Calculus of Variations
Summer Term 2014

Lecture 20

18. Juli 2014

Purpose of Lesson:

- To discuss several variational problems arising in image analysis.
- More precisely, we consider
  - Image restoration
  - Segmentation problem
  - Inpainting

§12. Variational problems in image processing

Image Restoration:

- Image restoration is historically one of the oldest concerns in image processing.
- It is still a necessary preprocessing step for many applications.
- Applications are based on images and then rely on their quality.
- Unfortunately, images are not always of good quality for various reasons (defects in the sensors, interference, transmission problems, etc.).

Image Restoration (cont.):

- It is well known that during formation, transmission, and recording processes images deteriorate.
- This degradation is the result of two phenomena :
  1. The first one is deterministic and is related to possible defects of the imaging systems (for example, blur created by an incorrect lens adjustment or by motion).
  2. The second phenomenon is random and corresponds to the noise coming from any signal transmission.
- Our aim to remove or diminish the effects of such degradation. We call this processing restoration.


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Image Restoration (cont.):

A commonly used model is the following:

Model of restoration

Let $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be an original image describing a real scene (the unknown), and let u_0 be the observed image of the same scene, i.e., a degradation of u (the data).

We assume that

$$u_0 = R(u) + \eta, \tag{20.1}$$

where η stands for a white additive Gaussian noise and where R is a linear operator representing the blur (usually a convolution).

Given u_0, the problem is then to reconstruct u knowing (20.1).

Image Restoration (cont.):

The problem of recovering u from u_0 can be approximated by the following variational problem:

$$J[u] = \frac{1}{2}\int\limits_{\Omega} |u_0 - R(u)|^2 dx + \lambda \int\limits_{\Omega} \phi(|\nabla u|) dx \to \min \qquad (20.2)$$
$$\mathbb{X} = W_2^1(\Omega),$$

where $\phi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing function having linear growth at infinity and allowing the preservation of discontinuites.

Remarks

- The choice of ϕ determines the smoothness of the resulting function u.
- The first term in $J[u]$ measures the fidelity to the data. The second is a smoothing term.
- In other words, we search for a u that best fits the data so that its gradient is low (so that noise will be removed).
- The parameter λ is a positive weighting constant.

Difficulties:

- The space $\mathbb{X} = W_2^1(\Omega)$ is not reflexive. In particular, we cannot say anything about minimizing sequences that are bounded in $\mathbb{X}$.
- To recover and preserve edges in an image, it would certainly be preferablr to impose a growth condition on ϕ of the type
$$\lim_{s \to +\infty} \phi(s) = \beta > 0.$$
In this case the contribution of the term $\phi(|\nabla u|)$ in $J[u]$ would not penalize the formation of strong gradients.
- Since we also want ϕ to have a quadratic behaviour near zero, then necessarily ϕ should have a nonconvex shape.

How to survive?

- We have to use the relaxation method and approximate both the functional space $\mathbb{X}$ and the functional J.

- $\mathbb{X} = W_2^1(\Omega)$ we replace by the space $BV(\Omega)$

$$BV(\Omega) = \left\{ u \in L_1(\Omega);\ \int\limits_{\Omega} |Du| < \infty \right\}$$

$$\|u\|_{BV(\Omega)} = \|u\|_{L_1(\Omega)} + \int\limits_{\Omega} |Du| dx$$

$$\int\limits_{\Omega} |Du| dx = \sup \left\{ \int\limits_{\Omega} u \, \mathrm{div}\varphi dx;\ \begin{array}{c} \varphi = (\varphi_1, \dots, \varphi_n) \in C_0^1(\Omega) \\ \|\varphi\|_{L_\infty(\Omega)} \leqslant 1 \end{array} \right\}$$

Remarks

- Surprisingly, we can observe very good numerical results using nonconvex functions ϕ, for example with

$$\phi(s) = \frac{s^2}{1+s^2}.$$

- Observe that the minimization problem

$$J[u] = \int\limits_{\Omega} |u - u_0|^2 dx + \lambda \int\limits_{\Omega} \frac{|\nabla u|^2}{1+|\nabla u|^2} dx \to \min$$

has no minimizer in $W_2^1(\Omega)$ and that the infimum is zero (if u_0 is not a constant).

