Calculus of Variations
Summer Term 2014

Lecture 17

4. Juli 2014

Purpose of Lesson:

- To continue the study of aerospace example
- Hamiltonian's formulation.

End-point conditions

Example 16.1 Launching a rocket-14

Final end-points conditions

$$
\begin{aligned}
T &= \text{ free} \\
z(T) &= h \\
u(T) &= u_0, \text{ orbital velocity} \\
v(T) &= \text{ free} \\
\theta(T) &= \text{ free} \\
\lambda_u &= \text{ free} \\
\lambda_v &= \text{ free} \\
\lambda_z &= \text{ free}
\end{aligned}
$$

Natural boundary conditions

Example 16.1 Launching a rocket-15

The free-end point boundary condition for

$$J[\mathbf{q}, \dot{\mathbf{q}}] = \int F(t, \mathbf{q}, \dot{\mathbf{q}}) dt$$

is

$$\left[\sum_{k=1}^{m} \delta q_k \frac{\partial F}{\partial \dot{q}_k} + \delta t \left(F - \sum_{k=1}^{m} \dot{q}_k \frac{\partial F}{\partial \dot{q}_k} \right) \right]_{t=T} = 0.$$

In our problem

$$\frac{\partial F}{\partial \dot{\lambda}_k} = 0, \quad \frac{\partial F}{\partial \dot{\theta}} = 0, \quad \frac{\partial F}{\partial \dot{u}} = \lambda_u, \quad \frac{\partial F}{\partial \dot{v}} = \lambda_v, \quad \frac{\partial F}{\partial \dot{z}} = \lambda_z$$

Natural boundary conditions

Example 16.1 Launching a rocket-16

Consider δq_k for each coordinate:

- for fixed coordinates u and z, we have $\delta q_k = 0$
- it is free for θ, λ_u, λ_v, λ_z, but in each case the corresponding $\frac{\partial F}{\partial \dot{q}_k} = 0$, so we can ignore these.
- only case where it matters is δv, which we can vary, and for which $\frac{\partial F}{\partial \dot{v}} = \lambda_v$.

Also δt is free, so we get two end-point conditions at $t = T$

$$\lambda_v(T) = 0$$
$$H(T) := [F - \dot{u}\lambda_u - \dot{v}\lambda_v - \dot{z}\lambda_z]_{t=T} = 0$$

Natural boundary conditions

Example 16.1 Launching a rocket-17

Given $\lambda_v(T) = 0$, and from previous work

$$\lambda_v = -\frac{\lambda_z v}{g} - \lambda_z t + b$$

we get

$$\begin{aligned} \frac{\lambda_z v(T)}{g} &= -\lambda_z T + b \\ &= \lambda_u \tan\theta(T) \end{aligned}$$

$$v(T) = \frac{\lambda_u g}{\lambda_z} \tan\theta(T)$$

Natural boundary conditions

Example 16.1 Launching a rocket-18

$$H(T) = [F - \dot{u}\lambda_u - \dot{v}\lambda_v - \dot{z}\lambda_z]_{t=T} = 0.$$

Substituting F and taking into account that $\lambda_v(T) = 0$ we get

$$a\lambda_u \cos\theta(T) + a\frac{\lambda_z v(T)}{g}\sin\theta(T) = 1$$

Combining the latter with $v(T) = \frac{\lambda_u g}{\lambda_z}\tan\theta(T)$ we arrive at

$$a\lambda_u \cos\theta(T) + a\lambda_u \tan\theta(T)\sin\theta(T) = 1$$

$$\lambda_u = \frac{\cos\theta(T)}{a}$$

Acceleration profile

Example 16.1 Launching a rocket-19

The next step depend on the acceleration profile $a(t)$, but lets take a simple case $a = const$.

First we can solve the DEs, with respect to θ, using the chain rule

$$\frac{dX}{dt} = \frac{dX}{d\theta}\frac{d\theta}{dt} = -\cos^2\theta\frac{\lambda_z}{\lambda_u}\frac{dX}{d\theta}$$

e.g. from the system DE $\dot{u} = a\cos\theta$

$$\dot{u} = -\cos^2\theta\frac{\lambda_z}{\lambda_u}\frac{du}{d\theta}$$

$$\frac{du}{d\theta} = -\frac{\lambda_u}{\lambda_z\cos^2\theta}\dot{u} = -\frac{a\lambda_u}{\lambda_z\cos\theta}$$

