DISCRETIZING MALLIAVIN CALCULUS

CHRISTIAN BENDER AND PETER PARCZEWSKI

Abstract. Suppose B is a Brownian motion and B^n is an approximating sequence of rescaled random walks on the same probability space converging to B pointwise in probability. We provide necessary and sufficient conditions for weak and strong L^2-convergence of a discretized Malliavin derivative, a discrete Skorokhod integral, and discrete analogues of the Clark-Ocone derivative to their continuous counterparts. Moreover, given a sequence (X^n) of random variables which admit a chaos decomposition in terms of discrete multiple Wiener integrals with respect to B^n, we derive necessary and sufficient conditions for strong L^2-convergence to a $\sigma(B)$-measurable random variable X via convergence of the discrete chaos coefficients of X^n to the continuous chaos coefficients of X. In the special case of binary noise, our results support the known formal analogies between Malliavin calculus on the Wiener space and Malliavin calculus on the Bernoulli space by rigorous L^2-convergence results.

MSC: 60H07, 60H05, 60F25

Keywords: Malliavin calculus, strong approximation, stochastic integrals, S-transform, chaos decomposition, invariance principle

1. Introduction

Let $B = (B_t)_{t\geq 0}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, where the σ-field $\mathcal{F}$ is generated by the Brownian motion and completed by null sets. Suppose ξ is a square-integrable random variable with zero expectation and variance one. As a discrete counterpart of B we consider, for every $n \in \mathbb{N} = \{1, 2, \ldots\}$, a random walk approximation

$$B_t^n := \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \xi_i^n \ , \ t \geq 0,$$

where $(\xi_i^n)_{i\in\mathbb{N}}$ is a sequence of independent random variables which have the same distribution as ξ. We assume that the approximating sequence B^n converges to B pointwise in probability, i.e.

$$\forall\ t \geq 0 \ : \quad \lim_{n\to\infty} B_t^n = B_t \ \text{ in probability.} \tag{1}$$

The aim of the paper is to provide L^2-approximation results for some basic operators of Malliavin calculus with respect to the Brownian motion B such as the chaos decomposition, the Malliavin derivative, and the Skorokhod integral by appropriate sequences of approximating operators based on the discrete time noise $(\xi_i^n)_{i\in\mathbb{N}}$. It turns out that in all our approximation results, the limits do not depend on the distribution of the discrete time noise, hence our results can be regarded as some kind of invariance principle for Malliavin calculus.

We briefly discuss our main convergence results in a slightly informal way:

(1) *Chaos decomposition:* The heuristic idea behind the chaos decomposition in terms of multiple Wiener integrals is to project a random variable $X \in L^2(\Omega, \mathcal{F}, P)$ on products of the white noise $\dot{B}_{t_1} \cdots \dot{B}_{t_k}$. This idea can be made rigorous with respect to the discrete noise $(\xi_i^n)_{i\in\mathbb{N}}$ by considering the discrete time functions

$$f_X^{n,k}(i_1, \ldots i_k) = \frac{n^{k/2}}{k!}\mathbb{E}\left[X \prod_{j=1}^{k} \xi_{i_j}^n\right]$$

Date: April 6, 2016.

for pairwise distinct $(i_1, \dots i_k) \in \mathbb{N}^k$. Our results show that, after a natural embedding as step functions into continuous time, the sequence $(f_X^{n,k})_{n\in\mathbb{N}}$ converges strongly in $L^2([0,\infty)^k)$ to the kth chaos coefficient of X, for every $k \in \mathbb{N}$ (Example 34). This is a simple consequence of a general Wiener chaos limit theorem (Theorem 28), which provides equivalent conditions for the strong $L^2(\Omega, \mathcal{F}, P)$-convergence of a sequence of random variables $(X^n)_{n\in\mathbb{N}}$ (with each X^n admitting a chaos decomposition via multiple Wiener integrals with respect to the discrete time noise $(\xi_i^n)_{i\in\mathbb{N}}$) in terms of the chaos coefficient functions. As a corollary, this Wiener chaos limit theorem lifts a classical result by Surgailis [28] on convergence in distribution of discrete multiple Wiener integrals to strong $L^2(\Omega, \mathcal{F}, P)$-convergence and adds the converse implication (in our setting, i.e. when the limiting multiple Wiener integral is driven by a Brownian motion).

(2) *Malliavin derivative:* With our weak moment assumptions on the discrete time noise, we cannot define a discrete Malliavin derivative in terms of a polynomial chaos as in the survey paper by [11] and the references therein. Instead we think of the discretized Malliavin derivative at time $j \in \mathbb{N}$ with respect to the noise $(\xi_i^n)_{i\in\mathbb{N}}$ as

$$D_j^n X = \sqrt{n}\mathbb{E}[\xi_j^n X | (\xi_i^n)_{i\in\mathbb{N}\setminus\{j\}}],$$

which is the gradient of the best approximation in $L^2(\Omega, \mathcal{F}, P)$ of X as a linear function in ξ_j^n with $\sigma(\xi_i^n,\ i \in \mathbb{N} \setminus \{j\})$-measurable coefficients. In the case of binary noise, this definition coincides with the standard notion of the Malliavin derivative on the Bernoulli space, see, e.g., [25]. Theorem 12 below implies that, if (X^n) converges weakly in $L^2(\Omega, \mathcal{F}, P)$ to X and the sequence of discretized Malliavin derivatives $(D_{\lceil n\cdot\rceil}^n X^n)_{n\in\mathbb{N}}$ converges weakly in $L^2(\Omega \times [0,\infty))$, then X belongs to the domain of the continuous Malliavin derivative and the continuous Malliavin derivative appears as the weak $L^2(\Omega \times [0,\infty))$-limit. As the Malliavin derivative is a closed, but discontinuous operator, this is the best type of approximation result which can be expected when discretizing the Malliavin derivative. Sufficient conditions for the strong convergence of a sequence of discretized Malliavin derivatives, which can be checked in terms of the discrete-time approximations, are presented in Theorems 16 and 35.

(3) *Skorokhod integral:* Defining the discrete Skorokhod integral as the adjoint operator to the discretized Malliavin derivative leads to

$$\delta^n(Z^n) := \lim_{M\to\infty} \sum_{i=1}^{M} \mathbb{E}[Z_i^n | (\xi_j^n)_{j\in\{1,\dots,M\}\setminus\{i\}}] \frac{\xi_i^n}{\sqrt{n}},$$

for a suitable class of discrete time processes Z^n, which is in line with the Riemann-sum approximation for Skorokhod integrals in terms of the driving Brownian motion in [24]. Analogous results for the 'closedness across the discretization levels' as in the case of the discretized Malliavin derivative and sufficient conditions for strong $L^2(\Omega, \mathcal{F}, P)$-convergence of a sequence of discrete Skorokhod integrals are provided in Theorems 8, 18 and 36. When restricted to predictable integrands, the convergence results for the Skorokhod integral give rise to necessary and sufficient conditions for strong and weak $L^2(\Omega, \mathcal{F}, P)$-convergence of a sequence of discrete Itô integrals (Theorem 20). This result can be applied to study different discretization schemes for the generalized Clark-Ocone derivative (which provides the integrand in the predictable representation of a square-integrable random variable as Itô integral with respect to the Brownian motion B). In this respect, Theorems 23 and 25 below complement related results in the literature such as [6, 20] and the references therein.

We note that related classical semimartingale limit theorems for stochastic integrals (with adapted integrands) [15, 19] and for multiple Wiener integrals [2, 3, 28], or robustness results for martingale representations [6, 14] are usually obtained in the framework of (or using techniques of) convergence in distribution (on the Skorokhod space). In contrast, we exploit that strong and weak convergence in $L^2(\Omega, \mathcal{F}, P)$ can be characterized in terms of the S-transform,

which is an important tool in white noise analysis, see e.g. [13,16,18], and corresponds to taking expectation under suitable changes of measure. We introduce a discrete time version of the S-transform in terms of the noise $(\xi_i^n)_{i\in\mathbb{N}}$ and show that strong and weak $L^2(\Omega,\mathcal{F},P)$-convergence can be equivalently expressed via convergence of the discrete S-transform to the continuous S-transform (Theorem 1). With this observation at hand, all our convergence results can be obtained in a surprisingly simple way by computing suitable $L^2(\Omega,\sigma(\xi_i^n)_{i\in\mathbb{N}},P)$-inner products and their limits as n tends to infinity.
The paper is organized as follows: In Section 2, we introduce the discrete S-transform and discuss the connections between weak (and strong) $L^2(\Omega,\mathcal{F},P)$-convergence and the convergence of the discrete S-transform to the continuous one. Equivalent conditions for the weak L^2-convergence of sequences of discretized Malliavin derivatives and discrete Skorokhod integrals to their continuous counterparts are derived in Section 3. By combining these weak L^2-convergence results with the duality between discrete Skorokhod integral and discretized Malliavin derivative, we also identify sufficient conditions for the strong L^2-convergence which can be checked solely in terms of the discrete time approximations. We are not aware of any such convergence results for general discrete time noise distributions in the literature. In Section 4, we specialize to the nonanticipating case and prove limit theorems for discrete Itô integrals and discretized Clark-Ocone derivatives. The strong L^2-Wiener chaos limit theorem is presented in Section 5, and is applied in order to provide equivalent conditions for the strong L^2-convergence of sequences of discretized Malliavin derivatives and discrete Skorokhod integrals in terms of tail conditions of the discrete chaos coefficients in Section 6. In the final Section 7, we first the consider the special case of binary noise (Subsection 7.1), in which discrete Malliavin calculus is very well studied, see e.g. the monograph [25]. We explain how our convergence results can be stated in a simplified way in this case and demonstrate by a toy example how to apply the results numerically in a Monte Carlo framework. In Subsection 7.2, we finally show, how our main convergence results can be translated into Donsker type theorems on convergence in distribution, when the discrete time noise is not necessarily assumed to be embedded into the driving Brownian motion. Two auxiliary results on the S-transform characterization of the Malliavin derivative and on the connection between strong L^2-convergence and convergence in distribution are postponed to the Appendix.

