Calculus of Variations
Summer Term 2014

Lecture 2

25. April 2014

Purpose of Lesson:

- To discuss the special cases of the E-L equation.
- To discuss the generalizations of the E-L equations to case of n functions and to the ones of higher order derivatives.
- To show that the E-L equation is a necessary, but not sufficient condition for a local extremum.

§2. Remarks on the Euler-Lagrange equation

The Euler–Lagrange Equation:

If y minimizes $J[y] = \int\limits_a^b F(x, y, y')dx$, then y must satisfy the equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] = 0$$

History of Leonhard Euler and Joseph-Louis Lagrange

Leonhard Euler (1707-1783)

Joseph-Louis Lagrange (1736-1813)

- **Euler** developed Euler's Equations for fluids flow ans Euler's formula

$$e^{ix} = cosx + i \sin x.$$

In 1744 Euler published the first book on Calculus of Variations.

- **Lagrange** developed Lagrange Mulipliers, Lagrangian Mechanics, and the Method of Variations of Parameters.

- In 1766 Lagrange succeeded Euler as the director of Mathematics at the Prussian Academy of Sciences in Berlin.
- In letters to Euler between 1754 and 1756 Lagrange shared his observation of a connection between minimizing functionals and finding extrema of a function.
- Euler was so impressed with Lagrange's simplification of his earlier analysis it is rumored he refrained from submitting a paper covering the same topics to give Lagrange more time.

Generalization to n functions:

- Let $F(x, y_1, \ldots, y_n, y_1', \ldots, y_n')$ be a function with continuous first and second partial derivatives. Consider the problem of finding necessary conditions for finding the extremum of the following functional

$$J[y_1, \ldots, y_n] = \int_a^b F(x, y_1, \ldots, y_n, y_1', \ldots, y_n')dx \to \min$$

which depends continuously on n continuously differentiable functions $y_1(x), ..., y_n(x)$ satisfying boundary conditions

$$y_i(a) = A_i, \quad y_i(b) = B_i \qquad \text{for} \quad i = 1, 2, ...n.$$

- One can derive that the necessary condition is a system of Euler-Lagrange Equations

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\frac{\partial F}{\partial y_i'} = 0$$

Generalization to Higher Order Derivatives:

- Let $F(x, y, y', y'', ..., y^{(n)})$ be a function with continuous first and second derivatives with respect to all arguments, and consider a functional of the form

$$J[y] = \int_a^b F(x, y, y', y'', ..., y^{(n)})dx \to \min$$

where the admissible class will be

$$\mathbb{A} = \{y(x) \in C^n[a, b]\}$$

$$y(a) = A_0, y'(a) = A_1, \ldots, y^{(n-1)}(a) = A_{n-1}$$

$$y(b) = B_0, y'(b) = B_1, \ldots, y^{(n-1)}(b) = B_{n-1}$$

- The necessary condition is the Euler-Lagrange Equation

$$F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} - \cdots + (-1)^n \frac{d^n}{dx^n}F_{y^{(n)}} = 0$$

Remark

For a functional of the form

$$J[y] = \int_a^b F(x, y, y')dx$$

the Euler-lagrange equation is in general a second-order differential equation, but it may turn out that the curve for which the functional has its extremum is not twice differentiable.

Example 2.1

Consider the functional

$$J[y] = \int_{-1}^{1} y^2(2x - y')^2 dx$$

where

$$y(-1) = 0, \qquad y(1) = 1.$$

- The minimum of $J[y]$ equals zero and is achieved for the function

$$y = y(x) = \begin{cases} 0 & \text{for} \quad -1 \leqslant x \leqslant 0, \\ x^2 & \text{for} \quad 0 < x \leqslant 1, \end{cases}$$

which has no second derivative for $x = 0$.

- Nevertheless, $y(x)$ satisfies the appropriate E-L equation.
- In fact, since in this case

$$F(x,y,y') = y^2(2x-y')^2$$

it follows that all the functions

$$F_y = 2y(2x-y')^2, \quad F_{y'} = -2y^2(2x-y'), \quad \frac{d}{dx}F_{y'}$$

vanish identically for $-1 \leqslant x \leqslant 1$.

- Thus, despite of the fact that the E-L equation is of the second order and $y''(x)$ does not exist everywhere in $[-1,1]$, substitution of $y(x)$ into E-L's equation converts it into an identity.

