Calculus of Variations
Summer Term 2014

Lecture 16

11. Juli 2014

Purpose of Lesson:

- To consider aerospace example

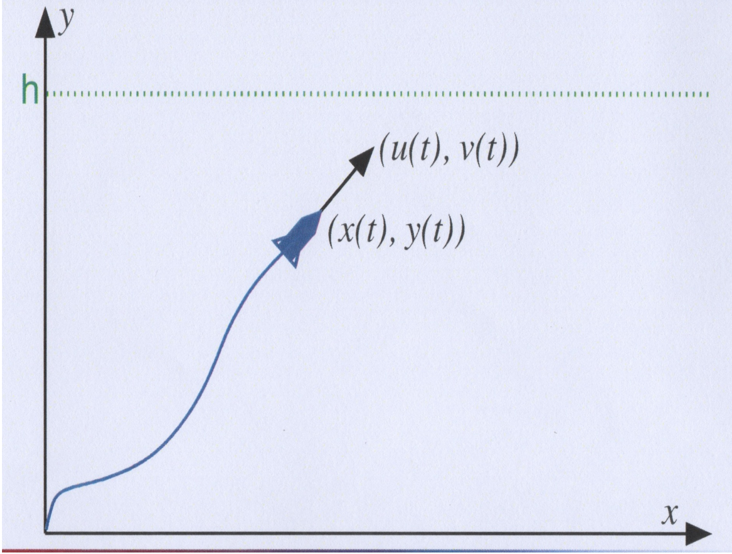
Example 16.1 Lauching a rocket

Launch a rocket (with one stage) to deliver its payload into Low-Earth Orbit (LEO) at some height h above the Earth's surface.

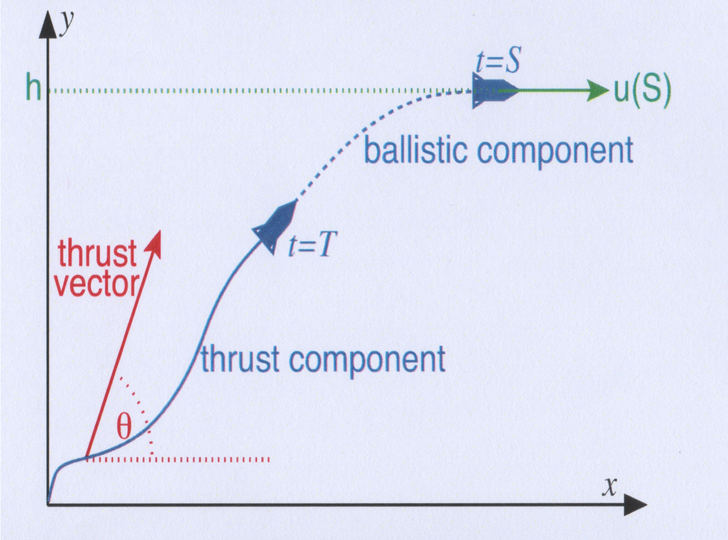
Assumptions:

- ignore drag, and curvature and rotation of Earth
- LEO so assume gravitational force at ground and orbit are approximately the same
- thrust will generate acceleration a, which is predefined by rocket parameters
- we thrust for some time T, then follow a ballistic trajectory until (hopefully) we reach height h, at zero vertical velocity, and with horizontal velocity matching the required orbital injection speed.

Example 16.1 Launching a rocket-2



Example 16.1 Launching a rocket-3



Example 16.1 Launching a rocket-4

Notation:

$$
\begin{aligned}
x &= \text{ horizontal position} \\
y &= \text{ vertical position} \\
u &= \text{ horizontal velocity} \\
v &= \text{ vertical velocity}
\end{aligned}
$$

Initial conditions $x(0) = y(0) = u(0) = v(0) = 0$. Thrust stops at time T, and then at some later time S, we reach the peak of the trajectory where

$$
\begin{aligned}
y(S) &= h \\
u(S) &= u_0, \text{ orbital velocity} \\
v(S) &= 0
\end{aligned}
$$

We don't actually care about the final position $x(S)$.

Example 16.1 Launching a rocket-5

- Control: thrust profile is pre-determined. The only thing we can control (in this problem) is the **angle** of thrust.
    - Thrust $a(t)$ is constant for our example.
    - Measure the angle of thrust $\theta(t)$ relative to horizontal.
- want to minimize fuel
    - but this is equivalent to minimizing time, e.g.,

$$J = \int\limits_0^t a dt = a \int_0^T 1 dt$$

    - need to get to height h
    - need to get to horizontal velocity u_0 to enter orbit.