2. Weak and strong L^2-convergence via discrete S-transforms

In this section, we study strong and weak $L^2(\Omega,\mathcal{F},P)$-convergence of a sequence (X^n) of random variables, where X^n is $\mathcal{F}^n := \sigma(\xi_i^n,\ i\in\mathbb{N})$-measurable, to an $\mathcal{F}$-measurable X. As a main result of this section (Theorem 1), we provide an equivalent criterion for this convergence, which only requires to compute a family of $L^2(\Omega,\mathcal{F}^n,P)$-inner products (hence, expectations which involve functionals of the discrete time noise $(\xi_i^n)_{i\in\mathbb{N}}$ only) and their limits as n tends to infinity.
Before doing so, let us recall that B^n can be constructed via a Skorokhod embedding of the random walk

$$\left(\sum_{i=1}^{j}\xi_i\right)_{j\in\mathbb{N}},\quad \xi_1,\xi_2,\dots \text{ independent and with the same distribution as } \xi,$$

into the rescaled Brownian motion $(\sqrt{n}B_{t/n})_{t\geq 0}$. In this way, one obtains, for every $n\in\mathbb{N}$, a sequence of stopping times $(\tau_i^n)_{i\in\mathbb{N}_0}$ with respect to the augmentation of the filtration generated by B such that

$$B^n := \left(B_{\tau^n_{\lfloor nt\rfloor}}\right)_{t\geq 0}$$

has the same distribution as $(\frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}\xi_i)_{t\geq 0}$ and converges to B uniformly on compacts in probability (see e.g. [22, Lemma 5.24 (b)]).
We now introduce the S-transform simultaneously in the continuous time setting and the discrete time setting, which turns ou to be the key tool for the proofs of our limit theorems. Recall,

that the mapping $\mathbf{1}_{(0,t]} \mapsto B_t$ can be extended to a continuous linear mapping from $L^2([0,\infty))$ to $L^2(\Omega, \mathcal{F}, P)$, which is known as the *Wiener integral*. We denote the Wiener integral of a function $f \in L^2([0,\infty))$ by $I(f)$. The *discrete Wiener integral* is given by

$$I^n(f^n) := \frac{1}{\sqrt{n}} \sum_{i=1}^{\infty} f^n(i)\xi_i^n.$$

Here, the discrete time function f^n is a member of

$$L_n^2(\mathbb{N}) := \left\{ f^n : \mathbb{N} \to \mathbb{R} : \ \|f^n\|^2_{L_n^2(\mathbb{N})} := \frac{1}{n} \sum_{i=1}^{\infty} (f^n(i))^2 < \infty \right\},$$

which obviously ensures that the series $I^n(f^n)$ converges (strongly) in $L^2(\Omega, \mathcal{F}^n, P)$.
The *Wick exponential* is, by definition, the stochastic exponential of a Wiener integral $I(f)$, i.e.,

$$\exp^\diamond(I(f)) := \exp\left(I(f) - 1/2 \int_0^\infty f^2(s)ds\right).$$

Hence, its discrete counterpart, the *discrete Wick exponential*, is given by

$$\exp^{\diamond_n}(I^n(f^n)) := \prod_{i=1}^{\infty}\left(1 + \frac{1}{\sqrt{n}} f^n(i)\xi_i^n\right).$$

In particular, by Fatou's lemma and the estimate $1 + x \le \exp(x)$,

$$\mathbb{E}[(\exp^{\diamond_n}(I^n(f^n)))^2] \le \exp(\|f^n\|^2_{L_n^2(\mathbb{N})}) < \infty. \tag{2}$$

Notice also that, by the martingale convergence theorem,

$$\begin{aligned} \exp^{\diamond_n}(I^n(f^n)) &= 1 + \sum_{i=1}^{\infty} \left(\mathbb{E}\left[\exp^{\diamond_n}(I^n(f^n)) | (\xi_j^n)_{j \le i}\right] - \mathbb{E}\left[\exp^{\diamond_n}(I^n(f^n)) | (\xi_j^n)_{j \le i-1}\right]\right) \\ &= 1 + \sum_{i=1}^{\infty} f^n(i) \exp^{\diamond_n}(I^n(f^n \mathbf{1}_{[1,i-1]})) \frac{\xi_i^n}{\sqrt{n}}, \end{aligned} \tag{3}$$

which is the discrete counterpart of the Doléans-Dade equation.
We finally recall that, for every $X \in L^2(\Omega, \mathcal{F}, P)$ and $f \in L^2([0,\infty))$, the *S-transform* is defined as

$$(SX)(f) := \mathbb{E}[X \exp^\diamond(I(f))].$$

Analogously, for every $X^n \in L^2(\Omega, \mathcal{F}^n, P)$ and $f^n \in L_n^2(\mathbb{N})$, we introduce the *discrete S-transform* as

$$(S^n X^n)(f^n) := \mathbb{E}[X^n \exp^{\diamond_n}(I^n(f^n))].$$

We emphasize that the S-transform is a powerful tool in the white noise analysis, see, e.g., [18], and has been successfully applied in the theory of stochastic partial differential equations, see [13]. To the best of our knowledge the discrete S-transform has, however, not been studied in the literature.
Let us next denote by $\mathcal{E}$ the set of step functions on left half-open intervals, i.e., functions of the form

$$g(x) = \sum_{j=1}^{m} a_j \mathbf{1}_{(b_j, c_j]}(x), \quad m \in \mathbb{N}, a_j, b_j, c_j \in \mathbb{R}.$$

As the set of Wick exponentials of step functions $\{\exp^\diamond(I(g)),\ g \in \mathcal{E}\}$ is total in $L^2(\Omega, \mathcal{F}, P)$, see e.g. [16, Corollary 3.40], every $L^2(\Omega, \mathcal{F}, P)$-random variable is uniquely determined by its S-transform. More precisely, if for $X, Y \in L^2(\Omega, \mathcal{F}, P)$, $(SX)(g) = (SY)(g)$ for every $g \in \mathcal{E}$, then $X = Y$ P-almost surely. We define the discretization of a step function $g \in \mathcal{E}$ as

$$\check{g}^n = (\check{g}^n(1), \check{g}^n(2), \ldots) := (g(1/n), g(2/n), \ldots),$$

and notice that
$$\{\breve{g}^n : g \in \mathcal{E}\} \subset L^2_n(\mathbb{N})$$
is the dense subspace of discrete time functions with finite support.
The convergence results of integral and derivative operators in this paper rely on the following characterization of $L^2(\Omega, \mathcal{F}, P)$-convergence in terms of convergence of the discrete S-transform to the continuous S-transform.

Theorem 1. *Suppose $X, X^n \in L^2(\Omega, \mathcal{F}, P)$ for every $n \in \mathbb{N}$, with X^n being $\mathcal{F}^n$-measurable. Then the following assertions are equivalent as n tends to infinity:*

(i) *$X^n \to X$ strongly (resp. weakly) in $L^2(\Omega, \mathcal{F}, P)$.*
(ii) *$(S^n X^n)(\breve{g}^n) \to (SX)(g)$ for every $g \in \mathcal{E}$, and additionally $\mathbb{E}[(X^n)^2] \to \mathbb{E}[X^2]$ in the case of strong convergence (resp. $\sup_{n\in\mathbb{N}} \mathbb{E}[(X^n)^2] < \infty$ in the case of weak convergence).*

In view of Lemma 3 below, the proof of Theorem 1 can be reduced to the following strong L^2-convergence result for (discrete) Wick exponentials.

Proposition 2. *Suppose $g \in \mathcal{E}$. Then, we have strongly in $L^2(\Omega, \mathcal{F}, P)$, as n tends to infinity:*
$$\exp^{\diamond_n}(I^n(\breve{g}^n)) \to \exp^{\diamond}(I(g)).$$

These type of convergence results for stochastic exponentials are somewhat standard and can be obtained in a much more general context by applying results on convergence in distribution for stochastic differential equations, see, e.g., [2, 19] and the references therein. For sake of completeness, we here provide an elementary proof.

