Calculus of Variations
Summer Term 2014

Lecture 7

23. Mai 2014

Purpose of Lesson:

- To discuss Newton's aerodynamical problem.

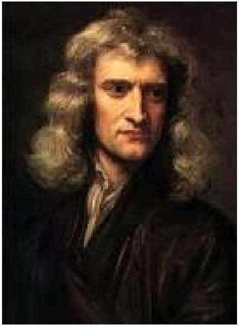
Problem 7-1 (Newton's aerodynamical problem)

Find extremal of "air resistance"

$$J[y] = \int\limits_0^R \frac{x}{1+y'^2}\,dx,$$

subject to $y(0) = L$ and $y(R) = 0$ and $y' \leqslant 0$ and $y'' \geqslant 0$.

- Finding the profile of a body that gives the minimal (aerodinamic or hydro- dynamic) resistance to the motion is one of the first problems in the theory of the calculus of variations.
- In 1685 Sir Isaac Newton studied this problem and presented a very simple model to compute the resistance of a body moving through an inviscid and incompressible medium.



Sir Isaac Newton (1642-1727)

In his words *"..If in a rare medium, consisting of equal particles freely disposed at equal distances from each other, a globe and a cylinder described on equal diameter move with equal velocities in the direction of the axis of the cylinder, (then) the resistance of the globe will be half as great as that of the cylinder. ...I reckon that this proposition will be not without application in the building of ships.."*

(I.Newton, Principia Mathematica)

- The Euler-Lagrange equation is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = -\frac{d}{dx}\left(\frac{2xy'}{(1+y'^2)^2}\right) = 0.$$

- Rearranging we get

$$2xy' = C\left(1 + y'^2\right)^2$$

which isn't much fun to solve directly.

- Alternative: define a new variable u, and constrain it

$$u = -y'$$

Add Lagrange multiplier $\lambda(x)$ to the functional

$$\mathcal{H}[y, u, \lambda] = \int_0^R \left(\frac{x}{1+u^2} + \lambda(y' + u)\right) dx.$$

- Now we solve this new problem with three dependent variables (y, u, λ) of x. So, we expect three Euler-Lagrange equations.

- The Euler-Lagrange equations

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$$
$$\frac{\partial H}{\partial u} - \frac{d}{dx}\left(\frac{\partial H}{\partial u'}\right) = 0$$
$$\frac{\partial H}{\partial \lambda} - \frac{d}{dx}\left(\frac{\partial H}{\partial \lambda'}\right) = 0$$

gives the DEs

$$\lambda = const$$
$$\lambda - \frac{2xu}{(1+u^2)^2} = 0 \qquad (7.1)$$
$$y' + u = 0$$

- If $\lambda = 0$, then for $x > 0$ we get $u = 0$, and hence $y = const$.
- If $\lambda \neq 0$, then the second equation in (7.1) implies

$$x(u) = \frac{C}{u}(1+u^2)^2 = C\left(\frac{1}{u} + 2u + u^3\right)$$

for C constant.

- From the last equation in (7.1) (which we insisted on at the start), we get

$$\frac{dy}{dx} = -u.$$

Now note that from the chain rule

$$\begin{aligned}\frac{dy}{du} &= \frac{dy}{dx}\frac{dx}{du} = -u\frac{dx}{du} \\ &= C\left(\frac{1}{u} - 2u - 3u^3\right)\end{aligned}$$

which we can integrate with respect to u to get

$$y(u) = const - C\left(-\ln u + u^2 + \frac{3}{4}u^4\right).$$

- Thus, we have

$$x(u) = C\left(\frac{1}{u} + 2u + u^3\right) = \frac{C}{u}\left(1 + u^2\right)^2 > 0 \quad \text{for all } u$$

$$y(u) = const - C\left(-lnu + u^2 + \frac{3}{4}u^4\right)$$

- End-point conditions take the form

$$y(u_1) = L \qquad\qquad y(u_2) = 0$$

$$x(u_1) = x_1 \qquad\qquad x(u_2) = R$$

but we don’t know x_1, u_1 or u_2.

$$y(u) = const - C\left(-lnu + u^2 + \frac{3}{4}u^4\right).$$

- At u_1 we have $y(u_1) = L$. For convenience we write

$$y(u) = L - C\left(-A - \ln u + u^2 + \frac{3}{4}u^4\right).$$

- So, at u_1 we get

$$L = L - C\left(-\ln u_1 - A + u_1^2 + \frac{3}{4}u_1^4\right)$$
$$0 = -C\left(-\ln u_1 - A + u_1^2 + \frac{3}{4}u_1^4\right)$$
$$A = -\ln u_1 + u_1^2 + \frac{3}{4}u_1^4$$

- Thus, we get

$$y(u) = L - C\left(-A - \ln u + u^2 + \frac{3}{4}u^4\right)$$
$$x(u) = \frac{C}{u}(1+u^2)^2$$

- Now at u_2 we have $x(u_2) = R$ and $y(u_2) = 0$. So

$$\begin{aligned} L &= C\left(-A - \ln u_2 + u_2^2 + \frac{3}{4}u_2^4\right) \\ R &= \frac{C}{u_2}\left(1+u_2^2\right)^2 \end{aligned} \tag{7.2}$$

- Dividing the first equation in (7.2) by the second one, we get.....

$$\frac{L}{R} = u_2 \left(-A - \ln u_2 + u_2^2 + \frac{3}{4} u_2^4 \right) (1 + u_2^2) - 2 \tag{7.3}$$

- The function on the RHS of (7.3) is increasing. So, we can solve (7.3) numerically and obtain a value for u_2.
- We can find C using $x(u_2) + R$.

$$R = \frac{C}{u_2} \left(1 + u_2^2\right)^2$$

$$C = \frac{u_2 R}{(1 + u_2^2)^2}$$

- All we need to know now is u_1, which gives us A and $x(u_1)$ which gives us u_2, which gives us C.