REPRESENTATIONS BY SPINOR GENERA OF TERNARY QUADRATIC FORMS

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§1. Introduction

It is our basic question to study the following Diophantine equations

$$ {}^tXAX = B \tag{1.1}$$

over the ring of integers $\mathbb{Z}$, where A and B are non-degenerate and symmetric matrices of size $m \times m$ and $n \times n$ over $\mathbb{Z}$ respectively, and A is indefinite with $m \geq 3$. It is a necessary condition for solubility of equation (1.1) that it is solvable over $\mathbb{Z}_p$ for all primes p and the real numbers $\mathbb{R}$. This necessary condition is already sufficient if $m - n \geq 3$ [Kn1,Hs]. However the equation (1.1) is no longer a purely local problem when $m - n \leq 2$. By the Hasse principle, the necessary condition implies there is a rational solution of (1.1). In the previous papers [CX] and [X1], one of us has given conditions that allow to decide for $m-n \leq 2$ whether the equation (1.1) is solvable over $\mathbb{Z}$ by looking at a given rational solution whose denominator is prime to the determinant of A. Can one also determine the solubility of (1.1) if the denominator of the rational solution is not prime to the determinant of A? In this note, we try to give such a condition.

Notation and terminology are standard if not explained, or adopted from [CX] and [X1]. Let V be a quadratic space over a number field F with a non-degenerate symmetric bilinear form $\langle x, y\rangle$, $Q(x) = \langle x, x\rangle$ be the quadratic map on V and $SO(V)$ be the special orthogonal group of V. A lattice in V means a finitely generated $\mathfrak{o}_F$ module in V such that it generates a non-degenerate quadratic subspace of V. A full lattice means a lattice which generates the whole space. For a full lattice L, $L^\sharp$ denotes the dual lattice of L. For two lattices K and L in V, $\langle K, L\rangle$ denotes the fractional ideal generated by $\langle x, y\rangle$ for $x \in K$ and $y \in L$. We use τ_z for the reflection if $Q(z) \neq 0$. We also denote $\mathfrak{n}(L)$ and $\mathfrak{s}(L)$ as norm and scale of a lattice L in the sense of [O] respectively.

For any prime $\mathfrak{p}$ of F, $V_\mathfrak{p}$ (resp. $F_\mathfrak{p}$, etc.) denotes the local completion of V (resp. F, etc.). Let $\mathfrak{o}_F$ be the ring of integers of F. If $\mathfrak{p}$ is a finite prime, the group of units of $\mathfrak{o}_{F_\mathfrak{p}}$ is denoted by $\mathfrak{u}_\mathfrak{p}$, and $\pi_\mathfrak{p}$ is a uniformizer of $F_\mathfrak{p}$. We use $\theta_\mathfrak{p}$ to denote the spinor norm map of $SO(V_\mathfrak{p})$. For a lattice $K_\mathfrak{p}$ and a full lattice $L_\mathfrak{p}$ in $V_\mathfrak{p}$, let

$$X(L_\mathfrak{p}/K_\mathfrak{p}) = \{\sigma \in SO(V_\mathfrak{p}) : \ K_\mathfrak{p} \subseteq \sigma L_\mathfrak{p}\}.$$

It is clear that $X(L_\mathfrak{p}/K_\mathfrak{p})$ is not empty if and only if $K_\mathfrak{p}$ is represented by $L_\mathfrak{p}$. Then there is $\sigma \in X(L_\mathfrak{p}/K_\mathfrak{p})$ such that $K_\mathfrak{p} \subseteq \sigma L_\mathfrak{p}$. By [HSX, Thm.2.1], one has that

$\theta_{\mathfrak{p}}(X(\sigma L_{\mathfrak{p}}/K_{\mathfrak{p}})$ is a group. It can be easily verified that this group is independent of choice of $\sigma \in X(L_{\mathfrak{p}}/K_{\mathfrak{p}})$. Then one can define

$$\theta_{\mathfrak{p}}(L_{\mathfrak{p}}, K_{\mathfrak{p}}) := \theta_{\mathfrak{p}}(X(\sigma L_{\mathfrak{p}}/K_{\mathfrak{p}})).$$

The invariant $\theta_{\mathfrak{p}}(L_{\mathfrak{p}}, K_{\mathfrak{p}})$ was introduced and computed in [SP1] for ternary forms with codimension two over non-dyadic and 2-adic local fields. For such case over general dyadic fields, this invariant is computed in [X]. For general high dimensional cases, the invariant is computed in [HSX] and [HSX1] over non-dyadic and 2-adic local fields respectively. Recently, Beli has announced a computation of $\theta_{\mathfrak{p}}(L_{\mathfrak{p}}, K_{\mathfrak{p}})$ over general dyadic local fields.

By the strong approximation theorem for spin groups, the solubility of (1.1) is reduced to determine in which coset of the quotient group $F_{\mathfrak{p}}^{\times}/\theta_{\mathfrak{p}}(L_{\mathfrak{p}}, K_{\mathfrak{p}})$ the set $\theta_{\mathfrak{p}}(X(L_{\mathfrak{p}}/K_{\mathfrak{p}}))$ is for all $\mathfrak{p}$. The results of [CX,X1] show that this coset depends only on $[K_{\mathfrak{p}} : L_{\mathfrak{p}} \cap K_{\mathfrak{p}}]$ if $\mathfrak{p}$ does not divide the discriminant of $L_{\mathfrak{p}}$. A question raised in [X1] is to find some invariants derived from $L_{\mathfrak{p}}$ and $K_{\mathfrak{p}}$ directly to describe to which coset $\theta_{\mathfrak{p}}(X(L_{\mathfrak{p}}/K_{\mathfrak{p}}))$ belongs if $\mathfrak{p}$ divides the discriminant of $L_{\mathfrak{p}}$.

As a first step, we show that (under a suitable primitivity condition in case that $rank(K_{\mathfrak{p}}) \geq 2$) the coset in question depends only on the image of (a suitable multiple of) $K_{\mathfrak{p}}$ in the discriminant group $L_{\mathfrak{p}}^{\#}/L_{\mathfrak{p}}$ of $L_{\mathfrak{p}}$.

