Calculus of Variations
Summer Term 2014

Lecture 12

26. Juni 2014

Purpose of Lesson:

- To discuss numerical solutions of the variational problems
- To introduce Euler's Finite Difference Method and Ritz's Method.

§9. Numerical Solutions

Numerical Solutions:

The Euler-Lagrange equations may be hard to solve.

Natural response is to find numerical methods.

1. Numerical solution of the Euler-Lagrange equations
   - We won't consider these here (see other courses)
2. Euler's finite difference method
3. Ritz (Rayleigh-Ritz)
   - In $2D$: Kantorovich's method

Euler's Finite Difference Method

- We can approximate our function (and hence the integral) onto a finite grid.
- In this case, the problem reduces to a standard multivariable maximization (or minimization) problem, and we find the solution by setting the derivatives to zero.
- In the limit as the grid gets finer, this approximates the Euler-Lagrange equations.

Numerical approximation of integrals:

- use an arbitrary set of mesh points

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

- approximate

$$y'(x_i) = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} = \frac{\Delta y_i}{\Delta x_i}$$

- rectangle rule

$$J[y] = \int\limits_a^b F(x, y, y')dx \simeq \sum_{i=0}^{n-1} F\left(x_i, y_i, \frac{\Delta y_i}{\Delta x_i}\right) \Delta x_i = \widehat{J}[\mathbf{y}]$$

$\widehat{J}[\cdot]$ is a function of the vector $\mathbf{y} = (y_1, y_2, \ldots, y_n)$.

- Treat this as a maximization of a function of n variables, so that we require

$$\frac{\partial \widetilde{J}}{\partial y_i} = 0$$

  for all $i = 1, 2, \ldots, n$.

- Typically use uniform grid so

$$\Delta x_i = \Delta x = \frac{b-a}{n}.$$

Example 12.1

Find extremals for

$$J[y] = \int_0^1 \left[\frac{1}{2}y'^2 + \frac{1}{2}y^2 - y\right] dx$$

with $y(0) = 0$ and $y(1) = 0$.

The Euler-Lagrange equation $y'' - y = -1$.

Example 12.1 (direct solution)

- E-L equation: $y'' - y = -1$
- Solution to homogeneous equation $y'' - y = 0$ is given by $e^{\lambda x}$ giving characteristic equation

$$\lambda^2 - 1 = 0,$$

so $\boxed{\lambda = \pm 1}$

- Particular solution $y = 1$.
- Final solution is

$$\boxed{y(x) = Ae^x + Be^{-x} + 1}.$$

Example 12.1 (direct solution)

- The boundary conditions $y(0) = y(1) = 0$ constrain

$$A + B = -1$$
$$Ae + Be^{-1} = -1$$

so $\boxed{A = \dfrac{1-e}{e^2-1}}$ and $\boxed{B = \dfrac{e-e^2}{e^2-1}}$.

- Then the exact solution to the extremal problem is

$$\boxed{y(x) = \frac{1-e}{e^2-1}e^x + \frac{e-e^2}{e^2-1}e^{-x} + 1}.$$

Example 12.1 (Euler's FDM)

Find extremals for

$$J[y] = \int\limits_0^1 \left[\frac{1}{2}y'^2 + \frac{1}{2}y^2 - y\right] dx$$

Euler's FDM:

- Take the grid $x_i = i/n$, for $i = 0, 1, \ldots, n$ so
  - end points $y_0 = 0$ and $y_n = 0$
  - $\Delta x = 1/n$ and $\Delta y_i = y_{i+1} - y_i$.
- So
  - $y_i' = \Delta y_i / \Delta x = n(y_{i+1} - y_i)$
  - and
$$y_i'^2 = n^2 \left(y_i^2 - 2y_i y_{i+1} + y_{i+1}^2\right).$$

Example 12.1 (Euler's FDM)

Find extremals for

$$J[y] = \int_0^1 \left[\frac{1}{2} y'^2 + \frac{1}{2} y^2 - y \right] dx$$

Its FDM approximation is

$$\begin{aligned}
\widetilde{J}[\mathbf{y}] &= \sum_{i=0}^{n-1} F(x_i, y_i, y_i') dx \\
&= \sum_{i=0}^{n-1} \frac{1}{2} n^2 \left(y_i^2 - 2 y_i y_{i+1} + y_{i+1}^2 \right) \Delta x + \left(y_i^2/2 - y_i \right) \Delta x \\
&= \sum_{i=0}^{n-1} \frac{1}{2} n \left(y_i^2 - 2 y_i y_{i+1} + y_{i+1}^2 \right) + \frac{y_i^2/2 - y_i}{n}.
\end{aligned}$$

Example 12.1 (end-conditions)

