Calculus of Variations
Summer Term 2014

Lecture 5

7. Mai 2014

Purpose of Lesson:

- To discuss catenary of fixed length.
- Consider possible pathologic cases, discuss rigid extremals and give interpretation of the Lagrange multiplier λ
- To solve the more general case of Dido's problem with general shape and parametrically described perimeter.

Example 5.1 (Catenary of fixed length)

- In Example 2.2 we computed the shape of a suspended wire, when we put no constraints on the length of the wire.

Picture: A hanging chain forms a catenary

- What happens to the shape of the suspended wire when we fix the length of the wire?

Example 5.1 (Catenary of fixed length)

- As before we seek a minimum for the potential energy

$$J[y] = \int_{x_0}^{x_1} y\sqrt{1+(y')^2}dx \to \min$$

but now we include the constraint that the lentgh of the wire is L, e.g.

$$\mathcal{G}[y] = \int_{x_0}^{x_1} \sqrt{1+(y')^2}dx = L$$

- We seek extremals of the new functional

$$\mathcal{H}[y] = \int_{x_0}^{x_1} (y+\lambda)\sqrt{1+(y')^2}\,dx.$$

- Notice that $H(x,y,y') = (y+\lambda)\sqrt{1+(y')^2}$ has no explicit dependence on x, and so we may compute

$$H - y'H_{y'} = \frac{(y+\lambda)(y')^2}{\sqrt{1+(y')^2}} - (y+\lambda)\sqrt{1+(y')^2} = const$$

- Perform the change of variables $u = y+\lambda$, and note that $u' = y'$ so that the above can be rewritten as

$$\frac{u(u')^2}{\sqrt{1+(u')^2}} - u\sqrt{1+(u')^2} = C_1. \tag{5.1}$$

- It is easy to see that Eq. (5.1) reduces to

$$\frac{u^2}{1+(u')^2} = C_1^2. \tag{5.2}$$

- Eq. (5.2) is exactly the same equation (in u) as we had previously for the catenary in y. So, the result is a catenary also, but shifted up or down by an amount such that the length of the wire is L.

$$y = u - \lambda = C_1 \cosh\left(\frac{x - C_2}{C_1}\right) - \lambda$$

- So, we have three constants to determine
  1. we have two end points
  2. we have the length constraint

- As in Example 2.2 we put $\boxed{C_2 = 0}$ and consider the even solution with $x_0 = -1$, $y(x_0) = 1$, $x_1 = 1$ and $y(1) = 1$.
- $$L = \int_{-1}^{1} \sqrt{1 + (y')^2} dx = \int_{-1}^{1} \cosh\left(\frac{x}{C_1}\right) dx$$
$$= C_1 \left[\sinh\left(\frac{x}{C_1}\right)\right]_{-1}^{1} = 2C_1 \sinh\left(\frac{1}{C_1}\right)$$
- Now we can calculate C_1 from the above equality.
- Once we know C_1 we can calculate λ to satisfy the end heights $y(-1) = y(1) = 1$.

Example 5.2 (cf. Example 5.1)

- From Example 5.1 we know that a solution of the catenary problem with length constraint has the form

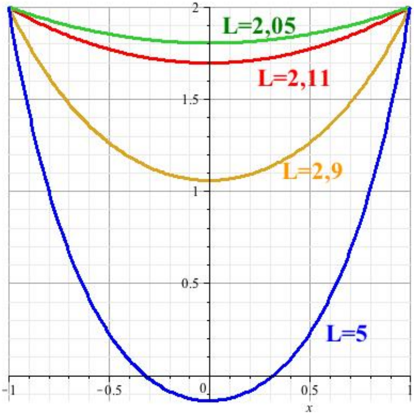
$$y = C cosh\left(\frac{x}{C}\right) - \lambda,$$

and y satisfy the additional conditions

$$y(-1) = y(1) = 2, \qquad L = \int\limits_{-1}^{1} \sqrt{1 + (y')^2} dx = 2C \sinh\left(\frac{x}{C}\right).$$

- Using Maple we calculate y for the natural catenary (without length constraint), as well as for $L = 2.05$ $L = 2.9$ and $L = 5$. See Worksheet 1 for the detailed calculation.