We focus then on the most interesting case that $rank(L) = 3$ and $rank(K) = 1$. The group $\theta_{\mathfrak{p}}(L_{\mathfrak{p}}, K_p)$ is then (as always in the case that the codimension of K in L is 2) a subgroup of index ≤ 2 of $F_{\mathfrak{p}}^{\times}$, and we consider the case that this subgroup is the group of units $\mathfrak{u}_{\mathfrak{p}}$ of $F_{\mathfrak{p}}$. For this unramified situation we obtain a complete answer in the special case $F = \mathbb{Q}$ (or more generally for $F/\mathbb{Q}$ such that 2 splits completely in F). In fact it turns out that the coset in question then depends only on $\langle K_{\mathfrak{p}}, L_{\mathfrak{p}}\rangle$ and (in the case of $\mathfrak{p} = (2)$) information about the primitivity of (a suitable multiple of) $K_{\mathfrak{p}}$ in the discriminant group. It appears that the case of dyadic $F_{\mathfrak{p}} \neq \mathbb{Q}_2$ is more complicated in the sense that one needs more information about the position of $K_{\mathfrak{p}}$ in the discriminant group in that case.

The case that $\theta_{\mathfrak{p}}(L_{\mathfrak{p}}, K_{\mathfrak{p}})$ is the norm group of a ramified quadratic extension of $F_{\mathfrak{p}}$ is apparently also quite a bit more complicated. We have no general result in that case but give some examples of the type of calculation arising in one case of the example below and in the final section dealing with the question of the existence of regular forms in an indefinite genus.

As an example for the usefulness of our (admittedly rather technical) results for getting concrete answers about the solubility of equation (1.1), we prove the following

Example 1.2. *Let m, n and k be positive integers. The Diophantine equation*

$$m^2x^2 + n^{2k}y^2 - nz^2 = 1$$

is solvable over $\mathbb{Z}$ if and only if $(m, n) = 1$ and

$$n \equiv \begin{cases} 1, 3, 7 \quad mod\ 8 & if\ \ 2 \mid m \ and\ 4 \nmid m \\ 1, 7 \quad mod\ 8 & if\ \ 4 \mid m. \end{cases}$$

We always assume that $X(L_{\mathfrak{p}}/K_{\mathfrak{p}})$ is not empty for all primes $\mathfrak{p}$. If $\mathfrak{n}(K_{\mathfrak{p}}) \subset \mathfrak{n}(L_{\mathfrak{p}})$, there is a full lattice $L_{\mathfrak{p}}' \subset L_{\mathfrak{p}}$ such that

$$\mathfrak{n}(K_{\mathfrak{p}}) \subseteq \mathfrak{n}(L_{\mathfrak{p}}') \subset \mathfrak{n}(L_{\mathfrak{p}})), \quad X(L_{\mathfrak{p}}'/K_{\mathfrak{p}}) = X(L_{\mathfrak{p}}/K_{\mathfrak{p}})$$

and

$$\theta_{\mathfrak{p}}(L'_{\mathfrak{p}}, K_{\mathfrak{p}}) = \theta_{\mathfrak{p}}(L_{\mathfrak{p}}, K_{\mathfrak{p}}).$$

Therefore we also assume that $\mathfrak{n}(K_{\mathfrak{p}}) = \mathfrak{n}(L_{\mathfrak{p}})$ by the above standard argument.

§2. An application of Witt's theorem

We recall that a nonzero submodule N of a module M over a ring R is called primitive if it is a direct summand of M.

Lemma 2.1. *Let R be a discrete valuation ring with field of fractions F and maximal ideal (π). Let M be a finitely generated free R-module with quadratic form $q : M \to R$ and associated symmetric bilinear form $\langle x, y\rangle = q(x + y) - q(x) - q(y)$ and let N_1, N_2 be two R-submodules of FM that are isometric with respect to the extension of q to FM.*

Assume that there is $j \in \mathbb{Z}$ such that $\pi^j N_\nu$ is a primitive submodule of $M^\# = \{y \in FM \mid \langle y, M\rangle \subseteq R\}$ for $\nu = 1, 2$. Assume moreover that there is an isometry $\tau : N_1 \to N_2$ such that $\tau(x) - x \in M$ for all $x \in \pi^j N_1$.

Then there is $\sigma \in O(M, q)$ with $\sigma(N_1) = N_2$.

Proof. For $x \in M$ and a submodule $N \subseteq M^\#$ let $\beta_N(x) \in N^*$ be the linear form on N given by $\beta_N(y) := \langle x, y\rangle$. It is then easily seen that a submodule N of $M^\#$ is a primitive submodule of $M^\#$ if and only if one has $\beta_N(M) = N^*$. A submodule of M satisfying this last property is called sharply primitive with respect to the symmetric bilinear form $\langle\ ,\ \rangle$ in [Kn3, Definition 2.17]. If in addition to our assumptions $\pi^j N_\nu \subseteq M$ is true for $\nu = 1, 2$, our assertion is therefore just Folgerung 4.4 in [Kn3]. An inspection of the proof given there shows that it remains valid in the present situation. □

Proposition 2.2. *Let $L_{\mathfrak{p}}$ be as in the introduction with $\frac{1}{2}\langle x, x\rangle \in \mathfrak{o}_{F_{\mathfrak{p}}}$ for all $x \in L_{\mathfrak{p}}$, let $K_{\mathfrak{p}}^{(1)}, K_{\mathfrak{p}}^{(2)}$ be isometric sublattices of $F_{\mathfrak{p}}L_{\mathfrak{p}}$. Assume that there are an isometry $\tau : K_{\mathfrak{p}}^{(1)} \to K_{\mathfrak{p}}^{(2)}$ and $j \in \mathbb{Z}$ such that $\mathfrak{p}^j K_{\mathfrak{p}}^{(1)}$ and $\mathfrak{p}^j K_{\mathfrak{p}}^{(2)}$ are primitive sublattices of $L_{\mathfrak{p}}^\#$ and such that $\tau(x) - x \in L_{\mathfrak{p}}$ for all $x \in \mathfrak{p}^j K_{\mathfrak{p}}^{(1)}$.*

Then one has $\theta_{\mathfrak{p}}(X(L_{\mathfrak{p}}/K_{\mathfrak{p}}^{(1)})) = \theta_{\mathfrak{p}}(X(L_{\mathfrak{p}}/K_{\mathfrak{p}}^{(2)}))$.

In particular, for one dimensional $K_{\mathfrak{p}} = \mathfrak{o}_{F_{\mathfrak{p}}}x$ as above with fixed $\langle x, x\rangle \in 2\mathfrak{o}_{F_{\mathfrak{p}}}$ the set $\theta_{\mathfrak{p}}(X(L_{\mathfrak{p}}/K_{\mathfrak{p}}))$ depends only on $\langle K_{\mathfrak{p}}, L_{\mathfrak{p}}\rangle$ and on the class of x modulo $(\langle K_{\mathfrak{p}}, L_{\mathfrak{p}}\rangle)^{-1}L_{\mathfrak{p}}$.

