Itô’s Formula for Gaussian Processes with Stochastic Discontinuities

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Abstract

We introduce a Skorokhod type integral and prove an Itô formula for a wide class of Gaussian processes which may exhibit stochastic discontinuities. Our Itô formula unifies and extends the classical one for general (i.e., possibly discontinuous) Gaussian martingales in the sense of Itô integration and the one for stochastically continuous Gaussian non-martingales in the Skorokhod sense, which was first derived in Alòs et al. (Ann. Probab. 29, 2001). A main observation is that the jump terms, which appear in the general Gaussian Itô formula, only depend on the deterministic times of stochastic discontinuities and not on the random pathwise jump times of the process.

1 Introduction

Since the pioneering work of Alòs et al. (2001), Itô’s formula for Gaussian processes in the sense of Skorokhod type integration has been developed in a series of papers. The generic formula reads as follows: If $X$ is a centered Gaussian process with variance function $V$, which is assumed to be of bounded variation and continuous, and $F$ is sufficiently smooth, then

$$F(X_T) = F(X_0) + \int_0^T F'(X_s)dX_s + \frac{1}{2} \int_0^T F''(X_s)dV(s).$$

(1)

This formula has been shown to been valid under structural assumptions on a kernel representation of $X$ with respect to a Brownian motion (Alòs et al., 2001; Mocioalca and Viens, 2005; Lebovits, 2017), on the covariance function (Kruk et al., 2007; Lei and Nualart, 2012; Hu et al., 2013; Alpay

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and Kipnis, 2013), on the associated Cameron-Martin space (Bender, 2014), and on the quadratic variation of $X$ (Nualart and Taqqu, 2006, 2008).

An important contribution by Mocioalca and Viens (2005) clarifies that the Itô formula (1) can even hold for certain classes of Gaussian processes with discontinuities in the paths, although the formula does not exhibit any jump components. Suppose, however, that $X$ is a Gaussian martingale with RCLL paths. Then, the classical semimartingale Itô formula implies that

$$F(X_T) = F(X_0) + \int_0^T F'(X_s-)dX_s + \frac{1}{2} \int_0^T F''(X_s-)dV^c(s) + \sum_{s\in(0,T]} \left(F(X_s) - F(X_{s-}) - F'(X_{s-})(X_s - X_{s-})\right),$$

(2)

(where we exploit that the continuous part $V^c$ of the variance coincides with the continuous part of the quadratic variation in the Gaussian case). Here, the integral with respect to $X$ is, of course, the usual stochastic Itô integral.

Comparing both Itô formulas (1) and (2), the following question arises: When, why, and how do jump terms appear in Itô’s formula for Gaussian processes? This question is addressed in the present paper. The key observation is the following: All works, which are concerned with deriving the Itô formula (1), impose assumptions on $X$, which guarantee that $X$ is stochastically continuous. It is, however, well-known, that all the jumps of a Gaussian martingale occur at the deterministic times, at which it is stochastically discontinuous, see e.g. Jain and Monrad (1982). In this paper, we prove a general Itô formula for centered Gaussian processes which may exhibit stochastic discontinuities, supposing that the variance of $X$ is of bounded variation and that the Cameron-Martin space of $X$ is separable and has a dense subset of functions, which satisfy a certain regularity property in terms of their quadratic variation. These assumptions can be verified in (almost) all the papers on the Gaussian Itô formula (1) in the Skorokhod sense, which we are aware of. If we additionally assume, for the sake of exposition, that $X$ is stochastically RCLL, then our Itô formula can be stated as follows:

$$F(X_T) = F(X_0) + \int_{0+}^T F'(X_s-)dX_s + \frac{1}{2} \int_{0+}^T F''(X_s-)dV^c(s)$$

\[+\sum_{s\in D_X \cap (0,T]} \left(F(X_s) - F(X_{s-}) - F'(X_{s-})(X_s - X_{s-})\right)
\]

\[+F''(X_{s-})E[(X_s - X_{s-})X_{s-}]]\].

(3)
Here, the sum runs over the set of deterministic time points in \((0, T]\), at which \(X\) exhibits a stochastic discontinuity. It can, hence, not be computed path-by-path in general, but is shown to converge unconditionally in \(L^2(\Omega)\) under appropriate conditions. Our new Itô formula, thus, contains the Skorokhod type Itô formula for stochastically continuous Gaussian processes \((1)\) and the Itô formula for Gaussian martingales \((2)\) as special cases, and extends them to a wide class of stochastically discontinuous Gaussian non-martingales.

The paper is organized as follows. In Section 2, we first recall some preliminaries on Gaussian processes and then introduce the concept of a weakly regulated process. It allows giving a meaning to the one-sided limits \(X_{s\pm}\) without imposing any path regularity assumption on \(X\), provided that the functions in the Cameron-Martin space of \(X\) are regulated. In Section 2, we also study the set of stochastic discontinuities for weakly regulated Gaussian processes. Section 3 is devoted to our notion of Wick-Skorokhod integration, which we define in terms of the \(S\)-transform and the Henstock-Kurzweil integral. This \(S\)-transform approach has already been adopted in Bender (2003b, 2014), and actually, can be viewed as a main tool for studying Hitsuda-Skorokhod integration in a white noise framework as in Lebovits (2017) and the references therein. We also show, that our integrals extends the classical stochastic Itô integral for predictable integrands to anticipating integrands in the case that \(X\) is a Gaussian martingale. After these preparations we can state and prove our Itô formula in its general form in Section 4. We finally explain how to check the required structural assumption on the Cameron-Martin space in Section 5 and compare our assumptions to the ones imposed in the existing literature in Section 6. In the appendix, we provide, following ideas of Norvaiša (2002), a chain rule for the Henstock-Kurzweil integral, which is required in our proof of Itô’s formula.

2 Weakly regulated processes

The main purpose of this section is to give a meaning to the one-sided limits \(X_{s\pm}\), which occur in the Itô formula, without imposing path regularity assumptions on \(X\). Before doing so, let us recall some facts on Gaussian processes for ready reference.

Suppose \((X_t)_{0 \leq t \leq T}\) is a centered Gaussian process on a complete probability space \((\Omega, \mathcal{F}, P)\) with variance function \(V(t) := E[X(t)^2]\). We denote by \(H_X\) the first chaos of \(X\), i.e. the Gaussian Hilbert space, which is obtained by taking the closure of the linear span of \(\{X_t; t \in [0, T]\}\) in \(L^2(\Omega, \mathcal{F}, P)\). To
each element $h \in H_X$, one can associate a function
\[ h : [0, T] \to \mathbb{R}, \quad t \mapsto E[X_t h]. \]
The space of functions
\[ CM_X := \{ h : h \in H_X \} \]
is called the Cameron-Martin space associated to $X$. As, by definition, the set \( \{X_t; t \in [0, T]\} \) is total in $H_X$, the map
\[ H_X \to CM_X, \quad h \mapsto h \]
is bijective. It becomes an isometry, if one equips the Cameron-Martin space with the inner product
\[ \langle h, g \rangle_{CM_X} := E[h g]. \]
The Wick exponential of $h \in H_X$ is defined to be
\[ \exp^\diamond(h) := \exp\{h - E[h^2]/2\}. \]
If $\mathcal{A}$ is a dense subset of $H_X$, then, by Corollary 3.40 in Janson (1997), the set
\[ \{\exp^\diamond(h); h \in \mathcal{A}\} \]
is total in $(L^2_X) := L^2(\Omega, \mathcal{F}^X, P)$, where $\mathcal{F}^X$ is the completion by the $P$-null sets of the $\sigma$-field generated by $X$. Hence, every random variable $\xi \in (L^2_X)$ is uniquely determined by its $S$-transform restricted to $\mathcal{A}$,
\[ (S\xi)(h) := E[\exp^\diamond(h)\xi], \quad h \in \mathcal{A}. \]
This means, the identity $\xi = \eta$ is valid in $(L^2_X)$, if and only if $(S\xi)(h) = (S\eta)(h)$ for every $h \in \mathcal{A}$. We also recall the following straightforward identities ($g, h \in H_X, \ t \in [0, T]$)
\[ \exp^\diamond(h)\exp^\diamond(g) = e^{E[gh]}\exp^\diamond(g + h), \quad (S \exp^\diamond(g))(h) = e^{E[gh]}, \]
\[ (Sg)(h) = E[gh], \quad (SX_t)(h) = h(t). \quad (4) \]
More, generally, we note that, by a classical result on Gaussian change of measure,
\[ (SG(g_1, \ldots, g_D))(h) = E[G(g_1 + E[g_1 h], \ldots, g_D + E[g_D h])] \quad (5) \]
for every $g_1, \ldots, g_D, h \in H_X$ and measurable $G : \mathbb{R}^D \to \mathbb{R}$, provided that the expectation on the right-hand side exists, see e.g. Janson (1997, Theorem 14.1).
After these preliminaries, we now define the notion of a weakly regulated process.
Definition 2.1. A stochastic process \( Y : [0, T] \to (L^2_X) \) is called weakly regulated, if for every \( s \in (0, T] \) there is a random variable \( Y_{s-} \in (L^2_X) \) such that for every sequence \( (s_n) \) which converges to \( s \) from the left
\[
\lim_{n \to \infty} Y_{s_n} = Y_{s-} \text{ weakly in } (L^2_X),
\]
and, if for every \( s \in [0, T) \) there is a random variable \( Y_{s+} \in (L^2_X) \) such that for every sequence \( (s_n) \) which converges to \( s \) from the right
\[
\lim_{n \to \infty} Y_{s_n} = Y_{s+} \text{ weakly in } (L^2_X),
\]
For a weakly regulated process, we shall apply the convention \( Y_0^- := Y_0 \) and \( Y_{T+} := Y_T \). Moreover, we write \( \Delta Y_s = Y_{s+} - Y_{s-} \), \( \Delta^+ Y_s = Y_{s+} - Y_s \), \( \Delta^- Y_s = Y_s - Y_{s-} \).

The analogous notation is used for the jumps of deterministic regulated functions.

Weakly regulated processes can be characterized via the \( S \)-transform as follows:

Proposition 2.2. Suppose \( Y : [0, T] \to (L^2_X) \). If \( Y \) is weakly regulated, then the map \( t \mapsto \mathbb{E}[Y^2_t] \) is bounded and, for every \( h \in H_X \), the map \( t \mapsto (SY_t)(h) \) is regulated (i.e., has limits from the left and from the right).

Conversely, assume that \( A \) is a dense subset of \( H_X \). If the map \( t \mapsto \mathbb{E}[Y^2_t] \) is bounded and, for every \( h \in A \), the map \( t \mapsto (SY_t)(h) \) is regulated, then \( Y \) is weakly regulated.

Before we prove this proposition, let us state the following corollary concerning the Gaussian process \( X \).

Corollary 2.3. The Gaussian process \( (X_t)_{t \in [0, T]} \) is weakly regulated, if and only if its variance function \( V \) is bounded and the Cameron-Martin space \( CM_X \) has a dense subset consisting of regulated functions. In this case the weak \( (L^2_X) \)-limits \( X_{s\pm} \) belong to the first chaos \( H_X \).

Proof. In view of Proposition 2.2, the first statement is an immediate consequence of (4). For the second one, let \( s \in [0, T) \) and denote by \( \pi_{H_X} \) the orthogonal projection from \( (L^2_X) \) on \( H_X \). Then,
\[
\mathbb{E}[[X_{s+} - \pi_{H_X}(X_{s+})]^2] = \mathbb{E}[X_{s+}(X_{s+} - \pi_{H_X}(X_{s+}))] = \lim_{n \to \infty} \mathbb{E}[X_{s+\frac{1}{n}}(X_{s+} - \pi_{H_X}(X_{s+}))].
\]
The right-hand side equals 0, because \( X_{s+\frac{1}{n}} \in H_X \). Hence, \( X_{s+} \in H_X \), and the left-sided limits can be handled analogously. \( \square \)
Proof of Proposition 2.2. Suppose first that $Y$ is weakly regulated. Fix $h \in H_X$ and $s \in (0, T]$. Then, for every sequence $(s_n)$ which converges to $s$ from the left,

$$
\lim_{n \to \infty} (SY_{s_n})(h) = \lim_{n \to \infty} E[Y_{s_n} \exp^\diamond(h)] = E[Y_{s_+} \exp^\diamond(h)]
$$

Hence, $t \mapsto (SY_t)(h)$ has a limit from the left at $s$. In the same way, one observes that it has a limit from the right at every $s \in [0, T)$. We next prove the boundedness of $t \mapsto E[Y_t^2]$ by contradiction. Thus, suppose to the contrary that this function is unbounded, and choose a sequence $(t_n)$ in $[0, T]$ such that $E[Y_{t_n}^2]$ converges to infinity. We can extract a subsequence $(t_{n_k})$ which converges to some $t \in [0, T]$ and satisfies (by passing to another subsequence, if necessary) $t_{n_k} < t$ or $t_{n_k} > t$ for every $k \in \mathbb{N}$. Then, the sequence $(X_{t_{n_k}})$ converges weakly in $(L^2_X)$ to $X_{t_+}$ (in the first case) or $X_{t_-}$ (in the second case). Hence, by Theorem V.1.1 in Yosida (1995), the sequence $(E[Y_{t_{n_k}}^2])_{k \in \mathbb{N}}$ is bounded, a contradiction.

