ARITHMETIC AND EQUIDISTRIBUTION OF MEASURES ON THE SPHERE

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Abstract. Motivated by problems of mathematical physics (quantum chaos) questions of equidistribution of eigenfunctions of the Laplace operator on a Riemannian manifold have been studied by several authors. We consider here, in analogy with arithmetic hyperbolic surfaces, orthonormal bases of eigenfunctions of the Laplace operator on the two dimensional unit sphere which are also eigenfunctions of an algebra of Hecke operators which act on these spherical harmonics. We formulate an analogue of the equidistribution of mass conjecture for these eigenfunctions as well as of the conjecture that their moments tend to moments of the Gaussian as the eigenvalue increases. For such orthonormal bases we show that these conjectures are related to the analytic properties of degree eight arithmetic L-functions associated to triples of eigenfunctions. Moreover we establish the conjecture for the third moments and give a conditional (on standard analytic conjectures about these arithmetic L-functions) proof of the equidistribution of mass conjecture.

1. Introduction

Let X be a Riemannian manifold of finite volume. Starting out from problems of theoretical physics (quantum chaos) several authors have recently studied questions of equidistribution of eigenfunctions of the Laplace operator.

In particular, precise versions of conjectures on equidistribution properties have been put forward by Rudnick and Sarnak [15] for arithmetic hyperbolic manifolds $X_\Gamma = \Gamma\backslash H$, where H is the upper half plane of the complex numbers and Γ an arithmetic subgroup of $SL_2(\mathbf{R})$ and for eigenfunctions of the Laplace operator, that are eigenfunctions of the (arithmetically defined) Hecke operators as well. The phenomenon of (conjectural) equidistribution of eigenfunctions in this arithmetical

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situation is one of the central problems in what has become known as arithmetic quantum chaos.

We investigate here the analogous question for the situation of the 2-dimensional unit sphere. Although the dynamics of geodesics for this manifold is certainly not chaotic it turns out that it nevertheless makes sense to look for an equidistribution property of eigenfunctions. At first sight, the well known fact that the usual spherical eigenfunctions $Y_{l,m}$ (see [20, Chapter III]) concentrate for $l = m \to \infty$ around the equator [3]seems to contradict the expectation of equidistribution, but since the eigenvalues occur on the sphere with multiplicities bigger than one, it makes sense to look into the question what happens if one varies the basis of eigenfunctions.

In this direction, it has been proved by Zelditch [22] that for a random orthonormal basis of eigenfunctions the equidistribution of mass conjecture is true.

We consider here, in analogy to the arithmetic hyperbolic surfaces, an orthonormal basis of eigenfunctions of the Laplace operator that are also eigenfunctions of an algebra of Hecke operators that acts on the space of spherical functions. The papers [11] and [19] also consider questions of the behaviour of eigenfunctions for such bases of spherical harmonics.

Concretely, a definite quaternion algebra such as the Hamilton Quaternions $\mathbb{H}$ over $\mathbb{Q}$, gives rise to Hecke operators on $L^2(S^2)$ (see [4], [13]). For

$$\alpha = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(\mathbb{R}),$$

let

$$S(\alpha) = \frac{1}{\sqrt{N(\alpha)}} \begin{bmatrix} x_0 + x_1 i & x_2 + x_3 i \\ -x_2 + x_3 i & x_0 - x_1 i \end{bmatrix} \in SU(2).$$

Here

$$N(\alpha) = \alpha\overline{\alpha} = x_0^2 + x_1^2 + x_2^2 + x_3^2.$$

For $n \geq 1$ an odd integer define the Hecke operator T_n on $L^2(S^2)$ by

$$(T_n\phi)(P) = \sum_{\substack{N(\alpha)=n \\ \alpha\in\mathbb{H}(\mathbb{Z})}} \phi(S(\alpha)P),$$

where $P \in S^2$ and $SU(2)$ acts on S^2 by isometries after one realizes S^2 as $\mathbb{C} \cup \{\infty\}$ via stereographic projection and $SU(2)$ acts by linear fractional transformations. The T_n's are selfadjoint, they commute with each other as well as with the Laplacian Δ on the round sphere. Thus the T_n's can be simultaneously diagonalized in each of the $2\nu + 1$ dimensional spaces H_ν, consisting of spherical harmonics on S^2 of degree ν (that is the restriction of harmonic polynomials in $\mathbb{R}^3$, homogeneous of degree ν). This algebra of Hecke operators arises naturally if one views the spherical harmonics as components at infinity of automorphic forms on the multiplicative group of the adelization of the rational

Hamilton quaternions. We denote by ψ_ν such a Hecke eigenform with ν indicating its degree (so that its Laplace eigenvalue is $\nu(\nu+1)/2$). The analogue of the equidistribution of mass conjecture [15] for the ψ_ν's is the following:

Conjecture 1. *Normalize ψ_ν on S^2 to have L^2-norm equal to 1, so that*

$$\mu_{\psi_\nu} := |\psi_\nu(P)|^2 dv(P)$$

is a probability measure. Then

$$\lim_{\nu\to\infty} \mu_{\psi_\nu} = \frac{dv}{2\pi},$$

in the sense of integration against continuous functions on S^2.

The analogue of the Gaussian equidistribution conjecture of Berry and others [7] in this context is as follows:

Conjecture 2. *Fix $q \geq 0$ an integer then*

$$\lim_{\nu\to\infty} \int_{S^2} \psi_\nu^q dv \longrightarrow \frac{c_q}{(2\pi)^{q/2}},$$

where c_q is the q-th moment of the Gaussian distribution.

