Calculus of Variations
Summer Term 2014

Lecture 21

23. Juli 2014

Purpose of Lesson:

- To discuss strong and weak topologies in real Banach spaces.
- To introduce Sobolev spaces which would be suitable functional spaces in direct methods

§13. Suitable Functional Spaces

Topologies on Banach Spaces

- Let $(\mathbb{X}, |\cdot|)$ denote a real Banach space.
- A Banach space is a complete, normed linear space. Complete means that any Cauchy sequence is convergent.
- A sequence $\{x_n\}$ of real numbers is called a Cauchy sequence, if for every positive real number ε, there is a positive integer N such that for all natural numbers $m, n > N$

$$|x_n - x_m| < \varepsilon.$$

- Examples:
  1. $\mathbb{R}^n$ with the norm defined by $\|x\| = \left(\sum_i |x_i|^2 \right)^{1/2}$.
  2. $C^k\left(\overline{\Omega}\right)$ with the norm defined by

$$\|u\|_{C^k} = \max_{0 \leqslant |\alpha| \leqslant k} \sup_{x \in \overline{\Omega}} |D^\alpha u(x)|$$

- We denote by $\mathbb{X}'$ the topological dual space of $\mathbb{X}$:

$$\mathbb{X}' = \left\{ l : \mathbb{X} \to \mathbb{R} \text{ linear such that } |l|_{\mathbb{X}'} = \sup_{x \neq 0} \frac{|l(x)|}{|x|_{\mathbb{X}}} < \infty \right\}.$$

- Classically, $\mathbb{X}$ can be endowed with two topologies.

Definition (topologies on $\mathbb{X}$)

(i) The strong topology, denoted by $x_n \underset{\mathbb{X}}{\to} x$, is defined by

$$|x_n - x|_{\mathbb{X}} \to 0 \qquad (n \to \infty).$$

(ii) The weak topology, denoted by $x_n \underset{\mathbb{X}}{\rightharpoonup} x$, is defined by

$$l(x_n) \to l(x) \qquad (n \to \infty) \quad \text{for every} \quad l \in \mathbb{X}'.$$

- Strong convergence implies weak convergence, but the converse is false in general.

Example 21.1 (Counterexample)

- Consider the sequence $f_n(x) = \sin(2\pi xn)$, $x \in (0,1)$, as $n \to \infty$.
- $\boxed{f_n \underset{L_2(\Omega)}{\rightharpoonup} 0}$. To prove weak convergence, we have for all $\varphi \in C^1(0,1)$, thanks to a classical integration by parts,

$$\int_0^1 \sin(2\pi xn)\varphi(x)dx = \frac{1}{2\pi n}[\varphi(0) - \varphi(1)] + \frac{1}{2\pi n}\int_0^1 \cos(2\pi xn)\varphi'(x)dx$$

Example 21.1 (continued)

- So, it is clear that $\langle f_n, \varphi \rangle \to 0$ as $n \to \infty$ for all $\varphi \in C^1(0,1)$, where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $L_2(0,1)$.
- By density, this result can be generalized for all $\varphi \in L_2(0,1)$.
- There is no strong convergence, i.e., $f_n(x) \underset{L_2(0,1)}{\longrightarrow} 0$ is not true.
- To prove that there is no strong convergence, we observe that

$$\int_0^1 \sin^2(2\pi xn)dx = \frac{1}{2}\int_0^1 (1 - \cos[(4\pi xn))\, dx = \frac{1}{2}.$$

- The dual space $\mathbb{X}'$ can also be endowed with the strong and the weak topologies.

Definition (topologies on $\mathbb{X}'$)

(i) The strong topology, denoted by $l_n \underset{\mathbb{X}'}{\rightarrow} l$, is defined by

$$|l_n - l|_{\mathbb{X}'} \to 0, \text{ or equivalently, } \sup_{x \neq 0} \frac{|l_n(x) - l(x)|}{|x|_{\mathbb{X}}} \to 0 \quad (n \to \infty).$$

(ii) The weak topology, denoted by $l_n \underset{\mathbb{X}'}{\rightharpoonup} l$, is defined by

$$z(l_n) \to z(l) \quad (n \to \infty) \quad \text{for every} \quad z \in \left(\mathbb{X}'\right)',$$

where $(\mathbb{X}')'$ denotes the bidual space of $\mathbb{X}$.