For the converse implication, fix $s \in [0, T)$ and a sequence $(s_n)$, which converges to $s$ from the right. As $\sup_{n \in \mathbb{N}} E[|Y_{s_n}|^2] \leq \sup_{t \in [0, T]} E[|Y_t|^2] < \infty$, there is, by the Banach-Alaoglu theorem (Yosida, 1995, Theorem V.2.1), a subsequence $(s_{n_k})$ such that $(Y_{s_{n_k}})$ converges weakly in $(L^2_X)$ to a limit, which we denote by $Y_{s_+}$. Define, for every $h \in A$, $\tilde{h} : [0, T] \to \mathbb{R}$, $t \mapsto (SY_t)(h)$. As, for every $h \in A$, $\tilde{h}$ is regulated, we get

$$
\tilde{h}(s+) = \lim_{k \to \infty} \tilde{h}(s_{n_k}) = \lim_{k \to \infty} E[Y_{s_{n_k}} \exp^\diamond(h)] = E[Y_{s_+} \exp^\diamond(h)], \quad h \in A.
$$

Now, for every sequence $(t_n)$ converging to $s$ from the right and every $h \in A$,

$$
E[Y_{t_n} \exp^\diamond(h)] = \tilde{h}(t_n) \to \tilde{h}(s+) = E[Y_{s_+} \exp^\diamond(h)].
$$

As, moreover, $\sup_{n \in \mathbb{N}} E[|Y_{t_n}|^2] < \infty$, we conclude thanks to Theorem V.1.3 in Yosida (1995) that $Y_{t_n} \to Y_{s_+}$ weakly in $(L^2_X)$.

An analogous argument shows, for every $s \in (0, T]$, existence of a weak limit $X_{s_-}$ in $(L^2_X)$ (as $u$ approaches $s$ from the left).

The following example is instructive and will be applied in the proof of Itô’s formula.
Example 2.4. Suppose that the variance function $V$ of $X$ is regulated and that the Cameron-Martin space $CM_X$ has a dense subset consisting of regulated functions. Then, by Corollary 2.3, $X$ is weakly regulated. We consider, for some continuous function $F : \mathbb{R} \to \mathbb{R}$, the process $Y_t := F(X_t)$. Assume that $F$ satisfies the following subexponential growth condition

$$|F(x)| \leq Ce^{\alpha x^2}, \quad x \in \mathbb{R},$$

for constants $C \geq 0$ and $0 \leq a < (4\lambda)^{-1}$ with $\lambda := \sup_{t \in [0,T]} V(t)$. This standard assumption guarantees that $t \mapsto \mathbb{E}[Y_t^2]$ is bounded. By (5), the $S$-transform of $Y_t$ is given by

$$(SF(X_t))(h) = \mathbb{E}[F(X_t + \bar{h}(t))] = \psi_F(V(t), \bar{h}(t)), \quad h \in H_X,$$

where

$$\psi_F : [0, \lambda] \times \mathbb{R} \to \mathbb{R}, \quad (t, x) \mapsto \int_{\mathbb{R}} F(x + \sqrt{t}y) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy.$$

Note that $\psi_F(0, x) = F(x)$.

By (7) and Proposition 2.2, $Y$ is weakly regulated and the weak limits $Y_{t+}$, $Y_{t-}$ satisfy

$$(SY_{t\pm})(h) = \psi_F(V(t\pm), \bar{h}(t\pm)), \quad h \in H_X.$$

Define $V^\pm(t) = E[X_{t\pm}^2]$. By Theorem V.1.1 in Yosida (1995), $V^\pm(t) \leq V(t\pm)$, and the inequality can be strict, as $X_{t\pm}$ are weak ($L^2_X$)-limits only. Then,

$$Y_{t+} = \psi_F(V(t+) - V^+(t), X_{t+}), \quad Y_{t-} = \psi_F(V(t-) - V^-(t), X_{t-}),$$

since, e.g., for every $h \in H_X$

$$S\left(\psi_F(V(t+) - V^+(t), X_{t+})\right)(h)$$

$$= \int_{\mathbb{R}^2} F(h(t+) + \sqrt{V(t+) - V^+(t)}u + \sqrt{V^+(t)}y) \frac{1}{2\pi} e^{-\left((u^2+y^2)/2\right)} d(u, y)$$

$$= \psi_F(V(t+), \bar{h}(t+)).$$

Thus, the identity $Y_{t+} = F(X_{t+})$ can, in general, only be expected to be valid at those $t \in [0, T)$, for which $V(t+) = V^+(t)$ (which is equivalent to saying that the convergence from the right to $X_{t+}$ takes place in probability).

We next study the set of stochastic discontinuities of $X$. 
**Definition 2.5.** The process $X$ is said to be *stochastically continuous at* $t \in [0, T]$, if for every sequence $(t_n)$ in $[0, T]$

$$t_n \to t \Rightarrow X_{t_n} \to X_t \text{ in probability.}$$

We denote by $C_X$ the set of points, at which $X$ is stochastically continuous, and by $D_X = [0, T] \setminus C_X$ the set of stochastic discontinuities of $X$.

**Proposition 2.6.** Suppose that the variance function $V$ of $X$ is regulated, $X$ is weakly regulated and $H_X$ is separable. Then the set $D_X$ of stochastic discontinuities of $X$ is at most countable.

**Proof.** We first apply the separability of $H_X$ and choose a countable dense subset $A'$ of $H_X$. Let

$$D = \{ s \in [0, T]; V \text{ is discontinuous at } s \} \cup \left( \bigcup_{h \in A'} \{ s \in [0, T]; h \text{ is discontinuous at } s \} \right).$$

As $V$ and all the $h$’s are regulated functions thanks to Proposition 2.2, the set $D$ is at most countable. If $t \in [0, T] \setminus D$, then we obtain, for every $h \in A'$

$$\lim_{s \to t} E[X_s h] = \lim_{s \to t} h(s) = h(t) = E[X_t h]$$
$$\lim_{s \to t} E[|X_s|^2] = \lim_{s \to t} V(s) = V(t) = E[|X_t|^2].$$

Now, Theorem V.1.3 in Yosida (1995) implies that

$$\lim_{s \to \infty} E[|X_s - X_t|^2] = 0,$$

and, hence, $X$ is stochastically continuous at $t$. In particular, $D_X \subset D$ is at most countable.

**Remark 2.7.** Note that the separability of $H_X$ cannot be dispensed with. Indeed, if $D_X$ is countable, then the rational span of

$$\{ X_t; t \in ([0, T] \cap \mathbb{Q}) \cup D_X \}$$

is a countable dense subset of $H_X$. 

8
3 Wick-Skorokhod integration

In this section, we introduce a class of generalized Skorokhod integrals, which is applied throughout the paper. We first define it via Wick-Stieltjes sums for a class of simple processes and then propose several extensions by means of the $S$-transform. Throughout this section, we assume that $X$ is weakly regulated.

Let us first recall that two random variables $\eta, \xi \in (L^2_X)$ are said to have a Wick product, if there is a random variable $\eta \diamond \xi \in (L^2_X)$ such that

$$S(\eta \diamond \xi)(h) = (S\eta)(h)(S\xi)(h)$$

for every $h \in H_X$. We denote by $D_{1,2}^X$ the subspace of all random variables $\xi \in (L^2_X)$ such that the Wick product $\xi \diamond g$ exists for every $g \in H_X$. It contains all Wick exponentials. Indeed, it is well-known and straightforward to verify by (4) that for every $f, g \in H_X$

$$\exp \diamond (f \circ g) = \exp \diamond (f) \cdot (g - E[g])$$

For a simple integrand of the form

$$Z := F_0 1_{\{0\}} + \sum_{i=1}^n (G_i 1_{(t_{i-1}, t_i]} + F_i 1_{\{t_i\}}),$$

$0 = t_0 < t_1 < \cdots < t_n = T$, $F_i, G_i \in D_{1,2}^X$, we define the Wick-Skorokhod integral of $Z$ with respect to $X$ by

$$\int_0^T Z_s d^\diamond X_s := F_0 \circ (\Delta^+ X_0) + \sum_{i=1}^n (G_i \circ (X_{t_i} - X_{t_{i-1}+}) + F_i \circ (\Delta X_{t_i})), \quad \text{(10)}$$

where we recall the convention $X_{T+} = X_T$ which entails that $\Delta X_T = \Delta^- X_T$. Note that this Wick-Itô integral for simple integrands can be characterized in terms of the $S$-transform by

$$S \left( \int_0^T Z_s d^\diamond X_s \right)(h) = (SF_0)(h) \cdot \Delta^+ h(0) + \sum_{i=1}^n (SG_i)(h) (h(t_i-) - h(t_{i-1}+)) + (SF_i)(h) \cdot \Delta h(t_i)$$

$$= \int_0^T (SZ_s)(h) \, dh(s)$$

for every $h \in H_X$, where the integral on the right-hand side can be understood in the sense of Henstock and Kurzweil.
Remark 3.1. The Henstock-Kurzweil integral can be applied to give a meaning to integrals of the form $\int_0^T u(s)dr(s)$ for suitably pairs of functions $u, r : [0, T] \to \mathbb{R}$, without assuming that the integrator $r$ is of bounded variation. We briefly recall the construction: A gauge function is any function $\delta : [0, T] \to (0, \infty)$. A tagged partition $\tau := \{(s_{i-1}, s_i), y_i; i = 1, \ldots, n\}$ of the interval $[0, T]$ is called $\delta$-fine, if $y_i - \delta(y_i) \leq s_{i-1} \leq y_i \leq s_i \leq y_i + \delta(y_i)$ for every $i = 1, \ldots, n$. The Riemann sum for the integral $\int_0^T u(s)dr(s)$ with respect to the tagged partition $\tau$ is given by

$$S_{RS}(u, r, \tau) := \sum_{i=1}^n u(y_i)(r(s_i) - r(s_{i-1})).$$

If there is an $I \in \mathbb{R}$ such that for every $\epsilon > 0$ there is a gauge function $\delta$ such that $|S_{RS}(u, r, \tau) - I| < \epsilon$ for every $\delta$-fine tagged partition $\tau$, then $I$ is uniquely determined, denoted by $\int_0^T u(s)dr(s)$ and called the Henstock-Kurzweil integral of $u$ with respect to $r$. A brief review of the relation between the Henstock-Kurzweil integral and other Stieltjes-type integrals can be found in Appendix F of Part I in Dudley and Norvaisa (1999). We just note the following important relation to the Lebesgue-Stieltjes integral: If $r$ is of bounded variation, then $r$ uniquely determines a signed measure $\mu_r$ via the relation

$$\mu_r([0, t]) := r(t+) - r(0), \quad 0 \leq t < T, \quad \mu_r([0, T]) = r(T) - r(0).$$

If the Lebesgue-Stieltjes integral $\int_0^T u(s)\mu_r(ds)$ exists, then so does the Henstock-Kurzweil integral $\int_0^T u(s)dr(s)$ and both integrals coincide.

In order to retain sufficient flexibility in the extension of the Wick-Skorokhod integral beyond simple integrands, it will be defined relative to a dense subset $A$ of the first chaos $H_X$ of $X$. Recall that the $S$-transform restricted to any such dense subset uniquely determines a random variable in $(L^2_X)$. 

Definition 3.2. Suppose $A$ is a dense subset of $H_X$. A process $Z : [0, T] \to (L^2_X)$ is said to be $A$-Wick-Skorokhod integrable with respect to $X$, if for every $h \in A$ the Henstock-Kurzweil integral $\int_0^T (SZ_s)(h) \, dh(s)$ exists and if there is a random variable $\int_0^T Z_s \, d_A^p X_s \in (L^2_X)$ such that for every $h \in A$

$$S \left( \int_0^T Z_s \, d_A^p X_s \right)(h) = \int_0^T (SZ_s)(h) \, dh(s).$$

We then call $\int_0^T Z_s \, d_A^p X_s$ the $A$-Wick-Skorokhod integral of $Z$ with respect to $X$. 

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We sometimes write \( \int_{0+}^{T} Z_s \, d\xi^A_s X_s := \int_{0}^{T} 1_{[0,T]}(s) Z_s \, d\xi^A X_s \), provided the integral on the right-hand side exists. The same notation will be applied for other types of integrals, when integration is understood over \((0,T]\) rather than \([0,T]\).

Note that this definition of a stochastic integral with respect to a Gaussian process generalizes the S-transform approach in Bender (2003b) beyond the fractional Brownian motion case. It is in the spirit of the white-noise approach to Hitsuda-Skorokhod integration, see Kuo (1996, Chapter 13) for the Brownian motion case and Lebovits (2017) and the references therein for generalizations to various classes of stochastically continuous Gaussian processes. It also generalizes the notion of an extended Skorokhod integral, which has been studied in the framework of Malliavin calculus e.g. in Cheridito and Nualart (2005); Mocioalca and Viens (2005); Lei and Nualart (2012).

