta'(x)\right] dx \end{aligned}$$

From integration by parts, we have

$$\int\limits_a^b \left\{\frac{\partial F}{\partial \bar{y}} - \frac{d}{dx}\left[\frac{\partial F}{\partial \bar{y}'}\right]\right\} \eta(x) dx = 0. \tag{1.2}$$

- Since the integral (1.2) is zero for any function $\eta(x)$ satisfying the BCs
$$\eta(a) = \eta(b) = 0,$$
it is fairly obvious that the remaining portion of the integrand must be zero; that is
$$\boxed{\frac{\partial F}{\partial \bar{y}} - \frac{d}{dx}\left[\frac{\partial F}{\partial \bar{y}'}\right] = 0}. \tag{1.3}$$

- Equation (1.3) is known as the Euler-Lagrange equation, and although it may look complicated in its general form, once we substitute in specific functions $F(x, y, y')$, we will see immediately that it is just a second-order, ODE in the dependent variable $\bar{y}$.

- In other words, we can solve (1.3) for the minimizing function $\bar{y}$.

So, what we have shown is that (we drop the notation $\bar{y}$ and jusy call it y)

If y minimizes $J[y] = \int\limits_a^b F(x, y, y')dx$, then y must satisfy the equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] = 0$$

Remarks

- The Euler-Lagrange equation is analogous to setting the derivative equal to zero in calculus. We don't always find the minimum (or maximum, for this matter) by this process. The same holds true with the Euler-Lagrange equation.
- We should think of Euler-Lagrange equation as a necessary condition that must be true for minimizing functions but may be true for other functions as well.
- Basic principles of physics are often stated in terms of minimizing principles rather than differential equations.

Remarks (cont.)

- Calculus of variations ideas can also be used to minimize important **multiple-integral functionals** like

$$J[u] = \int \int_D F(x, y, u, u_x, u_y) dxdy$$

where the corresponding Euler-Lagrange equation is

$$F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} = 0 \qquad (PDE)$$