By the work of Eichler [4] and of Jacquet/Langlands it is known that there is a correspondence between spherical harmonic polynomials and modular forms via the theory of theta series with spherical harmonics. This correspondence is Hecke-equivariant, and thus methods and results from the theory of modular forms, in particular from the theory of L-functions associated to Hecke eigenforms (or irreducible automorphic representations), can be used in the study of the spherical harmonics. The crucial point for our study of the integrals appearing in the equidistribution conjecture above is a formula proved in [2] that connects the integral of a product of 3 eigenfunctions over the sphere with the central critical value of the automorphic L-functions associated to a triple of modular Hecke eigenforms; this allows one to connect the equidistribution conjecture with conjectural properties of such automorphic L-functions. We note in passing that such integrals of products of 3 eigenfunctions of the Laplace operator on the sphere have been considered in various places in the physics literature, see [17].
The purpose of this note is to show that combining the main formula in [2] with the recent subconvex estimates for special values of L-functions of holomorphic modular forms [14] allows one to prove Conjecture 2 for $q = 3$ (the cases $q = 1$ and $q = 2$ are obvious). We also show that Conjecture 1 would follow from subconvex estimates for the degree 8 L-functions mentioned above. Such subconvex estimates are an immediate consequence of the Riemann Hypothesis for these L-functions. At

the present time such subconvex estimates are known only for special forms, see [16] and [10].
In his recent thesis [21], Watson has derived general explicit identities relating integrals of products of 3 Maass (or holomorphic) Hecke eigenforms on arithmetic surfaces, to special values of degree 8 L-functions. As a consequence he obtains similar results for "chaotic" eigenstates.

The estimates in this article depend on the results of [2], in some cases in corrected versions. A revised version of that article and a list of errata are available at
`www.math.uni-sb.de/~ag-schulze/Preprints`, the list of errata is also attached as an appendix to this article.
We would like to thank T. Ibukiyama for the permission to use the not yet published results in [9].

2. Equidistribution

Our first goal is to describe explicitly the connection between the central critical value of the triple product L-function associated to a triple of cusp forms on one side and integrals of harmonic polynomials over the unit sphere on the other side. We fix first some notations.
We consider a definite quaternion algebra D of discriminant N (where N is the product of the primes ramified in D) over $\mathbf{Q}$ and a maximal order R in D, we assume that the class number (i.e., the number of classes of left R-ideals) of D is 1; this restricts D to be one of the algebras of discriminant equal to $2, 3, 5, 7, 13$.
On D we have the involution $x \mapsto \overline{x}$, the (reduced) trace $\mathrm{tr}(x) = x + \overline{x}$ and the (reduced) norm $n(x) = x\overline{x}$.

For $\nu \in \mathbf{N}$ let $U_\nu^{(0)}$ be the space of homogeneous harmonic polynomials of degree ν on $\mathbf{R}^3$ and view $P \in U_\nu^{(0)}$ as a polynomial on $D_\infty^{(0)} = \{x \in D_\infty = D \otimes \mathbf{R} | \mathrm{tr}(x) = 0\}$ by putting $P(\sum_{i=1}^3 x_i e_i) = P(x_1, x_2, x_3)$ for an orthonormal basis $\{e_i\}$ of $D_\infty^{(0)}$ with respect to the norm form n. Integrating the polynomial in this identification over the set of $x \in D_\infty^{(0)}$ of norm 1 is the same as integrating the original polynomial in 3 real variables over the unit sphere S^2, we will freely use this identification below.
In the same way we fix an orthonormal basis of D_∞ extending the one from above and use it to identify (harmonic) polynomials in 4 variables with (harmonic) polynomial functions on D_∞.
The representation of $D_\infty^\times$ on $U_\nu^{(0)}$ by conjugation of the argument is denoted by τ_ν. By $\langle\langle \ , \ \rangle\rangle$ we denote the invariant scalar product in the representation space $U_\nu^{(0)}$ (where the choice of normalization will be discussed later). The $D^\times \times D^\times$-space $U_\nu^{(0)} \otimes U_\nu^{(0)}$ is isomorphic to the $D^\times \times D^\times$-space $U_{2\nu}$ of harmonic polynomials on D_∞ of degree 2ν, where $(d_1, d_2) \in D^\times \times D^\times$ acts by sending $P(x)$ to $P(d_1^{-1} x d_2)$. An

explicit isomorphism is given by mapping $P_1 \otimes P_2$ to the polynomial $P_1 \otimes P_2(d) := \langle\langle P_2(x), P_1(dx\bar{d})\rangle\rangle$. We will henceforth identify $U_\nu^{(0)} \otimes U_\nu^{(0)}$ with $U_{2\nu}$ using this isomorphism.

There is a Hecke action on $U_\nu^{(0)}$ which has been described by Eichler [4] in terms of Brandt matrices with polynomial entries, it is given by

$$\tilde{T}(p)P = \sum_{y \in R, n(y)=p} \tau_\nu(y)(P),$$

see also [13]. In particular the space $U_\nu^{(0)}$ has a basis consisting of eigenforms of all the $\tilde{T}(p)$ for the $p \nmid N$.

To $P_1 \in U_\nu^{(0)}$ we associate the theta series of R with harmonic polynomial $P_1 \otimes P_1$ given as usual as

$$f_{P_1}(z) := \frac{1}{|R^\times|} \sum_{x \in R} (P_1 \otimes P_1)(x) \exp(2\pi i n(x) z).$$

For this to be nonzero we have to restrict to polynomials P_1 that are invariant under the action of the group $R^\times$ of R, we will always do so in the sequel. The function f_{P_1} is then a cusp form for $\Gamma_0(N)$ of weight $2 + 2\nu$ if $\nu > 0$ and it is an eigenform for the Hecke operators $T(p)$ for $p \nmid N$ if P_1 is an eigenfunction of the $\tilde{T}(p)$ for the $p \nmid N$. In fact it is a normalized newform if $\langle\langle P_1, P_1\rangle\rangle = 1$, and it is a result of [4] that one gets all normalized newforms of level N, weight $2 + 2\nu$ and trivial character in this way (we will actually not use the latter fact). With these notations we can now formulate:

Proposition 2.1. *Let $P_1, P_2, P_3 \in U_{\nu_1}^{(0)}, U_{\nu_2}^{(0)}, U_{\nu_3}^{(0)}$ (with $\nu_1 = \nu_2, \nu_3 > 0$) be harmonic polynomials that are Hecke eigenforms as above, denote by f_1, f_2, f_3 the associated cusp forms of weights $k_1 = k_2 = 2+2\nu_1, k_3 = 2 + 2\nu_3$ and by $L(f_1, f_2, f_3; s)$ the triple product L-function associated to f_1, f_2, f_3, (as defined for the good primes e.g. in* [5]*, for the Euler factors at the bad primes we refer to* [2]*).*

Then one has for all $\epsilon > 0$:

(2.1)

$$L(f_1, f_2, f_3; \frac{2k_1 + k_3}{2} - 1) \geq C_1(N, \nu_3)\nu_1^{1-\epsilon}(\int_{S^2} P_1(x)P_2(x)P_3(x)dx)^2$$

with a poitive constant $C_1(N, \nu_3)$ depending only on N, ν_3 and ϵ.

If $\nu_1 = \nu_2 = \nu_3 =: \nu$, one has for all $\epsilon > 0$ (with $k := k_1 = k_2 = k_3 = 2 + 2\nu$):

$$L(f_1, f_2, f_3; 2 + 3\nu) \geq C_2(N)\nu^{2-\epsilon}(\int_{S^2} P_1(x)P_2(x)P_3(x)dx)^2 \qquad (2.2)$$

with a positive constant $C_2(N)$ depending only on N and ϵ.

Proof. According to [2] the central critical value is:

$$(2.3)\quad (-1)^{a'}2^{5+4a+3b-\omega(N)}\pi^{5+9a'+4b}\frac{(a'+1)^{[b]}}{2^{[a+b]}2^{[a']}(b+1)^{[a']}(1)^{[a']}} \times \langle f_1,f_1\rangle\langle f_2,f_2\rangle\langle f_3,f_3\rangle\big(T_0\big(\frac{1}{|R^\times|}P_1\otimes P_2\otimes P_3\big)\big)^2$$

with a certain trilinear form T_0 on $U_{\nu_1}^{(0)}\otimes U_{\nu_2}^{(0)}\otimes U_{\nu_3}^{(0)}$ whose definition is recorded below.
Here we have the following notations:

$$\alpha^{[\nu]} = \frac{\Gamma(\alpha+\nu)}{\Gamma(\alpha)} = \left\{\begin{array}{ll} 1 & \nu=0 \\ \alpha(\alpha+1)\dots(\alpha+\nu-1) & \nu>0 \end{array}\right\},$$

hence

$$\frac{(a'+1)^{[b]}}{2^{[a+b]}2^{[a']}(b+1)^{[a']}(1)^{[a']}} = \frac{b!}{(a'!)^2(a'+1)!(a+b+1)!}.$$

Unfortunately, [2] contains a mistake at this point, the correct value of the factor arising here is

$$(2.4)\qquad \frac{(a'+1+b)b!}{(a'!)^2(a'+1)!(3a'+b+1)!}.$$

Moreover, there should be an additional factor of N^{-1} in (2.3), the exponent at π should be $5+6a'+2b$ and the factor $(-1)^{a'}$ should be omitted.

The forms f_1,f_2,f_3 are normalized newforms of weights $k_1\ge k_2\ge k_3$ with $k_2+k_3>k_1$, in our case we have $k_1=k_2$. We write $k_1=k_2=2+a+b$ and $k_3=2+a, a=2a'$. For our purposes we can restrict to the case that both a' and b are even.
We normalize the invariant scalar product on the latter space in such a way that the Gegenbauer polynomial $G^{(\alpha)}(x,x')$ obtained from

$$G_1^{(\alpha)}(t) \;=\; 2^\alpha \sum_{j=0}^{[\frac{\alpha}{2}]}(-1)^j\,\frac{1}{j!(\alpha-2j)!}\,\frac{(\alpha-j)!}{2^{2j}}\,t^{\alpha-2j}$$

by

$$G^{(\alpha)}(x_1,x_2) = 2^\alpha(n(x_1)n(x_2))^{\alpha/2}G_1^{(\alpha)}\Big(\frac{\mathrm{tr}(x_1\overline{x_2})}{2\sqrt{n(x_1)n(x_2)}}\Big)$$

is a reproducing kernel.
The invariant scalar product on $U_\alpha^{(0)}$ is then normalized such that we obtain the product on $U_{2\alpha}$ given above under the identification $U_{2\alpha}=U_\alpha^{(0)}\otimes U_\alpha^{(0)}$.
Given this, the polynomials P_1,P_2,P_3 are normalized to

$$\langle\langle P_i,P_i\rangle\rangle = 1.$$

The normalization of the trilinear form is then as follows:
We have a harmonic polynomial $P \in U_{a+b} \otimes U_{a+b} \otimes U_a$ in three vector variables (each vector being a quaternion) derived from the action of a certain differential operator on an exponential in Section 1 of [2]. This gives an invariant trilinear form T on $U_{a+b}\otimes U_{a+b}\otimes U_a$ defined by taking the scalar product with $P_0 := (\pi)^{-3a'-b} i^{-3a'} P$ (notice that in [2] we write erroneously $\pi^{-3a-2b}P$). Using the identification $U_{2\alpha} = U_\alpha^{(0)} \otimes U_\alpha^{(0)}$ from above T decomposes as $T_0 \otimes T_0$.

