Calculus of Variations
Summer Term 2014

Lecture 4

2. Mai 2014

Purpose of Lesson:

- To discuss the generalization of the E-L equation to case of several variables. To illustrate the correspondence between the multivariable variational problems and PDEs.
- To consider a class of problems in which the functionals are required to conform with certain restrictions that are added to the usual continuity requirements and possible end-points conditions.

In this subsection we consider functionals of the type

$$J[u] = \int\limits_{\Omega} F(x, u(x), Du(x))dx,$$

where

- Ω is a bounded open set in $\mathbb{R}^n$,
- $u : \overline{\Omega} \to \mathbb{R}$ is a function of class C^2,
- $F(x, u, p)$ is a real valued function of class C^2 with respect of all its arguments.

Remark

- The integrand $F(x, u, p)$ is denoted as Lagrangian, or variational integrand, or Lagrange function.

- Consequently, the function

$$\Phi(\varepsilon) := J[u + \varepsilon\eta]$$

is defined for each $\eta \in C_0^\infty(\Omega, \mathbb{R})$ and for sufficiently small ε.

- Moreover, Φ is of class C^2 on some interval $(-\varepsilon_0, \varepsilon_0)$.

- We will call its derivative $\Phi'(0)$ at $\varepsilon = 0$ the first variation of J at u in direction of η and denote (sometimes)

$$\delta J[u, \eta] = \Phi'(0).$$

- A stright-forward computation yields

$$\delta J[u, \eta] = \Phi'(0) = \int\limits_\Omega \{F_u(x, u, Du)\eta + F_p(x, u, Du) \cdot D\eta\}\, dx$$

$$= \int\limits_\Omega \left\{ F_u(x, u, Du)\eta + \sum_{j=1}^n \frac{\partial F(x, u, Du)}{\partial p_j} D_j\eta \right\} dx$$

- Involving the integration by parts we end up with

$$
\begin{aligned}
0 &= F_u(x, u, Du) - \sum_{j=1}^{n} \frac{d}{dx_j} \left[\frac{\partial F(x, u, Du)}{\partial p_j} \right] \\
&= F_u - \sum_{j=1}^{n} \left[\frac{\partial^2 F}{\partial p_j \partial x_j} + \frac{\partial^2 F}{\partial p_j \partial u} D_j u + \sum_{k=1}^{n} \frac{\partial^2 F}{\partial p_j \partial p_k} D_k D_j u \right].
\end{aligned}
$$

Example 4.1 (The Laplace equation)

Consider the *Dirichlet integral* defined by

$$\mathcal{D}[u] := \frac{1}{2} \int\limits_{\Omega} |Du|^2 dx$$

The corresponding Euler-Lagrange equation has the form

$$\Delta u = 0 \quad \text{in} \quad \Omega.$$

Example 4.2 (The Poisson equation)

The integral

$$J[u] = \int\limits_{\Omega} \left[\frac{1}{2}|Du|^2 + f(x)u\right] dx$$

with the Lagrangian

$$F(x, u, p) = \frac{1}{2}|p|^2 + f(x)u$$

has the so-called *The Poisson equation*

$$\Delta u = f(x) \quad \text{in} \quad \Omega$$

as the corresponding E-L equation.

Example 4.3 (The nonlinear Poisson equation)

The nonlinear Poisson equation

$$\Delta u = f(u) \quad \text{in} \quad \Omega$$

is the Euler-Lagrange equation of the integral

$$J[u] = \int\limits_{\Omega} \left\{ \frac{1}{2} |Du|^2 + g(u) \right\} dx$$

where g is a primitive function of f, i.e., $g'(z) = f(z)$.

Example 4.4 (The wave equation)

Interpret $\mathbb{R}^4$ as space-time continuum of points $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$ where x is the position vector in $\mathbb{R}^3$ and t denotes the time.

