# Calculus of Variations
# Summer Term 2014

Lecture 13

26. Juni 2014

**Purpose of Lesson:**

- First application of the Ritz method.
- The Ritz method applied to the catenary gives additional insights.

## Example 13.1

Find extremals for

$$J[y] = \int_0^1 \left[\frac{1}{2}y'^2 + \frac{1}{2}y^2 - y\right] dx$$

with $y(0) = 0$ and $y(1) = 0$.

The Euler-Lagrange equation $y'' - y = 1$, but we shall bypass the Euler-Lagrange equation to use Ritz's method.

$$y_n(x) = \phi_0(x) + \sum_{i=1}^{n} c_i \phi_i(x)$$

where we take $\phi_0(x) = 0$ and $\phi_i(x) = x^i (1-x)^i$.

## Example 13.1

- Simple approximation $y_1 = c_1\phi_1(x)$ we get

$$J_1[c_1] = J[y_1] = \int_0^1 \left[\frac{1}{2}c_1^2\phi_1'^2 + \frac{1}{2}c_1^2\phi_1^2 - c_1\phi_1\right] dx.$$

- Now $\phi_1(x) = x(1-x)$ so $\phi_1' = 1 - 2x$, and

$$\begin{aligned} J_1[c_1] &= \int_0^1 \left[\frac{c_1^2}{2}(1-2x)^2 + \frac{c_1^2}{2}x^2(1-x)^2 - c_1 x(1-x)\right] dx \\ &= \frac{c_1^2}{2}\int_0^1 \left[1 - 4x + 5x^2 - 2x^3 + x^4\right] dx + c_1\int_0^1 \left[-x + x^2\right] dx \\ &= \frac{3c_1^2}{5} - \frac{c_1}{6}. \end{aligned}$$

## Example 13.1

- We solve for $c_1$ by setting

$$\frac{dJ_1}{dc_1} = \frac{6c_1}{5} - \frac{1}{6} = 0$$

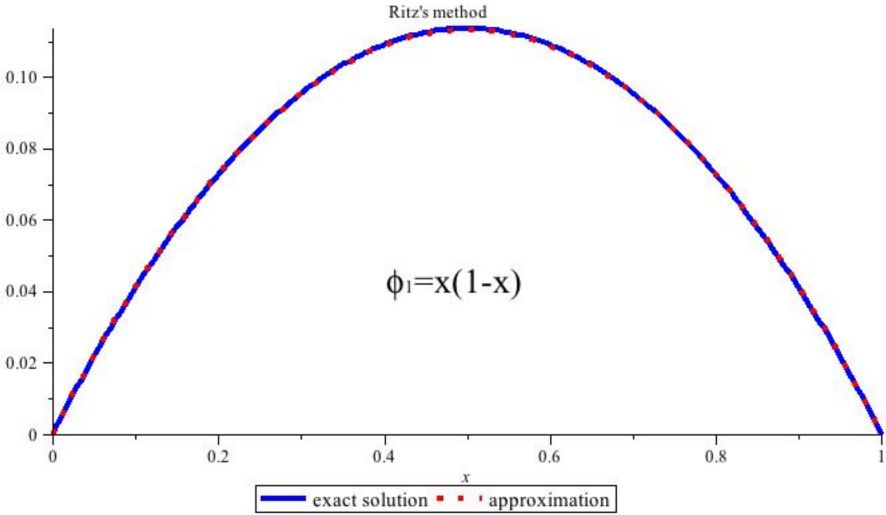
to get $c_1 = 5/36$, so the approximate extremal is

$$y_1(x) = \frac{5}{36}x(1-x).$$

- The value of the approximate functional at this point is

$$J_1[5/36] = \frac{3c_1^2}{5} - \frac{c_1}{6} = -0.01157407$$

which is an upper bound on the true value of the functional on the extremal.


Ritz's method
0.10
0.08
0.06
0.04
0.02
0
0
0.2
0.4
0.6
0.8
1
x
ϕ1=x(1-x)
exact solution
approximation

## Example 13.1 (alternate approach)

- Choose $\phi_1(x) = \sin(\pi x)$ (use the first element of a trigonometric series to approximate $y$).

- Then, $\phi_1'(x) = \pi \cos(\pi x)$, and so the functional is

$$J_1[c_1] = J[c_1\phi_1] = \int_0^1 \left[\frac{1}{2}c_1^2\phi_1'^2 + \frac{1}{2}c_1^2\phi_1^2 - c_1\phi_1\right] dx$$

$$= \int_0^1 \left[\frac{c_1^2\pi^2}{2}\cos^2(\pi x) + \frac{c_1^2}{2}\sin^2(\pi x) - c_1 \sin(\pi x)\right] dx.$$

- Observe that $\int_0^1 \cos^2(\pi x)dx = \int_0^1 \sin^2(\pi x)dx = 1/2$, and $\int_0^1 \sin(\pi x)dx = \left[-\frac{1}{\pi}\cos(\pi x)\right]_0^1 = -2/\pi$.