- All catenaries are valid, but one is natural



- The red curve shows the natural catenary (without length constraint), and the green, yellow and blue curves show other catenaries with different lengths.

Pathologies

Note that in both cases (”simple Dido’s problem” and ”catenary of fixed length”)

- the approach only works for certain ranges of L.
- If L is too small, there is no physically possible solution
  - e.g., if wire length $L < x_1 - x_0$
  - e.g., if oxhide length $L < x_1 - x_0$
- If L is too large in comparison to $y_1 = y(x_1)$, the solution may have our wire dragging on the ground.

A particular problem to watch for are rigid extremals

- Rigid extremals are extremals that cannot be perturbed, and stiil satisfy the constraint.

Example 5.3

- For example

$$\mathcal{G}[y] = \int_0^1 \sqrt{1 + (y')^2} dx = \sqrt{2}$$

  with the boundary constraints $y(0) = 0$ and $y(1) = 1$.

- The only possible y to satisfy this constraint is $y(x) = x$, so we cannot perturb around this curve to find conditions for viable extremals.

- Rigid extremals cases have some similarities to maximization of a function, where the constraints specify a single point:

Example 5.4

Maximize $f(x, y) = x + y$, under the constraint that $x^2 + y^2 = 0$.

- In Example 5.3, the constraint, and the end-point leave only one choice of function, $y(x) = x$.

Interpretation of λ:

Consider again to finding extremals for

$$\mathcal{H}[y] = J[y] + \lambda \mathcal{G}[y], \tag{5.3}$$

where we include $\mathcal{G}$ to meet an isoperimetric constraint $\mathcal{G}[y] = L$.

- One way to think about λ is to think of (5.3) as trying to minimize $J[y]$ and $\mathcal{G}[y] - L$.
  1. λ is a tradeoff between J and $\mathcal{G}$.
  2. If λ is big, we give a lot of weight to $\mathcal{G}$.
  3. If λ is small, then we give most weight to J.
- So, λ might be thought of as how hard we have to "pull" towards the constraint in order to make it.

Interpretation of λ (cont.)

For example,

- in the catenary problem, the size of λ is the amount we have to shift the cosh function up or down to get the right length.
- when $\lambda = 0$ we get the natural catenary,

  i.e., in this case, we didn't need to change anything to get the right shape, so the constraint had no affect.

Interpretation of λ (cont.)

Write the problem (including the constant) as minimize

$$\mathcal{H}[y] = \int F + \lambda(G - k)dx,$$

for the constant $k = \dfrac{L}{\int 1 dx}$, then

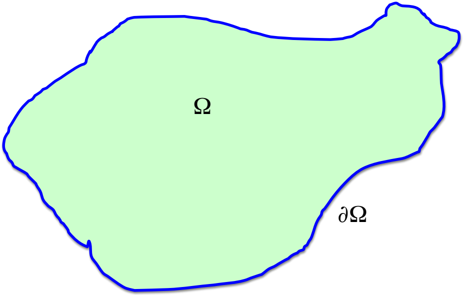
$$\frac{\partial \mathcal{H}}{\partial k} = \lambda,$$

- we can also think of λ as the rate of change of the value of the optimum with respect to k.
- when $\lambda = 0$, the functional $\mathcal{H}$ has a stationary point

  e.g., in the catenary problem this is a local minimum corresponding to the natural catenary.

Consider now the more general case of Dido's problem:

- a general shape,



- without a coast,

so that the perimeter must be parametrically described.

Dido's problem is usual posed as follows:

Problem 5-1 (Dido's problem -traditional)

To find the curve of length L which encloses the largest possible area, i.e., maximize

$$\text{Area} = \iint\limits_{\Omega} 1 dxdy$$

subject to the constraint

$$\oint\limits_{\partial\Omega} 1 ds = L$$

Of course Problem 5-1 is not yet in a convinient form.

Green's Theorem converts an integral over the area Ω to a contour integral around the boundary $\partial\Omega$.