For the intended application the form $T_0(Q_1^{(0)}, Q_2^{(0)}, Q_3^{(0)})$ should be replaced by the integral

$$\int Q_1^{(0)}(\mathbf{x}) Q_2^{(0)}(\mathbf{x}) Q_3^{(0)}(\mathbf{x}) d\mathbf{x}$$

over the unit sphere.

As a first step we compare $T(Q_1, Q_2, Q_3)$ with

$$\int Q_1(x) Q_2(x) Q_3(x) dx;$$

since both expressions give invariant trilinear forms they have to be proportional.
We compute T for special polynomials Q_i on the space of quaternions: Write $G_w^{(\alpha)}(x)$ for $G^{(\alpha)}(w, x)$.
Then we have

$$T(G_w^{(a+b)}, G_w^{(a+b)}, G_w^{(a)}) = P_0(w, w, w)$$

by the reproducing property of the $G^{(\alpha)}$.
On the other hand, by [20, p.490] the integral

$$\int G_w^{(a+b)}(x) G_w^{(a+b)}(x) G_w^{(a)}(x) dx$$

is equal to $\pi/2$ and hence

$$T(Q_1, Q_2, Q_3) = 2P_0(w, w, w) \int Q_1(\mathbf{x}) Q_2(\mathbf{x}) Q_3(\mathbf{x}) d\mathbf{x}. \tag{2.5}$$

We have to compute $P_0(w, w, w)$ explicitly. This looks at first sight rather awkward since our description in [2] gives us an explicit formula only for one coefficient of the polynomial.
Fortunately there are some results on such polynomials in forthcoming work of Ibukiyama and Zagier, see [9]:
For $n \in \mathbf{N}$ we denote by $\mathcal{H}_n(4)$ the space of harmonic homogeneous polynomials of degree n in 4 variables. For nonnegative integers μ_1, μ_2, μ_3 we then put

$$\mathcal{H}_{\mu_1,\mu_2,\mu_3}(4) := (\mathcal{H}_{\mu_2+\mu_3}(4) \otimes \mathcal{H}_{\mu_1+\mu_3}(4) \otimes \mathcal{H}_{\mu_1+\mu_2}(4))^{O(4)}$$

This space is then always one-dimensional and a nonzero element of $\mathcal{H}_{\mu_1,\mu_2,\mu_3}(4)$ is (explicitly!) given as the coefficient of $X_1^{\mu_1}X_2^{\mu_2}X_3^{\mu_3}$ in the formal power series

$$G_4(\mathbf{X},T) = G_4(X_1,X_2,X_3;T) = \frac{1}{\sqrt{\Delta(\mathbf{X},T)^2 - 4d(T)X_1X_2X_3}}$$

Here T is (twice of) a Gram matrix

$$T = \begin{pmatrix} 2m_1 & r_3 & r_2 \\ r_3 & 2m_2 & r_1 \\ r_2 & r_1 & 2m_3 \end{pmatrix} = 2\cdot \begin{pmatrix} (\mathbf{x},\mathbf{x}) & (\mathbf{x},\mathbf{y}) & (\mathbf{x},\mathbf{z}) \\ (\mathbf{y},\mathbf{x}) & (\mathbf{y},\mathbf{y}) & (\mathbf{y},\mathbf{z}) \\ (\mathbf{z},\mathbf{x}) & (\mathbf{z},\mathbf{y}) & (\mathbf{z},\mathbf{z}) \end{pmatrix} \quad (\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbf{C}^4)$$

and

$$d(T) = 4m_1m_2m_3 - m_1r_1^2 - m_2r_2^2 - m_3r_3^2 + r_1r_2r_3 = \frac{1}{2}\det(T),$$

$$\begin{aligned} \Delta(X_1,X_2,X_3;T) &= \Delta(\mathbf{X},T) \\ &= 1 - r_1X_1 - r_2X_2 - r_3X_3 + r_1m_1X_2X_3 + r_2m_2X_3X_1 + r_3m_3X_1X_2 \\ &\quad + m_1m_2X_3^2 + m_2m_3X_1^2 + m_3m_1X_2^2. \end{aligned}$$

We are interested in the coefficient of $X_1^{a'}X_2^{a'}X_3^{a'+b}$ which we call $\tilde{P}$ in the sequel. The coefficient of $m_1^0m_2^0m_3^0r_1^{a'}r_2^{a'}r_3^{a'+b}$ in the polynomial $\tilde{P}$ can be read off from the expression above (putting $m_1 = m_2 = m_3 = 0$); it is

$$\sum_{\alpha=0}^{a'} \binom{2a'-2\alpha}{a'-\alpha}\binom{2a'+b+\alpha}{3\alpha+b}\binom{3\alpha+b}{\alpha,\alpha,\alpha+b},$$

where we write

$$\binom{j}{\alpha,\beta,\gamma} := \frac{j!}{\alpha!\beta!\gamma!}.$$

This is known to be equal to

$$\frac{(2a')!(b+2a')!^2}{a'!^4(b+a')!^2};$$

an identity which can be reduced to a special case of an exercise on page 44 in [18] with hints to [1], who traces it back to "Saalschutz summation".