**Remark 3.3.** Suppose \( A_1, A_2 \) are dense subsets of \( H_X \). Obviously, the following relations are true:

(i) If \( A_1 \cap A_2 \) is dense in \( H_X \) and \( \int_{0}^{T} Z_s \, d^A_{A_1} X_s \) and \( \int_{0}^{T} Z_s \, d^A_{A_2} X_s \) both exist, then they coincide in \( (L^2_X) \).

(ii) If \( A_1 \subset A_2 \) and \( \int_{0}^{T} Z_s \, d^A_{A_2} X_s \) exists, then so does \( \int_{0}^{T} Z_s \, d^A_{A_1} X_s \) and both integrals are equal in \( (L^2_X) \).

If however, \( A_1 \cap A_2 \) is not dense in \( H_X \), the integrals \( \int_{0}^{T} Z_s \, d^A_{A_1} X_s \) and \( \int_{0}^{T} Z_s \, d^A_{A_2} X_s \) cannot be compared in general.

We next relate this notion of Wick-Skorokhod integration to the usual stochastic Itô integral (see e.g. Protter, 2005) in the martingale case:

**Theorem 3.4.** Let \( \mathbb{F}^X = (\mathbb{F}^X_t)_{t \in [0,T]} \) denote the augmentation of the filtration generated by \( X \). If \( X \) is an \( \mathbb{F}^X \)-martingale with RCLL paths and \( Z \) is \( \mathbb{F}^X \)-predictable and satisfies

\[
E \left[ \int_{0}^{T} |Z_s|^2 \, dV(s) \right] < \infty,
\]

then the Wick-Skorokhod integral \( \int_{0}^{T} Z_s \, d^A X_s \) exists for \( A = H_X \) and coincides with the stochastic Itô integral \( \int_{0+}^{T} Z_s \, dX_s \).

**Proof.** Note first that, by the martingale property of \( X \), the set \( D_X \) of stochastic discontinuities of \( X \) consists of the jump times of the nondecreasing RCLL variance function \( V \) of \( X \), and is thus at most countable. By Theorem 1.8 in Jain and Monrad (1982),

\[
X_t = X_t^c + \sum_{s \in D_X \cap (0,t]} (X_s - X_{s-}),
\]
where $X^c$ is a Gaussian martingale with continuous paths. Of course, in this decomposition, the pathwise left limits of the RCLL martingale $X$ coincide, for every $s \in (0, T]$, $P$-almost surely with the weak $(L^2_X)$-limits $X_{s-}$. We write $V^c$ and $V^d$ for the continuous and the discrete part of $V$. Then, $V^c$ is the quadratic variation of $X^c$, $V^d$ is the predictable compensator of $\sum_{s \in D_X \cap [0, \cdot]} (X_s - X_{s-})^2$, and, hence, $V$ is the predictable compensator of the quadratic variation $[X]$ of $X$. Thus, if $Z$ is predictable and satisfies the assumed integrability condition, then, by the isometry of the stochastic Itô integral, $\int_0^T Z_s dX_s$ exists in $(L^2_X)$ and satisfies

$$E \left[ \left( \int_0^T Z_s dX_s \right)^2 \right] = E \left[ \int_0^T |Z_s|^2 d[X]_s \right] = E \left[ \int_0^T |Z_s|^2 dV(s) \right].$$

In particular, the first chaos of $X$ can be represented as

$$H_X = \left\{ aX_0 + \int_{0+}^T h(s)dX_s; \quad a \in \mathbb{R}, h \in L^2([0, T], dV) \right\}.$$

We now fix a generic element $h = aX_0 + \int_{0+}^T h(s)dX_s$ of the first chaos of $X$ and define

$$\mathcal{E}_t := \exp \left\{ aX_0 - \frac{a^2}{2} V(0) \right\} \exp \left\{ \int_{0+}^t h(s)dX_s - \frac{1}{2} \int_{0+}^t |h(s)|^2 dV(s) \right\},$$

$$Y_t := \int_{0+}^t Z_s dX_s.$$

Note that $\mathcal{A}_t = aV(0) + \int_{0+}^t h(s)dV(s)$. Thus, in view of Remark 3.1, all we need to show is that

$$E[\mathcal{E}_T Y_T] = \int_{0+}^T E[\mathcal{E}_T Z_s] h(s) dV(s),$$

where the integral on the right-hand side is a Lebesgue-Stieltjes integral. Applying Itô’s formula for semimartingales (Protter, 2005, Theorem II.32), we obtain after some elementary manipulations

$$\mathcal{E}_t = \mathcal{E}_0 + \int_{0+}^t \mathcal{E}_{s-} h(s) dX_s^c + \sum_{s \in D_X \cap [0, t]} \mathcal{E}_{s-} \left( e^{h(s)} \Delta X_s - \frac{1}{2} h(s)^2 \Delta V(s) - 1 \right).$$

Integration by parts (for semimartingales) thus yields

$$\mathcal{E}_t Y_t = \int_{0+}^t \mathcal{E}_{s-} dY_s + \int_{0+}^t Y_{s-} d\mathcal{E}_s + [\mathcal{E}, Y]_t$$

(12)
where the quadratic covariation of $E$ and $Y$ is given by

$$
[\mathcal{E}, Y]_t = \int_{0^+}^t Z_s \mathcal{E}_{s-} \mathcal{H}(s) dV^c(s) + \sum_{s \in D_X \cap (0,t]} Z_s \mathcal{E}_{s-} \left( e^{h(s)} \Delta X_s - \frac{1}{2} h(s)^2 \Delta V(s) - 1 \right) \Delta X_s. \quad (13)
$$

Since $E$ and $Y$ are square-integrable RCLL martingales, we may conclude from Emery’s inequality (Protter, 2005, Theorem V.3) that both stochastic integrals in (12) are martingales (of class $\mathcal{H}_1$) and, consequently, have zero expectation. Taking into account, that, for every $s \in D_X$, the jumps sizes $\Delta X_s$ are independent of $\mathcal{F}_s$ and $Z_s$ is $\mathcal{F}_s$-measurable by predictability, we obtain, thanks to (12)-(13),

$$
E[\mathcal{E}_T Y_T] = \int_{0^+}^T E[Z_s \mathcal{E}_{s-}] \mathcal{H}(s) dV^c(s) + \sum_{s \in D_X \cap (0,T]} E[Z_s \mathcal{E}_{s-}] E \left[ \left( e^{h(s)} \Delta X_s - \frac{1}{2} h(s)^2 \Delta V(s) - 1 \right) \Delta X_s \right].
$$

As, by (4),

$$
E \left[ \left( e^{h(s)} \Delta X_s - \frac{1}{2} h(s)^2 \Delta V(s) - 1 \right) \Delta X_s \right] = h(s) \Delta V(s),
$$

we arrive at

$$
E[\mathcal{E}_T Y_T] = \int_{0^+}^T E[Z_s \mathcal{E}_{s-}] \mathcal{H}(s) dV(s).
$$

Now, since $Z_s$ is $\mathcal{F}_s$-measurable and $\mathcal{E}$ is a martingale, we finally obtain

$$
[Z_s \mathcal{E}_{s-}] = E[Z_s E[\mathcal{E}_T | \mathcal{F}_s]] = E[\mathcal{E}_s],
$$

which yields (11), and finishes the proof. \qed

In the proof we have seen the well-known fact that, in the martingale case, every element in the Cameron-Martin space is absolutely continuous with respect to the variance function $V$ (with square integrable density). For general Gaussian processes one cannot expect such nice regularity properties for all elements in $CM_X$. Thus, alternatively, one can fix certain regularity properties and consider the set of those $h \in H_X$, for which the associated elements in the Cameron-Martin space satisfy the given regularity, as the set $A$ in the definition of the Wick-Skorokhod integral. Of course, the requirement that $A$ must be dense in the first chaos of $X$, restricts, in dependence
of the imposed regularity conditions, the class of processes, for which the $A$-Wick-Skorokhod integral can then be defined.

For the statement and proof of the Itô formula, we formulate the regularity requirement in terms of the quadratic variation as follows: We denote by $W^*_2 = W^*_2([0, T])$ the set of regulated functions $u : [0, T] \to \mathbb{R}$ such that

$$
\sigma_2(u) := \sum_{s \in (0, T]} |\Delta^- u(s)|^2 + \sum_{s \in [0, T)} |\Delta^+ u(s)|^2 < \infty
$$

and such that, for every $\epsilon > 0$, there is a partition $\lambda$ of $[0, T]$ such that for all refinements $\kappa = \{0 = s_0 < s_1 < \cdots < s_n \leq T\}$ of $\lambda$

$$
\left| \left( \sum_{i=1}^n |u(s_i) - u(s_{i-1})|^2 \right)^{1/2} - \sigma_2(u) \right| < \epsilon.
$$

Roughly speaking, this requirement means that the continuous part of the quadratic variation (in the sense of stochastic analysis) must vanish.

The following structural assumption on the Cameron-Martin space will be assumed for the main results of this paper:

**H** The Cameron-Martin space $CM_X$ of $X$ is separable and $W^*_2 \cap CM_X$ is dense in $CM_X$.

We write $W^*_2$ for the set of those elements $h \in H_X$ such that $h \in W^*_2$. Then, under condition (H), $W^*_2$ is dense in $H_X$ and we can, thus, study the Wick-Skorokhod integral for $A = W^*_2$. In order to lighten the notation, we skip the subscript from the notation of the Wick-Skorokhod integral in this case and write

$$
\int_0^T Z_s d^{\diamond} X_s := \int_0^T Z_s d^{\diamond}_{W^*_2} X_s.
$$

**Remark 3.5.** (i) Recall that, for $p \geq 1$, the $p$-variation of a regulated function $u : [0, T] \to \mathbb{R}$ is defined to be

$$
v_p(u) := \sup\left\{ \sum_{j=1}^m |u_1(t_j) - u_1(t_{j-1})|^p ; m \in \mathbb{N}, 0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = T \right\}.
$$

A regulated function $u$ is said to belong to $W_p([0, T])$, if $v_p(u) < \infty$. By Lemmas II.2.3 and II.2.14 in Dudley and Norvaiša (1999), $W_p([0, T]) \subset W^*_2([0, T]) \subset W_2([0, T])$ for every $1 \leq p < 2$. With this notation, $W_1([0, T])$ is the space of bounded variation functions on $[0, T]$. 

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(ii) Assumption (H) will be discussed in some more detail in Section 5 below. We note that it is obviously satisfied when $X$ has RCLL paths (or, less restrictively, is stochastically RCLL) and, for every fixed $s \in [0,T]$, the covariance function $R(t,s) := E[X_t X_s]$ is of bounded variation as function in $t$. This already covers a large class of relevant Gaussian processes.

(iii) It is important, that the regularity requirement is imposed on the elements of the Cameron-Martin space and not on the paths of the process $X$. A classical example is fractional Brownian motion, which has continuous paths and, in dependence of the Hurst parameter $H \in (0,1)$, the covariance function

$$R(t,s) = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right).$$

Hence, it satisfies (H) for every $H \in (0,1)$. However, it is well-known that almost every path of the fractional Brownian motion belongs to $W^2_\alpha$, if and only if $H > 1/2$.

4 Itô’s formula

After these preparations on Wick-Skorokhod integration and weakly regulated processes, we can now state and prove a general Gaussian Itô formula in the presence of stochastic discontinuities.

Let us first recall that $V^\pm(t)$ denotes the variance of $X_t^\pm$ and that, for a sufficiently integrable function $F$, $\psi_F$ is given by the convolution

$$\psi_F(t,x) := \int_\mathbb{R} F(x + \sqrt{t}y) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy,$$

see (8).

**Theorem 4.1.** Suppose $X$ is a centered Gaussian process satisfying (H), the variance function $V$ of $X$ is of bounded variation and

$$\sum_{s \in D_X \cap [0,T]} \left( E[(\Delta^+ X_s)^2] + (V(s) - V^+(s)) \right)$$

$$+ \sum_{s \in D_X \cap (0,T]} \left( E[(\Delta^- X_s)^2] + (V(s) - V^-(s)) \right) < \infty. \quad (14)$$

Assume $F \in C^2(\mathbb{R})$ and $F,F',F''$ satisfy the growth condition (6) with $\lambda = \sup_{t \in [0,T]} V(t)$. Then, $\int_0^T F'(X_s) d^c X_s$ exists and the following Itô formula
holds in $(L^2_X)$:

$$F(X_T) - F(X_0) = \int_0^T F'(X_s) dX_s + \frac{1}{2} \int_0^T F''(X_s) dV(s)$$

$$+ \sum_{s \in D_X \cap [0,T]} \left( F(X_s) - \psi F(V(s) - V^-(s), X_{s-}) - F'(X_s) \Delta^- X_sight)$$

$$+ \frac{1}{2} F''(X_s)(E[(\Delta^- X_s)^2] + V(s) - V^-(s))$$

$$+ \sum_{s \in D_X \cap [0,T]} \left( \psi F(V(s+) - V^+(s), X_{s+}) - F(X_s) - F'(X_s) \Delta^+ X_sight)$$

$$- \frac{1}{2} F''(X_s)(E[(\Delta^+ X_s)^2] + V(s+) - V^+(s)).$$

Here, the set $D_X$ of stochastic discontinuities of $X$ is at most countable and both sums converge unconditionally in $(L^2_X)$.