Proof. Since $\theta_{\mathfrak{p}}(X(L_{\mathfrak{p}}/K_{\mathfrak{p}}))$ is (if $\theta_{\mathfrak{p}}(L_{\mathfrak{p}}, K_{\mathfrak{p}})$ is non-empty) a coset of $\theta_{\mathfrak{p}}(L_{\mathfrak{p}}, K_{\mathfrak{p}}) \supseteq \theta_{\mathfrak{p}}(SO(L_{\mathfrak{p}}))$ this is an immediate consequence of Lemma 2.1. □

Remark. If $\mathfrak{p}$ does not divide the discriminant of L and $K = \mathfrak{o}_F x$ is one dimensional, Proposition 2.2 implies the well known fact that $\theta_{\mathfrak{p}}(X(L_{\mathfrak{p}}/K_{\mathfrak{p}}))$ depends only on $\langle x, x\rangle$ and $\langle K_{\mathfrak{p}}, L_{\mathfrak{p}}\rangle$ or equivalently on $[K_{\mathfrak{p}} : K_{\mathfrak{p}} \cap L_{\mathfrak{p}}]$ (see [CX]). In the cases of $\mathfrak{p}$ dividing the discriminant of L discussed below we see that at least in the dyadic case it is sometimes necessary to use (part of) the additional information about the position of $K_{\mathfrak{p}}$ in the local discriminant group $L_{\mathfrak{p}}^\#/L_{\mathfrak{p}}$.

§3. Nondyadic primes

In this section, we assume that $\mathfrak{p}$ is non-dyadic.

Proposition 3.1. *Suppose* $\theta_{\mathfrak{p}}(L_{\mathfrak{p}}, K_{\mathfrak{p}}) = \mathfrak{u}_{\mathfrak{p}}$ *and* $\mathfrak{n}(K_{\mathfrak{p}}) = \mathfrak{n}(L_{\mathfrak{p}}) = \mathfrak{o}_{F_{\mathfrak{p}}}$. *Write* $\langle K_{\mathfrak{p}}, L_{\mathfrak{p}}\rangle = \mathfrak{p}^k$. *Then*

$$\theta_{\mathfrak{p}}(X(L_{\mathfrak{p}}/K_{\mathfrak{p}})) = \begin{cases} \theta_{\mathfrak{p}}(L_{\mathfrak{p}}, K_{\mathfrak{p}}) & \text{if } k \geq 0 \\ \pi^k \theta_{\mathfrak{p}}(L_{\mathfrak{p}}, K_{\mathfrak{p}}) & \text{if } k < 0. \end{cases}$$

Proof. Without loss of generality, one can write

$$L = \mathfrak{o}_{F_{\mathfrak{p}}} x \perp \mathfrak{o}_{F_{\mathfrak{p}}} y \perp \mathfrak{o}_{F_{\mathfrak{p}}} z \quad \text{with} \quad Q(x) = 1,\ Q(y) = \varepsilon_1 \pi^{r_1} \text{ and } \ Q(z) = \varepsilon_2 \pi^{r_2}$$

and $K = \mathfrak{o}_{F_{\mathfrak{p}}} w$ with $Q(w) = 1$. The condition $\theta_{\mathfrak{p}}(L_{\mathfrak{p}}, K_{\mathfrak{p}}) = \mathfrak{u}_{\mathfrak{p}}$ implies that $0 \leq r_1 \leq r_2$ and that both r_1 and r_2 are even. Let

$$w = \alpha x + \beta y + \gamma z \quad \text{with } \alpha,\ \beta \text{ and } \gamma \text{ in } F_{\mathfrak{p}}.$$

i) $k \geq 0$. Since $\theta_{\mathfrak{p}}(L_{\mathfrak{p}}, K_{\mathfrak{p}}) = \mathfrak{u}_{\mathfrak{p}}$, one has $ord(Q(v))$ is even for any vector v in $F_{\mathfrak{p}} y \perp F_{\mathfrak{p}} z$ and $F_{\mathfrak{p}} y \perp F_{\mathfrak{p}} z$ is anisotropic. It is clear that

$$\tau_{w-x}\tau_x \in X(L_{\mathfrak{p}}/K_{\mathfrak{p}}) \quad \text{and} \quad Q(w-x) = 2(1-\alpha).$$

Since $K_{\mathfrak{p}} \subset L_{\mathfrak{p}}^{\sharp}$, one has $ord(\alpha) \geq 0$ and

$$1 - \alpha^2 = Q(\beta y + \gamma z)$$

Then

$$ord(1-\alpha) = 0 \quad \text{or} \quad ord(1-\alpha) = ord(Q(\beta y + \gamma z))$$

which is even. Therefore $\theta_{\mathfrak{p}}(X(L_{\mathfrak{p}}/K_{\mathfrak{p}})) = \theta_{\mathfrak{p}}(L_{\mathfrak{p}}, K_{\mathfrak{p}})$.

ii) $k < 0$. One can assume that $F_{\mathfrak{p}} y \perp F_{\mathfrak{p}} z$ is anisotropic. It is clear that

$$k = min\{ord(\alpha), r_1 + ord(\beta), r_2 + ord(\gamma)\}.$$

Suppose $ord(\alpha) > min\{r_1 + ord(\beta), r_2 + ord(\gamma)\}$. Then one of $ord(\beta), ord(\gamma)$ is negative and smaller than $ord(\alpha)$. This implies

$$min\{ord(Q(\beta y)), ord(Q(\gamma z))\} < ord(1 - Q(\alpha x)) = ord(Q(\beta y + \gamma z)),$$

which is impossible since $F_{\mathfrak{p}} y + F_{\mathfrak{p}} z$ is anisotropic. Therefore

$$ord(\alpha) \leq min\{r_1 + ord(\beta), r_2 + ord(\gamma)\} \quad \text{and} \quad ord(\alpha) = k.$$

Since $\tau_{w-x}\tau_x \in X(L_{\mathfrak{p}}/K_{\mathfrak{p}})$ and

$$ord(Q(w-x)) = ord(2(1-\alpha)) = ord(\alpha) = k,$$

the proof is complete. □

§4. Over $\mathbb{Z}_2$

It becomes very complicated to study such a question over dyadic local fields. The simple result as in the previous section is no longer true over general dyadic local fields even if 2 is unramified. Therefore we restrict ourselves to $F_{\mathfrak{p}} = \mathbb{Q}_2$ and $\mathfrak{o}_{F_{\mathfrak{p}}} = \mathbb{Z}_2$. As pointed out in previous sections, one only needs to consider $\mathfrak{n}(K_{\mathfrak{p}}) = \mathfrak{n}(L_{\mathfrak{p}})$ and $\theta_{\mathfrak{p}}(L_{\mathfrak{p}}, K_{\mathfrak{p}}) = \mathfrak{u}_{\mathfrak{p}}$. According to the classification in [X], we only need to study Case II, III and IV in [X]. For convenience, we list them as follows by [X, Theorem 2.1], taking into account the additional restrictions imposed in these cases by the condition $\theta_2(L_2, K_2) = \mathfrak{u}_2$.