Proof. Let
$$g = \sum_{j=1}^m a_j \mathbf{1}_{(b_j, c_j]} \in \mathcal{E}.$$
We denote by C, N constants in $\mathbb{N}$ such that g is bounded by C and has support in $[0, N]$. It suffices to show

(i) $\lim_{n\to\infty} \mathbb{E}\left[(\exp^{\diamond_n}(I^n(\breve{g}^n)))^2\right] = \mathbb{E}\left[(\exp^{\diamond}(I(g)))^2\right]$,
(ii) $\exp^{\diamond_n}(I^n(\breve{g}^n)) \to \exp^{\diamond}(I(g))$ in probability.

(i) Due to $p < \lceil q \rceil \leq r \Leftrightarrow \lfloor p \rfloor < q \leq \lfloor r \rfloor$ for all $p, q, r \in \mathbb{R}$, we obtain for every $t \in (0, \infty)$,
$$\breve{g}^n(\lceil nt \rceil) \quad = \quad \sum_{j=1}^m a_j \mathbf{1}_{(\lfloor b_j n \rfloor/n, \lfloor c_j n \rfloor/n]}(t). \tag{4}$$
Hence,
$$\|g - \breve{g}^n(\lceil n \cdot \rceil)\|_{L^2([0,\infty))} \leq \sqrt{2} \left(\sum_{j=1}^m |a_j|\right) \frac{1}{\sqrt{n}} \to 0, \tag{5}$$
and in particular,
$$\sum_{i=1}^{Nn} (\breve{g}^n(i))^2 \frac{1}{n} = \|\breve{g}^n(\lceil n \cdot \rceil)\|^2_{L^2([0,\infty))} \to \int_0^\infty g(s)^2 ds.$$
Thus, by the independence of the centered random variables $(\xi_i^n)_{i\in\mathbb{N}}$ with unit variance and taking the boundedness of g into account, we get
$$\begin{aligned} \mathbb{E}\left[(\exp^{\diamond_n}(I^n(\breve{g}^n)))^2\right] &= \prod_{i=1}^{Nn} \mathbb{E}\left[(1 + \frac{1}{\sqrt{n}}\breve{g}^n(i)\xi_i^n)^2\right] = \prod_{i=1}^{Nn} \left(1 + \frac{1}{n}(\breve{g}^n(i))^2\right) \\ &\to \exp\left(\int_0^\infty g(s)^2 ds\right) = \mathbb{E}\left[(\exp^{\diamond}(I(g)))^2\right]. \end{aligned}$$

(ii) In order to treat the large jumps of B^n and the small ones separately, we consider

$$\xi_i^{n,1} := \xi_i^n \mathbf{1}_{\{|\xi_i^n| \le \frac{\sqrt{n}}{2C}\}}, \quad \xi_i^{n,2} := \xi_i^n \mathbf{1}_{\{|\xi_i^n| > \frac{\sqrt{n}}{2C}\}},$$

cp. also [27]. Then,

$$\exp^{\diamond_n}(I^n(\breve{g}^n)) = \prod_{i=1}^{Nn}\left(1 + \frac{1}{\sqrt{n}}\breve{g}^n(i)\xi_i^{n,1}\right)\prod_{i=1}^{Nn}\left(1 + \frac{1}{\sqrt{n}}\breve{g}^n(i)\xi_i^{n,2}\right) =: E^{n,1}\cdot E^{n,2}.$$

We note that, for every $\epsilon > 0$, by the independence of $(\xi_i^n)_{i\in\mathbb{N}}$,

$$P\left(\left\{\sup_{i=1,\dots,Nn}\frac{|\xi_i^n|}{\sqrt{n}} > \epsilon\right\}\right) = 1 - \left(1 - \frac{P(\{|\xi| > \epsilon\sqrt{n}\})Nn}{Nn}\right)^{Nn} \to 0, \tag{6}$$

because, by square-integrability of ξ, $P(\{|\xi| > \epsilon\sqrt{n}\})n \to 0$, see, e.g., [26, p. 208]. Hence, for every $\epsilon > 0$,

$$P(\{|E^{n,2} - 1| \ge \epsilon\}) \le P(\{\sup_{i=1,\dots,Nn}|\xi_i^{n,2}| > 0\}) = P\left(\left\{\sup_{i=1,\dots,Nn}\frac{|\xi_i^n|}{\sqrt{n}} > 1/(2C)\right\}\right) \to 0,$$

i.e., $(E^{n,2})_{n\in\mathbb{N}}$ converges to 1 in probability. By construction, each factor in $E^{n,1}$ is larger than 1/2. Applying a Taylor expansion to the logarithm, thus, yields

$$\log E^{n,1} = \sum_{i=1}^{Nn}\breve{g}^n(i)\frac{\xi_i^{n,1}}{\sqrt{n}} - \frac{1}{2}\sum_{i=1}^{Nn}(\breve{g}^n(i))^2\frac{(\xi_i^{n,1})^2}{n} + R_n$$

with a remainder term satisfying

$$|R_n| \le \frac{8C}{3}\left(\sup_{j=1,\dots,Nn}\frac{|\xi_j^n|}{\sqrt{n}}\right)\sum_{i=1}^{Nn}(\breve{g}^n(i))^2\frac{(\xi_i^{n,1})^2}{n}.$$

It, thus, suffices to show

(iii) $\sum_{i=1}^{Nn}\breve{g}^n(i)\frac{\xi_i^{n,1}}{\sqrt{n}} \to I(g)$ in probability,

(iv) $\sum_{i=1}^{Nn}(\breve{g}^n(i))^2\frac{(\xi_i^{n,1})^2}{n} \to \int_0^\infty g(s)^2ds$ in probability.

Indeed, by (6), the remainder term then vanishes in probability as n tends to infinity, and, thus,

$$E^{n,1} \to \exp\left(I(g) - \frac{1}{2}\int_0^\infty g(s)^2ds\right) \text{ in probability.}$$

The same argument, which was applied for the convergence of $E^{n,2}$, shows that we can (and shall) replace $\xi_i^{n,1}$ by ξ_i^n in (iii) and (iv). However, by (1) and (4),

$$\lim_{n\to\infty}\sum_{i=1}^{Nn}\breve{g}^n(i)\frac{\xi_i^n}{\sqrt{n}} = \lim_{n\to\infty}\sum_{j=1}^m a_j\left(B_{c_j}^n - B_{b_j}^n\right) = \sum_{j=1}^m a_j\left(B_{c_j} - B_{b_j}\right) = I(g)$$

in probability. Finally, by the law of large numbers, $\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}(\xi_i^n)^2$ converges to t in probability for every $t \ge 0$, and, hence, by (4),

$$\lim_{n\to\infty}\sum_{i=1}^{Nn}(\breve{g}^n(i))^2\frac{(\xi_i^n)^2}{n} = \sum_{j=1}^m a_j^2(c_j - b_j) = \int_0^\infty g(s)^2ds, \text{ in probability.}$$

□

The following lemma from functional analysis turns out to be useful.

Lemma 3. *Suppose H is a Hilbert space, A is an arbitrary index set, $\{x^a,\ a\in A\}$ is total in H, and, for every $a \in A$, $(x_n^a)_{n\in\mathbb{N}}$ is a sequence in H which converges strongly in H to x^a. Then, the following are equivalent, as n tends to infinity:*

(i) $x^n \to x$ *strongly (resp. weakly) in* H.
(ii) $\langle x_n, x_n^a\rangle_H \to \langle x, x^a\rangle_H$ *for every* $a \in A$, *and additionally* $\|x_n\|_H \to \|x\|_H$ *in the case of strong convergence (resp.* $\sup_{n\in\mathbb{N}} \|x_n\|_H < \infty$ *in the case of weak convergence).*

Proof. Firstly, we observe that $\sup_{n\in\mathbb{N}} \|x_n\|_H$ is finite, either by weak convergence [29, Theorem V.1.1] in (i) or by assumption (ii). Thus, for every $a \in A$, by the strong convergence of (x_n^a) to x^a,

$$|\langle x_n, x_n^a\rangle_H - \langle x_n, x^a\rangle_H| = |\langle x_n, x_n^a - x^a\rangle_H| \leq \sup_{m\in\mathbb{N}} \|x_m\|_H \|x_n^a - x^a\|_H \to 0. \tag{7}$$

Let us treat the case of weak convergence: If (i) holds, the term $\langle x_n, x^a\rangle_H$ in (7) converges to $\langle x, x^a\rangle_H$, and then so does $\langle x_n, x_n^a\rangle_H$, which implies (ii). Conversely, if (ii) holds, the first term $\langle x_n, x_n^a\rangle_H$ in (7) tends to $\langle x, x^a\rangle_H$, and then so does $\langle x_n, x^a\rangle_H$, which yields (i) in view of [29, Theorem V.1.3]. The case of strong convergence is an immediate consequence, as, in a Hilbert space, strong convergence is equivalent to weak convergence and convergence of the norms [29, Theorem V.1.8]. $\square$

In view of the definition of the (discrete) S-transform, and as the set of Wick exponentials of step functions $\{\exp^\diamond(I(g)),\ g \in \mathcal{E}\}$ is total in $L^2(\Omega, \mathcal{F}, P)$, Theorem 1 now turns out to be a direct consequence of Proposition 2 and Lemma 3 with $H = L^2(\Omega, \mathcal{F}, P)$.
We close this section with an example.