Note first that, under the assumptions of Theorem 4.1, $X$ is weakly regulated by Corollary 2.3 and has at most countably many stochastic discontinuities by Proposition 2.6.

We prove Theorem 4.1 by an $S$-transform approach, which originates in the work by Kubo (1983) on Itô’s formula for generalized functionals of a Brownian motion in the setting of white noise analysis. It has since then been successfully applied to wider classes of (stochastically) continuous Gaussian processes, see e.g. Bender (2003a,b); Alpay and Kipnis (2013); Lebovits and Vehel (2014); Lebovits (2017). In a closely related development, the Malliavin calculus approach to Itô’s formula, replaces the $S$-transform (and, thus, the pairing with Wick exponentials) by pairings with Hermite polynomials, see Al`os et al. (2001); Mocioalca and Viens (2005); Kruk et al. (2007); Lei and Nualart (2012).

The key idea of the $S$-transform approach to Itô’s formula is the following: While the process $F(X_t)$ may lack good path regularity, its $S$-transform $t \mapsto S(F(X_t))(\eta)$ is typically more regular and may be expanded via a ‘classical’ chain rule. In a second step, the resulting terms, which appear after application of the chain rule, must be identified as the $S$-transforms of the different terms in the Gaussian Itô formula. The main contribution to this technique of proof in the present paper is to deal with the jumps that occur at the times of stochastic discontinuities and to make this technique applicable under the very weak regularity assumption (H) on the Cameron-Martin space.
Expanding the $S$-transform of $F(X_t)$ via the chain rule in Theorem A.1 for the Henstock-Kurzweil integral leads to the following result.

**Proposition 4.2.** Let all assumptions of Theorem 4.1 be in force. Then, for every $h \in W^2$, $\int_0^T (SF'(X_s))(h) \, dh(s)$ exists as Henstock-Kurzweil integral and

\[
(S F(X_T))(h) = (S F(X_0))(h) + \int_0^T (S F'(X_s))(h) \, dh(s) + \frac{1}{2} \int_0^T (S F''(X_s))(h) \, dV(s) + \sum_{s \in D \cap (0,T]} \left( (S F'(X_s))(h) - S(\psi_F(V(s^-) - V^-(s), X_{s^-}))(h) \right) \\
- (S F'(X_s))\Delta^-(h) - \frac{1}{2} (S F''(X_s))\Delta^- V(s) \right) \\
+ \sum_{s \in D \cap (0,T]} \left( S(\psi_F(V(s^+) - V^+(s), X_{s^+}))(h) - (S F(X_s))(h) \right) \\
- (S F'(X_s))\Delta^+(h) - \frac{1}{2} (S F''(X_s))\Delta^+ V(s) \right),
\]

where the two sums converge absolutely.

**Proof.** Recall that, by (7),

\[
(S F(X_t))(h) = \psi_F(V(t), h(t)).
\]

So, we first check that $\psi_F$ satisfies the regularity requirements of Theorem A.1. By the subexponential growth condition (6), we can interchange differentiation and integration and obtain, for every $x \in \mathbb{R}$, $t \in [0,\lambda]$

\[
\frac{\partial}{\partial x}\psi_F(t,x) = \psi_F'(t,x), \quad \frac{\partial^2}{\partial x^2}\psi_F(t,x) = \psi_F''(t,x). \quad (15)
\]

As $\psi_F''$ is continuous on $[0,\lambda] \times \mathbb{R}$, the Lipschitz condition in the first line of (35) is clearly satisfied. Moreover, for $t \in (0,\lambda]$ and $x \in \mathbb{R}$, integration by parts yields

\[
\frac{\partial}{\partial t}\psi_F(t,x) = \frac{1}{2} \int_\mathbb{R} F'(x + \sqrt{\lambda}y) \frac{y}{\sqrt{2\pi t}} e^{-y^2/2} dy = \frac{1}{2} \psi_F''(t,x). \quad (16)
\]

This identity is also valid at $t = 0$, because for every $\epsilon > 0$, by a Taylor
expansion,
\[
\psi_F(\epsilon, x) - \psi_F(0, x) = \int_{\mathbb{R}} F'(x) \frac{y}{\sqrt{2\pi\epsilon}} e^{-y^2/2} dy + \frac{1}{2} \int_{\mathbb{R}} F''(x) \frac{y^2}{\sqrt{2\pi}} e^{-y^2/2} dy + R_x(\epsilon)
\]
\[= \frac{1}{2} F''(x) + R_x(\epsilon)
\]
with remainder term
\[
R_x(\epsilon) = \int_{\mathbb{R}} \int_0^1 (1 - v)(F''(x + v\sqrt{\epsilon}y) - F''(x)) dv \frac{y^2}{\sqrt{2\pi}} e^{-y^2/2} dy,
\]
and by dominated convergence \(R_x(\epsilon)\) tends to zero, as \(\epsilon\) goes to zero, for every \(x \in \mathbb{R}\). In order to show the Hölder-type condition in the second line of (35), we define
\[
K : \mathbb{R} \times [0, \lambda] \times [0, \lambda] \to \mathbb{R}_{\geq 0}, \quad (x, t, s) \mapsto \begin{cases} \frac{\left|\psi_F'(t, x) - \psi_F'(s, x)\right|}{|t - s|^{1/2}}, & t \neq s, \\ 0, & t = s. \end{cases}
\]
Then, by (15),
\[
\left|\frac{\partial}{\partial x} \psi_F(t, x) - \frac{\partial}{\partial x} \psi_F(s, x)\right| = K(x, t, s)|t - s|^{1/2},
\]
and it suffices to show that \(K\) is continuous. However,
\[
\psi_F'(t, x) - \psi_F'(s, x) = \int_{\mathbb{R}} \int_{x + \sqrt{\epsilon}y}^{x + \sqrt{\epsilon}y} F''(r) dr \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy = (\sqrt{t} - \sqrt{s})
\]
\[
\times \int_{\mathbb{R}} \int_0^1 \left(F''(x + y(\sqrt{s}(1 - v) + v\sqrt{t})) - F''(x)\right) dv \frac{y}{\sqrt{2\pi}} e^{-y^2/2} dy.
\]
Thus, for \(t \neq s\),
\[
K(x, t, s) = \frac{|t - s|^{1/2}}{\sqrt{t} + \sqrt{s}}
\]
\[
\times \left|\int_{\mathbb{R}} \int_0^1 \left(F''(x + y(\sqrt{s}(1 - v) + v\sqrt{t})) - F''(x)\right) dv \frac{y}{\sqrt{2\pi}} e^{-y^2/2} dy\right|,
\]
which, by dominated convergence, implies that \(K\) is continuous at every \((x_0, t_0, s_0) \in \mathbb{R} \times ([0, \lambda]^2 \setminus \{(0, 0)\})\). In order to show continuity at \((x_0, 0, 0)\)
for \( x_0 \in \mathbb{R} \), let \((x_n,t_n,s_n)\) be as sequence converging to \((x_0,0,0)\). Then,
\[
|K(x_n,t_n,s_n)| \leq \int_{\mathbb{R}} \int_0^1 \left| F''(x_n + y(\sqrt{s_n}(1 - v) + v\sqrt{t_n})) - F''(x_n) \right| \frac{|y|}{\sqrt{2\pi}} e^{-y^2/2} dy.
\]
which tends to zero by dominated convergence.

We can, thus, apply the chain rule in Theorem A.1 to \(\psi_F(V(t),\mathcal{h}(t))\) for \(h \in W_2^\infty\), and obtain, in view of (15)–(16),
\[
\psi_F(V(T),\mathcal{h}(T)) - \psi_F(V(0),\mathcal{h}(0)) = \int_0^T \psi_F'(V(s),\mathcal{h}(s)) \, dh(s) + \frac{1}{2} \int_0^T \psi_F''(V(s),\mathcal{h}(s)) \, dV(s) + \frac{1}{2} \sum_{s \in (0,T]} \psi_F(V(s^+),\mathcal{h}(s^+)) - \psi_F(V(s),\mathcal{h}(s)) - \psi_F'(V(s),\mathcal{h}(s)) \Delta^- \mathcal{h}(s) - \frac{1}{2} \psi_F''(V(s),\mathcal{h}(s)) \Delta^- V(s),
\]
including the existence of the integral with respect to \(\mathcal{h}\) as Henstock-Kurzweil integral and the absolute convergence of the two sums. As \(\mathcal{h}\) and \(V\) are continuous at \(s\), if \(X\) is stochastically continuous at \(s\), the two sums can be restricted to \((0,T] \cap D_X\) and \([0,T) \cap D_X\), respectively, without changing their values. Finally, each term in the above identity can be rewritten in terms of the \(S\)-transform by an application of Example 2.4, which yields the assertion. \(\square\)

We are now in the position to prove the Itô formula in Theorem 4.1.

**Proof of Theorem 4.1.** We first treat the term, which involves the jumps from the right. For \(s \in D_X \cap [0,T)\), we consider
\[
J_s^+ := \psi_F(V(s^+) - V^+(s),X_{s^+}) - F(X_s) - F'(X_s) \Delta^+ X_s - \frac{1}{2} F''(X_s)(E[(\Delta^+ X_s)^2] + V(s^+) - V^+(s)).
\]
Note that the subexponential growth condition (6) ensures that each \(J_s^+\) belongs to \((L^2_X)\). In order to compute the \(S\)-transform of \(J_s^+\), we note that,
for every $h \in H_X$,

$$S(F'(X_s)\Delta^+ X_s)(h) = E[F'(X_s + h(s))(\Delta^+ X_s + \Delta^+ h(s))]$$

$$= E[F'(X_s + h(s))X_s]E[X_s\Delta^+ X_s]/V(s) + E[F'(X_s + h(s))]\Delta^+ h(s)$$

$$= E[F''(X_s + h(s))]E[X_s\Delta^+ X_s] + E[F'(X_s + h(s))]\Delta^+ h(s)$$

$$= S(F''(X_s))(h)E[X_s\Delta^+ X_s] + S(F'(X_s))(h)\Delta^+ h(s). \quad (17)$$

Here, the first equality is due to (5), the second one follows by projecting $\Delta^+ X_s$ on $\{yX_s; \ y \in \mathbb{R}\}$, the third one by rewriting the expectation as integral with respect to the Gaussian density and by integration by parts. Note that this well-known identity can alternatively be derived from the relation between Wick product and Malliavin derivative, see Nualart (2006, Proposition 1.3.4). As

$$2E[X_s\Delta^+ X_s] + E[(\Delta^+ X_s)^2] + V(s+) - V^+(s) = V(s+) - V(s) = \Delta^+ V(s), \quad (18)$$

we obtain, for every $h \in H_X$

$$S(J^+_s)(h) = S(\psi_F(V(s+) - V^+(s), X_{s+}))(h) - (SF(X_s))(h)$$

$$- (S F'(X_s))\Delta^+ h(s) - \frac{1}{2}(SF''(X_s))\Delta^+ V(s). \quad (19)$$

We next show that the sum of $J^+_s$ converges unconditionally in $(L^2_X)$ as $s$ runs through $D_X \cap [0, T)$. To this end, we decompose $J^+_s = J^+_s,^1 - J^+_s,^2$, where

$$J^+_s,^1 := \psi_F(V(s+) - V^+(s), X_{s+}) - F(X_s) - F'(X_s)\Delta^+ X_s,$$

$$J^+_s,^2 := \frac{1}{2}F''(X_s)E[(\Delta^+ X_s)^2] + V(s+) - V^+(s)).$$

Taylor’s theorem yields for $s \in D_X \cap [0, T)$

$$J^+_s,^1$$

$$= \int_{\mathbb{R}} \left( F(X_{s+} + \sqrt{V(s+)} - V^+(s)y) - F(X_s) - F'(X_s)(\Delta^+ X_s + \sqrt{V(s+)} - V^+(s)y) \right) \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy$$

$$= \int_{\mathbb{R}} \int_0^1 (\Delta^+ X_s + \sqrt{V(s+)} - V^+(s)y)^2 \times (1 - u)F''((1 - u)X_s + u(X_{s+} + \sqrt{V(s+)} - V^+(s)y)) du \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy.$$
We now define $\epsilon := ((4a\lambda)^{-1} - 1)/2$, where $a$ is the constant in the sub-exponential growth condition, let $\epsilon^* := 1/\epsilon$, and abbreviate

$$l(s, y, u) := (1 - u)X_s + u(X_{s+} + \sqrt{V(s)} - V^+(s)y).$$

Then, by Jensen’s inequality, Fubini’s theorem, and Hölder’s inequality,

$$E[|J_s^{+1}|^2]^{1/2} \leq \left( \int_{\mathbb{R}} \int_{1}^1 E[(\Delta^+ X_s + \sqrt{V(s)} - V^+(s)y)^4 |F''(l(s, y, u))|^2] du \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy \right)^{1/2}$$

$$\leq \left( \int_{\mathbb{R}} \int_{0}^1 E[|F''(l(s, y, u))|^{2(1+\epsilon)}] du \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy \right)^{1/(1+\epsilon^*)} \times \left( \int_{\mathbb{R}} E[|\Delta^+ X_s + \sqrt{V(s)} - V^+(s)y|^{4(1+\epsilon^*)}] \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy \right)^{1/(1+\epsilon^*)}$$