Case II: $L_2 = \mathbb{Z}_2 x \perp \mathbb{Z}_2 y \perp \mathbb{Z}_2 z$ with

$$Q(x) = 1, \quad Q(y) = 2^{r_1}\gamma \quad \text{and} \quad Q(z) = 2^{r_2}\delta$$

where $0 < r_1 < r_2$, both r_1 and r_2 are even and $\gamma\delta = -5$.
Then $K_2 = \mathbb{Z}_2 w$ with $Q(w) = 1$.

Case III: $L_2 = \mathbb{Z}_2 x \perp (\mathbb{Z}_2 y + \mathbb{Z}_2 z)$ with

$$Q(x) = 1, \quad Q(y) = 2^{r_1+r_2}\gamma, \quad Q(z) = -2^{r_2-r_1+2}\gamma^{-1}, \quad \text{and} \quad \langle y, z\rangle = 2^{r_2}$$

where $r_2 \geq r_1 \geq 0$, $r_2 > 0$ and $r_1 \equiv r_2 \bmod 2$.
Then $K_2 = \mathbb{Z}_2 w$ with $Q(w) = 1$.

Case IV: If $\mathfrak{s}(L_2) = \mathfrak{n}(L_2)$, $L_2 = \mathbb{Z}_2 x \perp \mathbb{Z}_2 y \perp \mathbb{Z}_2 z$ with

$$Q(x) = 1, \quad Q(y) = \gamma \quad \text{and} \quad Q(z) = 2^{r_2}\delta$$

where $r_2 \geq 2$ and is even, $\gamma\delta = -5$ and $\gamma = -1$ or -5.
Then $K_2 = \mathbb{Z}_2 w$ with $Q(w) = 1$.
If $2\mathfrak{s}(L_2) = \mathfrak{n}(L_2)$, $L_2 = (\mathbb{Z}_2 x + \mathbb{Z}_2 y) \perp \mathbb{Z}_2 z$ with

$$Q(x) = 2, \quad Q(y) = 2\Gamma, \quad \langle x, y\rangle = 1 \quad \text{and} \quad Q(z) = \delta 2^{r_2}$$

where $r_2 \geq 3$ and is odd, $\delta = 1$ or 5 and

$$\Gamma = \begin{cases} -1 & \text{if } \delta = 1 \\ 0 & \text{if } \delta = 5. \end{cases}$$

Then $K_2 = \mathbb{Z}_2 w$ with $Q(w) = 2$.

§5. Over $\mathbb{Z}_2$, Rank of the first Jordan component equals one

In this section, we study the case that the first Jordan component of L_2 is of rank one which corresponds to the cases II and III in §4.

Proposition 5.1. *Suppose $\theta_2(L_2, K_2) = \mathfrak{u}_2$, $\mathfrak{s}(L_2) = \mathbb{Z}_2$ and the first Jordan component of L_2 is of rank one . Let $\langle K_2, L_2\rangle = 2^k\mathbb{Z}_2$.*
If $k > 0$, then $\theta_2(X(L_2/K_2)) = 2\theta_2(L_2, K_2)$.

If $k = 0$, then

$$\theta_2(X(L_2/K_2)) = \begin{cases} 2\theta_2(L_2, K_2) & \textit{if } (K_2 + L_2)/L_2 \textit{ is primitive in } L_2^\sharp/L_2, \\ \theta_2(L_2, K_2) & \textit{otherwise.} \end{cases}$$

If $k < 0$, then $\theta_2(X(L_2/K_2)) = 2^{k+1}\theta_2(L_2, K_2)$.

Proof. It is clear that one only needs to consider Case II and III in §4. We first consider Case II. Write $w = ax + by + cz$ where a, b and c are in $\mathbb{Q}_2$. Then

$$1 = a^2 + b^2Q(y) + c^2Q(z) \tag{5.2}$$

and

$$min\{ord(a), r_1 + ord(b), r_2 + ord(c)\} = k \tag{5.3}$$

It is clear that $\tau_{w-x}\tau_x \in X(L_2/K_2)$ and $Q(w - x) = 2(1 - a)$.

For $k > 0$, the result follows from the same argument as in Proposition 3.1 i).

For $k = 0$, we have the following two cases.

If $(K_2 + L_2)/L_2$ is primitive in $L_2^\sharp/L_2$, then $ord(b) = -r_1$ or $ord(c) = -r_2$. This implies

$$ord(Q(by + cz)) \leq -r_1 + 2 \leq 0,$$

since $\mathbb{Q}_2y + \mathbb{Q}_2z$ is anisotropic. By (5.2) one has $ord(Q(by + cz)) = 0$, and $a^2 + Q(by + cz) = 1$ implies $ord(a) > 0$ and hence $(1 - a) \in \mathbb{Z}_2^\times$ (the fact that we are working over $\mathbb{Z}_2$ will be used in a similar way several times in this section).

Otherwise, $ord(a) = 0$ by (5.3). The result follows clearly for $ord(1 - a) = 1$. We can assume that $ord(1 - a) > 1$. Then $ord(1 + a) = 1$, and $ord(1 - a^2) = ord(1 - a) + ord(1 + a) = ord(2(1 - a))$ is even since $\mathbb{Q}_2y + \mathbb{Q}_2z$ is anisotropic, and the result follows from (5.2).

For $k < 0$, we claim that $ord(a) = k$.

If $ord(a) > r_1 + ord(b)$, then

$$ord(a) \geq 3 + ord(b) \quad \text{and} \quad ord(a^2) \geq 4 + ord(Q(by)).$$

By (5.3)

$$min\{ord(Q(by)), ord(Q(cz))\} < 0.$$

Then

$$r_1 + 2ord(b) = ord(Q(by)) = ord(Q(cz)) = r_2 + 2ord(c)$$

by (5.2), and hence $ord(b) > ord(c)$ and $r_2 + ord(c) > r_1 + ord(b)$. Therefore we have

$$ord(b) + r_1 = k < 0 \quad \text{and} \quad ord(Q(by)) = 2k - r_1 \leq -4.$$

Since $\mathbb{Q}_2y \perp \mathbb{Q}_2z$ is anisotropic, one has

$$ord(Q(by) + Q(cz)) \leq 2 + ord(Q(by)).$$

A contradiction is derived from (5.2). This implies that $ord(a) \leq r_1 + ord(b) < r_2 + ord(c)$ and the claim follows.