Now we compare P_0 with Ibukiyama's polynomial $\tilde{P}$; it is enough to compare the coefficients in the monomial above. From Section 1 of [2] one reads off that the coefficient of P_0 in the same monomial is

$$\frac{2^b}{(b+1)!b!}.$$

Again, there is a mistake in [2] here. The correct value is

$$\frac{2^b 2^{4a'}\Gamma(a'+2)}{\Gamma(a'+b+2)b!}, \tag{2.6}$$

so that we arrive at

$$P_0 = 2^{b+4a'} \frac{a'!^4(b+a')!^2(a'+1)!}{(2a')!(b+2a')!^2(a'+b+1)!b!} \tilde{P}.$$

Next we have to evaluate $\tilde{P}$ at the triple (w,w,w), i. e., at the matrix T with $m_1 = m_2 = m_3 = 1, r_1 = r_2 = r_3 = 2$. We get in this case

$$G_4(\mathbf{X}, T) = (1 - \sum_{i=1}^{3} X_i)^{-2},$$

the coefficient of which at $X_1^{a'} X_2^{a'} X_3^{a'+b}$ is the value we try to compute. It is proved easily (Taylor expansion) that this is equal to

$$\frac{(3a'+b+1)!}{(a')!(a')!(a'+b)!},$$

which leads us to

$$\begin{aligned} P_0(w,w,w) &= 2^{b+4a'} \frac{(a'+1)!(3a'+b+1)!a'!^4(b+a')!^2}{a'!^2(a'+b)!(2a')!(b+2a')!^2(a'+b+1)!b!} \\ &= 2^{b+4a'} \frac{(3a'+b+1)!a'!^2(a'+1)!(a'+b)!}{(2a')!(b+2a')!^2 b!(a'+b+1)!}. \end{aligned}$$

For polynomials $Q_1^{(0)}, Q_2^{(0)} \in U_{a'+b/2}^{(0)}, Q_3^{(0)} \in U_{a'}^{(0)}$ we have by definition

$$(T_0(Q_1^{(0)}, Q_2^{(0)}, Q_3^{(0)}))^2 = T(Q_1^{(0)} \otimes Q_1^{(0)}, Q_2^{(0)} \otimes Q_2^{(0)}, Q_3^{(0)} \otimes Q_3^{(0)})$$

and hence (as a consequence of the discussion given above)

$$\begin{aligned} (T_0(Q_1^{(0)}, Q_2^{(0)}, Q_3^{(0)}))^2 &= 2^{b+4a'+1}\pi^{-1} \frac{(3a'+b+1)!a'!^2(a'+1)!(a'+b)!}{(2a')!(b+2a')!^2 b!(a'+b+1)!} \\ &\times \int (Q_1^{(0)} \otimes Q_1^{(0)})(x)(Q_2^{(0)} \otimes Q_2^{(0)})(x)(Q_3^{(0)} \otimes Q_3^{(0)})(x)dx, \end{aligned}$$

where the integration is over the 3-dimensional unit sphere.

Our next task is to relate the integral

$$\int (Q_1^{(0)} \otimes Q_1^{(0)})(x)(Q_2^{(0)} \otimes Q_2^{(0)})(x)(Q_3^{(0)} \otimes Q_3^{(0)})(x)dx \tag{2.7}$$

with $(\tilde{T}_0(Q_1^{(0)}, Q_2^{(0)}, Q_3^{(0)})^2$, where we put

$$\tilde{T}_0(Q_1^{(0)}, Q_2^{(0)}, Q_3^{(0)}) := \int Q_1^{(0)}(z)Q_2^{(0)}(z)Q_3^{(0)}(z)dz, \tag{2.8}$$

and where the integration is now over the 2-dimensional unit sphere (the factor of proportionality arising here depends on the identification between $U_{2\alpha}$ and $U_\alpha^{(0)} \otimes U_\alpha^{(0)}$ and hence on the degrees of the polynomials involved).

In order to do this we need again special polynomials which show us the normalization of our isomorphism. We recall first how this isomorphism is described:
Given $P_1^{(0)}, P_2^{(0)} \in U_\alpha^{(0)}$ we defined the polynomial $P_1^{(0)} \otimes P_2^{(0)}$ by $(P_1 \otimes P_2)(x) = \langle\langle P_1^{(0)}(d), P_2^{(0)}(xd\bar{x})\rangle\rangle_0$, where $\langle\langle \cdot, \cdot\rangle\rangle_0$ denotes the invariant scalar product chosen in $U_\alpha^{(0)}$.

We consider again the Gegenbauer polynomial $G^{(\alpha,0)}$ of degree α in $U_\alpha^{(0)} \otimes U_\alpha^{(0)}$, derived in the same way from the one-variable polynomial with indices $l = \alpha, p = 1/2$ given in [20] as we did it above for the G^β in U_β and let $\langle\langle \cdot, \cdot\rangle\rangle_0$ be normalized such that this polynomial is a reproducing kernel, this normalization determines then our choice of the isomorphism between $U_{2\alpha}$ and $U_\alpha^{(0)} \otimes U_\alpha^{(0)}$.

In order to relate the integrals in (2.7) and (2.8) we evaluate them for a special choice of polynomials: We put $Q_1^{(0)} = Q_2^{(0)} = G_z^{(a'+b/2,0)}$ and $Q_3^{(0)} = G_z^{(a',0)}$ for some quaternion z of norm 1 and trace 0.
The integral in (2.8) is then by [20, p.490] equal to

$$(2.9)\quad c_1 := \frac{\Gamma(\frac{3a'+b}{2}+1)\Gamma(\frac{a'+1}{2})^2\Gamma(\frac{a'+b+1}{2})}{\pi\Gamma(\frac{a'}{2}+1)^2\Gamma(\frac{a'+b}{2}+1)\Gamma(\frac{3a'+b+3}{2})}$$

$$(2.10)\quad = 16\frac{\Gamma(a')^2\Gamma(a'+b)\Gamma(\frac{3a'+b}{2}+1)^2}{\Gamma(3a'+b+2)\Gamma(\frac{a'}{2})^2\Gamma(\frac{a'}{2}+1)^2\Gamma(\frac{a'+b}{2})\Gamma(\frac{a'+b}{2}+1)}$$

(where the second form is derived from the first using the duplication formula for the Γ-function and where a factor $\Gamma(2w)/\Gamma(w)$ has to be replaced by 2 if $w = 0$).