Acceleration profile

Example 16.1 Launching a rocket-20

$$\frac{dX}{d\theta} = \frac{\frac{dX}{dt}}{\frac{d\theta}{dt}} = \frac{\frac{dX}{dt}}{-\cos^2\theta\frac{\lambda_z}{\lambda_u}}$$

The complete set of system DEs becomes

$$\begin{aligned}
\frac{du}{d\theta} &= -\frac{a\lambda_u}{\lambda_z\cos\theta} \\
\frac{dv}{d\theta} &= -\frac{a\lambda_u}{\lambda_z}\frac{\sin\theta}{\cos^2\theta} + \frac{g\lambda_u}{\lambda_z\cos^2\theta} \\
\frac{dz}{d\theta} &= -\frac{a\lambda_u}{g\lambda_z}\frac{\sin\theta}{\cos^2\theta}v(\theta)
\end{aligned}$$

These can just be integrated with respect to θ.

Acceleration profile

Example 16.1 Launching a rocket-21

The system DEs can be directly integrated (with respect to θ) including initial conditions $u(0) = v(0) = z(0) = 0$ to get

$$
\begin{aligned}
u(\theta) &= \frac{a\lambda_u}{\lambda_z} \log\left(\frac{\sec\theta_0 + \tan\theta_0}{\sec\theta + \tan\theta}\right) \\
v(\theta) &= \frac{a\lambda_u}{\lambda_z}\left(\sec\theta_0 - \sec\theta\right) - \frac{g\lambda_u}{\lambda_z}\left(\tan\theta_0 - \tan\theta\right) \\
z(\theta) &= \frac{a^2\lambda_u^2}{g\lambda_z^2}\sec\theta_1\left(\sec\theta_0 - \sec\theta\right) - \frac{a^2\lambda_u^2}{2g\lambda_z^2}\left(\tan^2\theta_0 - \tan^2\theta\right) \\
&\quad + \frac{a\lambda_u^2}{2\lambda_z^2}\left[\tan\theta_0\sec\theta_0 - \tan\theta\sec\theta + \log\left(\frac{\sec\theta_0 + \tan\theta_0}{\sec\theta + \tan\theta}\right)\right] \\
\theta &= \tan^{-1}\left(-\frac{\lambda_z t - b}{\lambda_u}\right)
\end{aligned}
$$

Calculating the constants

Example 16.1 Launching a rocket-22

There are five constants to calculate:

- θ_0 the initial angle of thrust
- θ_1 the final angle of thrust
- λ_u
- λ_z
- b

and also we need to calculate T.

Solving for end-point conditions is non-trivial, but a method that works well follows.

Calculating the constants

Example 16.1 Launching a rocket-23

Take the equation for v at time T, and substitute $\lambda_z v(T) = g\lambda_u \tan\theta_1$ to get

$$
\begin{aligned}
v(\theta_1) &= \frac{a\lambda_u}{\lambda_z}(\sec\theta_0 - \sec\theta_1) - \frac{g\lambda_u}{\lambda_z}(\tan\theta_0 - \tan\theta_1) \\
\frac{g\lambda_u}{\lambda_z}\tan\theta_1 &= \frac{a\lambda_u}{\lambda_z}(\sec\theta_0 - \sec\theta_1) - \frac{g\lambda_u}{\lambda_z}(\tan\theta_0 - \tan\theta_1) \\
\sec\theta_1 &= \sec\theta_0 - \frac{g}{a}\tan\theta_0
\end{aligned}
$$

which gives us a way to calculate θ_1 from θ_0.

Once we know θ_1 we can calculate λ_u using $a\lambda_u = \cos\theta_1$, and b from $\tan\theta = (-(\lambda_z t - b)/\lambda_u)$ at $t = 0$.

Then we can calculate λ_z from $u(\theta_1) = u_0$, the orbital injection velocity.

Calculating the constants

Example 16.1 Launching a rocket-24

So the only remaining question is how to calculate θ_0. We do so numerically, by

- take a range of θ_0
- calculate all of the above
- use this to calculate $z(T) = z_1$ as a function of θ_0
- look for the point where $z_1(\theta_0) = h$ the orbit height.

That gives us the θ_0, from which we can derive everything else.