Case III follows from similar arguments as above and the domination principle in [R].

The proof is complete. $\square$

Remark 5.4. *The case that $(K_2+L_2)/L_2$ is primitive in $L_2^\sharp/L_2$ can only happen for case II with $r_1=2$ and Case III with $r_2\le 2$.*

§6. Over $\mathbb{Z}_2$, Rank of the first Jordan component equals two

In this section, we study the case that the first Jordan component of L_2 is of rank two which corresponds to case IV in §4.

Proposition 6.1. *Suppose $\theta_2(L_2,K_2)=\mathfrak{u}_2$, $\mathfrak{s}(L_2)=\mathbb{Z}_2$ and the first Jordan component of L_2 is of rank two. Let $\langle K_2,L_2\rangle=2^k\mathbb{Z}_2$.*

If $k>0$, then $\theta_2(X(L_2/K_2))=2\theta_2(L_2,K_2)$.

If $k=0$, then $\theta_2(X(L_2/K_2))=\theta_2(L_2,K_2)$.

If $k<0$ and $\mathfrak{n}(L_2)=\mathbb{Z}_2$, then K_2 is a sublattice of $2^{k-r}L$ and

$$\theta_2(X(L_2/K_2))=\begin{cases}2^k\theta_2(L_2,K_2) & \text{if } K_2 \text{ is primitive in } 2^{k-r}L\\ 2^{k+1}\theta_2(L_2,K_2) & \text{otherwise.}\end{cases}$$

where $r=[\frac{1}{2}(ord(\mathfrak{n}(L_2^\sharp))]$.

If $k<0$ and $\mathfrak{n}(L_2)=2\mathbb{Z}_2$, then $\theta_2(X(L_2/K_2))=2^\mu\theta_2(L_2,K_2)$ where

$$\mu=\begin{cases}k & \text{when the first Jordan component of } L_2 \text{ is hyperbolic}\\ k+1 & \text{otherwise.}\end{cases}$$

Proof. It is clear that we only need to consider Case IV in §3. We first consider the case $\mathfrak{n}(L_2)=\mathbb{Z}_2$. Write $w=ax+by+cz$. Then

$$min\{ord(a),ord(b),r_2+ord(c)\}=k \tag{6.2}$$

and

$$1=a^2+\gamma b^2+\delta 2^{r_2}c^2. \tag{6.3}$$

One also has $\tau_{w-x}\tau_x\in X(L_2/K_2)$ and $Q(w-x)=2(1-a)$.

For $k>0$, the arguments in Proposition 3.1 i) are still valid.

For $k=0$, we claim that $ord(a)=0$. Otherwise

$$ord(b)=0 \ \text{ and } \ ord(\delta 2^{r_2}c^2)\ge 2$$

by (6.2) and (6.3) ($r_2+2ord(c)=0$ is impossible since then $a^2+\gamma b^2+\delta 2^{r_2}c^2$ would be even). A contradiction is derived by considering (6.3) modulo 4.
Then the result follows from the same arguments as in Proposition 5.1 for $k=0$ in the case $ord(a)=0$.

For $k<0$, (6.3) and (6.2) imply $r_2+2ord(c)\ge min\{0,2ord(a),2ord(b)\}\ge 2k$, and hence (with $r=\frac{r_2}{2}$)

$$ord(\delta 2^{r_2}c^2)\ge 2k \ \text{ and } \ ord(c)\ge k-r.$$

Therefore K is a sublattice of $2^{k-r}L$ and

$$min\{ord(a), ord(b)\} = k.$$

If K is primitive in $2^{k-r}L$, then $ord(c) = k - r$ and hence

$$ord(\delta 2^{r_2} c^2) = 2k = min\{ord(a^2), ord(\gamma b^2)\}.$$

$ord(a) = k$ would (since $\delta = 1$ or 5) imply $ord(a^2 + \delta 2^{r_2} c^2) = 2k + 1 \neq ord(\gamma b^2)$, and (6.3) leads to a contradiction for $ord(b) = k$ as well as for $ord(b) > k$.
We have therefore $ord(a) > k, ord(b) = k$ and $ord(\gamma b^2 + \delta 2^{r_2} c^2) = 2 + 2k$ since $\gamma\delta = -5$. Equation (6.3) implies then $ord(a) > 0$ for $k = -1$ and $ord(a) = k + 1$ for $k < -1$, and hence $ord(Q(w - x)) = 1 + ord(1 - a)$ is congruent to k modulo 2 in both cases.

If K is not primitive in $2^{k-r}L$ we have $ord(c) > k - r$, and (6.3) implies $ord(a) = ord(b) = k$ and hence

$$ord(Q(w - x)) = 1 + ord(1 - a) = k + 1.$$

Now we consider the case $\mathfrak{n}(L_2) = 2\mathbb{Z}_2$. Write $w = ax + by + cz$. Then

$$min\{ord(a), ord(b), ord(c) + r_2\} = k \tag{6.4}$$

and

$$2 = 2a^2 + 2ab + 2\Gamma b^2 + \delta 2^{r_2} c^2. \tag{6.5}$$

It is clear that $\tau_{x-w}\tau_x \in X(L_2/K_2)$ and $Q(w - x) = 2(2(1 - a) - b)$.

For $k > 0$, one has $ord(a) > 0$ and $ord(b) > 0$. We claim that $ord(b) \geq 2$.
If $\delta = 1$, one has

$$ord(a^2 + ab - b^2) = min\{2ord(a), 2ord(b)\} \geq 3$$

by (6.5). Therefore $ord(b) \geq 2$.
If $\delta = 5$, then $ord(a^2 + ab) = 2$ by (6.5). This also implies that $ord(b) \geq 2$, and we get $ord(Q(x) + ord(Q(w - x)) = 3$ as asserted in this case.

For $k = 0$, $ord(b) = 0$ implies $ord(Q(w - x)) = 1 = ord(Q(x))$ and hence $\theta_2(X(L_2)/K_2) = \theta(L_2, K_2)$. We can therefore assume $ord(b) \geq 1$ which implies $ord(a) = 0$.
Then

$$ord(\delta 2^{r_2} c^2) \geq 3 \quad \text{and} \quad ord(b) \geq 2$$

by (6.5). In fact, one can further assume that $ord(2(1 - a) - b) > 2$. Then

$$ord(2(1 - (a + 2^{-1}b))) = ord(1 - (a + 2^{-1}b)^2) = ord((2^{-1}b)^2(4\Gamma - 1) + \delta 2^{r_2 - 1} c^2)$$

by (6.5), and this order is always even.