- We know the end conditions $y(0) = y(1) = 0$, which imply that

$$y_0 = y_n = 0.$$

- Include them into the objective using Lagrange multipliers

$$\mathcal{H}[\mathbf{y}] = \sum_{i=0}^{n-1} \frac{1}{2} n \left(y_i^2 - 2 y_i y_{i+1} + y_{i+1}^2 \right) + \frac{y_i^2/2 - y_i}{n} + \lambda_0 y_0 + \lambda_n y_n.$$

Example 12.1 (Euler's FDM)

- Taking derivatives, note that y_i only appears in two terms of the FDM approximation

$$\mathcal{H}[\mathbf{y}] = \sum_{i=0}^{n-1} \frac{1}{2} n \left(y_i^2 - 2y_i y_{i+1} + y_{i+1}^2 \right) + \frac{y_i^2/2 - y_i}{n} + \lambda_0 y_0 + \lambda_n y_n$$

$$\frac{\partial \mathcal{H}[\mathbf{y}]}{\partial y_i} = \begin{cases} n(y_0 - y_1) + \dfrac{y_0 - 1}{n} + \lambda_0 & \text{for} \quad i = 0 \\ n(2y_i - y_{i+1} - y_{i-1}) + y_i/n - 1/n & \text{for } i = 1, \ldots, n-1 \\ n(y_n - y_{n-1}) + \lambda_n & \text{for} \quad i = n \end{cases}$$

- We need to set the derivatives to all be zero, so we now have $n = 3$ linear equations, including $y_0 = y_n = 0$, and $n + 3$ variables including the two Lagrange multipliers.

- We can solve this system numerically using, e.g., Maple.

Example 12.1 (Euler's FDM)

Example: $n = 4$, solve

$$A\mathbf{z} = \mathbf{b}$$

where

$$A = \begin{pmatrix} 1.00 & & & & & & \\ 4.25 & -4.00 & & & & 1 & \\ -4.00 & 8.25 & -4.00 & & & & \\ & -4.00 & 8.25 & -4.00 & & & \\ & & -4.00 & 8.25 & -4.00 & & \\ & & & -4.00 & 4.00 & & 1.00 \\ & & & & 1.00 & & \end{pmatrix}$$

and

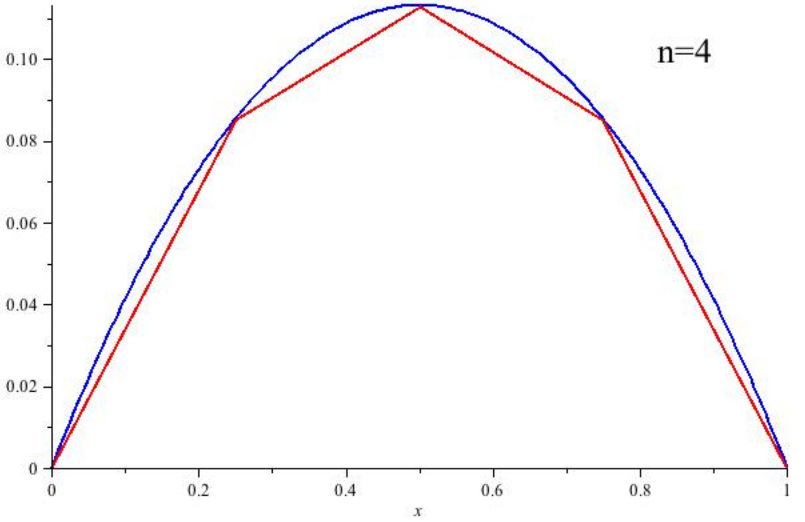
$$\mathbf{b} = (0.00,\ 0.25,\ 0.25,\ 0.25,\ 0.25,\ 0.00,\ 0.00)^T$$

Example 12.1 (Euler's FDM)

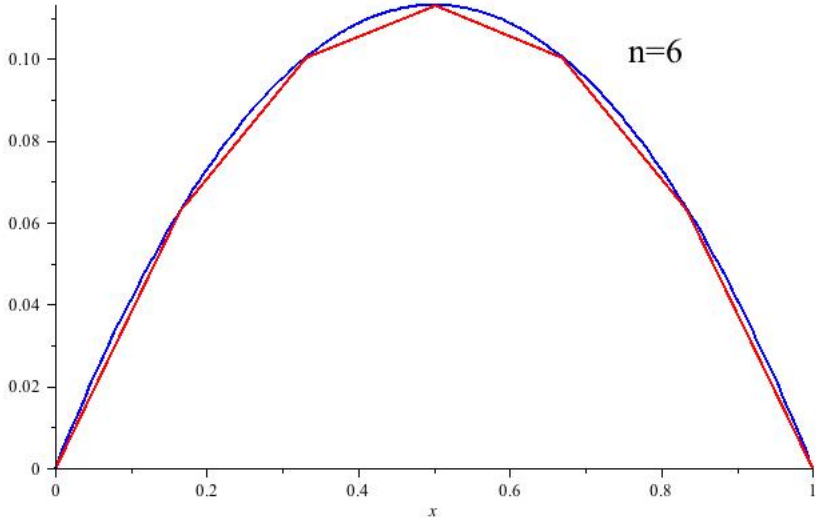
- First $n+1$ terms of $\mathbf{z}$ give $\mathbf{y}$
- Last two terms of $\mathbf{z}$ give the Lagrange multipliers λ_0 and λ_n.
- Solving the system we get for $n = 4$

$$y_1 = y_3 = 0.08492201040, \qquad y_2 = 0.1126516464$$

Example 12.1 (results)