The second factor is the square of the $L^{4(1+\epsilon^*)}$-norm of a centered Gaussian random variable with variance $E[(\Delta^+ X_s)^2] + V(s) - V^+(s)$. Hence, there is a constant $d_{\epsilon^*}$ such that

$$\left( \int_{\mathbb{R}} E\left[|\Delta^+ X_s + \sqrt{V(s)} - V^+(s)y|^{4(1+\epsilon^*)}\right] \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy \right)^{2(1+\epsilon^*)} \leq d_{\epsilon^*}(E[(\Delta^+ X_s)^2] + V(s) - V^+(s)).$$

The first factor is bounded by a constant $K_{\epsilon}$ independent of $s$ by the sub-exponential growth condition. Indeed, by convexity,

$$\int_{\mathbb{R}} \int_{0}^1 E[|F''(l(s, y, u))|^{2(1+\epsilon)}] du \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy \leq C^{2(1+\epsilon)} \int_{\mathbb{R}} \max_{u \in \{0, 1\}} E\left[e^{2(1+\epsilon)a|l(s, y, u)|^2}\right] \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy$$

$$\leq C^{2(1+\epsilon)} \left( E\left[e^{2(1+\epsilon)a|X_s|^2}\right] + \int_{\mathbb{R}} E\left[e^{2(1+\epsilon)a|X_{s+} + \sqrt{V(s)} - V^+(s)y|^2}\right] \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy \right)$$

$$\leq C^{2(1+\epsilon)} \sup_{t \in [0, T]} \int_{\mathbb{R}} \left( e^{2(1+\epsilon)aV(t)z^2} + e^{2(1+\epsilon)aV(t+z)^2}\right) \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz =: K_{\epsilon}^{2(1+\epsilon)} < \infty.$$
Thus,
\[ E[J_s^+ 1^2]/2 \leq K_\epsilon d_\epsilon \cdot (E[(\Delta^+ X_s)^2] + V(s) - V^+(s)). \] (20)

Moreover, we clearly observe that
\[ E[J_s^+ 2^2]/2 \leq \frac{1}{2} \left( \sup_{t \in [0,T]} E[[F''(X_t)]^2] \right) (E[(\Delta^+ X_s)^2] + V(s) - V^+(s)). \] (21)

We can now deduce from (20), (21), and (14) that the sum \( \sum_{s \in D_X \cap [0,T]} J_s^+ \) converges absolutely and, hence, unconditionally, in \( L^2 \). In particular,
\[
S \left( \sum_{s \in D_X \cap [0,T]} J_s^+ \right) (h) = \sum_{s \in D_X \cap [0,T]} S(J_s^+)(h), \quad h \in W^2_2.
\]

Let us summarize the foregoing: The sum \( \sum_{s \in D_X \cap [0,T]} J_s^+ \) converges unconditionally in \( L^2 \) and, due to (19), its S-transform is given by
\[
\sum_{s \in D_X \cap [0,T]} \left( S(\psi_F(V(s) - V^+(s), X_{s+}) - F(X_s) - F'(X_s)\Delta^+ X_s - \frac{1}{2} F''(X_s)(E[(\Delta^+ X_s)^2] + V(s) - V^+(s))) \right) \] (22)

for \( h \in W^2_2 \). The jumps from the left can be treated in the same way. The only difference is that we apply
\[ 2E[X_s \Delta^- X_s] - E[(\Delta^- X_s)^2] - (V(s) - V^-(s)) = V(s) - V(s-) = \Delta^- V(s) \]
instead of (18), which explains the change of sign in front of the second derivative term compared to the corresponding term resulting in the case of the jumps from the right. We finally obtain that
\[
\sum_{s \in D_X \cap [0,T]} \left( F(X_s) - \psi_F(V(s-) - V^-(s), X_{s-}) - F'(X_s)\Delta^- X_s + \frac{1}{2} F''(X_s)(E[(\Delta^- X_s)^2] + V(s-) - V^-(s)) \right) \] (24)

for \( h \in W^2_2 \).
converges unconditionally in \((L^2_X)\) and its \(S\)-transform is given by

\[
\sum_{s \in D_X \cap [0,T]} \left( SF(X_s)(h) - S(\psi_F(V(s-)-V^-(s),X_{s-}))(h) - (SF'(X_s))\Delta^- h(s) - \frac{1}{2}(SF''(X_s))\Delta^- V(s) \right)
\]

for \(h \in W^*_2\).

We next discuss the integral with respect to the variance \(V\). The subexponential growth condition (6) again ensures that

\[
\int_0^T E[|F''(X_s)|^2]d|V|(s) < \infty,
\]

where \(|V|\) denotes the total variation of \(V\). Thus, by Fubini’s theorem and Hölder’s inequality,

\[
\int_0^T F''(X_s)dV(s) \in (L^2_X).
\]

Another application of Fubini’s theorem then yields

\[
S(\int_0^T F''(X_s)dV(s))(h) = \int_0^T (SF''(X_s))(h) dV(s),
\]

for every \(h \in W^*_2\).

We can finally combine (22)–(26) with Proposition 4.2 in order to show that the \(S\)-transform of

\[
F(X_T) - F(X_0) - \frac{1}{2} \int_0^T F''(X_s)dV(s)
\]

at every \(h \in W^*_2\) is given by the Henstock-Kurzweil integral

\[
\int_0^T (SF'(X_s))(h) dh(s).
\]

Hence, the Wick-Skorokhod integral \(\int_0^T F'(X_s)d^\diamond X_s\) and the asserted Itô formula is valid. \(\square\)
We close this section with a simplified version of the Itô formula in Theorem 4.1 as announced in (3). To this end, we assume that $X$ is stochastically RCLL, i.e. for every $t \in [0, T)$ and every sequence $(t_n)$ converging to $t$ from the right, $X_{t_n}$ converges to $X_t$ in probability, and, moreover: For every $t \in (0, T]$ there is a random variable $X_t - \Delta X_t$ such that for every sequence $(t_n)$ converging to $t$ from the left, $X_{t_n}$ converges to $X_t - \Delta X_t$ in probability. By Gaussianity, both limits also hold strongly in $(L^2_X)$. In particular, $X$ is weakly regulated with $X_{t+} = X_t$.

**Corollary 4.3.** Suppose the centered Gaussian process $X$ satisfies the following assumptions: $X$ is stochastically RCLL, $W^*_2$ is dense in $H_X$, the variance function $V$ of $X$ is of bounded variation, and

$$\sum_{s \in D_X} E[(\Delta^- X_s)^2] < \infty.$$  

Assume $F \in C^2(\mathbb{R})$ and $F', F''$ satisfy the growth condition (6) with $\lambda = \sup_{t \in [0, T]} V(t)$. Then, $\int_{0+}^T F'(X_s) d\Delta X_s$ exists and the following Itô formula holds in $(L^2_X)$:

$$F(X_T) = F(X_0) + \int_{0+}^T F'(X_s) d\Delta X_s + \frac{1}{2} \int_{0+}^T F''(X_s) dV^c(s)$$

$$+ \sum_{s \in D_X \cap (0, T]} \left( F(X_s) - F(X_{s-}) - F'(X_{s-}) \Delta^- X_s 
+F''(X_{s-}) E[X_{s-}(\Delta^- X_s)] \right).$$

Here, $V^c$ denotes the continuous part of $V$, the set $D_X$ of stochastic discontinuities is at most countable and the sum converges unconditionally in $(L^2_X)$.

Note that $E[\Delta^- X_s] = 0$, if $X$ is martingale. Hence, in view of Theorem 3.4, Corollary 4.3 contains the classical Itô formula (2) for Gaussian martingales as special case. (Recall that the pathwise jumps of a Gaussian martingale occur only at the deterministic times of stochastic discontinuities).

**Proof of Corollary 4.3.** As $X$ is stochastically right-continuous, the rational span of

$$\{X_t; t \in ([0, T) \cap \mathbb{Q}) \cup \{T\}\}$$

is dense in $H_X$. Hence, condition (H) holds for $X$. Moreover, $V^\pm(t) = V(t \pm)$, because $X$ is stochastically RCLL, as already observed at the end.
of Example 2.4. Finally, \( \Delta^+ X_t = 0 \) for every \( t \in [0,T] \). Consequently, Theorem 4.1 is applicable and simplifies in the following way:

\[
F(X_T) - F(X_0) = \int_0^T F'(X_s)d^*X_s + \frac{1}{2} \int_0^T F''(X_s)dV(s)
+ \sum_{s \in D_X \cap (0,T]} (F(X_s) - F(X_s^-) - F'(X_s)\Delta^- X_s + \frac{1}{2} F''(X_s)E[(\Delta^- X_s)^2]).
\]

As \( V \) is rightcontinuous, \( \mu_V \) has no atom at 0. Hence,

\[
\frac{1}{2} \int_0^T F''(X_s)dV(s) = \frac{1}{2} \int_{0^+}^T F''(X_s)dV(s)
= \frac{1}{2} \int_{0^+}^T F''(X_{s^-})dV^c(s) + \frac{1}{2} \sum_{s \in D_X \cap (0,T]} F''(X_s)\Delta^- V(s).
\]

Noting that \( \frac{1}{2}(E[(\Delta^- X_s)^2] + \Delta^- V(s)) = E[X_s\Delta^- X_s] \), we obtain

\[
F(X_T) - F(X_0) - \frac{1}{2} \int_{0^+}^T F''(X_{s^-})dV^c(s)
- \sum_{s \in D_X \cap (0,T]} (F(X_s) - F(X_{s^-}) - F'(X_{s^-})\Delta^- X_s
+ F''(X_{s^-})E[X_{s^-}(\Delta^- X_s)])
= \int_0^T F'(X_s)d^*X_s - \sum_{s \in D_X \cap (0,T]} \left( (F'(X_s) - F'(X_{s^-}))\Delta^- X_s
- F''(X_{s^-})E[X_{s^-}\Delta^- X_s] - F''(X_s)E[X_s\Delta^- X_s] - F''(X_{s^-})E[X_{s^-}\Delta^- X_s]) \right),
\]

where, by the local Lipschitz continuity of \( F' \) and the growth condition on \( F'' \), the sum on the right-hand side can be seen to converge unconditionally in \( (L^2_X) \). We now compute the \( S \)-transform of the right-hand side at \( h \in W_2^* \).

Applying the rightcontinuity of \( h \) and the analogue of (17), we observe that it is given by

\[
\int_{0^+}^T (S F'(X_s))(h) dh(s) - \sum_{s \in D_X \cap (0,T]} (S(F'(X_s) - F'(X_{s^-}))(h)\Delta^- h(s)
= \int_{0^+}^T (S F'(X_{s^-}))(h) dh(s).
\]

Again, we may conclude that the Wick-Skorokhod integral \( \int_{0^+}^T F'(X_{s^-})d^*X_s \) exists and that the asserted Itô formula holds. \( \square \)
5 Regularity in the Cameron-Martin space

In this section, we study the regularity of the elements of the Cameron-Martin space of $X$ as required in condition (H). We will present sufficient conditions in terms of the quadratic variation of $X$ and in terms of integral representations of $X$ with respect to sufficiently regular Gaussian processes. Simple sufficient conditions in terms of the covariance function of $X$ have already been stated in Remark 3.5 (ii).

**Definition 5.1.** We say that $X$ is $W^*_2$-regular, if $CM_X \subset W^*_2([0,T])$, i.e. if every function $h$ in the Cameron-Martin space of $X$ belongs to $W^*_2([0,T])$. Moreover, we call $X$ $W^*_2$-dense, if $CM_X \cap W^*_2([0,T])$ is dense in $CM_X$.

Recall that condition (H) requires $X$ to be $W^*_2$-dense and $H_X$ to be separable. If $W^*_2$-denseness is replaced by the stronger condition of $W^*_2$-regularity, then the Wick-Itô integral in the Itô formula (Theorem 4.1) can be defined with respect to the first chaos, and not only with respect to a dense subset.

As a first basic example we note that every Gaussian martingale $X$ with RCLL paths is $W^*_2$-regular by Remark 3.5 (i). This is, because all elements of its Cameron-Martin space are absolutely continuous with respect to its nondecreasing variance function and, thus, of bounded variation, see the proof of Theorem 3.4.

We next provide a sufficient condition for $X$ to be $W^*_2$-regular in terms of the planar quadratic variation of the covariance function of $X$.