Restricting choice of θ_0

Example 16.1 Launching a rocket-25

Calculating the range of θ_0 to search

- The maximum (reasonable) value for θ_0 is $\pi/2$.
- The minimum value of θ_0 will be determined by the minimum possible value of θ_1, i.e., $\theta_1 = 0$

$$\sec\theta_1 = \sec\theta_0 - \frac{g}{a}\tan\theta_0$$

$$\sec 0 = 1 = \sec\theta_0 - \frac{g}{a}\tan\theta_0$$

$$1 = \frac{1+\tan^2(\theta_0/2)}{1-\tan^2(\theta_0/2)} - \frac{g}{a}\frac{2\tan(\theta_0/2)}{1-\tan^2(\theta_0/2)}$$

$$1-\tan^2(\theta_0/2) = 1+\tan^2(\theta_0/2) - \frac{2g}{a}\tan(\theta_0/2)$$

Restricting choice of θ_0

Example 16.1 Launching a rocket-26

$$
\begin{aligned}
1 - \tan^2(\theta_0/2) &= 1 + \tan^2(\theta_0/2) - \frac{2g}{a}\tan(\theta_0/2) \\
2\tan^2(\theta_0/2) - \frac{2g}{a}\tan(\theta_0/2) &= 0 \\
\tan(\theta_0/2)\left(\tan(\theta_0/2) - \frac{g}{a}\right) &= 0.
\end{aligned}
$$

Now θ_0 can't be zero, so the last step implies that the minimum value of θ_0 is

$$\theta_0 = 2\tan^{-1}\left(\frac{g}{a}\right).$$

Note the existence of a minimum critical h below which we can't find a trajectory of this type.

Parameters

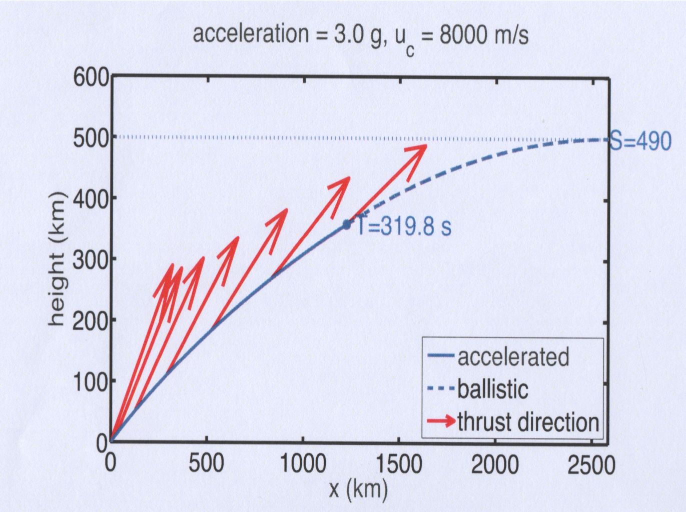
Parameters of previous example consistent with a LEO.

$$h = 500km$$
$$u_0 = 8000m/s$$
$$g = 9.8m/s^2$$
$$a = 3g$$

Derived constants

$$\theta_0 = 0.234\pi \qquad \theta_1 = 0.0973\pi$$
$$\lambda_u = 0.0324 \qquad \lambda_z = 6.0257e - 0.5$$
$$b = -0.0295$$
$$T = 319.8 \; seconds$$
$$S = 489.6 \; seconds$$

Trajectory


acceleration = 3.0 g, u_c = 8000 m/s
height (km)
x (km)
0
100
200
300
400
500
600
0
500
1000
1500
2000
2500
S=490
T=319.8 s
accelerated
ballistic
thrust direction

Generalizations

More realistic assumptions

- non-zero drag (depends on velocity and height)
- Thrust is constant, but rocket mass changes, so that acceleration isn't constant
- multiple stages
- centripetal forces

Hamiltonian's formulation

- We've seen the Hamiltonian $\mathbb{H}$ earlier an, but haven't explored its full power.
- Using $\mathbb{H}$ can often result in a simpler approach than solving the E-L equatons, e.g., where F has no dependence on x, or where there is more than one dependent variable.
- Hamiltonian's formulation can lead to an understanding of how symmetries in the problem of interest lead to conservation laws.