For $k < 0$, (6.5) and (6.4) imply $ord(\delta 2^{r_2-1}c^2) \geq min\{2ord(a), 2ord(b)\} \geq 2k$ and hence

$$ord(\delta 2^{r_2-1}c^2) \geq 2k \quad \text{and} \quad ord(c) \geq k - \frac{1}{2}(r_2 - 1)$$

by (6.4) and (6.5).

If the first Jordan component of L_2 is not a hyperbolic plane, then

$$\delta = 1 \quad \text{and} \quad ord(c) = k - \frac{1}{2}(r_2 - 1)$$

by (6.4), (6.5) and the domination principle in [R]. We have

$$\tau_{2^{-\frac{1}{2}(r_2-1)}z-w}\tau_x \in X(L_2/K_2),$$

and

$$ord(Q(2^{-\frac{1}{2}(r_2-1)}z - w)) = ord(4(1 - 2^{\frac{1}{2}(r_2-1)}c)) = k + 2.$$

If the first Jordan component is a hyperbolic plane, then $\delta = 5$ and $\Gamma = 0$.
If $ord(c) = k - \frac{1}{2}(r_2 - 1)$, one has $ord(a) = ord(a + b) = k$ and

$$ord(a^2 - \delta 2^{r_2-1}c^2) = 2k + 2.$$

by (6.4) and (6.5). Since

$$a(2a + b) = a^2 + a(a + b) = a^2 - \delta 2^{r_2-1}c^2 + 1$$

by (6.5), one has

$$ord(2a + b) = \begin{cases} > 1 & \text{if } k = -1 \\ k + 2 & \text{if } k < -1. \end{cases}$$

Therefore $ord(Q(w - x)) = k + 3$.
Otherwise $ord(c) > k - \frac{1}{2}(r_2 - 1)$. By (6.4) and (6.5), we have

$$ord(a) \neq ord(a + b) \quad \text{and} \quad min\{ord(a), ord(a + b)\} = k.$$

Therefore $ord(Q(w - x)) = k + 1$. The proof is complete. □

§7. Proof of Example 1.2

Before we give a proof of Example 1.2, we recall some basic facts (see [Kn1, SP1, HSX]):

To a lattice K in FL of codimension 2 that is represented locally everywhere by L one associates the quadratic extension $E = F(\sqrt{d})$ of F where d is the discriminant of $FK^{\perp}$ and puts $N_{\mathfrak{p}}(E) := N_{F_{\mathfrak{p}}}^{E_{\mathfrak{P}}}(E_{\mathfrak{P}}^{\times})$ for any place $\mathfrak{p}$ of F with a place $\mathfrak{P}$ in E dividing $\mathfrak{p}$. The lattice K is then represented by all classes in the genus of L if and only if either d is a square or there is some $\mathfrak{p}$ such that $\theta_{\mathfrak{p}}(L_{\mathfrak{p}}, K_{\mathfrak{p}}) \neq N_{\mathfrak{p}}(E)$ (notice that $\theta_{\mathfrak{p}}(L_{\mathfrak{p}}, K_{\mathfrak{p}}) \supseteq N_{\mathfrak{p}}(E)$ is true automatically). If this is not the case, K is represented by precisely half the classes in the genus of L; such K are called exceptional for $gen(L)$.

If K is exceptional for $gen(L)$ and $\sigma = (\sigma_{\mathfrak{p}})_{\mathfrak{p}}$ is in the adelic special orthogonal group of FL such that K is represented by σL, then K is represented by L if and only if

$$(\theta_{\mathfrak{p}}(\sigma_{\mathfrak{p}}))_{\mathfrak{p}} \subseteq F^{\times} \cdot \prod_{\mathfrak{p}} N_{\mathfrak{p}}(E).$$

We have the following proposition which explains that the solubility of (1.1) for the case of codimension two depends only on the local computation of $\theta_{\mathfrak{p}}(X(L_{\mathfrak{p}}/K_{\mathfrak{p}}))$. For any two lattices K and L, let

$$P(L,K) = \{\mathfrak{p} \text{ prime}: \ \theta_{\mathfrak{p}}(X(L_{\mathfrak{p}}/K_{\mathfrak{p}})) \neq \theta_{\mathfrak{p}}(L_{\mathfrak{p}}, K_{\mathfrak{p}})\}.$$

It is clear that $\mid P(L,K) \mid$ is finite.

Proposition 7.1. *If K is an exceptional lattice for $gen(L)$ with codimension two, then K is represented by L if and only if $\mid P(L,K) \mid$ is even.*

Proof. As above let $\sigma = (\sigma_{\mathfrak{p}})_{\mathfrak{p}}$ be in adelic special orthogonal group of FL such that $K \subseteq \sigma L$. By Hilbert's reciprocity law

$$(\theta_{\mathfrak{p}}(\sigma_{\mathfrak{p}}))_{\mathfrak{p}} \subseteq F^{\times} \cdot \prod_{\mathfrak{p}} N_{\mathfrak{p}}(E).$$

is true if and only if $\theta_{\mathfrak{p}}(\sigma_{\mathfrak{p}}) \nsubseteq N_{\mathfrak{p}}(E)$ holds for an even number of places $\mathfrak{p}$, i. e., if and only if $|P(L,K)|$ is even. $\square$

Proof of Example 1.2. Let $L = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3$ be a quadratic lattice such that

$$Q(xe_1 + ye_2 + ze_3) = m^2x^2 + n^{2k}y^2 - nz^2$$

and $K = \mathbb{Z}(m^{-1}e_1)$.

It is easily seen that 1 is represented by $gen(L)$ if and only if $(m,n) = 1$. We have then $K_p \subseteq L_p$ for all $p \nmid m$, and it is clear that $\langle L, K\rangle = m\mathbb{Z}$.

We can conclude from Proposition 3.1 that $\theta_{\mathfrak{p}}(X(L_p/K_p)) = \theta_{\mathfrak{p}}(L_p, K_p)$ for all odd $p \mid m$, hence 1 is certainly represented by L if m is odd. If m is even (which we assume from now on) and $4 \nmid m$ we see from [SP1,X] that K is not exceptional if n is congruent to 3 or 7 modulo 8. In all other cases of even m, the lattice K is exceptional, and in these remaining cases 1 is represented by L if and only if $\theta_2(X(L_2/K_2)) = \theta_2(L_2, K_2)$ by Proposition 7.1.

If $n \equiv 1 \bmod 8$ there is nothing to check at $p = 2$ and the representability of 1 by L follows. If $n \equiv 5 \bmod 8$ Proposition 6.1 applies and yields $\theta_2(X(L_2/K_2)) = 2\theta_2(L_2, K_2)$, thus $P(L,K) = \{2\}$ and 1 is not represented by L.