## Example 13.1 (alternate approach)

- So

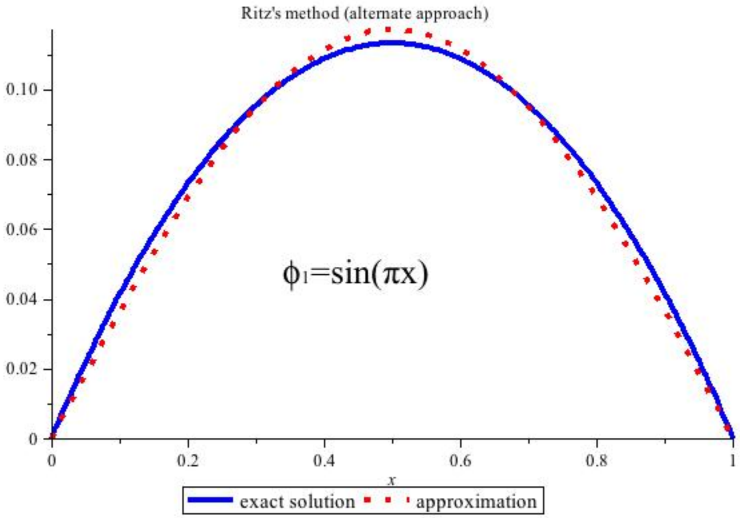
$$J_1[c_1] = \frac{c_1^2}{4}\left[\pi^2 + 1\right] - \frac{2}{\pi}c_1.$$

- Once again we solve for $c_1$ by setting

$$\frac{dJ_1}{dc_1} = \frac{c_1}{2}\left[\pi^1 + 1\right] - \frac{2}{\pi} = 0$$

to get $c_1 = \frac{4}{\pi(\pi^2+1)}$, so the approximate extremal is

$$y_1(x) = \frac{4}{\pi(\pi^2+1)}\sin(\pi x).$$


Ritz's method (alternate approach)
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ϕ1=sin(πx)
exact solution
approximation

## Example 13.2 (the catenary, again)

The functional of interest (the potential energy) is

$$J_p[y] = mg \int_{x_0}^{x_1} y\sqrt{1+y'^2}\,dx.$$

- Take symmetric problem with fixed end points

$$y(-1) = a \quad \text{and} \quad y(1) = a$$

and we know the solution looks like

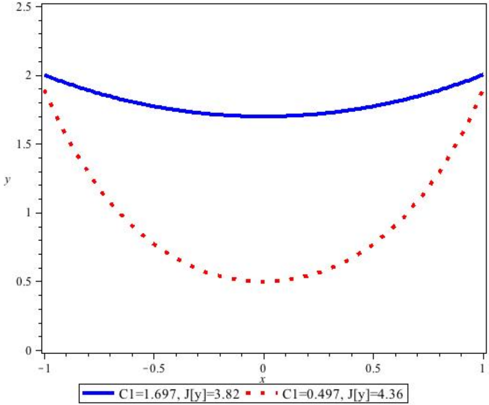
$$y(x) = c_1 \cosh\left(\frac{x}{c_1}\right)$$

where $c_1$ is chosen to match the end points.

## Example 13.2 (the catenary, again)

$y(1) = 2$ gives $c_1 = 0.47$ or $c_1 = 1.697$

- Are they both local minima?



## Example 13.2 (Ritz and the Catenary)

- Lets try approximating the curve by a polynomial

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

- Note that symmetry of problem implies $y$ is an even function, and hence the odd terms

$$a_1 = a_3 = \dots = 0.$$

- So, to second order we can approximate

$$y(x) \simeq a_0 + a_2 x^2.$$

- We have fixed $y(1) = y_1$, so we can simplify to get

$$y(x) \simeq a_0 + (y_1 - a_0)x^2.$$

## Example 13.2 (Ritz and the Catenary)

$$y \simeq a_0 + (y_1 - a_0)x^2$$
$$y' \simeq 2(y_1 - a_0)x$$

- Taking into account $y(1) = 2$ we get $a_0 + a_2 = 2$. We can substitute into the functional

$$J_p[y] = mg \int_{x_0}^{x_1} y\sqrt{1 + y'^2}dx$$

  and integrate to get a function $J_p[a_2]$ with respect to $a_2$.

- But this function is pretty complicated.

