Calculus of Variations
Summer Term 2014

Lecture 14

20. Juni 2014

Purpose of Lesson:

- Kantorovich's method generalizes Ritz to $2D$ functions.

$2D$ Case:

We are approximating a surface with series of functions, e.g.

$$z(x,y) \simeq z_n(x,y) = \phi_0(x,y) + \sum_{i=1}^{n} c_i \phi_i(x,y)$$

where

- $\phi_0(x,y)$ satisfies the boundary conditions, e.g.

$$\phi_0(x,y) = z_0(x,y) \quad \text{for} \quad (x,y) \in \partial\Omega,$$

  the boundary of the region on interest Ω,

- and the $\phi_i(x,y)$ satisfy the homogeneous boundary conditions

$$\phi_i(x,y) = 0 \quad \text{for} \quad (x,y) \in \partial\Omega.$$

$2D$ Case:

- As before, we approximate the functional by

$$J[z] \simeq J[z_n] = J_n(c_1, \dots, c_n).$$

- As before we determine the c_j by requiring that the partial derivatives are zero, e.g.

$$\frac{\partial J_n}{\partial c_i} = 0$$

for all $i = 1, 2, \dots, n$.

Kantorovich's Method

- Approximate with

$$z(x, y) \simeq z_n(x, y) = \phi_0(x, y) + \sum_{i=1}^{n} c_i(x)\phi_i(x, y).$$

- Again the ϕ_i are suitable chosen, but the c_i are no longer constants, but rather functions of one independent variable.
- This allows a larger class of functions to be used.

Kantorovich's Method

- Note that the integral function

$$J[z_n] = \iint_{\Omega} z_n(x,y)dxdy = \sum_{i=0}^{n} \int c_i(x) \left[\int_{y_0(x)}^{y_1(x)} \phi_i(x,y)dy \right] dx$$

- We integrate the inner integral, and get

$$J[z_n] = \sum_{i=0}^{n} \int c_i(x)\Phi_i(x)dx.$$

- Now we just have a function of x, and so we may apply the Euler-Lagrange machinery.
- The method approx. separates the variables x and y.

Example 14.1

Find the extremals of

$$J[z(x,y)] = \int_{-b}^{b}\int_{-a}^{a}\left(z_x^2 + z_y^2 - 2z\right)dxdy$$

with $z = 0$ on the boundary.

- The Euler-Lagrange equation reduces to the Poisson equation, e.g.

$$F_z - \frac{d}{dx}F_{z_x} - \frac{d}{dy}F_{z_y} = 0$$

$$-2 - \frac{d}{dx}(2z_x) - \frac{d}{dy}(2z_y) = 0$$

$$z_{xx} + z_{yy} = -1$$

Example 14.1

- Approximate

$$z_1(x,y) = c(x)\left(b^2 - y^2\right)$$

- Note $z_1(x, \pm b) = 0$ (as required) and

$$\begin{aligned}
\left(\frac{\partial z_1}{\partial x}\right)^2 &= \left[c'(x)\left(b^2 - y^2\right)\right]^2 \\
&= c'(x)^2\left[b^4 - 2b^2y^2 + y^4\right], \\
\left(\frac{\partial z_1}{\partial y}\right)^2 &= [c(x)2y]^2 \\
&= 4c(x)^2y^2
\end{aligned}$$

Example 14.1

Hence, we approximate

$$J[z(x,y)] \simeq J[z_1(x,y)] = \int_{-b}^{b}\int_{-a}^{a} \left(z_x^2 + z_y^2 - 2z\right) dxdy$$

$$= \int_{-a}^{a} \left[\int_{-b}^{b} \left[c'(x)^2 \left(b^2 - y^2\right)^2 + 4c(x)^2 y^2 - 2c(x) \left(b^2 - y^2\right) \right] dy \right] dx$$

$$= \int_{-a}^{a} \Big[c'(x)^2 \left(b^4 y - 2b^2 y^3/3 + y^5/5\right) + 4c(x)^2 y^3/3$$

$$+2c(x) \left(b^2 y - y^3/3\right) \Big]_{-b}^{b} dx$$

$$= \int_{-a}^{a} \left[\frac{16}{15} b^5 c'(x)^2 + \frac{8}{3} b^3 c(x)^2 - \frac{8}{3} b^3 c(x) \right] dx$$

Example 14.1

- So we can write

$$J[z(x,y)] \simeq J[z_1(x,y)] = J[c(x)] = \int_{-a}^{a} F(x,c,c')dx$$

- We can use the simple Euler-Lagrange equation, where

$$F(x,c,c') = \frac{16}{15}b^5c'(x)^2 + \frac{8}{3}b^3c(x)^2 - \frac{8}{3}b^3c(x)$$

$$\frac{\partial F}{\partial c} = \frac{16}{3}b^3c(x) - \frac{8}{3}b^3$$

$$\frac{\partial F}{\partial c'} = \frac{32}{15}b^5c'(x) \qquad \frac{d}{dx}\frac{\partial F}{\partial c'} = \frac{32}{15}b^5c''(x)$$

Example 14.1

- The Euler-Lagrange equation

$$\frac{16}{3}b^3c(x) - \frac{8}{3}b^3 - \frac{32}{15}b^5c''(x) = 0$$
$$c''(x) - \frac{5}{2b^2}c(x) = -\frac{5}{4b^2}$$

- Solutions

$$c(x) = k_1 \cosh\left(\sqrt{\frac{5}{2}}\frac{x}{b}\right) + k_2 \sinh\left(\sqrt{\frac{5}{2}}\frac{x}{b}\right) + \frac{1}{2}$$

Example 14.1

- Note that the function must be zero on the boundary, so $z(\pm a, y) = 0$.
- We look for an even function $c(x)$, and so $k_2 = 0$.
- Also $c(\pm a) = 0$, so

$$c(a) = k_1 \cosh\left(\sqrt{\frac{5}{2}}\frac{a}{b}\right) + \frac{1}{2}$$

$$-\frac{1}{2} = k_1 \cosh\left(\sqrt{\frac{5}{2}}\frac{a}{b}\right)$$

$$k_1 = -\frac{1}{2\cosh\left(\sqrt{\frac{5}{2}}\frac{a}{b}\right)}$$

Example 14.1

- Solution

$$z_1(x, y) = \frac{1}{2}(b^2 - y^2)\left(1 - \frac{cosh\left(\sqrt{\frac{5}{2}}\frac{x}{b}\right)}{cosh\left(\sqrt{\frac{5}{2}}\frac{a}{b}\right)}\right)$$

- If we want a more exact approximation, we could try

$$z_2(x, y) = (b^2 - y^2)c_1(x) + (b^2 - y^2)^2 c_2(x).$$

Remarks

- Obviously, quality of solution depends on
  - family of functions chosen
  - number of terms used, n
- Could test convergence by increasing n and seeing the difference in

$$|J[y_{n+1}] - J[y_n]|,$$

  but this is not guaranteed to be a good indication.
- A better way to assess convergence is to have a lower bound

$$\text{lower bound } \leqslant J[y] \leqslant \text{ upper bound}$$