- In some cases it is more convenient to equip $\mathbb{X}'$ with a third topology:

  The weak* topology, denoted by $l_n \underset{\mathbb{X}'*}{\rightharpoonup} l$, is defined by

$$l_n(x) \to l(x) \quad (n \to \infty) \quad \text{for every} \quad x \in \mathbb{X}.$$

- The space $\mathbb{X}$ is called reflexive if $(\mathbb{X}') = \mathbb{X}$.
- $\mathbb{X}$ is called separable if it contains a countable dense set.
- Examples:
  1. $\mathbb{X} = L_p(\Omega)$ is reflexive for $1 < p < \infty$ and separable for $1 \leqslant p < \infty$. The dual space of $L_p(\Omega)$ is $L_{p'}(\Omega$ for $1 \leqslant p < \infty$ with $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.
  2. $\mathbb{X} = L_1(\Omega)$ is nonreflexive and $\mathbb{X}' = L_\infty(\Omega)$.

The main properties associated with these different topologies are summarized in the following theorem :

Theorem 21.1 (weak sequential compactness)

(i) Let $\mathbb{X}$ be a reflexive Banach space, $K > 0$, and $x_n \in \mathbb{X}$ a sequence such that $|x_n|_{\mathbb{X}} \leqslant K$.

Then there exist $x \in \mathbb{X}$ and a subsequence x_{n_j} of x_n such that

$$x_{n_j} \underset{\mathbb{X}}{\rightharpoonup} x \qquad (n \to \infty).$$

(ii) Let $\mathbb{X}$ be a separable Banach space, $K > 0$, and $l_n \in \mathbb{X}'$ such that $|l_n|_{\mathbb{X}'} \leqslant K$.

Then there exist $l \in \mathbb{X}'$ and a subsequence l_{n_j} of l_n such that

$$l_{n_j} \underset{\mathbb{X}'*}{\rightharpoonup} l \qquad (n \to \infty).$$

Sobolev spaces

- Before giving the definition of Sobolev spaces, we need to weaken the notion of derivative.
- In doing so we want to keep the right to intergate by parts; this is one of the reasons of the following definition.

Definition (weak derivative)

Let $\Omega \subset \mathbb{R}^n$ be open and $u \in L_{1,loc}(\Omega)$.

We say that $v \in L_{1,loc}(\Omega)$ is the weak partial derivative of u with respect to x_i if

$$\int_{\Omega} v(x)\varphi(x)dx = -\int_{\Omega} u(x)\frac{\partial \varphi(x)}{\partial x_i}dx, \quad \forall \varphi \in C_0^{\infty}(\Omega).$$

By abuse of notations we write $v = \dfrac{\partial u}{\partial x_i}$ or u_{x_i}.

We say that u is weakly differentiable of the weak partial derivatives $u_{x_1}, \dots, u_{x_n}$ exist.

Remarks

- All the usual rules of differentiation are easily generalized to the present context of weak differentiability.
- If a function is C^1, then the usual notion of derivative and the weak one coincide.
- Not all measurable functions can be differentiated weakly. In particular, a discontinuous function of $\mathbb{R}$ cannot be differentiated in the weak sense.

Definition (Sobolev spaces)

Let $\Omega \subset \mathbb{R}^n$ be an open set and $1 \leqslant p \leqslant \infty$

We let $W_p^1(\Omega)$ be the set of functions $u : \Omega \to \mathbb{R}$, $u \in L_p(\Omega)$, whose weak partial derivatives $u_{x_i} \in L_p(\Omega)$ for every $i = 1, \dots, n$.