**Proposition 5.2.** Suppose that the variance function $V$ of $X$ is regulated, $X$ is weakly regulated and $H_X$ is separable. Moreover, assume that

$$\sum_{s \in D_X} \left( E[|\Delta^+ X_s|^2] + E[|\Delta^- X_s|^2] \right) < \infty.$$  

Then, $X$ is $W^*_2$-regular, if one of the following equivalent conditions holds:

**(PQV)** For every $\epsilon > 0$ there is a partition $\tau$ of $[0,T]$ such that for every refinement $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ of $\tau$

$$E \left[ \left| \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})^2 - E \left[ (X_{t_i} - X_{t_{i-1}})^2 \right] \right| \right] \leq \epsilon.$$  

**(PQV')** For every $\epsilon > 0$ there is a partition $\tau$ of $[0,T]$ such that for every refinement $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ of $\tau$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} E[(X_{t_i} - X_{t_{i-1}})(X_{t_j} - X_{t_{j-1}})] \leq \epsilon. \quad (27)$$  

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Remark 5.3. Suppose that $X$ is stochastically continuous and recall that we denote the covariance of $X_t$ and $X_s$ by $R(t, s)$. Then, (27) can be rephrased as

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (R(t_i, t_j) + R(t_{i-1}, t_{j-1}) - R(t_i, t_{j-1}) - R(t_{i-1}, t_j))^2 \leq \epsilon.$$ 

Hence, (PQV') means that the planar quadratic variation of $R$ vanishes along the direction of refinement of partitions. The importance of the concept of planar quadratic variation for the development of a stochastic calculus beyond semimartingales has first been emphasized in Russo and Vallois (2000), who introduce the planar quadratic variation in the sense of regularization. Nualart and Taqqu (2008) prove Itô’s formula for centered Gaussian processes with continuous paths by a Wick-Riemann sum approach under the key condition, that the covariance function $R$ of $X$ has zero planar quadratic variation along subclasses of sufficiently uniform partitions, as the mesh size tends to zero.

Proof of Proposition 5.2. Recall that $D_X$ is at most countable by Proposition 2.6. Given an element $g \in H_X$, we denote by $P_2(g) := g^2 - E[g^2]$ the Hermite polynomial of $g$ of degree 2. We first show the equivalence of the two conditions. By hypercontractivity (Janson, 1997, Theorem 5.10), there is a constant $c$ independent of $X$ and $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ such that

$$E \left[ \sum_{i=1}^{n} P_2((X_{t_i} - X_{t_{i-1}+}))^2 \right]^{1/2} \leq c E \left[ \sum_{i=1}^{n} P_2((X_{t_i} - X_{t_{i-1}+})) \right].$$

Hence, the $L^1(\Omega, \mathcal{F}^X, P)$-norm can equivalently be replaced by the $(L^2_X)$-norm in (PQV). However,

$$E \left[ \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}+})^2 - E[(X_{t_i} - X_{t_{i-1}})^2] \right]^2$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} E[P_2((X_{t_i} - X_{t_{i-1}+}))P_2((X_{t_j} - X_{t_{j-1}+}))]$$

$$= 2 \sum_{i=1}^{n} \sum_{j=1}^{n} E[(X_{t_i} - X_{t_{i-1}+})(X_{t_j} - X_{t_{j-1}+})]^2,$$
where the \((L_2^{X})\)-inner product of two Hermite polynomials can be computed by Lemma 1.1.1 in Nualart (2006). Consequently, (PQV) and (PQV') are equivalent.

We now fix \(h \in H_X\). Given a partition \(\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}\), we obtain, by (7), Hölder’s inequality, and (28),

\[
\sum_{i=1}^{n} (h(t_i) - h(t_{i-1})^2 = \sum_{i=1}^{n} E[\exp^{\circ}(h) P_2((X_{t_i} - X_{t_{i-1}}))]
\]

\[
\leq E(|\exp^{\circ}(h)|^2)^{1/2} E \left[ \sum_{i=1}^{n} P_2((X_{t_i} - X_{t_{i-1}})) \right]^{1/2}
\]

\[
\leq c E(|\exp^{\circ}(h)|^2)^{1/2} E \left[ \sum_{i=1}^{n} P_2((X_{t_i} - X_{t_{i-1}})) \right].
\]

Hence, by (PQV), for every \(\epsilon > 0\) there is a partition \(\tau\) of \([0, T]\) such that for every refinement \(\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}\) of \(\tau\)

\[
\sum_{i=1}^{n} (h(t_i) - h(t_{i-1}))^2 \leq \epsilon.
\]

By the equivalent characterization of \(W_2^*\) in Norvaiša (2002, Lemma 4.1), it now suffices to show that

\[
\sum_{s \in (0,T)} |\Delta^{-} h(s)|^2 + \sum_{s \in [0,T)} |\Delta^{+} h(s)|^2 < \infty. 
\]

Both sums can be handled in the very same way, so we only consider the jumps from the left. We first note that

\[
E \left[ \sum_{s \in D_X \cap (0,T)} P_2(\Delta^{-} X_s) \right]^{2} \leq E \left[ \sum_{s \in D_X \cap (0,T)} P_2(\Delta^{-} X_s) \right]^{2}^{1/2}
\]

\[
= \sqrt{2} \sum_{s \in D_X \cap (0,T)} E[(\Delta^{-} X_s)^2] < \infty.
\]

Recalling that the first sum in (31) can be restricted to \(s \in D_X \cap (0,T]\), because \(h\) is continuous at the points of stochastic continuity of \(X\), the same argument as in (29) yields

\[
\sum_{s \in (0,T]} |\Delta^{-} h(s)|^2 \leq E(|\exp^{\circ}(h)|^2)^{1/2} E \left[ \sum_{s \in D_X \cap (0,T]} P_2(\Delta^{-} X_s) \right]^{2}^{1/2} < \infty.
\]
Corollary 5.4. Suppose $X$ is stochastically continuous and the covariance function $R$ of $X$ has bounded planar variation, i.e. there is a constant $K > 0$ such that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} |R(t_i, t_j) + R(t_{i-1}, t_{j-1}) - R(t_{i-1}, t_j) - R(t_i, t_{j-1})| \leq K$$

for every partition $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ of $[0, T]$. Then, $X$ is $W_{2}^{*}$-regular.

Proof. Stochastic continuity of $X$ implies that the covariance function $R : [0, T]^2 \to \mathbb{R}$ is continuous. Hence, for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\max_{t_i, t_j = 1, \ldots, n} |R(t_i, t_j) + R(t_{i-1}, t_{j-1}) - R(t_i, t_{j-1}) - R(t_{i-1}, t_j)| \leq \epsilon/K,$$

if the mesh $|\pi|$ of $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ is lesser than $\delta$. For such partitions $\pi$, we clearly obtain that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (R(t_i, t_j) + R(t_{i-1}, t_{j-1}) - R(t_i, t_{j-1}) - R(t_{i-1}, t_j))^2 \leq \epsilon.$$

Hence, Proposition 5.2 concludes in conjunction with Remark 5.3. □

Remark 5.5. In view of Remark 3.1 in Kruk et al. (2007), all the examples in Section 4.1–4.5 of the latter reference satisfy the conditions of Corollary 5.4. These include bifractional Brownian motion in the parameter range $2HK \geq 1$ and continuous Gaussian processes with stationary increments, whose variance function $V$ is differentiable with absolutely continuous derivative and satisfies $V(0) = 0$.

The next theorem states that an RCLL centered Gaussian process is $W_{2}^{*}$-regular, if the continuous part of the quadratic variation is deterministic and it has fixed discontinuities only. Here is the precise statement.

Theorem 5.6. Suppose $X$ has RCLL paths and that jumps of the paths of $X$ only take place at the times of stochastic discontinuities $s \in D_X$. If

$$\sum_{s \in D_X \cap (0, T]} E[(\Delta X_s)^2] < \infty$$

then $X$ is $W_{2}^{*}$-regular.

□
and there is a deterministic function $v : [0, T] \to \mathbb{R}$ such that for every $t \in [0, T]$

$$\sum_{i=1}^{n} (X_{t_i \wedge t} - X_{t_{i-1} \wedge t})^2 \to v(t) + \sum_{s \in (0, t]} (\Delta X_s)^2$$

in probability as the mesh size of the partition $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ tends to zero, then $X$ is $W^*_2$-regular.

Proof. The assumptions guarantee that

$$\sum_{s \in (0, t]} (\Delta X_s)^2 = \sum_{s \in (0, t] \cap D_X} (\Delta X_s)^2$$

and that this sum converges absolutely in $(L_2^X)$, because

$$\sum_{s \in (0, t] \cap D_X} E[|\Delta X_s|^2]^2]^{1/2} = \sqrt{3} \sum_{s \in (0, t] \cap D_X} E[|\Delta X_s|^2] < \infty.$$

Let $(\pi_k)$ be a sequence of partitions of $[0, T]$, whose mesh size tends to zero. Then, by Theorem 3.50 in Janson (1997)

$$\sum_{t_i \in \pi_k \setminus \{0\}} (X_{t_i} - X_{t_{i-1}})^2 - \sum_{s \in D_X \cap (0, T]} (\Delta X_s)^2 \to v(T) \quad (32)$$

in $(L_2^X)$ as $k$ goes to infinity. We now again fix some arbitrary $h \in H_X$. As shown in the proof of Proposition 5.2, it holds that

$$\sum_{s \in (0, T]} |\Delta h(s)|^2 = \sum_{s \in (0, T] \cap D_X} |\Delta h(s)|^2 < \infty.$$

Arguing as in (29), we thus obtain

$$\left| \sum_{t_i \in \pi_k \setminus \{0\}} (h(t_i) - h(t_{i-1}))^2 - \sum_{s \in (0, T]} |\Delta h(s)|^2 \right|$$

$$= \left| E \left[ \exp^\diamond(h) \left( \sum_{t_i \in \pi_k \setminus \{0\}} P_2(X_{t_i} - X_{t_{i-1}}) - \sum_{s \in (0, T] \cap D_X} P_2(\Delta X_s) \right) \right] \right|$$

$$\leq E[|\exp^\diamond(h)|^2]^{1/2}$$

$$\times E \left[ \left( \sum_{t_i \in \pi_k \setminus \{0\}} P_2(X_{t_i} - X_{t_{i-1}}) - \sum_{s \in (0, T] \cap D_X} P_2(\Delta X_s) \right)^2 \right]^{1/2}.$$
Taking expectation in (32) yields
\[
\sum_{t_i \in \pi_k \setminus \{0\}} E[(X_{t_i} - X_{t_{i-1}})^2] - \sum_{s \in D_X \cap (0, T]} E[(\Delta X_s)^2] \to v(T)
\]
and, hence,
\[
\sum_{t_i \in \pi_k \setminus \{0\}} P_2(X_{t_i} - X_{t_{i-1}}) - \sum_{s \in (0, T]} P_2(\Delta X_s) \to 0
\]
in \((L^2_X)\). Thus,
\[
\lim_{k \to \infty} \sum_{t_i \in \pi_k \setminus \{0\}} (h(t_i) - h(t_{i-1}))^2 = \sum_{s \in (0, T]} |\Delta h(s)|^2 = \sigma^2_h,
\]
where we apply the right-continuity of \(h\) for the last identity. This convergence along the direction of mesh size clearly implies convergence along the direction of refinement of partitions, as required in the definition of \(W_2^\ast\).

We now derive sufficient conditions for \(X\) to be \(W_2^\ast\)-dense in terms of integral representations with respect to another Gaussian process \(W\). To this end we assume that \((\Omega, \mathcal{F}, P)\) carries \((X_t)_{t \in [0,T]}\) and another centered Gaussian process \((W_t)_{t \in \mathbb{R}}\) such that \(W_{t_0} = 0\) for some \(t_0 \in \mathbb{R}\). We denote by \(H_W\) the first chaos of \(W\), i.e. the closure of \(\text{span}(W_t, t \in \mathbb{R})\) in \(L^2(\Omega, \mathcal{F}, P)\). Following the standard construction of the Wiener integral with respect to \(W\), we endow the set of simple functions \(E := \text{span}\{1_{(a,b]}: a, b \in \mathbb{R}\}\) with the inner product induced by \(\langle 1_{(a,b]}, 1_{(c,d]} \rangle_{H_W} := E[(W_b - W_a)(W_d - W_c)]\), see e.g. Huang and Cambanis (1978). Here, we identify, of course, \(f \in E\) with the equivalence class \([f] := \{g \in E: \langle f - g, f - g \rangle_{H_W} = 0\}\). We call the closure \(\mathcal{H}_W\) of \(E\) with respect to this inner product the space of deterministic integrands with respect to \(W\). Then, the mapping
\[
1_{(a,b]} \mapsto W_b - W_a
\]
extends to a linear isometry \(I_W : \mathcal{H}_W \to H_W\), which is known as the Wiener integral with respect to \(W\).

For the remainder of this section we shall assume that there are deterministic integrands \(g_t \in \mathcal{H}_W\), \(t \in [0, T]\), such that \(X\) admits the following representation as Wiener integral with respect to \(W\).
\[
X_t = I_W(g_t), \quad t \in [0, T]. \tag{33}
\]
In particular, $H_X$ is then a closed subspace of $H_W$ and we denote by $\pi_{H_X}$ the orthogonal projection on $H_X$. We also extend $g_t$ to $t \in \mathbb{R}$ by setting $g_t := g_T$ for $t > T$ and $g_t := g_0$ for $t < 0$.

