Calculus of Variations
Summer Term 2014

Lecture 10

4. Juni 2014

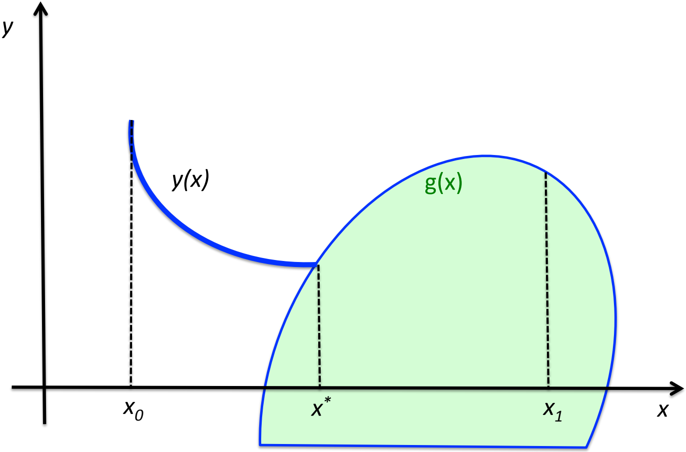
Purpose of Lesson:

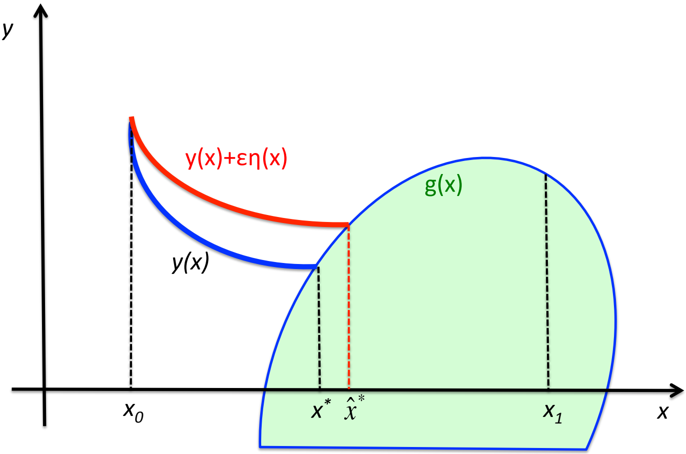
- To prove a general result about problems with inequality constraints

§6. Inequality constraints (cont.)

General result

If $F_{y'}$ depends on y', then at the point where the extremal transfers from the Euler-Lagrange curve to the domain boundary the tangent varies continuously.


y
y(x)
g(x)
x
x_0
x^*
x_1


y
y(x)+εη(x)
g(x)
y(x)
x0
x*
x̂*
x1
x

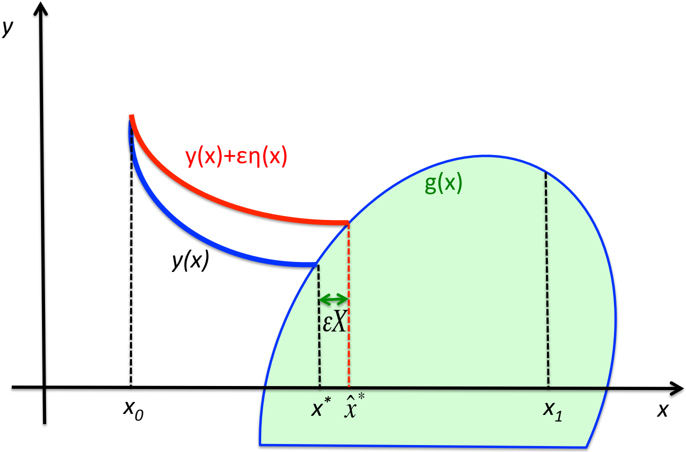
- We break the integral into two parts:

$$J[y] = J_1[y] + J_2[y] = \int_{x_0}^{x^*} F(x, y, y')dx + \int_{x^*}^{x_1} F(x, y, y')dx$$

- we assume the shape of the curve on the RHS of x^* fits the boundary, e.g. $y(x) = g(x)$, and the LHS follows the Euler-Lagrange equations

$$J[y] = J_1[y] + J_2[y] = \int_{x_0}^{x^*} F(x, y, y')dx + \int_{x^*}^{x_1} F(x, g, g')dx$$

- So, we study the functional $J_1[y]$ with free right-end x^* satisfying the condition $y(x^*) = g(x^*)$.