Example 4. *(i) In this example, we provide a simple proof, that, for every* $X \in L^2(\Omega, \mathcal{F}, P)$, $X^n := \mathbb{E}[X|\mathcal{F}^n]$ *converges to* X *strongly in* $L^2(\Omega, \mathcal{F}, P)$*. Indeed, by Proposition 2, for every* $g \in \mathcal{E}$,

$$\begin{aligned}(S^n X^n)(\breve{g}^n) &= \mathbb{E}\left[\mathbb{E}[X|\mathcal{F}^n]\exp^{\diamond_n}(I^n(\breve{g}^n))\right] \\ &= \mathbb{E}\left[X\exp^{\diamond_n}(I^n(\breve{g}^n))\right] \to \mathbb{E}\left[X\exp^{\diamond}(I(g))\right] = (SX)(g).\end{aligned}$$

As $\mathbb{E}[(X^n)^2] \leq \mathbb{E}[X^2]$, *Theorem 1 implies weak* $L^2(\Omega, \mathcal{F}, P)$*-convergence of* (X^n) *to* X*. The same theorem finally yields strong* $L^2(\Omega, \mathcal{F}, P)$*-convergence, since, by the already established weak convergence,*

$$\mathbb{E}[(X^n)^2] = \mathbb{E}\left[\mathbb{E}\left[X^n|\mathcal{F}^n\right]X\right] = \mathbb{E}[X^n X] \to \mathbb{E}[X^2].$$

We note that this result can also be derived by the uniform integrability of $((X^n)^2)$ *via the concept of convergence of filtrations making use of [8, Proposition 2].*

(ii) Denote by $(\mathcal{F}_t)_{t\geq 0}$ *the augmented Brownian filtration and let* $\mathcal{F}_i^n = \sigma(\xi_1^n, \ldots, \xi_i^n)$*. We assume* $X \in L^2(\Omega, \mathcal{F}_T, P)$*. Then, one can always approximate* X *by a sequence* (X_T^n) *strongly in* $L^2(\Omega, \mathcal{F}, P)$*, where* X_T^n *is measurable with respect to* $\mathcal{F}_{\lfloor nT\rfloor}^n$*. Indeed, take any sequence* (X^n) *of* $\mathcal{F}^n$*-measurable random variables which converges strongly in* $L^2(\Omega, \mathcal{F}, P)$ *to* X*, and define* $X_T^n = \mathbb{E}[X^n|\mathcal{F}_{\lfloor nT\rfloor}^n]$*. Then, for every* $g \in \mathcal{E}$*, by Proposition 2,*

$$\begin{aligned}(S^n X_T^n)(\breve{g}^n) &= \mathbb{E}\left[X^n \prod_{i=1}^{\lfloor nT\rfloor}\left(1 + \frac{1}{\sqrt{n}}g(i/n)\xi_i^n\right)\right] \\ &= \mathbb{E}\left[X^n \exp^{\diamond_n}(I^n((g\mathbf{1}_{(0,T]})^{\breve{}\,n}))\right] \to \mathbb{E}\left[X\exp^\diamond(I(g\mathbf{1}_{(0,T]}))\right] \\ &= \mathbb{E}\left[X\mathbb{E}[\exp^\diamond(I(g))|\mathcal{F}_T]\right] = (SX)(g).\end{aligned}$$

Moreover,

$$\sup_{n\in\mathbb{N}} \mathbb{E}[(X_T^n)^2] \leq \sup_{n\in\mathbb{N}} \mathbb{E}[(X^n)^2] < \infty.$$

Hence, (X_T^n) *converges weakly in* $L^2(\Omega, \mathcal{F}, P)$ *to* X *by Theorem 1. Then, strong* $L^2(\Omega, \mathcal{F}, P)$*-convergence follows by Theorem 1 as well, because*

$$\mathbb{E}[(X_T^n)^2] = \mathbb{E}[X_T^n X] + \mathbb{E}[X_T^n(X^n - X)] \to \mathbb{E}[X^2].$$

3. Weak L^2-approximation of the Skorokhod integral and the Malliavin derivative

In this section, we first discuss weak L^2-approximations of the Skorokhod integral and the Malliavin derivative via appropriate discrete-time counterparts. We then show how to lift these results from weak convergence to strong convergence via duality under appropriate conditions which can be formulated in terms of the discrete-time approximations.
While most presentations of Malliavin calculus first introduce the Malliavin derivative and then define the Skorokhod integral as adjoint operator of the Malliavin derivative, we shall here employ the following equivalent characterization of the Skorokhod integral in terms of the S-transform, cp. [16, Theorem 16.46, Theorem 16.50].

Definition 5. *$Z \in L^2(\Omega\times[0,\infty)) := L^2(\Omega\times[0,\infty),\mathcal{F}\otimes\mathcal{B}([0,\infty)),P\otimes\lambda_{[0,\infty)})$ is said to belong to the domain $D(\delta)$ of the Skorokhod integral, if there is an $X\in L^2(\Omega,\mathcal{F},P)$ such that for every $g\in\mathcal{E}$*

$$(SX)(g) = \int_0^\infty (SZ_t)(g)g(t)dt.$$

In this case, X is uniquely determined and $\delta(Z) := X$ is called the Skorokhod integral of Z.

For the discrete-time approximation we first introduce the space

$$L^2_n(\Omega\times\mathbb{N}) := \left\{ Z^n : \mathbb{N}\to L^2(\Omega,\mathcal{F}^n,P),\ \|Z^n\|^2_{L^2_n(\Omega\times\mathbb{N})} := \frac{1}{n}\sum_{i=1}^\infty \mathbb{E}[(Z_i^n)^2] < \infty \right\}.$$

Moreover, we recall the definitions

$$\mathcal{F}^n := \sigma(\xi_j^n, j\in\mathbb{N}), \qquad \mathcal{F}^n_M := \sigma(\xi_1^n,\ldots,\xi_M),$$

and introduce the shorthand notations

$$\mathcal{F}^n_{-i} := \sigma(\xi_j^n, j\in\mathbb{N}\setminus\{i\}), \qquad \mathcal{F}^n_{M,-i} := \sigma(\xi_j^n, j\in\{1,\ldots,M\}\setminus\{i\}).$$

Definition 6. *We say, $Z^n\in L^2_n(\Omega\times\mathbb{N})$ belongs to the domain $D(\delta^n)$ of the discrete Skorokhod integral, if*

$$\delta^n(Z^n) := \lim_{M\to\infty}\sum_{i=1}^M \mathbb{E}[Z_i^n|\mathcal{F}^n_{M,-i}]\frac{\xi_i^n}{\sqrt{n}}. \tag{8}$$

exists strongly in $L^2(\Omega,\mathcal{F},P)$. If this is the case, $\delta^n(Z^n)$ is called the discrete Skorokhod integral of Z^n.

We note that, by the independence of $\mathbb{E}[Z_i^n|\mathcal{F}^n_{M,-i}]$ and ξ_i^n, each summand on the right-hand side of (8) is indeed a member of $L^2(\Omega,\mathcal{F},P)$. Moreover, the martingale convergence theorem implies that, for every $Z^n\in L^2_n(\Omega\times\mathbb{N})$ and $N\in\mathbb{N}$, $Z^n\mathbf{1}_{[1,N]}\in D(\delta^n)$ and

$$\delta^n(Z^n\mathbf{1}_{[1,N]}) = \sum_{i=1}^N \mathbb{E}[Z_i^n|\mathcal{F}^n_{-i}]\frac{\xi_i^n}{\sqrt{n}}. \tag{9}$$

Hence, the discrete Skorokhod integral is densely defined from $L^2_n(\Omega\times\mathbb{N})$ to $L^2(\Omega,\mathcal{F},P)$. We will show in Proposition 13 below that it is a closed operator.

Remark 7. *This definition of the discrete Skorokhod integral closely resembles the following Riemann-sum approximation of the Skorokhod integral by [24], who show that under appropriate conditions on Z,*

$$\mathbb{E}\left[n\int_{\frac{i}{n}}^{\frac{i+1}{n}} Z_s ds \,\middle|\, (B_s, B_1 - B_r)_{0\le s\le \frac{i}{n}\le\frac{i+1}{n}\le r\le 1}\right]\left(B_{(i+1)/n} - B_{i/n}\right)$$

converges strongly in $L^2(\Omega,\mathcal{F},P)$ to $\delta(Z\mathbf{1}_{[0,1]})$.

As a first main result of this section we are going to show the following weak approximation theorem for Skorokhod integrals.

Theorem 8. *Suppose $Z^n \in D(\delta^n)$ for every $n \in \mathbb{N}$, and $(Z^n_{\lceil n\cdot\rceil})_{n\in\mathbb{N}}$ converges to Z weakly in $L^2(\Omega\times[0,\infty))$. Then, the following assertions are equivalent:*

(i) $\sup_{n\in\mathbb{N}} \mathbb{E}[|\delta^n(Z^n)|^2] < \infty$.

(ii) *$Z \in D(\delta)$ and $(\delta^n(Z^n))$ converges to $\delta(Z)$ weakly in $L^2(\Omega,\mathcal{F},P)$ as n tends to infinity.*

As a first tool for the proof we state the discrete S-transform of a discrete Skorokhod integral.