**Proposition 5.7.** Suppose $\mathfrak{A}$ is a dense subset of $\mathcal{H}_W$. Then, the functions of the form
\[
[0, T] \to \mathbb{R}, \ t \mapsto E[X_t I_W(f)], \ f \in \mathfrak{A}
\]
constitute a dense subset of the Cameron-Martin space $CM_X$ of $X$.

**Proof.** For $f \in \mathfrak{A}$ let $f := \pi_{H_X} (I_W(f))$. Then, $E[X_t I_W(f)] = E[X_t f]$ is indeed an element of the Cameron-Martin space of $X$. As $\{I_W(f); f \in \mathfrak{A}\}$ is dense in $H_W$, by the isometry for the Wiener integral, then clearly $\{\pi_{H_X} (I_W(f)); f \in \mathfrak{A}\}$ is dense in $H_X$, which completes the proof.

The following theorem provides a sufficient condition for $X$ to be $W_2^*$-dense in terms of the integral kernel $g_t$ in the representation (33), provided the integrator $W$ is $W_2^*$-regular.

**Theorem 5.8.** Suppose $(W_t)_{t \in \mathbb{R}}$ is a centered Gaussian process with $W_{t_0} = 0$ for some $t_0 \in \mathbb{R}$. Moreover assume that the restriction $(W_t)_{t \in [0, T]}$ is $W_2^*$-regular and $X = (X_t)_{t \in [0, T]}$ admits an integral representation of the form (33). Then $X$ is $W_2^*$-dense, if one of the following two equivalent conditions is satisfied:

1. **(IR)** The linear map $K : \mathcal{H}_W \supset \mathcal{E} \to \mathcal{H}_W$, $1_{(a, b]} \mapsto g_b - g_a$ is closable, i.e.:
   
   If a sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{E}$ satisfies $f_n \to 0$ in $\mathcal{H}_W$ and $K f_n \to f$ in $\mathcal{H}_W$, then $f = 0$.

2. **(IR')** The adjoint operator $K^*$ of the linear map $K : \mathcal{H}_W \supset \mathcal{E} \to \mathcal{H}_W$, $1_{(a, b]} \mapsto g_b - g_a$ is densely defined in $\mathcal{H}_W$.

**Proof.** Theorem VII.2.3 in Yosida (1995) yields the equivalence of conditions (IR) and (IR'). Now suppose that (IR') is in force. Then, the set $D(K^*)$ consisting of those $f \in \mathcal{H}_W$, such that there is an $K^* f \in \mathcal{H}_W$ satisfying
\[
\langle K^* f, h \rangle_{\mathcal{H}_W} = \langle f, K^* h \rangle_{\mathcal{H}_W}
\]
for every $h \in \mathcal{E}$, is dense in $\mathcal{H}_W$. Then, for every $f \in D(K^*)$ and $t \in [0, T]$, by the isometry for Wiener integrals,
\[
E[X_t I_W(f)] - E[X_0 I_W(f)] = \langle K 1_{(0,t]}, f \rangle_{\mathcal{H}_W} = \langle 1_{(0,t]}, K^* f \rangle_{\mathcal{H}_W}
= E[W_t I_W(K^* f)] - E[W_0 I_W(K^* f)].
\]
The map
\[ [0, T] \to \mathbb{R}, \; t \mapsto E[W_t I_W(K^*f)] + (E[X_0 I_W(f)] - E[W_0 I_W(K^*f)]) \]
is up to a constant an element of the Cameron-Martin space of \((W_t)_{t \in [0,T]} \) (see Proposition 5.7) and, thus, by assumption of class \(W_2^2\). As \(D(K^*)\) is dense in \(\mathcal{H}_W\), Proposition 5.7 concludes.

**Remark 5.9.** A typical application of Theorem 5.8 is when \(X\) has an integral representation
\[ X_t = \int_{0+}^T g_t(s) dM_s, \quad t \in [0, T] \quad (34) \]
on the interval \([0, T]\) with respect to an RCLL Gaussian martingale \(M\) with \(M_0 = 0\). Then we can set \(W_t = 1_{[0,T]}(t)M_t\) and obtain \(\mathcal{H}_W = L^2([0,T],dV_M)\), where \(V_M\) denotes the RCLL nondecreasing variance function of \(M\). In this case, (IR') can be reformulated as follows: The set \(D(K^*)\), consisting of those \(f \in L^2([0,T],dV_M)\) such that there is an \(K^*f \in L^2([0,T],dV_M)\) satisfying
\[ \int_0^T (g_t(s) - g_0(s))f(s) dV_M(s) = \int_0^t K^*f(s) dV_M(s) \]
for every \(t \in [0, T]\), is dense in \(L^2([0,T],dV_M)\). Hence, a dense subset of the Cameron-Martin space of \(X\) is absolutely continuous with respect to \(dV_M\) with a square integrable density with respect to \(dV_M\). We leave it to the reader to simplify the closability condition (IR) in the obvious way.

### 6 Comparison to the literature

We finally explain how condition (H) can be verified in the literature on Itô’s formula for Gaussian processes in the Skorokhod sense. In all the cases discussed below, the authors assume or show that the variance function \(V\) of \(X\) is of bounded variation and continuous:

1. Alòs et al. (2001): The authors assume or show that the variance function \(V\) of \(X\) is of bounded variation and continuous:
   - **1.** Alòs et al. (2001): The authors assume an integral representation of the form \(X_t = \int_0^T g_t(s) dM_s\) with respect to a Brownian motion \(M\). In the more general ‘singular’ case (Alòs et al., 2001, Theorem 1), condition \((K3)\) entails that \(t \mapsto E[X_t I_W(f)]\) is continuous and of bounded variation for every step function \(f\). Hence, by Proposition 5.7, \(X\) is \(W_2^2\)-dense and, moreover, \(X\) is stochastically continuous (Janson, 1997, Theorem 8.21). In particular, condition (H) is satisfied.
2. Mocioalca and Viens (2005): Again an integral representation of the form (34) with respect to a Brownian motion $M$ is supposed. Proposition 15 in Mocioalca and Viens (2005) shows that condition (IR') of Theorem 5.8 is satisfied in the variant of Remark 5.9. Hence, a dense subset of the Cameron-Martin space of $X$ is absolutely continuous with respect to the Lebesgue measure with square integrable density. This again implies that (H) holds and $X$ is stochastically continuous.

3. Nualart and Taqqu (2006): In this reference $X$ is supposed to have continuous paths. The key assumption on $X$ is stated in terms of the planar quadratic variation. As indicated in Remark 5.3, their condition is very closely related (although not identical) to our condition (PQV') in Proposition 5.2, which together with the path continuity implies that $X$ is stochastically continuous and satisfies (H). In our general framework of Gaussian processes with stochastic discontinuities it does not seem to be possible to restrict to sufficiently uniform partitions as suggested in Nualart and Taqqu (2006) in the continuous case.

4. Kruk et al. (2007): The authors assume that $X$ is stochastically continuous and has a covariance measure. By their Remark 3.1 the latter property is equivalent to the property that the covariance function of $X$ is of bounded planar variation. By Corollary 5.4, $X$ is $W^*_2$-regular and, in particular, satisfies (H).

5. Nualart and Taqqu (2008): Here the authors assume that $X$ has continuous paths and satisfies the quadratic variation property in our Theorem 5.6. Hence, $X$ is stochastically continuous, $W^*_2$-regular, and satisfies (H).

6. Lei and Nualart (2012) and Hu et al. (2013): The assumptions in these two references include that, for every $s \in [0, T]$, the covariance function $t \mapsto R(t, s)$ of $X$ is absolutely continuous with respect to the Lebesgue measure. Thus, $X$ is stochastically continuous and satisfies (H) by Remark 3.5 (ii).

7. Alpay and Kipnis (2013): The authors consider a class of Gaussian stationary increment processes and show in their equation (5.1) that the $S$-transform $t \mapsto (SX_t)(h)$ of $X$ is absolutely continuous with respect to the Lebesgue measure for a dense subspace of $H_X$. Hence, by (4), $X$ is $W^*_2$-dense and satisfies (H), because it also is stochastically continuous.
8. Lebovits (2017): $X$ is defined in terms of an integral representation of the form (33) with respect to a two-sided Brownian motion $W$. The conditions on the kernel $g_t$ are such that $t \mapsto E[X_t I_W(f)]$ is absolutely continuous with respect to the Lebesgue measure for every Schwartz function $f$, see Remark 1 in Lebovits (2017). As the Schwartz functions are dense in $\mathcal{D}_W = L^2(\mathbb{R})$, Proposition 5.7 again implies that $X$ is stochastically continuous and that (H) is satisfied.

We emphasize again that in all these references, the Gaussian process $X$ is stochastically continuous and, hence, Itô’s formula has the form (1) without jump terms, while the main contribution of the present paper is to understand the influence of the stochastic discontinuities on the Gaussian Itô formula.

### A A chain rule for the Henstock-Kurzweil integral

In this appendix, we prove a chain rule for the Henstock-Kurzweil integral. The general lines of proof closely follow the arguments in Norvaiša (2002). We have, however, to deal with a case of ‘mixed’ regularity which is not covered there. Here is what we are going to show.

**Theorem A.1.** Suppose $u_1 \in W^*_2([0,T])$, $u_2 \in W_1([0,T])$ (i.e, of bounded variation), and let $u := (u_1, u_2)$. Denote

$$S_i := \left[ \inf_{t \in [0,T]} u_i(t), \sup_{t \in [0,T]} u_i(t) \right], \quad i = 1, 2.$$

Suppose $G \in C^1(S_1 \times S_2; \mathbb{R})$ and such that there is a constant $K_1 \geq 0$ and a continuous function $K: S_1 \times S_2 \times S_2 \to \mathbb{R}_{\geq 0}$ satisfying $K(x_1, x_2, x_2) = 0$ and

$$\left| \frac{\partial G}{\partial x_1} (x_1, x_2) - \frac{\partial G}{\partial x_1} (y_1, x_2) \right| \leq K_1 |x_1 - y_1|,$$

$$\left| \frac{\partial G}{\partial x_1} (x_1, x_2) - \frac{\partial G}{\partial x_1} (x_1, y_2) \right| \leq K(x_1, x_2, y_2) |x_2 - y_2|^{1/2}$$

for every $x_1, y_1 \in S_1$ and $x_2, y_2 \in S_2$. Then, $\int_0^T \frac{\partial G}{\partial x_1}(u(s)) \, du_1(s)$ exists as
Henstock-Kurzweil integral and

\[ G(u(T)) - G(u(0)) = \int_0^T \frac{\partial G}{\partial x_1}(u(s)) \, du_1(s) + \int_0^T \frac{\partial G}{\partial x_2}(u(s)) \, du_2(s) \]

\[ + \sum_{s \in (0,T]} G(u(s)) - G(u(s-)) - \frac{\partial G}{\partial x_1}(u(s)) \Delta^- u_1(s) - \frac{\partial G}{\partial x_2}(u(s)) \Delta^- u_2(s) \]

\[ + \sum_{s \in [0,T)} G(u(s+)) - G(u(s)) - \frac{\partial G}{\partial x_1}(u(s)) \Delta^+ u_1(s) - \frac{\partial G}{\partial x_2}(u(s)) \Delta^+ u_2(s). \]

Here, both sums converge absolutely.

Note first that the integral with respect to \( u_2 \) exists as Lebesgue-Stieltjes integral, because \( u_2 \) is of bounded variation and the integrand is regulated. The chain rule above, thus, holds as a consequence of Theorem 4.2 (with \( \alpha = 1 \)) in Norvaiša (2002) under the stronger Lipschitz condition

\[ \left| \frac{\partial G}{\partial x_1}(x_1, x_2) - \frac{\partial G}{\partial x_1}(y_1, y_2) \right| + \left| \frac{\partial G}{\partial x_2}(x_1, x_2) - \frac{\partial G}{\partial x_2}(y_1, y_2) \right| \leq K_1(|x_1 - y_1| + |x_2 - y_2|), \]

which is not satisfied in our application of this chain rule in Section 4.

As in Norvaiša (2002), we shall show that the chain rule is valid in the sense of Young-Stieltjes integration and then exploit the relationship between the Young-Stieltjes integral and the Henstock-Kurzweil integral as stated in Dudley and Norvaiša (1999, Part I, Theorem F.2). We, thus, first recall the construction of the Young-Stieltjes integral.

A Young-tagged partition \( \tau := \{(s_{i-1}, s_i), y_i; i = 1, \ldots, n\} \) of the interval \([0,T]\), by definition, satisfies \( 0 = s_0 < s_1 < \ldots < s_n = T \) and \( y_i \in (s_{i-1}, s_i) \), whereas in a tagged partition the tag point \( y_i \) lies in the closed interval \([s_{i-1}, s_i]\). Given such a Young-tagged partition \( \tau \) and regulated functions \( u, r : [0,T] \to \mathbb{R} \), one considers the Young-Stieltjes sums

\[ S_{YS}(u, r, \tau) := \sum_{i=1}^n u(s_{i-1}) \Delta^+ r(s_{i-1}) + u(y_i)(r(s_i) - r(s_{i-1} + )) + u(s_i) \Delta^- r(s_i) \]

The Young-Stieltjes integral of \( u \) with respect to \( r \) is said to exist, if there is a real number \( I \) such that, for every \( \epsilon > 0 \), there is a Young-tagged partition \( \chi \) such that for every refinement \( \tau \) of \( \chi \)

\[ |S_{YS}(u, r, \tau) - I| < \epsilon. \]
In this case, \( I \) is defined to be the value of this integral. In the above, a Young-tagged partition \( \tau = \{(s_{i-1}, s_i), y_i); i = 1, \ldots, n\} \) is said to be a refinement of a Young-tagged partition \( \chi \), if the partition \( \kappa(\tau) := \{s_0, \ldots, s_n\} \) is a refinement of the partition \( \kappa(\chi) \), i.e. \( \kappa(\tau) \supset \kappa(\chi) \).