We endow this space with the following norm

$$\|u\|_{W_p^1} = \left(\|u\|_p^p + \|\nabla u\|_p^p\right)^{1/p} \quad \text{if} \quad 1 \leqslant p < \infty$$
$$\|u\|_{W_\infty^1} = \max\left\{\|u\|_\infty, \|\nabla u\|_\infty\right\} \quad \text{if} \quad p = \infty.$$

Here

$$\|u\|_p := \left(\int\limits_\Omega |u|^p dx\right)^{1/p} \qquad \text{if} \quad 1 \leqslant p < \infty$$
$$\|u\|_\infty := \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| = \inf\left\{\alpha : |u(x)| \leqslant \alpha \text{ a.e.in } \Omega\right\}.$$

Remarks

- By abuse of notations we write $W_p^0 = L_p$.
- If $1 \leqslant p < \infty$, the set $W_{p,0}^1(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$-functions in $W_p^1(\Omega)$.
- We often say, if Ω is bounded, that $u \in W_{p,0}^1(\Omega)$ is such that $u \in W_p^1(\Omega)$ and $u = 0$ on $\partial\Omega$.
- We also write $u \in u_0 + W_{p,0}^1(\Omega)$ meaning that $u, u_0 \in W_p^1(\Omega)$ and $u - u_0 \in W_{p,0}^1(\Omega)$.
- We let $W_{\infty,0}^1(\Omega) = W_\infty^1(\Omega) \cap W_{1,0}^1(\Omega)$.
- Note that if Ω is bounded, then

$$C^1(\overline{\Omega}) \subsetneqq W_\infty^1(\Omega) \subsetneqq W_p^1(\Omega) \subsetneqq L_p(\Omega) \qquad \text{for every} \quad 1 \leqslant p < \infty.$$

Analogously we define the Sobolev spaces with higher derivatives as folows.

Definition (Sobolev spaces with higher derivatives)

Let $\Omega \subset \mathbb{R}^n$ be an open set and $1 \leqslant p \leqslant \infty$

If $k > 0$ is an integer we let $W_p^k(\Omega)$ to be the set of functions $u : \Omega \to \mathbb{R}$, whose weak partial derivatives $D^\alpha u \in L_p(\Omega)$, for every multi-index $\alpha \in \mathcal{A}_m$ with

$$\mathcal{A}_m := \left\{ \alpha = (\alpha_1, \dots \alpha_n), \alpha_j \geqslant 0 \text{ an integer and } \sum_{j=1}^{n} \alpha_j = m \right\},$$

$0 \leqslant m \leqslant k$.

Definition (Sobolev spaces with higher derivatives - cont.)

The norm in $W_p^k(\Omega)$ is given by

$$\|u\|_{W_p^k} = \begin{cases} \left(\sum\limits_{0 \leqslant |\alpha| \leqslant k} \|D^\alpha u\|_p^p \right)^{1/p} & \text{if} \quad 1 \leqslant p < \infty \\ \max\limits_{0 \leqslant |\alpha| \leqslant k} \{\|D^\alpha u\|_\infty\} & \text{if} \quad p = \infty \end{cases}$$

Remark

If we denote by $I = (a, b)$, we have, for $p \geqslant 1$,

$$C_0^\infty(I) \subset \cdots \subset W_p^2(\Omega) \subset C^1(\bar{I}) \subset W_p^1(I)$$
$$\subset C(\bar{I}) \subset L_\infty(I) \subset \cdots \subset L_2(I) \subset L_1(I)$$

Theorem 21.2

Let $\Omega \subset \mathbb{R}^n$ be an open set, $1 \leqslant p \leqslant \infty$ and $k \geqslant 1$ an integer.

$W_p^k(\Omega)$ equipped with its norm $\| \cdot \|_{W_p^k}$ is a Banach space which is separable if $1 \leqslant p < \infty$ and reflexive if $1 < p < \infty$.

Remarks

- Note that the space W_1^1 is not reflexive.
- This is the main source of difficulties in the minimal surface problem.