Example 12.1 (results)



Convergence of Euler's FDM

$$\widehat{J}[\mathbf{y}] = \sum_{i=0}^{n-1} F\left(x_i, y_i, \frac{\Delta y_i}{\Delta x_i}\right)\Delta x \quad \text{and} \quad \Delta y_i = y_{i+1} - y_i$$

Only two terms in the sum involve y_i, so

$$\begin{aligned}
\frac{\partial \widehat{J}}{\partial y_i} &= \frac{\partial}{\partial y_i} F\left(x_{i-1}, y_{i-1}, \frac{\Delta y_{i-1}}{\Delta x}\right) + \frac{\partial}{\partial y_i} F\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right) \\
&= \frac{1}{\Delta x}\frac{\partial F}{\partial y_i'}\left(x_{i-1}, y_{i-1}, \frac{\Delta y_{i-1}}{\Delta x}\right) \\
&\quad + \frac{\partial F}{\partial y_i}\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right) - \frac{1}{\Delta x}\frac{\partial F}{\partial y_i'}\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right) \\
&= \frac{\partial F}{\partial y_i}(x_i, y_i, y_i') - \frac{\frac{\partial F}{\partial y_i'}\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right) - \frac{\partial F}{\partial y_i'}\left(x_{i-1}, y_{i-1}, \frac{\Delta y_{i-1}}{\Delta x}\right)}{\Delta x}
\end{aligned}$$

Convergence of Euler's FDM

$$\frac{\partial \widehat{J}}{\partial y_i} = \frac{\partial F}{\partial y_i}(x_i, y_i, y_i') - \frac{\frac{\partial F}{\partial y_i'}\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right) - \frac{\partial F}{\partial y_i'}\left(x_{i-1}, y_{i-1}, \frac{\Delta y_{i-1}}{\Delta x}\right)}{\Delta x} = 0.$$

In limit $n \to \infty$, then $\Delta x \to 0$, and so we get

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

which are the Euler-Lagrange equations.

- i.e., the finite difference solution converges to the solution of the Euler-Lagrange equations.

Remarks

- There are lots of ways to improve Euler's FDM
  - use a better method of numerical quadrature (integration)
    - trapezoidal rule
    - Simpson's rule
    - Romberg's method
  - use a non-uniform grid
    - make it finer where there is more variation
- We can use a different approach that can be even better.

Ritz's Method

- In Ritz's method (called Kantorovich's method where there is more than one independent variable), we approximate our functions (the extremal in particular) using a family of simple functions.
- Again we can reduce the problem into a standard multivariable maximization problem, but now we seek coefficients for our approximation.

Assume we can approximate $y(x)$ by

$$y(x) = \phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x)$$

where we choose a convenient set of functions $\phi_j(x)$ and find the values of c_j which produce an extremal.

For fixed end-points problem:

- Choose $\phi_0(x)$ to satisfy the end conditions.
- Then $\phi_j(x_0) = \phi_j(x_1) = 0$ for $j = 1, 2, \ldots, n$

The ϕ can be chosen from standard sets of functions, e.g. power series, trigonometric functions, Bessel's functions, etc. (but must be linearly independent).

- Select $\{\phi_j\}_{j=0}^{n}$
- Approximate

$$y_n(x) = \phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x)$$

- Approximate $J[y] \simeq J[y_n] = \int\limits_{x_0}^{x_1} F(x, y_n, y_n')dx$.
- Integrate to get $J[y_n] = J_n(c_1, c_2, \dots, c_n)$.
- J_n is a known function of n variables, so we can maximize (or minimize) it as usual by

$$\frac{\partial J_n}{\partial c_i} = 0$$

for all $i = 1, 2, \dots, n$.

- Assume the extremal of interest is a minimum, then for the extremal

$$J[y] < J[\hat{y}]$$

for all $\hat{y}$ within the neighborhood of y.

- Assume our approximating function y_n is close enough to be in that neighborhood, then

$$J[y] \leqslant J[y_n] = J_n[\mathbf{c}]$$

so the approximation provides an upper bound on the minimum $J[y]$.

- Another way to think about it is that we optimize on a smaller set of possible functions y, so we can't get quite as good a minimum.