We now give conditions guaranteeing that a solution of the E-L equation has a second derivative:

Theorem 2.1

Suppose $y = y(x)$ has a continuous first derivative and satisfy the E-L equation

$$F_y - \frac{d}{dx} F_{y'} = 0.$$

Then, if the function $F(x, y, y')$ has continuous first and second derivatives with respect to all its arguments, $y(x)$ has a continuous second derivative at all points (x, y) where

$$F_{y'y'}(x, y(x), y'(x)) \neq 0.$$

We now indicate some special cases where the Euler-Lagrange equation can be reduced to a first-order differential equation, or where its solution can be obtained by evaluating integrals.

1. Suppose the integrand does not depend on y, i.e., let the functional under consideration have the form

$$J[y] = \int_a^b F(x, y')dx,$$

where F does not contain y explicitly.

- In this case, the E-L equation becomes $\frac{d}{dx}F_{y'} = 0$, which obviously has the first integral

$$F_{y'} = C, \tag{2.1}$$

where C is a constant. This is a first-oder ODE which does not contain y. Solving (2.1) for y', we obtain an equation of the form

$$\boxed{y' = f(x, C)}.$$

2. If the integrand does not depend on x, i.e., if

$$J[y] = \int_a^b F(y, y')dx,$$

then

$$F_y - \frac{d}{dx}F_{y'} = F_y - F_{y'y}y' - F_{y'y'}y''. \tag{2.2}$$

Multiplying (2.2) by y', we obtain

$$F_y y' - F_{y'y}y'^2 - F_{y'y'}y'y'' = \frac{d}{dx}(F - y'F_{y'}).$$

Thus, in this case the E-L equation has the first integral

$$\boxed{F - y'F_{y'} = C}$$

where C is a constant.

3. If F does not depend on y', the E-L equation takes the form

$$F_y(x, y) = 0,$$

and hence is not a differential equation, but a *finite*, whose solution consists of one or more curves $y = y(x)$

4. In a variety of problems, one encounters functionals of the form

$$J[y] = \int\limits_a^b f(x,y)\sqrt{1+y'^2}dx,$$

representing the integral of a function $f(x,y)$ with respect to the arc length s ($ds = \sqrt{1+y'^2}dx$).

- In this case, the E-L equation can be transformed into

$$F_y - \frac{d}{dx}F_{y'} = f_y(x,y)\sqrt{1+y'^2} - \frac{d}{dx}\left[f(x,y)\frac{y'}{\sqrt{1+y'^2}}\right]$$
$$= \frac{1}{\sqrt{1+y'^2}}\left[f_y - f_x y' - f\frac{y''}{1+y'^2}\right] = 0$$

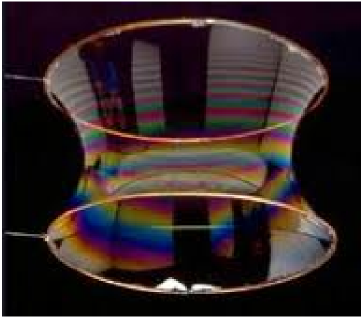
i.e.,

$$\boxed{f_y - f_x y' - f\frac{y''}{1+y'^2} = 0}.$$

Remark

Note that the Euler-Lagrange Equation is a necessary, but not sufficient condition for a local extremum.

Next we will consider the famous example of the minimal surface area for a soap film.



Example 2.2

We want to minimize the following functional

$$J[y] = 2\pi \int_{x_0}^{x_1} y\sqrt{1+(y')^2}dx \to \min$$

according to the boundary conditions $y(x_0) = y_0$, $y(x_1) = y_1$.

- If we use the Euler-Lagrange equation and solve it for $y(x)$ we find the Caternary function

$$y(x) = C_1 \cosh\left\{\left(\frac{x+C_2}{C_1}\right)\right\}$$

- Consider the special case where $C_2 = 0$ and require that $y(x) = C_1 \cosh\left\{\left(\frac{x}{C_1}\right)\right\}$ pass through $(-x_1, 1)$ and $(x_1, 1)$ where x_1 is a constant.
- So C_1 will satisfy

$$1 = C_1 \cosh\left\{\left(\frac{x}{C_1}\right)\right\} \tag{2.1}$$

- Compare $y = 1$ and (2.1) versus C_1

1. For $x_1 = 1$ there is **No Solutions**;
2. For $x_1 = 0.7$ there is exactly One Solutions;
3. For $x_1 = 0.4$ there are Two Solutions.