y
y(x)+εη(x)
g(x)
y(x)
εX
x0
x*
x̂*
x1
x

- As before, differentiating the function $\phi_1(\varepsilon) = J_1[y + \varepsilon\eta]$ with respect to ε, taking into account the formula of differentiation of the integral (cf exercise 2) and setting $\varepsilon = 0$, we arrive at

$$0 = \frac{d\phi_1(\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int\limits_{x_0}^{\hat{x}^*} F(x, y + \varepsilon\eta, y' + \varepsilon\eta')dx$$

$$= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int\limits_{x_0}^{x^*+\varepsilon X} F(x, y + \varepsilon\eta, y' + \varepsilon\eta')dx$$

$$= XF(x, y, y')\big|_{x=x^*} + \int\limits_{x_0}^{x^*} \left(F_y\eta + F_{y'}\eta'\right) dx$$

$$= XF(x, y, y')\big|_{x=x^*} + F_{y'}\eta\big|_{x=x^*} + \int\limits_{x_0}^{x^*} \left(F_y - \frac{d}{dx}F_{y'}\right) \eta dx$$

- Note that $\left[\eta F_{y'}\right]_{x=x^*}$ is no longer simple to calculate because we don't fix x^*.
- How can we learn x^*?
- We need a new natural boundary condition that will give us this.
- The perturbed point $(\hat{x}^*, \hat{y}^*)$ and perturbation function η must satisfy certain conditions to be compatible.
- Remember that

$$\hat{x}^* = x^* + \varepsilon X$$
$$\hat{y}^* = y^* + \varepsilon Y$$

- Notice that

$$\hat{y}^* = y(x^* + \varepsilon X) + \varepsilon \eta(x^* + \varepsilon X).$$

- From Taylor's theorem, for small ε

$$\begin{aligned} y(x^* + \varepsilon X) &= y(x^*) + \varepsilon X y'(x^*) + O(\varepsilon^2) \\ &= y^* + \varepsilon X y'(x^*) + O(\varepsilon^2) \\ \varepsilon\eta(x^* + \varepsilon X) &= \varepsilon\eta(x^*) + O(\varepsilon^2) \end{aligned}$$

- So

$$\begin{aligned} y^* + \varepsilon Y &= y^* + \varepsilon X y'(x^*) + \varepsilon\eta(x^*) + O(\varepsilon^2) \\ \varepsilon Y &= \varepsilon X y'(x^*) + \varepsilon\eta(x^*) + O(\varepsilon^2) \\ \eta(x^*) &= Y - X y'(x^*) + O(\varepsilon) \end{aligned}$$

- Thus, we have

$$\boxed{\eta(x^*) = Y - X y'(x^*) + O(\varepsilon)} \tag{10.1}$$

- Substituting the compatibility constraint (10.1) into the our first variation we get

$$0 = \left[XF + F_{y'}\eta\right]_{x=x^*} + \int\limits_{x_0}^{x^*} \left(F_y - \frac{d}{dx}F_{y'}\right)\eta dx$$

$$= XF\Big|_{x=x^*} + \left[Y - Xy'(x^*)\right]F_{y'}\Big|_{x=x^*} + \int\limits_{x_0}^{x^*} \left(F_y - \frac{d}{dx}F_{y'}\right)\eta dx$$

$$= X\left[F - y'F_{y'}\right]_{x=x^*} + YF_{y'}\Big|_{x=x^*} + \int\limits_{x_0}^{x^*} \left(F_y - \frac{d}{dx}F_{y'}\right)\eta dx$$

- So, we get an integral term which results in the E-L equation, plus the additional constraint

$$X\left[F - y'F_{y'}\right]_{x=x^*} + YF_{y'}\Big|_{x=x^*} = 0 \qquad (10.2)$$

- Due to condition $y(x^*) = g(x^*)$ we cannot consider arbitrary (X, Y). In fact

$$\hat{y}^* = g(\hat{x}^*) = g(x^* + \varepsilon X), \qquad \hat{y}^* = y^* + \varepsilon Y$$
$$y^* = g(x^*)$$

Therefore,

$$\varepsilon Y = g(x^* + \varepsilon X) - g(x^*) = g'(x^*)\varepsilon X + O(\varepsilon^2)$$
$$Y = g'(x^*)X$$

- Assuming that $\dfrac{dg}{dx}$ is defined and substituting $Y = g'(x)X$ into (10.2) we get the condition

$$X\left\{g'F_{y'} + F - y'F_{y'}\right\}\Big|_{x=x^*} = 0,$$

and, consequently,

$$\boxed{\left\{g'F_{y'} + F - y'F_{y'}\right\}\Big|_{x=x^*} = 0}. \qquad (10.3)$$

- From (10.3) it follows that we may write the condition in x^* in terms of limits from the left and right, e.g.