Let Ω be some bounded domain in $\mathbb{R}^4$, and consider the function $u(x, t)$ of x and t. Then the integral

$$J[u] = \int\limits_{\Omega} \frac{1}{2} \left\{ u_t^2 - |Du|^2 \right\} dxdt$$

has the wave equation

$$\Box u := u_{tt} - \Delta u = 0 \quad \text{in} \quad \Omega$$

as the Euler-Lagrange equation.

Example 4.5 (The minimal surface equation)

The *area functional*

$$\mathcal{A}[u] = \int_{\Omega} \sqrt{1 + |Du|^2} dx$$

for hypersurfaces $z = u(x)$, $x \in \Omega \subset \mathbb{R}^n$, in $\mathbb{R}^{n+1}$ yields the minimal surface equation

$$\operatorname{div} Tu = 0 \quad \text{in} \quad \Omega, \qquad Tu := \frac{Du}{\sqrt{1 + |Du|^2}}$$

as the Euler-Lagrange equation, which we can also write as

$$\sum_{i=1}^{n} D_i \left(\frac{D_i u}{\sqrt{1 + |Du|^2}} \right) = 0.$$

Example 4.6 (Digital Image Processing)

We consider a distorted Black-White-Image which is described by a function $f : \mathbb{R}^2 \supset \Omega \to \mathbb{R}$, here $\Omega = [0, a] \times [0, b]$.

The value $f(x)$ corresponds the darkness of gray color in the image point x (the brighter corresponds the greater value of f). In order to eliminate the interference as much as possible we minimize the functional

$$E_f[u] := \int\limits_{\Omega} \left\{ (u - f)^2 + \alpha |Du|^2 \right\} dx, \qquad \alpha > 0.$$

Here $(f - u)^2$ stands for the difference with the original image and $|Du|$ is a measure of the smoothness of the denoised image. The Integral $E_f[u]$ has the following E-L equation

$$u - \Delta u = f \quad \text{in} \quad \Omega.$$

§3. Isoperimetric problems

We now include additional constarints into our minimizations problems:

- Integral constraints of the form

$$\int G(x, y, y')dx = const$$

  e.g., the Isoperimetric problem.

- Holonimic constraints, e.g., $G(x, y) = 0$
- Non-holonomic constraints, e.g., $G(x, y, y') = 0$
- We won't consider inequality constraints until later.

The standard example of a problem with **integral constraints** is *Dido's problem*.

Dido's problem

- This is probably one of the oldest problem in the Calculus of Variations.
- Dido (Carthaginian queen) founded the city of Carthage, in Tunisia.
- According to legend, she arrived at the site with her entourage, a refugee from a power struggle with her brother in Tyre in the Lebanon.
- She asked the locals for as much land as could be bound by a bull's hide.
- She cut the hide into a long thin strip and bounded the maximum possible area with this.

Dido Purchases Land for the Foundation of Carthage. Engraving by Matthäus Merian the Elder, in *Historische Chronica*, Frankfurt a.M., 1630. Dido's people cut the hide of an ox into thin strips and try to enclose a maximal domain.

Dido's problem falls into the class of isoperimetric problems.

- iso- (from same) and perimetric (from perimeter), roughly meaning "same perimeter".
- In general, such problems involve a constraint
  - e.g., the length of the bull's hide strip.
  - But the constraints is not always to fix the perimeter length.
  - Sometimes the constraint does not even involve a length.
  - But the term isoperimetric is still used.

We can write the isoperimetric problems in the following form:

The simple isoperimetric problem:

We are looking for the extremals of the functional

$$J[y] = \int_a^b F(x, y, y')dx \to \min$$

with all the usual conditions (e.g. on end points, and continuous derivatives) but in addition we must satisfy the extra functional constraint

$$\mathcal{G}[y] = \int_a^b G(x, y, y')dx = L$$

A simplified form of Dido's problem:

Imagine that the two end-points are fixed, along the cost (Carthage was a great sea power), and we wish to encompass the largest possible area inland with a fixed length L.