# Example 13.2 (Ritz and the Catenary)

From Maple we have the value for $J_p[a_2]$, $(a := a_2)$

```
> f(x) := (2 - a + a·x^2)·sqrt(1 + 4·a^2·x^2) :
> int(f(x), x=-1..1)
```

$$\frac{1}{64}\frac{1}{a^2}\Big(\big(16\,a^2\ln\big(\big(-2\,a+\sqrt{1+4\,a^2}\,\mathrm{csgn}(a)\big)\,\mathrm{csgn}(a)\big)+128\sqrt{1+4\,a^2}\,a^2\,\mathrm{csgn}(a) - 64\,a^3\sqrt{1+4\,a^2}\,\mathrm{csgn}(a) - 32\,a\ln\big(\big(-2\,a+\sqrt{1+4\,a^2}\,\mathrm{csgn}(a)\big)\,\mathrm{csgn}(a)\big)+\ln\big(\big(-2\,a+\sqrt{1+4\,a^2}\,\mathrm{csgn}(a)\big)\,\mathrm{csgn}(a)\big)-4\sqrt{1+4\,a^2}\,a\,\mathrm{csgn}(a)+8\,(1+4\,a^2)^{3/2}\,a\,\mathrm{csgn}(a)-16\,a^2\ln\big(\big(2\,a+\sqrt{1+4\,a^2}\,\mathrm{csgn}(a)\big)\,\mathrm{csgn}(a)\big)+32\,a\ln\big(\big(2\,a+\sqrt{1+4\,a^2}\,\mathrm{csgn}(a)\big)\,\mathrm{csgn}(a)\big)-\ln\big(\big(2\,a+\sqrt{1+4\,a^2}\,\mathrm{csgn}(a)\big)\,\mathrm{csgn}(a)\big)\big)\,\mathrm{csgn}(a)\Big) \quad \mathbf{(1)}$$

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>
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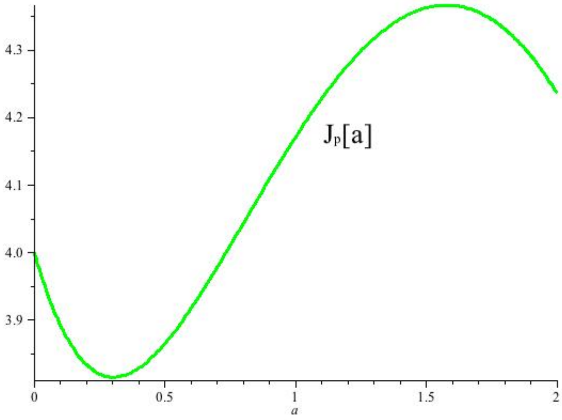
## Example 13.2 (Ritz and the Catenary)

- Its a pain to find the zeros of $\frac{dJ_p}{da}$, but its easy to plot, and find them numerically.

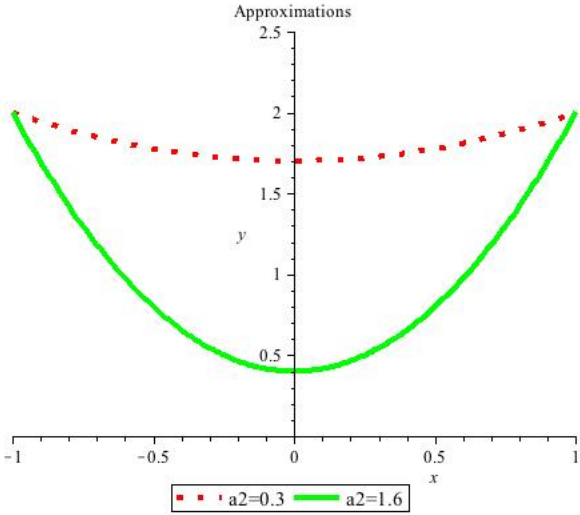
# Example 13.2 (Ritz and the Catenary)



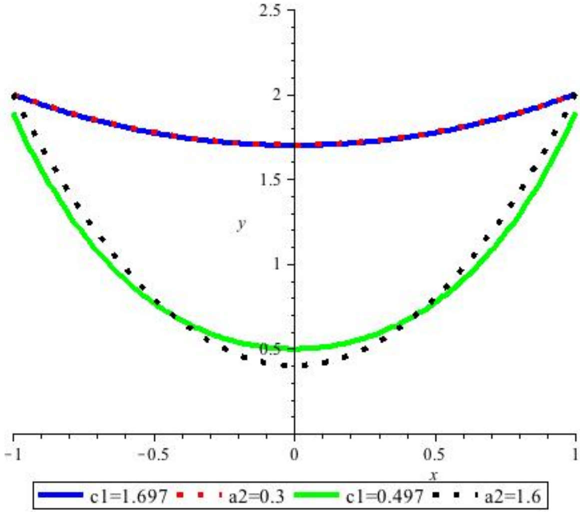

Stationary points

- local max: $a = a_2 \simeq 1.6$
- local min: $a = a_2 \simeq 0.3$

## Example 13.2 (Ritz and the Catenary)



## Example 13.2 (Ritz and the Catenary)



## Ritz and the Catenary

Doesn't just give us an approximation to the extremal curves, its also give us some insight into the nature of these extremals. If

- approximations are near to the actual extrema
- There are no other extrema so close by
- The functional is smooth (it can't have jumps either)

Then the type of extrema we get for the approximation will be the same for the real extrema, i.e.,

- local max: $a_2 \simeq 1.6 \Rightarrow$ local max for $c_1 = 0.497$
- local min: $a_2 \simeq 0.3 \Rightarrow$ local min for $c_1 = 1.697$