Green's Theorem

$$\iint\limits_{\Omega} \left(\frac{\partial \phi}{\partial x} + \frac{\partial \varphi}{\partial y} \right) dxdy = \oint\limits_{\partial\Omega} \phi dy - \varphi dx$$

for $\phi, \varphi : \Omega \to \mathbb{R}$ such that ϕ, φ, ϕ_x and φ_y are continuous.

This converts an area integral over a region into a line integral around the boundary.

- The area of a region is given by

$$\text{Area} = \iint_{\Omega} 1 dxdy.$$

- In Green's theorem choose $\phi = \dfrac{x}{2}$ and $\varphi = \dfrac{y}{2}$, so that we get

$$\text{Area} = \iint_{\Omega} 1 dxdy = \frac{1}{2} \oint_{\partial\Omega} xdy - ydx$$

- Previous approach to Dido, was to use $y = y(x)$, but in more general case where the boundary must be closed, we can't define y as a function of x (or visa versa).
- So, we write the boundary curve parametrically as $(x(t), y(t))$.

- If the boundary $\partial\Omega$ is represented parametrically by $(x(t), y(t))$ then

$$\text{Area} = \iint_{\Omega} 1 dxdy = \frac{1}{2}\oint_{\partial\Omega} xdy - ydx = \frac{1}{2}\oint_{\partial\Omega} (x\dot{y} - y\dot{x})\, dt$$

- So, now the problem is written in terms of

$$\begin{aligned} \text{one independent variable } &= t \\ \text{two dependent variables } &= (x, y). \end{aligned}$$

- Previously we wrote the isoperimetric constraint as

$$\mathcal{G}[y] = \int 1 ds = \int_{x_0}^{x^1} \sqrt{1 + (y')^2} dx = L$$

- Now we must also modify the constraint using

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

to get

$$\mathcal{G}[y] = \oint 1 ds = \oint \sqrt{\dot{x}^2 + \dot{y}^2} dt = L$$

- Hence, we look for extremals of

$$\mathcal{H}[x,y] = \oint \left(\frac{1}{2}(x\dot{y} - y\dot{x}) + \lambda\sqrt{\dot{x}^2 + \dot{y}^2} \right) dt$$

- So, $H(t,x,y,\dot{x},\dot{y}) = \frac{1}{2}(x\dot{y} - y\dot{x}) + \lambda\sqrt{\dot{x}^2 + \dot{y}^2}$, and there are two dependent variables, with derivatives

$$\frac{\partial H}{\partial x} = \frac{1}{2}\dot{y} \qquad \frac{\partial H}{\partial \dot{x}} = -\frac{1}{2}y + \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

$$\frac{\partial H}{\partial y} = -\frac{1}{2}\dot{x} \qquad \frac{\partial H}{\partial \dot{y}} = \frac{1}{2}x + \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

- Leading to the 2 Euler-Lagrange equations

$$\frac{d}{dt}\left[-\frac{1}{2}y + \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right] = \frac{1}{2}\dot{y}$$
$$\frac{d}{dt}\left[\frac{1}{2}x + \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right] = -\frac{1}{2}\dot{x}$$

- Integrate

$$-\frac{1}{2}y + \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} = \frac{1}{2}y + A$$
$$\frac{1}{2}x + \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} = -\frac{1}{2}x - B$$

- After simplification we get

$$\frac{\lambda \dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} = y + A$$
$$\frac{\lambda \dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} = -x - B$$

- Now square the both equations, and add them to get

$$\lambda^2 \frac{\dot{x}^2+\dot{y}^2}{\dot{x}^2+\dot{y}^2} = (y+A)^2 + (x+B)^2$$

- Or, more simply

$$(y+A)^2 + (x+B)^2 = \lambda^2,$$

the equation os a circle with center $(-B, -A)$ and radius $|\lambda|$.

Remarks

- Note, we can't set value at end points arbitrarily.
- If $x(t_0) = x(t_1)$, and $y(t_0) = y(t_1)$, then we get a closed curve, obviously a circle.
- These conditions only amount to setting one constant, λ.
- On the other hand, if we specify different end-points, we are really solving a problem such as the simplified problem considered in Lecture 4.