It remains to consider the case that m is even and $\mathbb{Q}_2(\sqrt{n})/\mathbb{Q}_2$ is ramified, i.e., $n \equiv 3 \bmod 8$ or $n \equiv 7 \bmod 8$; by the argument above we need to discuss this only for $4 \mid m$.

It is clear that

$$\tau_{\frac{1}{n^k}e_2 - \frac{1}{m}e_1}\tau_{e_2} \in X(L_2/K_2)$$

and

$$Q(\frac{1}{n^k}e_2 - \frac{1}{m}e_1) = 2.$$

We have $2 \notin \theta_2(L_2, K_2) = N_2(\mathbb{Q}(\sqrt{n}))$ for $n \equiv 3 \bmod 8$ and $2 \in \theta_2(L_2, K_2) = N_2(\mathbb{Q}(\sqrt{n}))$ for $n \equiv 7 \bmod 8$, which finishes the proof. $\square$

§8. Regularity of indefinite ternary forms

In this section, we discuss the regularity of indefinite ternary forms in the sense of Dickson [Di].

Definition 8.1. *A quadratic form is called regular if it represents all integers which are locally represented.*

It is well known that there are only finitely many regular positive definite ternary quadratic forms by the results of Watson. Jagy, Kaplansky and Schiemann [JaKaSc] try to give a complete list of such positive definite ternary forms. For indefinite ternary forms, it is easy to see that there are infinitely many genera consisting only of one class and hence infinitely many regular forms. In [BH] an example was given of genera with arbitrary large class number such that all forms in the genus are regular. We consider here some examples of other types of behaviour.
Let L be an indefinite lattice of rank 3.

Definition 8.2. *([BH]) A set of exceptional square classes $\{c_1, \cdots, c_n\}$ for $gen(L)$ is called independent if $\{-c_1 det(L), \cdots, -c_n det(L)\}$ are linearly independent vectors in the $\mathbb{F}_2$ vector space $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ (or equivalently, if the fields $\mathbb{Q}(\sqrt{-c_i det(L)})$ are linearly disjoint).*
It is called complete if in addition 2^n is the number of spinor genera in $gen(L)$.

The following proposition is already known in [BH]. Here we give a proof which also provides a method of explicit construction.

Proposition 8.3. *If the set of all exceptional square classes for $gen(L)$ is independent, then there is a regular form in $gen(L)$.*

Proof. Let $\{c_1, \cdots, c_n\}$ be all exceptional square classes for $gen(L)$ and

$$Gal(\mathbb{Q}(\sqrt{-c_i det(L)})/\mathbb{Q}) = \{1, g_i\}$$

for $1 \leq i \leq n$. Define

$$h_i = \begin{cases} 1 & \text{if } L \text{ represents } c_i, \\ g_i & \text{otherwise.} \end{cases}$$

Since $\{-c_1 det(L), \cdots, -c_n det(L)\}$ are linearly independent, by Kummer theory and Cebotarev density Theorem, there is a prime $l \nmid 2det(L)$ such that

$$l \text{ is unramified in } Gal(\mathbb{Q}(\sqrt{-c_i det(L)})/\mathbb{Q})$$

and the Frobenius of l is h_i for $1 \leq i \leq n$. Then the l-neighbor of L in the sense of Kneser [Kn2,SP2] is in $gen(L)$ and regular. $\square$

Corollary 8.4. *If $gen(L)$ has only two exceptional square classes, then there is a regular form in $gen(L)$.*

Proof. It is clear that any two exceptional square classes are independent. $\square$

Example 8.5. *Let p, q and r be primes such that $p \equiv 1 \ mod\ 8$, $q \equiv r \equiv 3 \ mod\ 8$ and $(\frac{q}{p}) = (\frac{r}{p}) = 1$. Let*

$$f(x, y, z) = x^2 - pqry^2 + p^3q^2r^2z^2$$

Then 1 *and* p *are the only two exceptional square classes and* f *is regular.*

Proof. It is clear that 1 and p are represented by $gen(f)$ and are also exceptional integers for $gen(f)$. In fact, they are the only exceptional square classes. Indeed, let a be an exceptional integer for $gen(f)$. By [CX, (3.2)] at ∞, one has that a is positive. Write $a = a_0 b^2$ where both a_0 and b are positive integers and a_0 is square-free. Besides 1 and p, the other possible values of a_0 are q, r, pq, pr and pqr.

Since q and r are symmetric, one can assume $(\frac{r}{q}) = 1$ and $(\frac{q}{r}) = -1$. By a simple computation of Hasse symbols, one has that a is not represented by f over $\mathbb{Q}_q$ if $a_0 = q$, pq or pqr. For $a_0 = r$ or pr, the relative spinor norm is $\mathbb{Q}_r^\times$ at r but r is not split in the corresponding quadratic field $\mathbb{Q}(\sqrt{q})$ or $\mathbb{Q}(\sqrt{pq})$ respectively. Then a is not exceptional.

In order to prove that f is regular, one only needs to show that p is represented by f. Equivalently, one only needs to show that the following equation

$$px^2 - qry^2 + p^2q^2r^2z^2 = 1$$

is solvable over $\mathbb{Z}$. Let $L = \mathbb{Z}e_1 \perp \mathbb{Z}e_2 \perp \mathbb{Z}e_3$ be a quadratic lattice such that

$$Q(xe_1 + ye_2 + ze_3) = px^2 - qry^2 + p^2q^2r^2z^2$$

and $K = \mathbb{Z}\frac{1}{pqr}e_3$. It is clear that the primes in $P(L, K)$ in §7 are from p, q and r.

At p, one has

$$\tau_{\sqrt{-qr}^{-1}e_2-(pqr)^{-1}e_3}\tau_{e_2} \in X(L_p/K_p)$$

and

$$Q(\sqrt{-qr}^{-1}e_2 - (pqr)^{-1}e_3) = 2 \in (Q_p^\times)^2$$

Therefore $p \notin P(L, K)$.

At q, one has

$$\tau_{\sqrt{p}^{-1}e_1-(pqr)^{-1}e_3}\tau_{e_1} \in X(L_q/K_q)$$

and

$$Q(\sqrt{p}^{-1}e_1 - (pqr)^{-1}e_3) = 2 \notin (\mathbb{Z}_q^\times)^2$$

Therefore $q \in P(L, K)$. Similarly one has $r \in P(L, K)$. The result follows from Prop.7.1. $\square$

If there are more than two exceptional square classes, the set of all exceptional square classes is not necessarily independent and there may be no regular forms in the given genus.