$$\left[g' F_{y'} + F - y' F_{y'}\right]_{x^*-} - \left[g' F_{y'} + F - y' F_{y'}\right]_{x^*+} = 0$$

- Taking into account that $y' = g'$ on the RHS of x^* we get

$$\begin{aligned} 0 &= \left[g' F_{y'} + F - y' F_{y'}\right]_{x^*-} - \left[g' F_{y'} + F - g' F_{y'}\right]_{x^*+} \\ &= \left[(g' - y') F_{y'} + F\right]_{x^*-} - F\big|_{x^*+} \end{aligned}$$

or

$$\left[(g' - y') F_{y'}\right]_{x^*-} = F\big|_{x^*+} - F\big|_{x^*-}. \tag{10.4}$$

- Consider the term $\left\{ F\Big|_{x^{*+}} - F\Big|_{x^{*-}} \right\}$.
- Note that at the "join" $y(x^*) = g(x^*)$, so if the two limits of F differ it is because of a difference in y' on either side of the join.
- Treat F as a function of just y', i.e.,

$$F(x, y, y') = q_{x,y}(y') = q(y').$$

- Taking $q(y') = F(x, y, y')$ we get

$$\frac{d}{dz} q(z) = \frac{\partial F}{\partial y'}(x, y, y')\Big|_{y'=z}.$$

So

$$q'(c) = \frac{\partial F}{\partial y'}(x^*, y^*, c).$$

- Hence

$$F\Big|_{x^*+} - F\Big|_{x^*-} = q(g'(x^*)) - q(y'(x^*)) = \left[g'(x^*) - y'(x^*)\right] q'(c) = \left[g'(x^*) - y'(x^*)\right] \frac{\partial F}{\partial y'}(x^*, y^*, c)$$

- So, the condition (10.4) can be rewritten as follows

$$\left[(g' - y')\frac{\partial F}{\partial y'}\right]_{x^*-} = \left[g'(x^*) - y'(x^*)\right] \frac{\partial F}{\partial y'}(x^*, y^*, c)$$

- Hence

$$\left[(g' - y')\left(\frac{\partial F}{\partial y'}(x, y, y') - \frac{\partial F}{\partial y'}(x, y, c)\right)\right]_{x=x^*} = 0$$

for some c between $g'(x^*)$ and $y'(x^*)$.

$$(g'(x^*) - y'(x^*)) \left(\frac{\partial F}{\partial y'}(x^*, y(x^*), y'(x^*)) - \frac{\partial F}{\partial y'}(x^*, y(x^*), c) \right) = 0$$

So, there are two possibilities

- $g'(x^*) = y'(x^*)$, which means that y meets the boundary at a tangent to the boundary.
- $F_{y'}(x, y, y') - F_{y'}(x, y, c) = 0$. This latter condition holds when $F_{y'}$ is constant with respect to y', i.e.,

$$\frac{\partial^2 F}{\partial y'^2} = 0.$$

Remark

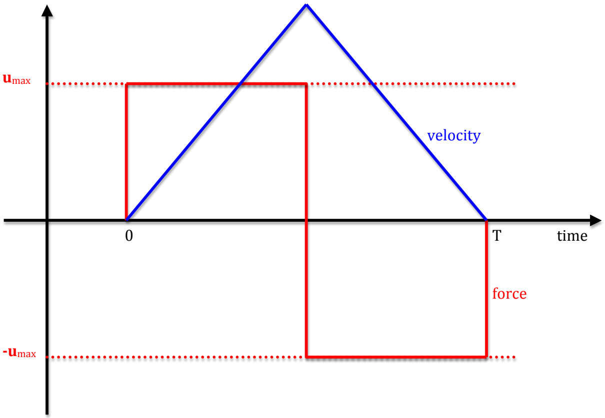
In the lake example, $F_{y'y'} \neq 0$.

Example 10.1 : parking a car (see Example 9.1)

- Revisit the problem of parking a car.
- If we think about the problem, it makes no sense unless there is maximum force u_{max}.
  - Otherwise we move from A to B arbitrarily fast.
- There are no valid E-L equation solutions.
- We must end-up in the boundary domain, e.g. $u = \pm u_{max}$.
  - Obvious solution is to accelerate as fast as possible until we get half-way, and then to decelerate as fast as possible.
  - $\dfrac{\partial F}{\partial \dot{u}} = 0$, so we don't have to stress about continuity (u is not continuous either).

Example 10.1: parking a car (cont.)

- Our solution is in the boundary domain, e.g. $u = \pm u_{max}$



- called a bang-bang controller.