**Proof of Theorem A.1.** Define \( S := S_1 \times S_2, S' := S_1 \times S_2 \times S_2 \). We consider for every Young-tagged partition \( \tau := \{(s_{i-1}, s_i), y_i); i = 1, \ldots, n\} \)

\[
V^+(	au) = \sum_{i=0}^{n-1} G(u(s_{i+})) - G(u(s_i)) - \frac{\partial G}{\partial x_1}(u(s_i))\Delta^+ u_1(s_i)
- \frac{\partial G}{\partial x_2}(u(s_i))\Delta^+ u_2(s_i)
\]

\[
V^-(\tau) = \sum_{i=1}^{n} G(u(s_i)) - G(u(s_i-)) - \frac{\partial G}{\partial x_1}(u(s_i))\Delta^- u_1(s_i)
- \frac{\partial G}{\partial x_2}(u(s_i))\Delta^- u_2(s_i)
\]

\[
R(\tau) = \sum_{i=1}^{n} G(u(s_i-)) - G(u(s_i-1+)) - \frac{\partial G}{\partial x_1}(u(y_i))
\times (u_1(s_i-)) - u_1(s_i-1+)) - \frac{\partial G}{\partial x_2}(u(y_i))(u_2(s_i-) - u_2(s_i-1+))
\]

Then,

\[
S_{YS}\left(\frac{\partial G}{\partial x_1}(u), u_1, \tau\right) = G(u(T)) - G(u(0)) - S_{YS}\left(\frac{\partial G}{\partial x_2}(u), u_2, \tau\right)
- V^+(\tau) - V^-(\tau) - R(\tau).
\]  

We fix an arbitrary \( \epsilon > 0 \). As \( \frac{\partial G}{\partial x_2}(u) \) is regulated and \( u_2 \) is of bounded variation, the Young-Stieltjes integral of \( \frac{\partial G}{\partial x_2} \circ u \) with respect to \( u_2 \) exists and coincides with the Henstock-Kurzweil integral and the Lebesgue-Stieltjes integral by Theorems I.4.2, I.F.2 and Corollary II.3.20 in Dudley and Norvaiša (1999). Hence, there is a Young-tagged partition \( \chi \) of \([0, T]\) such that for all refinements \( \tau \) of \( \chi \)

\[
\left| S_{YS}\left(\frac{\partial G}{\partial x_2}(u), u_2, \tau\right) - \int_0^T \frac{\partial G}{\partial x_2}(u(s)) du_2(s) \right| < \epsilon/4.
\]  

We next treat the jumps from the right. Suppose \( s \in [0, T] \) such that \( \Delta^+ u_l(s) \neq 0 \) for \( l = 1 \) or \( l = 2 \). Note that the set of such time points \( s \) is at most countable, since \( u_1 \) and \( u_2 \) are regulated. By the mean value
theorem there are \( \theta_l \in [u_l(s) \wedge u_l(s+), u_l(s) \vee u_l(s+)] \), \( l = 1, 2 \) such that for \( \theta = (\theta_1, \theta_2) \)

\[
G(u(s+)) - G(u(s)) = \frac{\partial G}{\partial x_1}(\theta)\Delta^+ u_1(s) + \frac{\partial G}{\partial x_2}(\theta)\Delta^+ u_2(s).
\]

Define

\[
V_s^+ := G(u(s+)) - G(u(s)) - \frac{\partial G}{\partial x_1}(u(s))\Delta^+ u_1(s) - \frac{\partial G}{\partial x_2}(u(s))\Delta^+ u_2(s).
\]

Let \( K_2 := \max_{(x_1,x_2,\tilde{x}_2) \in S'} K(x_1, x_2, \tilde{x}_2) \) and \( K_3 := 2 \max_{(x_1,x_2) \in S} |\frac{\partial G}{\partial x_2}(x_1, x_2)| \).

Then, by (35) and Young’s inequality,

\[
|V_s^+| \leq |\Delta^+ u_1(s)|(K_1|u_1(s) - \theta_1| + K_2|u_2(s) - \theta_2|^{1/2}) + K_3|\Delta^+ u_2(s)|
\leq (K_1 + K_2/2)|\Delta^+ u_1(s)|^2 + (K_3 + K_2/2)|\Delta^+ u_2(s)|.
\]

Hence, for some constant \( \tilde{K} > 0 \)

\[
\sum_{s \in [0,T)} |V_s^+| \leq \tilde{K} \left( \sum_{s \in [0,T)} |\Delta^+ u_1(s)|^2 + \sum_{s \in [0,T)} |\Delta^+ u_2(s)| \right) < \infty,
\]

because \( u_1 \in W_2^s \) and \( u_2 \) is of bounded variation. Thus, the sum over the jumps from the right in the asserted chain rule converges absolutely. We can, then, find a finite subset \( \mu \subset [0,T) \) such that

\[
\sum_{s \in [0,T) \setminus \mu} |V_s^+| \leq \epsilon/4.
\]

By passing to a refinement, if necessary, we can assume without loss of generality, that \( \mu \subset \kappa(\chi) \). Then, for every refinement \( \tau \) of \( \chi \)

\[
\left| V^+(\tau) - \sum_{s \in [0,T)} G(u(s+)) - G(u(s)) - \frac{\partial G}{\partial x_1}(u(s))\Delta^+ u_1(s) \right.
\left. - \frac{\partial G}{\partial x_2}(u(s))\Delta^+ u_2(s) \right| \leq \epsilon/4.
\]

(38)

By the same argument, the sum over the jumps from the left converges absolutely and for every refinement \( \tau \) of \( \chi \)

\[
\left| V^-(\tau) - \sum_{s \in [0,T]} G(u(s)) - G(u(s-)) - \frac{\partial G}{\partial x_1}(u(s))\Delta^- u_1(s) \right.
\left. - \frac{\partial G}{\partial x_2}(u(s))\Delta^- u_2(s) \right| \leq \epsilon/4.
\]

(39)
It remains to treat the remainder term $R(\tau)$. Again, by the mean value theorem, there are $\theta_{l,i} \in [u_l(s_i-) \wedge u_l(s_{i-1}+), u_l(s_i-) \lor u_l(s_{i-1}+)]$, $l = 1, 2$, $i = 1, \ldots, n$, such that for $\theta_i := (\theta_{1,i}, \theta_{2,i})$, $i = 1, \ldots, n$,

$$R(\tau) = \sum_{i=1}^{n} \left( \frac{\partial G}{\partial x_1}(\theta_i) - \frac{\partial G}{\partial x_1}(u(y_i)) \right) (u_1(s_i-) - u_1(s_{i-1}+))$$

$$+ \sum_{i=1}^{n} \left( \frac{\partial G}{\partial x_2}(\theta_i) - \frac{\partial G}{\partial x_2}(u(y_i)) \right) (u_2(s_i-) - u_2(s_{i-1}+))$$

$$=: (I) + (II).$$

In order to estimate these two terms separately, we need some extra notation. We write

$$\text{Osc}(u, E) := \max_{i=1,2} \sup \{|u_i(s) - u_i(t)|; s, t \in E\}, \quad E \subset [0,T],$$

for the oscillation of $u$ over the set $E$. We denote the total variation of $u_2$ over $[0,T]$ by $v_1(u_2)$ and the 2-variation of $u_1$ over the open interval $(s_{i-1}, s_i)$ by $v_2(u_1, (s_{i-1}, s_i))$, i.e.

$$v_2(u_1, (s_{i-1}, s_i))$$

$$:= \sup \left\{ \sum_{j=1}^{m} |u_1(t_j) - u_1(t_{j-1})|^2; m \in \mathbb{N}, s_{i-1} < t_0 < t_1 < \cdots < t_m < s_i \right\}.$$

Noting that, for every $l = 1, 2$ and $i = 1, \ldots, n$ and $p \geq 1$,

$$|u_l(y_i) - \theta_{l,i}|^p \leq \max \{|u_l(s_i-) - u_l(y_i)|^p, |u_l(y_i) - u_l(s_{i-1}+)|^p\},$$
we obtain, by (35) and Young’s inequality,

\[ |(I)| \leq \sum_{i=1}^{n} K_1 |\theta_{1,i} - u_1(y_i)| |u_1(s_i) - u_1(s_{i-1})| \]

\[ + \sum_{i=1}^{n} K(\theta_{1,i}, \theta_{2,i}, u_2(y_i)) |\theta_{2,i} - u_2(y_i)|^{1/2} |u_1(s_i) - u_1(s_{i-1})| \]

\[ \leq \sum_{i=1}^{n} \frac{K_1 + K_2}{2} |u_1(s_i) - u_1(s_{i-1})|^2 + \frac{K_1}{2} |\theta_{1,i} - u_1(y_i)|^2 \]

\[ + \frac{1}{2} K(\theta_{1,i}, \theta_{2,i}, u_2(y_i)) |\theta_{2,i} - u_2(y_i)| \]

\[ \leq \frac{2K_1 + K_2}{2} \sum_{i=1}^{n} v_2(u_1, (s_{i-1}, s_i)) \]

\[ + \frac{v_1(u_2)}{2} \sup \{ K(x_1, x_2, \tilde{x}_2); \}

\[ (x_1, x_2, \tilde{x}_2) \in S', |x_2 - \tilde{x}_2| \leq \max_{j=1,...,n} \text{Osc}(u, (s_{i-1}, s_i)) \}. \]

Moreover,

\[ |(II)| \leq v_1(u_2) \sup \{ \frac{\partial G}{\partial x_2}(z_1) - \frac{\partial G}{\partial x_2}(z_2); \]

\[ z_1, z_2 \in S, |z_1 - z_2|_\infty \leq \max_{j=1,...,n} \text{Osc}(u, (s_{i-1}, s_i)) \}, \]

where \(| \cdot |_\infty\) denotes the maximum norm in \(\mathbb{R}^2\). By uniform continuity of \(\frac{\partial G}{\partial x_2}\)

on \(S\) and of \(K\) on \(S'\), there is a \(\delta > 0\) such that

\[ \max \left\{ K(x_1, x_2, \tilde{x}_2), \left| \frac{\partial G}{\partial x_2}(x_1, x_2) - \frac{\partial G}{\partial x_2}(\tilde{x}_1, \tilde{x}_2) \right| \right\} \leq \frac{\varepsilon}{16v_1(u_2)} \]

for every \((x_1, x_2), (\tilde{x}_1, \tilde{x}_2) \in S\) with \(|(x_1, x_2) - (\tilde{x}_1, \tilde{x}_2)|_\infty \leq \delta\). As \(u_1\) and \(u_2\)

are regulated, there is a partition \(\lambda = \{ t_0, \ldots, t_m \}\) of \([0, T]\) such that

\[ \max_{j=1,...,m} \text{Osc}(u, (t_{j-1}, t_j)) \leq \delta. \]

Finally, by the equivalent characterization of \(W^*_2\) in Lemma 4.1 of Norvaïsa (2002), there is a partition \(\tilde{\lambda}\) of \([0, T]\) such that for every refinement \(\{x_1, \ldots, x_k\}\) of \(\tilde{\lambda}\)

\[ \sum_{j=1}^{k} v_2(u_1, (x_{j-1}, x_j)) \leq \frac{\varepsilon}{8K_1 + 4K_2}. \]
We may and shall again assume that $\lambda \cup \tilde{\lambda} \subset \kappa(\chi)$, by passing to a refinement of $\chi$, if necessary. Then, for every refinement $\tau$ of $\chi$, 
\[
|R(\tau)| \leq \frac{\epsilon}{4}.
\] (40)

Gathering (36)–(40), we observe that $\int_0^T \frac{\partial G}{\partial x_1}(u(s)) \, du_1(s)$ exists as Young-Stieltjes integral and satisfies the asserted chain rule. By Theorem F.2 in Part I of Dudley and Norvaiša (1999), this integral exists as Henstock-Kurzweil integral and coincides with the Young-Stieltjes integral, provided the integrand belongs to $W^2([0,T])$, i.e. has finite 2-variation over $[0,T]$. However, by (35), there is a constant $\tilde{K} \geq 0$ such that for every $m \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = T$,
\[
\sum_{j=1}^m \left| \frac{\partial G}{\partial x_1}(u(t_j)) - \frac{\partial G}{\partial x_1}(u(t_{j-1})) \right|^2 \leq \tilde{K}(v_2(u_1) + v_1(u_2)) < \infty,
\]
recalling that the $p$-variation was defined in Remark 3.5. 

\[\square\]

References