We can write this problem as maximize the area

$$J[y] = \int_a^b y dx \to \max$$

encompassed by the curve y, such that the curve y has the fixed length L, e.g., as before the length of the curve is

$$\mathcal{G}[y] = \int_a^b \sqrt{1 + y'^2} dx = L$$

subject to the end-pont conditions $y(a) = y(b) = 0$.

A simplified form of Dido's problem:

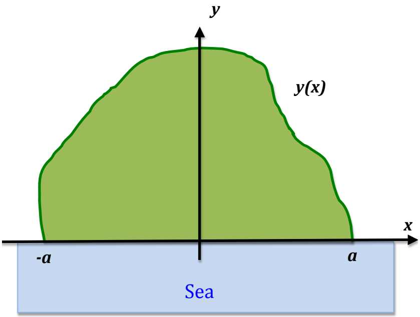
$$J[y] = \int_{-a}^{a} y\,dx \longrightarrow \max$$

subject to

$$G[y] = \int_{-a}^{a} \sqrt{1+(y')^2}\,dx = L$$

and

$$y(-a) = y(a) = 0$$



For simplicity take

$$\boxed{2a < L \le \pi a}$$

As before

- we perturb the curve, and consider the first variation
- but we cannot perturb by an arbitrary function $\varepsilon\eta$. because then the constraint

$$\mathcal{G}[y + \varepsilon\eta] = L$$

might be violated.

- **solution:** use the same approach as in constrained maximization, e.g. use Lagrange multipliers

Problem

To find the minimum (or maximum) of $f(x)$ for $x \in \mathbb{R}^n$ subject the constraints

$$g_i(x) = 0, \qquad i = 1, \dots, m < n \tag{4.1}$$

- Solution requires Lagrange Multipliers.
- Minimize (ot maximize) a new function (of $m + n$ variables)

$$h(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x),$$

  where λ_i are the undetermined Lagrange multipliers.
- The constants $\lambda_1, \dots, \lambda_m$ are evaluated by means of the set of equations consisting of (4.1) and

$$\frac{\partial h(x, \lambda)}{\partial x_j} = 0, \qquad j = 1, \dots, n$$

To maximize

$$J[y] = \int_a^b F(x, y, y')dx$$

subject to

$$\mathcal{G}[y] = \int_a^b G(x, y, y')dx = L$$

we instead consider the problem of finding extremals of

$$\mathcal{H}[y] = \int_a^b H(x, y, y')dx = \int_a^b \{F(x, y, y') + \lambda G(x, y, y')\} \, dx$$

The Euler-Lagrange equations become

$$\boxed{\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0}$$

where $H = F + \lambda G$, and λ is the unknown Lagrange multiplier.

Example 4.7 (Simple Dido's problem)

$$\mathcal{H}[y] = \int\limits_{-a}^{a} \left(y + \lambda\sqrt{1+(y')^2} \right) dx$$

so

$$\frac{\partial H}{\partial y} = 1$$

$$\frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = \frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+(y')^2}}\right)$$

and the Euler-Lagrange equation is

$$\frac{d}{dx}\frac{\lambda y'}{\sqrt{1+(y')^2}} = 1$$

Example 4.7 (Simple Dido's problem)