On the other hand, the definition of the isomorphism between $U_{2\alpha}$ and $U_\alpha^{(0)} \otimes U_\alpha^{(0)}$ and the reproducing property of the Gegenbauer polynomials imply that

$$(G_z^{(\alpha,0)} \otimes G_z^{(\alpha,0)})(x) = G_z^{(\alpha,0)}(\bar{x}zx),$$

and hence that

$$(2.11)\quad \int (G_z^{((a'+b/2),0)} \otimes G_z^{((a'+b/2),0)})(x)(G_z^{((a'+b/2),0)} \otimes G_z^{((a'+b/2),0)})(x) \times (G_z^{(a',0)} \otimes G_z^{(a',0)})(x)dx$$
$$= \mathrm{vol}(\mathrm{Stab}(z)) \int G_z^{((a'+b/2),0)}(z')G_z^{((a'+b/2),0)}(z')G_z^{(a',0)}(z')dz',$$

where $\mathrm{Stab}(z)$ is the set of x of norm 1 with $\bar{x}zx = z$.
The normalizations of the integrals over the 3-sphere and over the 2-sphere in [20] are such that $\mathrm{vol}(\mathrm{Stab}(z)) = \frac{\pi}{4}$ holds.

Taken together we obtain

$$\int (Q_1^{(0)} \otimes Q_1^{(0)})(x)(Q_2^{(0)} \otimes Q_2^{(0)})(x)(Q_3^{(0)} \otimes Q_3^{(0)})(x)dx = \frac{\pi}{4c_1}\left(\int Q_1^{(0)}(z)Q_2^{(0)}(z)Q_3^{(0)}(z)dz\right)^2, \tag{2.12}$$

where c_1 is the constant computed in (2.9).

This gives us the first formula for the central critical value of the triple product L-function.

$$N^{-1}2^{12a'+4b-4-\omega(N)}\pi^{5+6a'+2b}\langle f_1, f_1\rangle\langle f_1, f_1\rangle\langle f_3, f_3\rangle \times \frac{(b+a')a'\Gamma(\frac{a'+b}{2})^2\Gamma(\frac{a'}{2})^4\Gamma(3a'+b+2)}{\Gamma(a')^2\Gamma(2a')\Gamma(\frac{3a'+b}{2}+1)^2\Gamma(a'+b)\Gamma(2a'+b+1)^2} \times \Big(\int P_1(x)P_1(x)P_3(x)dx\Big)^2 \tag{2.13}$$

In this we replace the Petersson product $\langle f_i, f_i\rangle$ by

$$(4\pi)^{1-k_i}\Gamma(k_i)D_{f_i}(k-1)$$

(where D_{f_i} denotes the symmetric square L-function of f_i), which leads us to

$$2^{-9-\omega(N)}\pi^2 D_{f_1}(k_1-1)D_{f_1}(k_1-1)D_{f_3}(k_3-1) \times \frac{(a')^2(2a'+1)(a'+b)(b+2a'+1)^2\Gamma(\frac{a'+b}{2})^2\Gamma(3a'+b+2)\Gamma(\frac{a'}{2})^4}{\Gamma(a')^2\Gamma(\frac{3a'+b}{2}+1)^2\Gamma(a'+b)} \times \Big(\int P_1(x)P_1(x)P_3(x)dx\Big)^2 \tag{2.14}$$

Here the factor $D_{f_1}(k_1-1)D_{f_1}(k_1-1)D_{f_3}(k_3-1)$ does not contribute in an essential way to the asymptotics as $k_1 \to \infty$ since it is well known that $k_i^{-\delta} << D_{f_i}(k_i-1) << k_i^{\delta}$ for all $\delta > 0$, see e.g. [8].
We analyze the total factor on the right hand side in front of

$$D_{f_1}(k_1-1)D_{f_1}(k_1-1)D_{f_3}(k_3-1)\Big(\int P_1(x)P_2(x)P_3(x)dx\Big)^2$$

with Stirling's formula: For the first assertion of the proposition we fix ν_3 and let ν_1 tend to infinity; we find that the factor from above can for all $\epsilon > 0$ be bounded from below by

$$c' b^{3-\epsilon}$$

for some nonzero constant c' depending on a', ϵ and the level N as b tends to infinity.

For the second part of the proposition we have all the ν_i equal, which implies $b = 0$ in the notation used above; we find that the factor can (for all $\epsilon > 0$) be bounded from below by

$$c'' a^{5-\epsilon}$$

for some nonzero constant c'' depending on ϵ and the level N. We have to adjust a final normalization: The P_i were normalized to have

$$\langle\langle P_i, P_i\rangle\rangle_0 = 1$$

whereas we want them to have L^2-norm 1.
A comparison of $\langle\langle P_i, P_i\rangle\rangle_0$ with the scalar product on the space of P derived from the L^2-norm with the help of [20, p. 461] shows that we have

$$P_1 = \sqrt{a' + (b+1)/2}^{\,-1}\ \tilde{P}_1, \quad P_3 = \sqrt{a' + 1/2}^{\,-1}\ \tilde{P}_3,$$

where the $\tilde{P}_i$ are L^2-normalized.
This multiplies the last formula with

$$(a' + (b+1)/2)^{-2}(a' + 1/2)^{-1}$$

in the first case and leads to an expression that is bounded from below for every $\epsilon > 0$ by

$$C_1 b^{1-\epsilon}$$

for some nonzero constant C_1 depending on ϵ, a' and the level N as b tends to infinity.
In the second case the formula gets multiplied with $(a' + 1/2)^{-3}$ and leads to an expression that is bounded from below for every $\epsilon > 0$ by

$$C_2 a^{2-\epsilon}$$

for some nonzero constant C_2 depending on ϵ and the level N as a tends to infinity. This finishes the proof of the Proposition.