Proposition 9. *Suppose $Z^n \in D(\delta^n)$. Then, for every $g \in \mathcal{E}$,*

$$(S^n\delta^n(Z^n))\,(\check{g}^n) = \frac{1}{n}\sum_{i=1}^{\infty}(S^n Z^n_i)(\check{g}^n\mathbf{1}_{\mathbb{N}\setminus\{i\}})\check{g}^n(i).$$

This result is a special case of the more general Proposition 13 below, to which we refer the reader for the proof.

The second tool for the proof of Theorem 8 is the following variant of Theorem 1 for stochastic processes.

Theorem 10. *Suppose $Z \in L^2(\Omega\times[0,\infty))$, $(Z^n)_{n\in\mathbb{N}}$ satisfies $Z^n \in L^2_n(\Omega\times\mathbb{N})$ for every $n\in\mathbb{N}$. Then the following assertions are equivalent as n tends to infinity:*

(i) *$(Z^n_{\lceil n\cdot\rceil})$ converges strongly (resp. weakly) to Z in $L^2(\Omega\times[0,\infty))$.*

(ii) *For every $g, h \in \mathcal{E}$*

$$\frac{1}{n}\sum_{i=1}^{\infty}(S^n Z^n_i)(\check{g}^n)\check{h}^n(i) \to \int_0^\infty (SZ_s)(g)h(s)ds.$$

and, additionally, $\mathbb{E}[\int_0^\infty (Z^n_{\lceil ns\rceil})^2 ds] \to \mathbb{E}[\int_0^\infty Z_s^2 ds]$ in the case of strong convergence (resp. $\sup_{n\in\mathbb{N}}\mathbb{E}[\int_0^\infty (Z^n_{\lceil ns\rceil})^2 ds] < \infty$ in the case of weak convergence).

Here, $\frac{1}{n}\sum_{i=1}^\infty (S^n Z^n_i)(\check{g}^n)\check{h}^n(i)$ can be replaced by $\frac{1}{n}\sum_{i=1}^\infty (S^n Z^n_i)(\check{g}^n\mathbf{1}_{\mathbb{N}\setminus\{i\}})\check{h}^n(i)$ in (ii).

Proof. We wish to apply Lemma 3 in order to prove the equivalence of (i) and (ii). As $L^2(\Omega\times[0,\infty)) = L^2(\Omega,\mathcal{F},P)\otimes L^2([0,\infty))$ (with the tensor product in the sense of Hilbert spaces), the set $\{\exp^\diamond(I(g))h;\, g,h\in\mathcal{E}\}$ is total in $L^2(\Omega\times[0,\infty))$. By Proposition 2 and (5), $(\exp^{\diamond_n}(I^n(\check{g}^n))\check{h}^n(\lceil n\cdot\rceil))_{n\in\mathbb{N}}$ converges to $\exp^\diamond(I(g))h$ strongly in $L^2(\Omega\times[0,\infty))$ for every $g,h\in\mathcal{E}$. As

$$\frac{1}{n}\sum_{i=1}^{\infty}(S^n Z^n_i)(\check{g}^n)\check{h}^n(i) = \left\langle Z^n_{\lceil n\cdot\rceil}, e^{\diamond_n I^n(\check{g}^n)}\check{h}^n(\lceil n\cdot\rceil)\right\rangle_{L^2(\Omega\times[0,\infty))},$$

Lemma 3 applies indeed. We finally note, that the modified assertion is an immediate consequence of the Cauchy-Schwarz inequality and the estimate

$$\begin{aligned}\mathbb{E}\left[\left(\exp^{\diamond_n}(I^n(\check{g}^n)) - \exp^{\diamond_n}(I^n(\check{g}^n\mathbf{1}_{\mathbb{N}\setminus\{i\}}))\right)^2\right] &= \mathbb{E}\left[\left(\exp^{\diamond_n}(I^n(\check{g}^n\mathbf{1}_{\mathbb{N}\setminus\{i\}}))\right)^2\right]\mathbb{E}\left[\left(\check{g}^n(i)\xi^n_i/\sqrt{n}\right)^2\right] \\ &\le \exp(\|\check{g}^n\|^2_{L^2_n(\mathbb{N})})\sup_{j\in\mathbb{N}}|g(j)|^2/n \to 0,\end{aligned}$$

making use of (2) in the last line. □

We are now ready to give the proof of Theorem 8.

Proof of Theorem 8. As the implication '$(ii) \Rightarrow (i)$' is trivial, we only have to show the converse implication. To this end, note first that, by Proposition 9 and Theorem 10, for every $g\in\mathcal{E}$,

$$\lim_{n\to\infty}(S^n\delta^n(Z^n))(\check{g}^n) = \int_0^\infty (SZ_t)(g)g(t)dt. \tag{10}$$

As the sequence $(\delta^n(Z^n))_{n\in\mathbb{N}}$ is norm bounded by (i), it has a weakly convergent subsequence [29, Theorem V.2.1]. We denote its limit by X. Then, applying Theorem 1 and (10) along the subsequence, we obtain, for every $g \in \mathcal{E}$,

$$(SX)(g) = \int_0^\infty (SZ_t)(g)g(t)dt. \tag{11}$$

Hence, by Definition 5, $Z \in D(\delta)$ and $\delta(Z) = X$. Finally, by Theorem 1 and (10)–(11), weak $L^2(\Omega, \mathcal{F}, P)$-convergence of $(\delta^n(Z^n))_{n\in\mathbb{N}}$ to $\delta(Z)$ holds along the whole sequence, and not only along the subsequence. □

We now turn to the weak approximation of the Malliavin derivative. Again, we apply a definition in terms of the S-transform, which we show to be equivalent to the more classical one in terms of the chaos decomposition in the Appendix.

Definition 11. *A random variable $X \in L^2(\Omega, \mathcal{F}, P)$ is said to belong to the domain $\mathbb{D}^{1,2}$ of the Malliavin derivative, if there is a stochastic process $Z \in L^2(\Omega \times [0,\infty))$ such that for every $g, h \in \mathcal{E}$,*

$$\int_0^\infty (SZ_s)(g)h(s)ds = \mathbb{E}\left[X \exp^\diamond(I(g)) \left(I(h) - \int_0^\infty g(s)h(s)ds\right)\right].$$

In this case, Z is unique and $DX := Z$ is called the Malliavin derivative X.

For every $X \in L^2(\Omega, \mathcal{F}, P)$ we define the discretized Malliavin derivative of X at $j \in \mathbb{N}$ with respect to $(\xi_i^n)_{i\in\mathbb{N}}$ by

$$D_j^n X := \sqrt{n}\mathbb{E}[\xi_j^n X|\mathcal{F}_{-j}^n].$$

We note that, for fixed j, D_j^n is a continuous linear operator from $L^2(\Omega, \mathcal{F}, P)$ to $L^2(\Omega, \mathcal{F}, P)$, because by Hölder's inequality for conditional expectations and the independence of the family $(\xi_i^n)_{i\in\mathbb{N}}$,

$$|D_j^n X|^2 \le n\mathbb{E}[X^2|\mathcal{F}_{-j}^n]\,\mathbb{E}[(\xi_j^n)^2|\mathcal{F}_{-j}^n] = n\mathbb{E}[X^2|\mathcal{F}_{-j}^n].$$

We say that X belongs to the domain $\mathbb{D}_n^{1,2}$ of the discretized Malliavin derivative, if the process $D^n X := (D_i^n X)_{i\in\mathbb{N}}$ is a member of $L_n^2(\Omega\times\mathbb{N})$. In this case $D^n X$ is called the *discretized Malliavin derivative* of X with respect to $(\xi_i^n)_{i\in\mathbb{N}}$. As D_j^n is continuous for fixed j, it is easy to check that the discretized Malliavin derivative is a densely defined closed operator from $L^2(\Omega, \mathcal{F}, P)$ to $L_n^2(\Omega \times \mathbb{N})$.
In the following theorem and in the remainder of the paper we use the convention $Z_0^n = 0$ for $Z^n \in L_n^2(\Omega \times \mathbb{N})$.

Theorem 12. *Suppose $(X^n)_{n\in\mathbb{N}}$ converges to X weakly in $L^2(\Omega, \mathcal{F}, P)$ and $X^n \in \mathbb{D}_n^{1,2}$ for every $n \in \mathbb{N}$. Then, the following are equivalent:*

(i) $\sup_{n\in\mathbb{N}} \frac{1}{n}\sum_{i=1}^\infty \mathbb{E}[(D_i^n X^n)^2] < \infty$.
(ii) *$X \in \mathbb{D}^{1,2}$ and $(D_{\lceil n\cdot\rceil}^n X^n)_{n\in\mathbb{N}}$ converges to DX weakly in $L^2(\Omega \times [0,\infty))$.*

The proof is prepared by two propositions. The first one contains the duality relation between the discrete Skorokhod integral and discretized Malliavin derivative.