- Integrating with respect to x we get

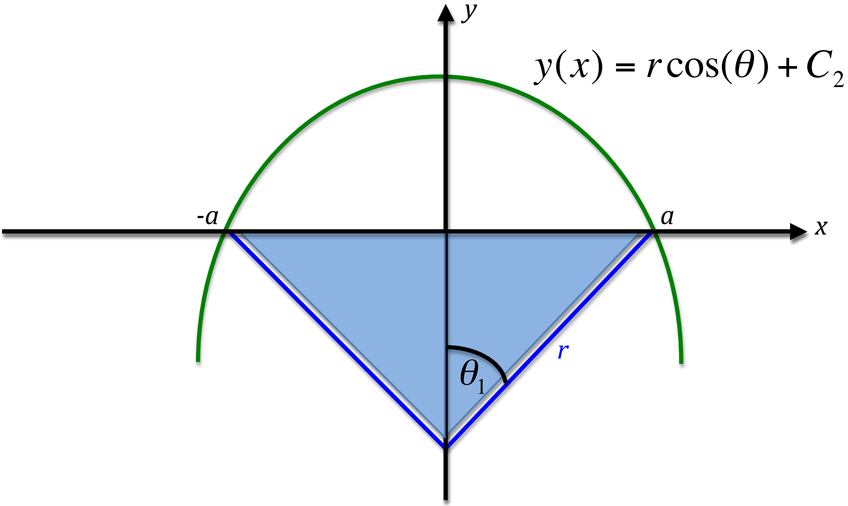
$$\begin{aligned} x + C_1 &= \lambda \sin(\theta) \\ y &= -\lambda \cos(\theta) + C_2 \end{aligned}$$

  where λ, C_1 and C_2 are determined by the two end-points, and the length of the curve L.

- We may draw a sketch of the solution, and clearly we can identify $-\lambda = r$ the radius of a circle, of which our region is a segment.

- Note we deliberately started with

$$2a < L \leqslant \pi a.$$


y
$y(x) = r\cos(\theta) + C_2$
-a
a
x
r
θ_1

Example 4.7 (Simple Dido's problem)

- We can see that the arc length of the enclosing curve will be

$$L = 2\theta_1 r$$

and the the value on the right-end determines that

$$r = \frac{a}{\sin(\theta_1)}$$

- Therefore, we have

$$L = \frac{2a\theta_1}{\sin(\theta_1)}$$

from which we may determine θ_1.

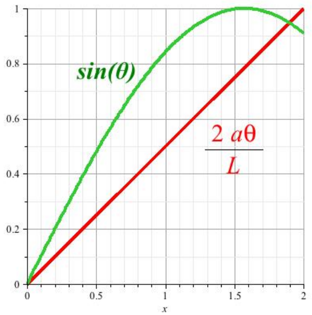
Example 4.7 (Simple Dido's problem)

- Since we determine θ_1 from

$$\sin(\theta_1) = \frac{2a}{L}\theta_1$$

we may compute

$$r = \frac{a}{\sin(\theta_1)}.$$



Example 4.7 (Simple Dido's problem)

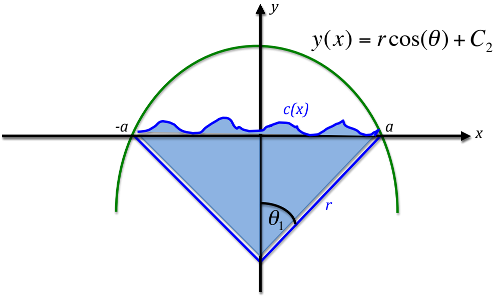
- From the conditions $y(\pm a) = 0$ it follows that

$$C_2 = -\cos(\theta_1).$$

- The maximum possible area bounded by a curve of fixed length is a circle. So the city of Carthage is circular in shape.
- The story of Carthage isn't quite true (see picture below).

What effect would a realistic coastline have?

- Coast $c(x)$.



- Area $= \int_{-a}^{a} (y - c)\, dx$
- Note that c doesn't depend on y or y', so the Euler-Lagrange equations are unchanged, provided $c(x) < y(x)$ for the extremal.

What effect would a realistic coastline have?

- If the condition $c(x) < y(x)$ is not satisfied then the area integral includes negative components, so the problem we are maximizing is not really Dido's problem any more (she can't own negative areas).

- We really want to maximize

$$\text{Area} = \int_a^b [y - c]^+ \, dx$$

where

$$[x]^+ = \begin{cases} x, & \text{for } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

- Note that the function $[x]^+$ does not have a derivative at $x = 0$.