On the other hand one can investigate the dependence of the central critical value $L(f_1, f_1, f_3; \frac{2k_1+k_3}{2} - 1)$ on the level and weights with analytic methods from the theory of L-functions.
We do this first for the equidistribution of mass conjecture, i. e., for the situation in which ν_3 is fixed and $\nu_1 = \nu_2$ tends to infinity:
As usual in the theory of L-functions the first step is to establish the convexity bound.

Lemma 2.2. *Let f_1, f_3 be newforms of level N and weights k_1, k_3 as above. Then*

$$L(f_1, f_1, f_3; \frac{2k_1 + k_3}{2} - 1) = O_N(k_1^{1+\epsilon})$$

for all $\epsilon > 0$.

Proof. In [10] a general description of the convexity bound of (standard) automorphic L-functions for $GL(n)$ is given. That bound is also applicable for our triple L-function.
We recall the (normalized) functional equation of the triple L-function (quoting from [6] for weight 2 and more generally from[2] ; we restrict ourselves to the case where all the cusp forms involved are newforms of the same -squarefree- level): Putting

$$\Lambda(s) = \Gamma_{\mathbb{C}}(s + k_1 + \frac{k_3}{2})\Gamma_{\mathbb{C}}(s + 1 + \frac{k_3}{2})\Gamma_{\mathbb{C}}(s + 1 + \frac{k_3}{2}) \times \Gamma_{\mathbb{C}}(s + 1 + k_1 - \frac{k_3}{2})\mathcal{L}(f_1, f_2, f_3, s) \tag{2.15}$$

with

$$\mathcal{L}(f_1, f_2, f_3, s) = L(f_1, f_2, f_3, s + \frac{k_1 + k_2 + k_3 - 3}{2}) \tag{2.16}$$

and $\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s)$ we get the functional equation

$$\Lambda(1 - s) = (N^5)^{s-\frac{1}{2}} w\Lambda(s)$$

Under such circumstances, the convexity bound (as described in [10]) is

$$\mathcal{L}(f_1, f_2, f_3, \frac{1}{2} + it) <<_\epsilon (C(f_1, f_2, f_3, t))^{\frac{1}{4}+\epsilon}$$

Here $C(f_1, f_2, f_3, t)$ is given in terms of the gamma factors; in our case this means

$$C(f_1, f_2, f_3, t) = (N^5)(1+ \mid it - k_1 - \frac{k_3}{2} \mid^2) \times (1+ \mid it - 1 - \frac{k_3}{2} \mid^2)^2(1+ \mid it - 1 - k_1 + \frac{k_3}{2} \mid^2) \tag{2.17}$$

In our special case (i.e.$k_1 = k_2$ and k_3 fixed) this implies an estimate of type

$$\mathcal{L}(f_1, f_1, f_3, \frac{1}{2}) << k_1^{1+\epsilon}$$

or

$$\mathcal{L}(f_1, f_1, f_3, \frac{1}{2}) << b^{1+\epsilon}$$

For our intended application this result is just too weak.
We will therefore assume henceforth that one can break convexity for the estimate of $L(f_1, f_1, f_3; \frac{2k_1+k_3}{2} - 1)$ in the k_1-aspect (see [10] for a survey of subconvex estimates).

Subconvexity hypothesis. *Fix f_3 as above. There is $\delta > 0$ such that for all f_1 as above*

$$\mathcal{L}(f_1, f_1, f_3, \frac{1}{2}) = O_{f_3}(k_1^{1-\delta}).$$

Then the result from our proposition immediately translates into a statement about equidistribution of measures on the unit sphere that are associated to Hecke eigenfunctions on a quaternion algebra which proves the last assertion in the introduction (concerning Conjecture 1).

Proposition 2.3. *Let D be as above, let $S^2 \subseteq \mathbb{R}^3$ be identified with $\{x \in D_\infty^{(0)} \mid n(x) = 1\}$ as above. For a harmonic polynomial $P \in \mathbb{C}[X_1, X_2, X_3]$ let the measure μ_P on S^2 be defined by*

$$\int_{S^2} f(x) d\mu_P(x) := \int_{S^2} |P(x)|^2 f(x) dx.$$

Then under our subconvexity hypothesis the measures μ_P become equidistributed if P runs through Hecke eigenfunctions of degree ν for $\nu \longrightarrow \infty$, i.e., one has

$$\lim_{\nu(P)\to\infty} \int_{S^2} f(x) d\mu_P(x) = \int_{S^2} f(x) dx \qquad (*)$$

for all continuous f on S^2.

Proof. We have to check $(*)$ only for Hecke eigenfunctions $f = Q$, as these form a Hilbert space basis of $L^2(S^2)$.
Then for $Q \neq 1$ the right hand side of $(*)$ is zero, for $Q = 1$ it is 1.
For $Q = 1$ we have equality in $(*)$ for all P (of L^2-norm 1).
For $Q \neq 1$, Proposition 2 together with the convexity breaking assumption implies that

$$\lim_{\nu\to\infty} \int_{S^2} Q(x)|P(x)|^2 dx = 0,$$

which proves the assertion.

We end with the statement and proof of Conjecture 1 for the case $q = 3$

Proposition 2.4. *Let D and P (harmonic of degree $\nu(P)$) be as in Proposition 2.3. Then*

$$\lim_{\nu(P)\to\infty} \int_{S^2} P(x)^3 dx = 0 \tag{2.18}$$

Proof. According to (2.2) we have for $\epsilon > 0$

$$|\int_{S^2} P(x)^3 dx|^2 << \nu(P)^{\epsilon-2} L(f, f, f; 2 + 3\nu(P)) \tag{2.19}$$

Now with the notation from (2.16) (i. e. denoting by $\mathcal{L}$ an L-function normalized to have functional equation under $s \mapsto (1 - s)$) we have

$$\mathcal{L}(f, f, f, s) = \mathcal{L}(\mathrm{Sym}^3 f, s)(\mathcal{L}(f, s))^2. \tag{2.20}$$

Furthermore $\mathcal{L}(\mathrm{Sym}^3 f, s)$ is the L-function of a GL_4-cusp form [12], so we may apply the general Molteni subconvexity bound (see [10]) to

$\mathcal{L}(\mathrm{Sym}^3 f, \frac{1}{2})$. This combined with the subconvex bound for $\mathcal{L}(f, \frac{1}{2})$ due to Peng [14] shows that

$$\nu(P)^{\epsilon-2} L(f, f, f; 2 + 3\nu(P)) = O(\nu^{-\delta})$$

for a fixed $\delta > 0$. This proves the Proposition.