- It is still open question to understand the numerical solution.

The Segmentation Problem:

- By segmentation we mean that we wish to know the constituent parts of an image rather than to improve its quality (which is a restoration task).
- The aim is to get a simplified version of the original image, compounded of homogeneous regions separated by sharp edges.
- Image segmentation plays a very important role in many applications.
- The main difficulty is that one needs to manipulate objects of different kinds: functions, domains in $\mathbb{R}^2$, and curves.


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The Segmentation Problem:

The Mumford-Shah Functional

Let Ω be a bounded open set of $\mathbb{R}^2$ or $\mathbb{R}^3$, and u_0 be the initial image.

Without loss of generality we can always assume that $0 \leqslant u_0(x) \leqslant 1$ a.e. $x \in \Omega$.

We search for a pair (u, K), where $u : \Omega \to \mathbb{R}$ is a function and $K \subset \Omega$ is the set of discontinuities, minimizing

$$J[u, K] = \int\limits_{\Omega - K} (u - u_0)^2 dx + \alpha \int\limits_{\Omega - K} |\nabla u|^2 dx + \beta \int\limits_{K} d\sigma, \tag{20.3}$$

where α and β are nonnegative constants and $\int\limits_{K} d\sigma$ is the length of K.

Remarks

- The first term in (20.3) measures the fidelity of the data.
- The second term in (20.3) imposes the condition that u be smooth in the region $\Omega - K$.
- The third term in (20.3) establishes that the discontinuity set has minimal length and therefore is as smooth as possible.
- This type of functional forms part of a wider class of problems called free discontinuity problems.

Difficulties:

- $J[u, K]$ involves two unknowns u and K of different natures: u is a function defined on an N-dimensional space ($N = 2$ or $N = 3$), while K is an $(N-1)$-dimensional set.

How to survive?

- It is necessary to find another formulation of $J[u, K]$
- The idea is to identify the set of edges K with the jump set S_u of u, which allows us to eliminate the unknown K.
- So, the idea is to consider the functional

$$\hat{J}[u] = \int_{\Omega} (u - u_0)^2 dx + \alpha \int_{\Omega} |\nabla u|^2 dx + \beta \mathcal{H}^{N-1}(S_u).$$

Inpainting:

- The goal of inpainting is to restore a damaged or corrupted image in which part of the information has been lost.
- Such degradation of an image may have different origins, such as image transmission problems and degradation in real images due to storage conditions or manipulation.
- Inpainting may also be an interesting tool for graphics people who need to remove artificially some parts os an image such as overlapping text or to implement tricks used in special effects.
- In any case, the restoration of missing parts has to be done so that the final image looks unaltered to an observer who does not know the original image.

Inpainting (cont.):

The problem can be described as follows:

Mathematical description

Given a domain image D, a hole $\Omega \subset D$, and an intensity u_0 known over $D - \Omega$.

We want to find an image u, an inpainting of u_0, that matches u_0 outside the hole and has "meaningful" content inside the hole Ω.

This can be achieved by examining the intensity around Ω and propagating it in Ω.

Inpainting (cont.):

Example of the variational approach

$$J[u] = \int\limits_{\overline{\Omega}} |\nabla u| \left(1 + \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)^p\right) dx \to \min$$
$$u = u_0 \quad \text{on} \quad D - \overline{\Omega} \tag{20.4}$$
$$\mathbb{X} = C^2(\overline{\Omega})$$

- Of course, in order to get the existence of a minimizer for $J[u]$ we need to enlarge the space of admissible functions, and consider a relaxed version of $J[u]$ defined on $L_1(D)$ by

$$RJ[u, D] = \inf\{\liminf J[u_n],\ u_n \to u \text{ in } L_1(D),\ n \to \infty\}$$
$$u = u_0 \quad \text{on} \quad D - \overline{\Omega}$$