Constrained equations

Example 16.1 Launching a rocket-6

Thrust component: $t \leqslant T$	Ballistic component: $T < t \leqslant S$
$\dot{x} = u$ $\dot{y} = v$ $\dot{u} = a\cos\theta$ $\dot{v} = a\sin\theta - g$	$\dot{x} = u$ $\dot{y} = v$ $\dot{u} = 0$ $\dot{v} = -g$
Initial point $x(0) = y(0) = u(0) = v(0) = 0.$	Initial point: fixed $x(T),\ y(T),\ u(T),\ v(T)$
Final point: free	Final point: $x(S)$ free $y(S) = h,\ v(S) = 0,\ u(S) = u_0$

1st consider ballistic component

Example 16.1 Launching a rocket-7

For $t \in [T, S]$ we have no control, and

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{u} &= 0 \\ \dot{v} &= -g \end{aligned}$$

we can calculate the top of the resulting parabola as

$$\begin{aligned} u(S) &= u(T) \\ v(S) &= 0 \\ y(S) &= y(T) + (v(T))^2 / 2g \end{aligned}$$

and $x(T)$ and $x(S)$ are free.

Coordinate transform

Example 16.1 Launching a rocket-8

So we can change variables: make the final point $t = T$, and take variables u, v as before, and

$$z = y + \frac{v^2}{2g}.$$

We can differentiate this and combine with previous results to get the new **system DEs**

$$\begin{aligned}
\dot{u} &= acos\theta \\
\dot{v} &= a\sin\theta - g \\
\dot{z} &= \dot{y} + v\dot{v}/g \\
&= v\left(1 + \dot{v}/g\right) = \frac{av}{g}\sin\theta
\end{aligned}$$

Optimization functional

Example 16.1 Launching a rocket-9

Time minimization problem

$$T = \int_0^T 1 \, dt.$$

Including Lagrange multipliers for the 3 system constraints we aim to minimize

$$\int_0^T \left(1 + \lambda_u \left(\dot{u} - a\cos\theta\right) + \lambda_v \left(\dot{v} - a\sin\theta + g\right) + \lambda_z \left(\dot{z} - \frac{av}{g}\sin\theta\right)\right) dt$$

subject to $u(0) = v(0) = z(0) = 0$, $\theta(0) =$ *free*, $u(T) = u_0$, $v(T) =$ *free*, $z(T) = h$, $\theta(T) =$ *free*.

Euler-Lagrange equations

Example 16.1 Launching a rocket-10

$$
\begin{aligned}
u: &\quad \frac{\partial h}{\partial u} - \frac{d}{dt}\frac{\partial h}{\partial \dot{u}} = 0 \quad \Rightarrow \quad \dot{\lambda}_u = 0 \\
v: &\quad \frac{\partial h}{\partial v} - \frac{d}{dt}\frac{\partial h}{\partial \dot{v}} = 0 \quad \Rightarrow \quad \dot{\lambda}_v = -\lambda_z \frac{a}{g}\sin\theta \\
z: &\quad \frac{\partial h}{\partial z} - \frac{d}{dt}\frac{\partial h}{\partial \dot{z}} = 0 \quad \Rightarrow \quad \dot{\lambda}_z = 0 \\
\theta: &\quad \frac{\partial h}{\partial \theta} - \frac{d}{dt}\frac{\partial h}{\partial \dot{\theta}} = 0 \quad \Rightarrow
\end{aligned}
$$

$$
a\lambda_u \sin\theta - \lambda_v a\cos\theta - \lambda_z \frac{av}{g}\cos\theta = 0
$$

(λ equations give back systems DEs).

Solving the E-L equations

Example 16.1 Launching a rocket-11

Take the v equation, and noting that $\dot{v} = a\sin\theta - g$

$$\dot{\lambda}_v = -\lambda_z \frac{a}{g}\sin\theta = -\frac{\lambda_z}{g}(\dot{v} + g),$$

$$\lambda_v = -\frac{\lambda_z}{g}(v + gt + c) = -\frac{\lambda_z v}{g} - \lambda_z t + b.$$

Solving the E-L equations

Example 16.1 Launching a rocket-12

Substitute

$$\lambda_z = -\frac{\lambda_z v}{g} - \lambda_z t + b$$

into the θ E-L equation (dropping the common factor a)

$$\lambda_u \sin\theta - \lambda_v \cos\theta - \lambda_z \frac{v}{g} \cos\theta = 0$$

and we get

$$\lambda_u \sin\theta + \left(\frac{\lambda_z v}{g} + \lambda_z t - b\right) \cos\theta - \lambda_z \frac{v}{g} \cos\theta = 0$$

$$\lambda_u \sin\theta + (\lambda_z t - b) \cos\theta = 0$$

$$\tan\theta = -\frac{\lambda_z t - b}{\lambda_u}$$

Solution

Example 16.1 Launching a rocket-13

Remember that λ_u and λ_v and b are all constants, so the equation

$$\boxed{\tan\theta = -\frac{\lambda_z t - b}{\lambda_u}}$$

- angle of thrust now specified

$$\theta = \tan^{-1}\left(-\frac{\lambda_z t - b}{\lambda_u}\right)$$

- but we need to determine constants