Calculus of Variations
Summer Term 2014

Lecture 3

2. Mai 2014

Purpose of Lesson:

- To discuss the simplest variational problems involving undetermined end points.

Problem 3-1

We seek to minimize the integral

$$J[y] = \int_a^b F(x, y, y')dx \to \min$$

with respect to functions that attain the value A for $x = a$, but for which no value is prescribed at $x = b$.

Question:

What is the arc of quickest descent from a fixed point to a vertical line?

- To find the minimizing function, we as before introduce a small variation from $y(x)$, namely

$$y(x) + \varepsilon\eta(x)$$

where ε is a small parameter and $\eta(x)$ is a smooth curve satisfying the BC

$$\eta(a) = 0.$$

- We take the derivative of

$$\phi(\varepsilon) = J[y + \varepsilon\eta]$$

with respect to ε, evaluate it $\varepsilon = 0$, and set this equal to zero; that is,

$$\begin{aligned} \frac{d\phi(\varepsilon)}{d\varepsilon} &= \frac{d}{d\varepsilon} J[y + \varepsilon\eta]\Big|_{\varepsilon=0} \\ &= \int_a^b \left[\frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\eta'(x) \right] dx \end{aligned}$$

- Integration by parts gives

$$\left[\frac{\partial F}{\partial y'}\eta(x)\right]^{x=b} + \int_a^b \left\{\frac{\partial F}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right]\right\}\eta(x)dx = 0. \qquad (3.1)$$

- Since (3.1) must hold for all choices of $\eta(x)$ satisfying

$$\eta(a) = 0,$$

it must in particular hold for those η for which $\eta(b) = 0$. For such $\eta(x)$ the first term in (3.1) disappeared and as before we end up with

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] = 0}. \qquad (3.2)$$

- So, with result (3.2), and for general $\eta(x)$ once again, the second member of (3.1) reduces to its first term

$$\left[\frac{\partial F}{\partial y'}\eta(x)\right]^{x=b} = 0.$$

- Now, by choosing $\eta(b) = 1$, the vanishing for all η of the term remaining requires fulfillment of the end-point condition

$$\boxed{\left.\frac{\partial F}{\partial y'}\right|_{x=b} = 0}. \qquad (3.3)$$

- The two constants of integration obtained in the solution of (3.2), a second-order equation, are determined by the end-point condition $y(a) = A$ and (3.3) - provided, of course, a solution of the problem exists.

Problem 3-2

We seek to minimize the integral

$$J[y] = \int_a^{x^*} F(x, y, y')dx \to \min$$

with respect to functions which attain the value A for $x = a$ and which satisfy the given relation

$$g(x, y) = 0$$

at the upper limit of integration, as yet undetermined.

Question:

- What is the arc of quickest descent from a fixed point to a given curve?

- To find the minimizing function, we as before introduce a small variation from $y(x)$, namely

$$y(x) + \varepsilon\eta(x)$$

where ε is a small parameter and $\eta(x)$ is a smooth curve satisfying the BC

$$\eta(a) = 0.$$

- The point of intersection of our small variation $y(x) + \varepsilon\eta(x)$ with the given curve $g(x, y) = 0$ is denoted by (x^*, y^*). We thus have

$$\boxed{g(x^*, y^*) = 0, \qquad y^* = y(x^*) + \varepsilon\eta(x^*)}. \tag{3.4}$$

- We take the derivative of

$$\phi(\varepsilon) = J[y + \varepsilon\eta]$$

with respect to ε, evaluate it $\varepsilon = 0$, and set this equal to zero; that is,

$$\frac{d\phi(\varepsilon)}{d\varepsilon} = \frac{d}{d\varepsilon} J[y + \varepsilon\eta]\Big|_{\varepsilon=0}$$

$$= \int\limits_a^{x^*} \left[\frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\eta'(x)\right] dx + F(x^*, y^*, (y^*)')\frac{dx^*}{d\varepsilon}\Big|_{\varepsilon=0}$$

- Since relations (3.4) hold for all ε, we have that the total derivative of $g(x^*, y^*)$ with respect to ε must vanish.

- From (3.4) we therefore obtain, on noting that x^* is a function of ε for any given $\eta(x)$,

$$\begin{aligned} 0 &= \frac{\partial g}{\partial x^*}\frac{dx^*}{d\varepsilon} + \frac{\partial g}{\partial y^*}\frac{dy^*}{d\varepsilon} \\ &= \frac{\partial g}{\partial x^*}\frac{dx^*}{d\varepsilon} + \frac{\partial g}{\partial y^*}\left[y'(x^*)\frac{dx^*}{d\varepsilon} + \eta(x^*) + \varepsilon\eta'(x^*)\frac{dx^*}{d\varepsilon}\right]. \end{aligned}$$

- Solving the above equality, with $\varepsilon = 0$, for $\left.\frac{dx^*}{d\varepsilon}\right|_{\varepsilon=0}$ we obtain

$$\left.\frac{dx^*}{d\varepsilon}\right|_{\varepsilon=0} = -\frac{\eta(x^*)\frac{\partial g}{\partial y^*}}{\frac{\partial g}{\partial x^*} + y'(x^*)\frac{\partial g}{\partial y^*}} \tag{3.5}$$

- With the aid of (3.5) and integration by parts we get

$$\int_a^{x^*}\left[\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta dx+\eta(x^*)\left[\frac{\partial F}{\partial y'}-\frac{F\frac{\partial g}{\partial y^*}}{\frac{\partial g}{\partial x^*}+(y^*)'\frac{\partial g}{\partial y^*}}\right]^{x=x^*}=0$$

- Repeating the line of arguments carried out in Problem 3-1 above we conclude that $y=y(x)$ satisfies the Euler-Lagrange equation

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right]=0$$

and, in addition to the BC $y(a)=A$, the right-hand end-point condition

$$\boxed{\left[\frac{\partial F}{\partial y'}-\frac{F\frac{\partial g}{\partial y^*}}{\frac{\partial g}{\partial x^*}+(y^*)'\frac{\partial g}{\partial y^*}}\right]^{x=x^*}=0}.$$