Proposition 13. *For every $n \in \mathbb{N}$, the discrete Skorokhod integral is the adjoint operator of the discretized Malliavin derivative. In particular, δ^n is closed and, for every $X \in \mathbb{D}_n^{1,2}$ and $Z^n \in D(\delta^n)$,*

$$\frac{1}{n}\sum_{i=1}^\infty \mathbb{E}\left[Z_i^n D_i^n X\right] = \mathbb{E}[\delta^n(Z^n)X].$$

We emphasize that, choosing $X = \exp^{\diamond_n}(I^n(\check{g}^n))$, $g \in \mathcal{E}$, in Proposition 13, we obtain the assertion of Proposition 9. Indeed, for every $f^n \in L_n^2(\mathbb{N})$,

$$D_i^n \exp^{\diamond_n}(I^n(f^n)) = f^n(i) \exp^{\diamond_n}(I^n(f^n \mathbf{1}_{\mathbb{N}\setminus\{i\}})).$$

Proof. Suppose first, that $Z^n \in D(\delta^n)$ and $X \in \mathbb{D}_n^{1,2}$. Then, for every $M \in \mathbb{N}$, and $i \in \mathbb{N}$,

$$\mathbb{E}\left[\left|\sqrt{n}\mathbb{E}[\xi_i^n X|\mathcal{F}_{M,-i}^n]\right|^2\right] \le \mathbb{E}\left[|D_i^n X|^2\right].$$

Hence, by the martingale convergence theorem and dominated convergence,

$$\lim_{M\to\infty} \frac{1}{n}\sum_{i=1}^{\infty} \mathbb{E}\left[\left|\sqrt{n}\mathbb{E}[\xi_i^n X|\mathcal{F}_{M,-i}^n] - D_i^n X\right|^2\right] = 0.$$

Consequently,

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{\infty} \mathbb{E}\left[Z_i^n D_i^n X\right] &= \lim_{M\to\infty} \frac{1}{n}\sum_{i=1}^{M} \mathbb{E}\left[Z_i^n \sqrt{n}\mathbb{E}[\xi_i^n X|\mathcal{F}_{M,-i}^n]\right] = \lim_{M\to\infty} \frac{1}{\sqrt{n}}\sum_{i=1}^{M} \mathbb{E}\left[X\xi_i^n \mathbb{E}[Z_i^n|\mathcal{F}_{M,-i}^n]\right] \\ &= \lim_{M\to\infty} \mathbb{E}\left[X\sum_{i=1}^{M} \mathbb{E}[Z_i^n|\mathcal{F}_{M,-i}^n]\frac{\xi_i^n}{\sqrt{n}}\right] = \mathbb{E}[X\delta^n(Z^n)].\end{aligned}$$

Conversely, suppose that Z^n is in the domain of the adjoint operator of the discretized Malliavin derivative, i.e., there is an $Y^n \in L^2(\Omega, \mathcal{F}, P)$ such that for every $X \in \mathbb{D}_n^{1,2}$,

$$\frac{1}{n}\sum_{i=1}^{\infty} \mathbb{E}\left[Z_i^n D_i^n X\right] = \mathbb{E}[Y^n X]. \tag{12}$$

We first note that, by construction, $X \in \mathbb{D}_n^{1,2}$ if and only $\mathbb{E}[X|\mathcal{F}^n] \in \mathbb{D}_n^{1,2}$, and, if this is the case, both random variables have the same discretized Malliavin derivative. Hence, applying the duality relation (12), with X and $\mathbb{E}[X|\mathcal{F}^n]$, we obtain $Y^n = \mathbb{E}[Y^n|\mathcal{F}^n]$. Now suppose that $X \in L^2(\Omega, \mathcal{F}_M^n, P)$. Then $X \in \mathbb{D}_n^{1,2}$, $D_i^n X = \sqrt{n}\mathbb{E}[\xi_i^n X|\mathcal{F}_{M,-i}^n]$ for every $i \le M$, and $D_i^n X = 0$ for $i > M$. Hence, (12) and the same manipulations as above imply

$$\mathbb{E}[Y^n X] = \mathbb{E}\left[X\sum_{i=1}^{M} \mathbb{E}[Z_i^n|\mathcal{F}_{M,-i}^n]\frac{\xi_i^n}{\sqrt{n}}\right],$$

i.e.

$$\mathbb{E}[Y^n|\mathcal{F}_M^n] = \sum_{i=1}^{M} \mathbb{E}[Z_i^n|\mathcal{F}_{M,-i}^n]\frac{\xi_i^n}{\sqrt{n}}.$$

By the martingale convergence theorem, $(\mathbb{E}[Y^n|\mathcal{F}_M^n])_{M\in\mathbb{N}}$ converges strongly in $L^2(\Omega, \mathcal{F}, P)$ to $\mathbb{E}[Y^n|\mathcal{F}^n] = Y^n$. Hence, $Z^n \in D(\delta^n)$ and $\delta^n(Z^n) = Y^n$. Now, closedness is a general property of adjoint operators, see [29, p. 196]. ☐

The next proposition is a consequence of the weak convergence result for discrete Skorokhod integrals in Theorem 8.

Proposition 14. *For every $g, h \in \mathcal{E}$,*

$$\lim_{n\to\infty} \delta^n(\exp^{\diamond_n}(I^n(\check{g}^n))\,\check{h}^n) = \exp^{\diamond}(I(g))\left(I(h) - \int_0^{\infty} g(s)h(s)ds\right)$$

strongly in $L^2(\Omega, \mathcal{F}, P)$.

Proof. Notice first that, for fixed $n \in \mathbb{N}$, $\exp^{\diamond_n}(I^n(\check{g}^n))\,\check{h}^n \in D(\delta^n)$, because $\check{h}^n(i)$ vanishes, if i is sufficiently large. A direct computation, making use of (9), shows

$$\delta^n(\exp^{\diamond_n}(I^n(\check{g}^n))\,\check{h}^n) = \sum_{i=1}^{\infty} \exp^{\diamond_n}(I^n(\check{g}^n\mathbf{1}_{\mathbb{N}\setminus\{i\}}))\check{h}^n(i)\frac{\xi_i^n}{\sqrt{n}}.$$

For $i \ne j$ we obtain, by independence of $(\xi_k^n)_{k\in\mathbb{N}}$,

$$\mathbb{E}\left[\exp^{\diamond_n}(I^n(\check{g}^n\mathbf{1}_{\mathbb{N}\setminus\{i\}}))\exp^{\diamond_n}(I^n(\check{g}^n\mathbf{1}_{\mathbb{N}\setminus\{j\}}))\xi_i^n\xi_j^n\right] = \check{g}^n(i)\frac{1}{\sqrt{n}}\check{g}^n(j)\frac{1}{\sqrt{n}}\prod_{k\in\mathbb{N}\setminus\{i,j\}}\left(1 + \check{g}^n(k)^2\frac{1}{n}\right).$$

Combining this with an analogous calculation for the case $i=j$ yields

$$\mathbb{E}\left[\left|\delta^n(\exp^{\diamond_n}(I^n(\check{g}^n))\,\check{h}^n)\right|^2\right] = \frac{1}{n^2}\sum_{i,j=1,\, i\neq j}^{\infty} \check{h}^n(i)\check{g}^n(i)\check{h}^n(j)\check{g}^n(j)\prod_{k\in\mathbb{N}\setminus\{i,j\}}\left(1+\check{g}^n(k)^2\frac{1}{n}\right)$$
$$+\frac{1}{n}\sum_{i=1}^{\infty}\check{h}^n(i)^2\prod_{k\in\mathbb{N}\setminus\{i\}}\left(1+\check{g}^n(k)^2\frac{1}{n}\right).$$

As g and h are bounded with compact support, it is straightforward to check in view of (5) that

$$\lim_{n\to\infty}\mathbb{E}\left[\left|\delta^n(\exp^{\diamond_n}(I^n(\check{g}^n))\,\check{h}^n)\right|^2\right] = e^{\int_0^\infty g(s)^2ds}\left(\left(\int_0^\infty h(s)g(s)ds\right)^2+\int_0^\infty h(s)^2ds\right). \quad (13)$$

Thus, $(\delta^n(\exp^{\diamond_n}(I^n(\check{g}^n))\,\check{h}^n))_{n\in\mathbb{N}}$ converges to $\delta(\exp^{\diamond}(I(g))\,h)$ weakly in $L^2(\Omega,\mathcal{F},P)$ by Theorem 8. The identity

$$\delta(\exp^{\diamond}(I(g))\,h) = \exp^{\diamond}(I(g))\left(I(h)-\int_0^\infty g(s)h(s)ds\right)$$

can either be derived by a direct computation making use of the S-transform definition of the Skorokhod integral (Definition 5) or alternatively is a simple consequence of [23, Proposition 1.3.3] in conjunction with Definition 1.2.1 in the same reference. Applying the Cameron-Martin shift [16, Theorem 14.1] twice, we observe

$$\mathbb{E}\left[\left(e^{\diamond I(g)}\left(I(h)-\int_0^\infty g(s)h(s)ds\right)\right)^2\right] = e^{\int_0^\infty g(s)^2ds}\mathbb{E}\left[e^{\diamond I(g)}I(h)^2\right]$$
$$= e^{\int_0^\infty g(s)^2ds}\mathbb{E}\left[\left(I(h)+\int_0^\infty g(s)h(s)ds\right)^2\right]$$
$$= e^{\int_0^\infty g(s)^2ds}\left(\left(\int_0^\infty h(s)g(s)ds\right)^2+\int_0^\infty h(s)^2ds\right).$$

Thanks to (13), this turns weak into strong convergence. □

The proof of Theorem 12 is now analogous to that of Theorem 8.