Remark. We see presently no way to extend the statement of Proposition 2.1 (and hence our arguments in this article) to a product of more than three polynomials, since our proofs here and in [2] use several special features of the case of three polynomials. In particular our proofs depend on

- the existence of an integral representation for the triple product L-function using an Eisenstein series whose special value is expressed by theta series with spherical harmonics
- the existence and uniqueness of trilinear forms on tensor products of spaces of harmonic polynomials (and their explicit description by the generating series in [9]).

Remark. In the case that the class number of the quaternion algebra is $h \neq 1$ Hecke eigenforms (on the adelic quaternion algebra) give rise to h-tuples of harmonic polynomials, they should then be viewed as functions on the disjoint sum of h copies of the unit sphere. All arguments from above can be carried out for such h-tuples of harmonic polynomials (resp. functions on the disjoint sum of h copies of the unit sphere).

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Appendix

S. Böcherer, R. Schulze-Pillot: Corrections to our article "On the central critical value of the triple product L-function" (reference [2] of this article)

- **p.5:**
  l.8 f.a.: read $\left(L_\alpha^{(b)} f\right)$ rather than $\left(L_\alpha^{(b)}\right)$
- **p.5:**
  l.9 f.b. read $\partial_{12}X_2$ rather than $\partial_{12}X_1$
- **p.7:**
  l.12 f.b.: read "(1.9)" rather than "(1.10)"
- **p.7:**
  Skip the sentences "It is easy..." (l.8 f.b.) until "...polynomial in $\partial_{12}, \partial_{13}, \partial_{23}$ (l.5 f.b.)
- **p.7:**
  l.4.f.b.: read a' rather than a
- **p.7:**
  l.3 f.b.:Formula (1.11) should read
  $$\frac{2^b 2^{a'} (2i)^{3a'}}{(\alpha + a')^{[b]} b!} \left(\partial_{12}\partial_{13}\partial_{23}\right)^{a'} \left(\partial_{12}X_2 + \partial_{13}X_3\right)^b$$
- **p.9:**
  l.11 f.a.: read a' rather than a
- **p.12**
  l.8 f.b.: the formula should be
  $$A(s,b) = (-1)^b \cdot \frac{2^{-a}(2s-2)^{[a]}(s)^{[a']}(2s+a-2)^{[b]}}{b!(s+a'-1)^{[b]}}$$
- **p.13:**
  in formula (2.11)
  $$(g < Z >^*) \qquad \text{rather than} \qquad \times < g < Z >^* >$$
  $$\det{}^{k+a} \qquad \text{rather than} \qquad \overset{k+a}{\det}$$
- **p.17:**
  l.7 f.b.: read h' rather than h
- **p.18:**
  delete factor 2 in formula (2.28)
- **p.32**
  in formula (4.1) read $L_p(f, \phi, \psi, s + 3a' + 2r + b - 2)$
- **p.33**
  Formula (4.3) is correct only under the additional assumption $r = 2$

- **p.33**
  In (4.3) read ζ rather than zeta
- **p.33**
  In the functional equation (4.4) the exponent of N should be $-4(s-\frac{k_1+k_2+k_3}{2}+1)$; the exponent of $gcd(N_f,N_\phi,N_\psi)$ should be
  $-(s-\frac{k_1+k_2+k_3}{2}+1)$
- **p.34:**
  l.10: read allows for $p \mid N$ rather than allows for $p \nmid N$
- **p.35:**
  l.5.f.a.: read (2.1) rather than (2.2)
- **p.37**:
  l.9 f.b. read $i^{3a'}\pi^{3a'+b}$ rather than π^{3a+2b}
- **p.37:**
  l.7 f.b: read $\left(i^{3a'}\pi^{3a'+b}\right)^{-1}$ rather than π^{-3a-2b}
- **p.38:**
  in formula 5.4: read $i^{3a'}\pi^{3a'+b}$ rather than π^{3a+2b}
- **p.42:**
  In Lemma 5.5 and in line 19 read
  $$\left(\sum_{i=1}^{h}\frac{T_0(\varphi_f(y_i)\otimes\varphi_\phi(y_i)\otimes\varphi_\psi(y_i))}{e_i}\right)$$
- **p.44:**
  The first line of formula (5.9) should read
  $$(-1)^{\omega(N)+\omega(M_1,M_2)}2^{5+3b+8a'-\omega(gcd(N_f,N_\phi,N_\psi)}\pi^{5+6a'+2b}$$
  $$\times N^2(M_1M_2)^{-3}M_3^{-6}\frac{1}{\binom{b}{\nu_2}}\frac{b!(a'+1)^{[b]}}{2^{[a]}2^{[a']}(a+2)^{[b]}}$$
  $$\times\frac{(a'+b+1)\Gamma(2a'+b+2)}{\Gamma(3a'+b+2)\Gamma(a'+\nu_2+1)\Gamma(a'+\nu_3+1)}$$
- **p.45:**
  l.8.f.a.: Nagoya Math.J.147(1997), 71-106
- **p.45**
  l.3 f.b.: Comm.Math.Univ.S.Pauli 48(1999), 103-118