Legendre transformation

- transformation that depends on the derivatives of a variable
- simple one variable Legendre transform of

$$y : [x_0, x_1] \longrightarrow \mathbb{R},$$

by defining new variable p, by

$$p(x) = y'(x)$$

- provided $y''(x) > 0$ we can define x in terms of p, by introducing the Hamiltonian

$$\mathbb{H}(p) = px - y(x)$$

Legendre transformation

Assume for convenience that y is convex, e.g. $y'' > 0$ for $x \in [x_0, x_1]$. Then

$$\begin{aligned}
\frac{d\mathbb{H}}{dp} &= \frac{d}{dp}(xp) - \frac{dy}{dp} \\
&= p\frac{dx}{dp} + x - \frac{dp}{dy} \\
&= p\frac{dx}{dp} + x - \frac{dy}{dx}\frac{dx}{dp} \\
&= \left(p - \frac{dy}{dx}\right)\frac{dx}{dp} + x \\
&= x
\end{aligned}$$

and also note $px - \mathbb{H} = y$, so from the pair $(p, \mathbb{H})$ we can recover the original pair (x, y), by a Legendre transform.

Hamiltonian's formulation

Refer back to problems with more than one dependent variable, or where F has no dependence on x.

Define generalized coordinates $\mathbf{q} : [t_0, t_1] \to \mathbb{R}^n$.

- i.e. take a set of n functions $q_k(t)$, with two continuous derivatives with respect to t, and put them into a vector $\mathbf{q}(t)$
- dot notation

$$\dot{q}_k = \frac{dq_k}{dt}, \quad \ddot{q}_k = \frac{d^2 q_k}{dt^2} \quad \text{and} \quad \dot{\mathbf{q}} = \left(\frac{dq_1}{dt}, \frac{dq_2}{dt}, \ldots, \frac{dq_n}{dt} \right)$$

- Lagrangian $L(t, \mathbf{q}, \dot{\mathbf{q}})$

Hamilton's formulation

The extremal of the functional

$$J[\mathbf{q}] = \int_{t_0}^{t_1} L(t, \mathbf{q}, \dot{\mathbf{q}}) dt$$

satisfy the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} = 0$$

for all k.

Hamilton's formulation

Legendre transform introduces the conjugate variables

$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

Suppose these equations can be solved to write $\dot{q}_i$ as a function of (t, q_i, p_i), then the Hamiltonian is

$$\mathbb{H}(t, q_1, \dots, q_n, p_1, \dots, p_n) = \sum_{i=1}^{n} p_i \dot{q}_i - L(t, \mathbf{q}, \dot{\mathbf{q}}).$$

- the p_i are called generalized momenta

Hamilton's formulation

$$\mathbb{H}(t, q_1, \ldots, q_n, p_1, \ldots, p_n) = \sum_{i=1}^{n} p_i \dot{q}_i - L(t, \mathbf{q}, \dot{\mathbf{q}}).$$

So

$$\frac{\partial \mathbb{H}}{\partial p_i} = \dot{q}_i$$
$$\frac{\partial \mathbb{H}}{\partial q_i} = -\frac{\partial L}{\partial q_i}$$

Given the E-L equations, the second equation gives

$$\frac{\partial \mathbb{H}}{\partial q_i} = -\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = -\frac{dp_i}{dt}.$$

Canonical Euler-Lagrange equations

$$\frac{\partial \mathbb{H}}{\partial p_i} = \frac{dq_i}{dt}$$
$$\frac{\partial \mathbb{H}}{\partial q_i} = -\frac{dp_i}{dt}$$

- called Hamiltonian's equations or canonical Euler-Lagrange equations.
- The n E-L DEs converted into $2n$ first-order DEs
- derivatives are now uncoupled
  - therefore may be easier to solve

Canonical Euler-Lagrange equations

We can get the same canonical E-L equations from finding extremals of the functional of $2n$ variables

$$\widehat{J}[q_1, \ldots, q_n, p_1, \ldots, p_n] = \int_a^b \left[\sum_{i=1}^n p_i \dot{q}_i - \mathbb{H} \right] dx$$

E.G.

$$\left(\frac{\partial}{\partial q_i} - \frac{d}{dt} \frac{\partial}{\partial \dot{q}_i} \right) \left[\sum_{i=1}^n p_i \dot{q}_i - \mathbb{H} \right] = 0$$

$$\left(\frac{\partial}{\partial p_i} - \frac{d}{dt} \frac{\partial}{\partial \dot{p}_i} \right) \left[\sum_{i=1}^n p_i \dot{q}_i - \mathbb{H} \right] = 0$$

Hamilton's formulation

- J and $\widehat{J}$ are equivalent under the Legendre transformation
  - make q and p independent, whereas before it was a bit of trick to pretend q and $\dot{q}_i$ were independent
- If L does not depend on t, then it should be clear from the Legendre transformation that $\mathbb{H}$ won't depend on t
  - the system will be conservative
  - i.e. $\mathbb{H}$ is a conserved (constant) quantity