Remark 8.6. *It should be pointed out that the term "regular spinor genus" in [BH] denotes a spinor genus which represents a maximal independent set of exceptional square classes but not necessarily all exceptional square classes.*

In fact, we'll construct a genus in the following steps with only three exceptional square classes which are not independent and with arbitrarily large class number but such that there is no regular form in the genus.

Step 1): Suppose $p_1, \cdots, p_n$ are odd primes with $p_i \equiv 1 \bmod 8$ for $1 \le i \le n$ and $(\frac{p_i}{p_j}) = 1$ for $i \ne j$. Let

$$f(x,y,z) = x^2 - (p_1 \cdots p_n)y^2 + (p_1 \cdots p_n)^2 z^2.$$

It is clear that the class number of f is 2^n and there are $2^n - 1$ exceptional square classes for $gen(f)$ which are the classes of 1 and of the $p_{i_1} \cdots p_{i_s}$ for $i_1 < \cdots < i_s$ with $1 \le s \le n-1$. The form f is equivalent to

$$f'(x,y,z) = (p_1 \cdots p_k)^2 x^2 - (p_1 \cdots p_n)y^2 + (p_{k+1} \cdots p_n)^2 z^2.$$

To see equivalence of these forms, one can write $L = \mathbb{Z}e_1 \perp \mathbb{Z}e_2 \perp \mathbb{Z}e_3$ such that

$$Q(xe_1 + ye_2 + ze_3) = (p_1 \cdots p_k)^2 x^2 - (p_1 \cdots p_n)y^2 + (p_{k+1} \cdots p_n)^2 z^2.$$

Then we only need to show that L is equivalent to $L' = \mathbb{Z}(p_k^{-1}e_1) \perp \mathbb{Z}e_2 \perp \mathbb{Z}(p_k e_3)$. It is clear that $L_p = L'_p$ if $p \ne p_k$. At $p = p_k$, one can check that

$$\tau_{\frac{1}{p_1 \cdots p_k}e_1 - \frac{1}{p_{k+1} \cdots p_n}e_3}(L_{p_k}) = L'_{p_k}$$

and

$$Q(\frac{1}{p_1 \cdots p_k}e_1 - \frac{1}{p_{k+1} \cdots p_n}e_3) = 2 \in (\mathbb{Q}_{p_k}^{\times})^2.$$

Step 2): To decide if f represents $(p_1 \cdots p_k)$ is equivalent to determine if the following Diophantine equation

$$(p_1 \cdots p_k)x^2 - (p_{k+1} \cdots p_n)y^2 + (p_1 \cdots p_k)(p_{k+1} \cdots p_n)^2 z^2 = 1$$

is solvable over $\mathbb{Z}$. Let $L = \mathbb{Z}e_1 \perp \mathbb{Z}e_2 \perp \mathbb{Z}e_3$ such that

$$Q(xe_1 + ye_2 + ze_3) = (p_1 \cdots p_k)x^2 - (p_{k+1} \cdots p_n)y^2 + (p_1 \cdots p_k)(p_{k+1} \cdots p_n)^2 z^2.$$

Write

$$p_1 \cdots p_k = m^2 + n^2$$

and

$$K = \mathbb{Z}(\frac{m}{p_1 \cdots p_k}e_1 + \frac{n}{p_1 \cdots p_n}e_3).$$

Since p_i with $1 \le i \le k$ splits in $\mathbb{Q}(\sqrt{p_{k+1} \cdots p_n})$, $P(L,K)$ in §7 contains only primes from $\{p_{k+1}, \ldots, p_n\}$.

Step 3): For any $p \in \{p_{k+1}, \ldots, p_n\}$, $(p_1 \cdots p_k)$ is a square in $\mathbb{Q}_p$. It is clear that

$$\tau_{(\sqrt{p_1 \cdots p_k})^{-1}e_1 - u}\tau_{e_1} \in X(L_p/K_p)$$

where $u = \frac{m}{p_1 \cdots p_k}e_1 + \frac{n}{p_1 \cdots p_n}e_3$. Therefore $p \in P(L,K)$ if and only if

$$Q((\sqrt{p_1 \cdots p_k})^{-1}e_1 - u) = 2(1 - m(\sqrt{p_1 \cdots p_k})^{-1})$$

is a non-square unit or p times a non-square unit up to a square in $\mathbb{Q}_p$.

Step 4): Let $n = 2$ and $p_1 = 17$ and $p_2 = 89$ (this is the first pair of primes satisfying the conditions above).

To see if f represents 17, one can write $17 = 1^2 + 4^2$ and have

$$1 - \sqrt{17}^{-1} \equiv 1 - 33 \equiv -2^5 \quad mod\ 89.$$

Therefore $P(L, K)$ is empty and f represents 17.

To see if f represents 89, one can write $89 = 8^2 + 5^2$ and have

$$1 - 8(\sqrt{89})^{-1} \equiv 1 - 8 \times 2^{-1} \equiv -3 \quad mod\ 17.$$

Therefore $P(L, K) = \{17\}$ and f does not represent 89.

By the Cebotarev density theorem, for any given class in $gen(f)$ there are primes $l \nmid 2 \cdot 17 \cdot 89$ such that the l-neighbor of f is equivalent to this given class.

However, there are no primes $l \nmid 2 \cdot 17 \cdot 89$ such that Frobenius of l is trivial at $Gal(\mathbb{Q}(\sqrt{89})/\mathbb{Q})$ and $Gal(\mathbb{Q}(\sqrt{17 \cdot 89})/\mathbb{Q})$ but non-trivial at $Gal(\mathbb{Q}(\sqrt{17})/\mathbb{Q})$. Hence no form in the genus of f can represent all of $1, 17, 89$, i. e., there are no regular forms in $gen(f)$.

Step 5): Based on the above construction, one can have

$$g(x, y, z) = x^2 - 17 \cdot 89 \cdot (q_1 \cdots q_m)^2 y^2 + (17 \cdot 89)^2 (q_1 \cdots q_m)^4 z^2$$

where q_i are odd primes with $q_i \equiv 1 \bmod 8$ and different from 17 and 89, and $(\frac{17}{q_i}) = (\frac{89}{q_i}) = 1$ for $1 \le i \le m$ and $(\frac{q_i}{q_j}) = 1$ for $i \neq j$.

If there were an exceptional integer a that is divisible by one of the q_i it had to be divisible by q^3 in order to be locally representable at q_i. But then it is seen from [SP1] that a can not be exceptional. All the exceptional squares for $gen(g)$ are therefore 1, 17 and 89. By the same argument as above, g represents 1 and 17 but not 89 and there are no regular forms in $gen(g)$. At the same time, the class number in $gen(g)$ can be arbitarily large. $\square$

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