Proof of Theorem 12. '(ii) ⇒ (i)' is obvious, since

$$\frac{1}{n}\sum_{i=1}^{\infty}\mathbb{E}[(D_i^nX^n)^2] = \int_0^\infty \mathbb{E}[(D^n_{\lceil ns\rceil}X^n)^2]ds.$$

'(i) ⇒ (ii)': Notice first that, for every $g,h\in\mathcal{E}$, by Proposition 13 with $Z^n=\exp^{\diamond_n}(I^n(\check{g}^n))\,\check{h}^n$ and Proposition 14,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{\infty}(S^nD_i^nX^n)(\check{g}^n)\check{h}^n(i) = \lim_{n\to\infty}\mathbb{E}[X^n\delta^n(\exp^{\diamond_n}(I^n(\check{g}^n))\check{h}^n)]$$
$$= \mathbb{E}\left[X\exp^{\diamond}(I(g))\left(I(h)-\int_0^\infty g(s)h(s)ds\right)\right], \quad (14)$$

since (X^n) converges to X weakly in $L^2(\Omega,\mathcal{F},P)$. The sequence $(D^n_{\lceil n\cdot\rceil}X^n)_{n\in\mathbb{N}}$ is norm bounded in $L^2(\Omega\times[0,\infty))$ by (i), and, hence, it has a weakly convergent subsequence. We denote its limit by Z. Applying (14) and Theorem 10 along this subsequence, we conclude

$$\int_0^\infty (SZ_s)(g)h(s)ds = \mathbb{E}\left[X\exp^{\diamond}(I(g))\left(I(h)-\int_0^\infty g(s)h(s)ds\right)\right]. \quad (15)$$

Hence, $X\in\mathbb{D}^{1,2}$ and $DX=Z$ by Definition 11. Finally, applying (14)–(15) and Theorem 10 along the whole sequence $(D^n_{\lceil n\cdot\rceil}X^n)_{n\in\mathbb{N}}$, shows that this sequence converges weakly in $L^2(\Omega\times[0,\infty))$ to DX. □

In order to check the assumptions of Theorem 8, we consider the space $\mathbb{L}_n^{1,2}$, which consists of processes $Z^n \in L^2_n(\Omega\times\mathbb{N})$ such that $Z_i^n \in \mathbb{D}_n^{1,2}$ for every $i\in\mathbb{N}$ and

$$\frac{1}{n^2}\sum_{i,j=1,\, i\neq j}^{\infty} \mathbb{E}\left[|D_j^n Z_i^n|^2\right] < \infty.$$

Proposition 15. *For every $n\in\mathbb{N}$, $\mathbb{L}_n^{1,2}\subset D(\delta^n)$ and, for $Z^n\in\mathbb{L}_n^{1,2}$,*

$$\delta^n(Z^n) = \sum_{i=1}^{\infty}\mathbb{E}[Z_i^n|\mathcal{F}_{-i}^n]\frac{\xi_i^n}{\sqrt{n}}, \quad \text{(strong } L^2(\Omega,\mathcal{F},P)\text{-convergence)}, \tag{16}$$

$$\mathbb{E}\left[(\delta^n(Z^n))^2\right] = \frac{1}{n}\sum_{i=1}^{\infty}\mathbb{E}\left[\mathbb{E}\left[Z_i^n|\mathcal{F}_{-i}^n\right]^2\right] + \frac{1}{n^2}\sum_{i,j=1,\, i\neq j}^{\infty}\mathbb{E}\left[(D_i^nZ_j^n)(D_j^nZ_i^n)\right]. \tag{17}$$

In particular, in the context of Theorem 8, assertion (i) is equivalent to

(i') $\sup_{n\in\mathbb{N}}\frac{1}{n^2}\sum_{i,j=1,\, i\neq j}^{\infty}\mathbb{E}\left[(D_i^nZ_j^n)(D_j^nZ_i^n)\right] < \infty$,

if we additionally assume that $Z^n\in\mathbb{L}_n^{1,2}$ for every $n\in\mathbb{N}$.

Proof. Fix $N_1 < N_2\in\mathbb{N}$. Then,

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{i=N_1}^{N_2}\mathbb{E}[Z_i^n|\mathcal{F}_{-i}^n]\frac{\xi_i^n}{\sqrt{n}}\right)^2\right] &= \frac{1}{n}\sum_{i=N_1}^{N_2}\mathbb{E}\left[\mathbb{E}[Z_i^n|\mathcal{F}_{-i}^n]^2\,(\xi_i^n)^2\right]\\ &\quad+\frac{1}{n}\sum_{i,j=N_1,\, i\neq j}^{N_2}\mathbb{E}\left[\mathbb{E}[Z_i^n|\mathcal{F}_{-i}^n]\mathbb{E}[Z_j^n|\mathcal{F}_{-j}^n]\xi_i^n\xi_j^n\right]\\ &= (I)_{N_1,N_2} + (II)_{N_1,N_2}.\end{aligned}$$

By the independence of the discrete-time noise $(\xi_i^n)_{i\in\mathbb{N}}$ and as the conditional expectation has norm 1, we obtain as N tends to infinity,

$$(I)_{1,N} = \frac{1}{n}\sum_{i=1}^{N}\mathbb{E}\left[\mathbb{E}[Z_i^n|\mathcal{F}_{-i}^n]^2\right] \to \frac{1}{n}\sum_{i=1}^{\infty}\mathbb{E}\left[\mathbb{E}[Z_i^n|\mathcal{F}_{-i}^n]^2\right] < \infty,$$

and $(I)_{N_1,N_2}\to 0$ as N_1,N_2 tend to infinity. In order to treat $(II)_{N_1,N_2}$, we first note that for any random variable $X^n\in L^1(\Omega,\mathcal{F}^n,P)$ and $i\neq j\in\mathbb{N}$, by Fubini's theorem,

$$\mathbb{E}\left[\mathbb{E}\left[X^n|\mathcal{F}_{-i}^n\right]\middle|\,\mathcal{F}_{-j}^n\right] = \mathbb{E}\left[\mathbb{E}\left[X^n|\mathcal{F}_{-j}^n\right]\middle|\,\mathcal{F}_{-i}^n\right]. \tag{18}$$

Hence, for $i\neq j\in\mathbb{N}$,

$$\begin{aligned}&\mathbb{E}\left[\mathbb{E}[Z_i^n|\mathcal{F}_{-i}^n]\mathbb{E}[Z_j^n|\mathcal{F}_{-j}^n]\xi_i^n\xi_j^n\right] = \mathbb{E}\left[\mathbb{E}[Z_i^n\xi_j^n|\mathcal{F}_{-i}^n]\mathbb{E}[Z_j^n\xi_i^n|\mathcal{F}_{-j}^n]\right]\\ =\;& \mathbb{E}\left[\mathbb{E}\left[\mathbb{E}[Z_i^n\xi_j^n|\mathcal{F}_{-i}^n]\middle|\,\mathcal{F}_{-j}^n\right]Z_j^n\xi_i^n\right] = \mathbb{E}\left[\mathbb{E}\left[\mathbb{E}[Z_i^n\xi_j^n|\mathcal{F}_{-j}^n]\middle|\,\mathcal{F}_{-i}^n\right]Z_j^n\xi_i^n\right]\\ =\;& \mathbb{E}\left[\mathbb{E}[Z_i^n\xi_j^n|\mathcal{F}_{-j}^n]\mathbb{E}[Z_j^n\xi_i^n|\mathcal{F}_{-i}^n]\right] = \frac{1}{n}\mathbb{E}\left[(D_i^nZ_j^n)(D_j^nZ_i^n)\right].\end{aligned}$$

Consequently, by Young's inequality,

$$n\left|\mathbb{E}\left[\mathbb{E}[Z_i^n|\mathcal{F}_{-i}^n]\mathbb{E}[Z_j^n|\mathcal{F}_{-j}^n]\xi_i^n\xi_j^n\right]\right| \le \frac{1}{2}\mathbb{E}\left[(D_i^nZ_j^n)^2\right] + \frac{1}{2}\mathbb{E}\left[(D_j^nZ_i^n)^2\right].$$

The $\mathbb{L}_n^{1,2}$-assumption, thus, ensures that

$$\lim_{N\to\infty}(II)_{1,N} \;=\; \frac{1}{n^2}\sum_{i,j=1,i\neq j}^{\infty}\mathbb{E}\left[(D_i^nZ_j^n)(D_j^nZ_i^n)\right] < \infty$$

and $(II)_{N_1,N_2}\to 0$ as N_1,N_2 tend to infinity. Hence, by (9), the sequence $(\delta^n(Z^n\mathbf{1}_{[1,N]}))_{N\in\mathbb{N}}$ is Cauchy in $L^2(\Omega,\mathcal{F},P)$. By the closedness of the discrete Skorokhod integral, $Z^n\in D(\delta^n)$ and we obtain $\mathbb{L}_n^{1,2} \subset D(\delta^n)$, (16) and (17). We finally suppose that the assumptions of Theorem 8 are in force and that $Z^n \in \mathbb{L}_n^{1,2}$ for every $n \in \mathbb{N}$. Then,

$$\sup_{n\in\mathbb{N}} \frac{1}{n} \sum_{i=1}^{\infty} \mathbb{E}\left[\mathbb{E}[Z_i^n|\mathcal{F}_{-i}^n]^2\right] < \infty,$$

because of the assumed weak convergence of the sequence $(Z^n_{\lceil n\cdot\rceil})_{n\in\mathbb{N}}$. Thus, the sequence $(\delta^n(Z^n))_{n\in\mathbb{N}}$ is norm bounded in $L^2(\Omega,\mathcal{F},P)$, if and only if (i') holds. □

As a consequence of the previous proposition, we obtain the following strong $L^2(\Omega,\mathcal{F},P)$-convergence results to the Malliavin derivative.

Theorem 16. *Suppose $(X^n)_{n\in\mathbb{N}}$ converges to X strongly in $L^2(\Omega,\mathcal{F},P)$. Moreover assume that $X^n \in \mathbb{D}_n^{2,2}$ for every $n \in \mathbb{N}$, i.e.*

$$\frac{1}{n}\sum_{i=1}^{\infty} \mathbb{E}\left[(D_i^n X^n)^2\right] + \frac{1}{n^2}\sum_{i,j=1,\, i\neq j}^{\infty} \mathbb{E}\left[(D_j^n D_i^n X^n)^2\right] < \infty.$$

Then, the following assertions are equivalent:

(i) $\sup_{n\in\mathbb{N}} \left(\frac{1}{n}\sum_{i=1}^{\infty} \mathbb{E}\left[(D_i^n X^n)^2\right] + \frac{1}{n^2}\sum_{i,j=1,\, i\neq j}^{\infty} \mathbb{E}\left[(D_j^n D_i^n X^n)^2\right]\right) < \infty.$

(ii) *$X \in \mathbb{D}^{1,2}$, $DX \in D(\delta)$, $(D^n_{\lceil n\cdot\rceil} X^n)_{n\in\mathbb{N}}$ converges to DX strongly in $L^2(\Omega\times[0,\infty))$, and $(\delta^n(D^n X^n))_{n\in\mathbb{N}}$ converges to $\delta(DX)$ weakly in $L^2(\Omega,\mathcal{F},P)$.*

Remark 17. *Recall that $L = -\delta \circ D$ is the infinitesimal generator of the Ornstein-Uhlenbeck semigroup, see [23, Section 1.4], and is sometimes called Ornstein-Uhlenbeck operator (cf. also [16, Example 4.7]). So the previous theorem provides, at the same time, sufficient conditions for the strong convergence to the Malliavin derivative and the weak convergence to the Ornstein-Uhlenbeck operator.*

Proof. Let $Z_i^n = D_i^n X^n$. Then, $X^n \in \mathbb{D}_n^{2,2}$ implies $Z^n \in \mathbb{L}_n^{1,2}$. Note that, for $i \neq j$, by (18),

$$D_j^n Z_i^n = D_j^n D_i^n X = D_i^n D_j^n X = D_i^n Z_j^n,$$

i.e. $(D_j^n Z_i^n)(D_i^n Z_j^n) = (D_j^n D_i^n X)^2$. Hence, by Theorem 12 and Theorem 8 in conjunction with Proposition 15, assertion (i) is equivalent to

(ii') $X \in \mathbb{D}^{1,2}$, $DX \in D(\delta)$, $(D^n_{\lceil n\cdot\rceil} X^n)_{n\in\mathbb{N}}$ converges to DX weakly in $L^2(\Omega\times[0,\infty))$, and $(\delta^n(D^n X^n))_{n\in\mathbb{N}}$ converges to $\delta(DX)$ weakly in $L^2(\Omega,\mathcal{F},P)$.

So we only need to show that under (ii') the convergence of $(D^n_{\lceil n\cdot\rceil} X^n)_{n\in\mathbb{N}}$ to DX holds true in the strong topology. However, by the duality relation in Proposition 13, the weak $L^2(\Omega,\mathcal{F},P)$-convergence of $(\delta^n(D^nX^n))_{n\in\mathbb{N}}$ and the strong $L^2(\Omega,\mathcal{F},P)$-convergence of $(X^n)_{n\in\mathbb{N}}$,

$$\int_0^{\infty} \mathbb{E}[(D^n_{\lceil n\cdot\rceil} X^n)^2]dt = \mathbb{E}[\delta^n(D^nX^n)X^n] \to \mathbb{E}[\delta(DX)X] = \int_0^{\infty} \mathbb{E}[(D_tX)^2]dt,$$

making use of the continuous time duality between Skorokhod integral and Malliavin derivative in the last step. □

The analogous result for the Skorokhod integral reads as follows.

Theorem 18. *Suppose $(Z^n_{\lceil n\cdot\rceil})_{n\in\mathbb{N}}$ converges strongly to Z in $L^2(\Omega\times[0,\infty))$ and assume that $Z^n \in \mathbb{L}_n^{2,2}$, i.e., for every $n \in \mathbb{N}$,*

$$\frac{1}{n^2}\sum_{i,j=1,\, i\neq j}^{\infty} \mathbb{E}\left[|D_i^n Z_j^n|^2\right] + \frac{1}{n^3}\sum_{i,j,k=1,\, |\{i,j,k\}|=3}^{\infty} \mathbb{E}\left[|D_i^n D_j^n Z_k^n|^2\right] < \infty.$$

Then the following assertions are equivalent:

(i) $\sup_{n\in\mathbb{N}}\left(\frac{1}{n^2}\sum_{i,j=1,\,i\neq j}^{\infty}\mathbb{E}\left[(D_i^n Z_j^n)(D_j^n Z_i^n)\right]\right)<\infty$ *and*

$$\sup_{n\in\mathbb{N}}\left(\frac{1}{n^2}\sum_{i,j=1,\,i\neq j}^{\infty}\mathbb{E}\left[\mathbb{E}[D_i^n Z_j^n|\mathcal{F}_{-j}^n]^2\right]+\frac{1}{n^3}\sum_{i,j,k=1,\,|\{i,j,k\}|=3}^{\infty}\mathbb{E}\left[(D_i^n D_j^n Z_k^n)(D_k^n D_j^n Z_i^n)\right]\right)<\infty.$$

(ii) $Z\in D(\delta)$, $\delta(Z)\in\mathbb{D}^{1,2}$, $(\delta^n(Z^n))_{n\in\mathbb{N}}$ *converges to* $\delta(Z)$ *strongly in* $L^2(\Omega,\mathcal{F},P)$ *and* $(D^n_{\lceil n\cdot\rceil}\delta^n(Z^n))_{n\in\mathbb{N}}$ *converges to* $D\delta(Z)$ *weakly in* $L^2(\Omega\times[0,\infty))$.

As a preparation we explain how to compute the dicretized Malliavin derivative of a discrete Skorokhod integral, which is analogous to the continuous-time situation, cp. e.g. [23, Proposition 1.3.8].

Proposition 19. *Suppose* $Z^n\in\mathbb{L}_n^{1,2}$. *Then* $(D_i^n Z^n)\mathbf{1}_{\mathbb{N}\setminus\{i\}}\in D(\delta^n)$ *for every* $i\in\mathbb{N}$, *and*

$$D_i^n\delta^n(Z^n)=\mathbb{E}[Z_i^n|\mathcal{F}_{-i}^n]+\delta^n(D_i^n Z^n\mathbf{1}_{\mathbb{N}\setminus\{i\}}).$$

Proof. By (16) and the continuity of D_i^n,

$$D_i^n\delta^n(Z^n)=D_i^n\left(\mathbb{E}[Z_i^n|\mathcal{F}_{-i}^n]\frac{\xi_i^n}{\sqrt{n}}\right)+\sum_{j=1,\,j\neq i}^{\infty}D_i^n\left(\mathbb{E}[Z_j^n|\mathcal{F}_{-j}^n]\frac{\xi_j^n}{\sqrt{n}}\right),$$

(including the strong convergence of the series on the right-hand side in $L^2(\Omega,\mathcal{F},P)$). By (18), for $i\neq j$,

$$\mathbb{E}[\xi_i^n\mathbb{E}[Z_j^n|\mathcal{F}_{-j}^n]\xi_j^n|\mathcal{F}_{-i}^n]=\mathbb{E}[\mathbb{E}[\xi_i^n Z_j^n|\mathcal{F}_{-j}^n]|\mathcal{F}_{-i}^n]\xi_j^n=\mathbb{E}[\mathbb{E}[\xi_i^n Z_j^n|\mathcal{F}_{-i}^n]|\mathcal{F}_{-j}^n]\xi_j^n.$$

Moreover,

$$\mathbb{E}[(\xi_i^n)^2\mathbb{E}[Z_i^n|\mathcal{F}_{-i}^n]|\mathcal{F}_{-i}^n]=\mathbb{E}[Z_i^n|\mathcal{F}_{-i}^n].$$

Hence,

$$D_i^n\delta^n(Z^n)=\mathbb{E}[Z_i^n|\mathcal{F}_{-i}^n]+\sum_{j=1,\,j\neq i}^{\infty}\mathbb{E}[D_i^n Z_j^n|\mathcal{F}_{-j}^n]\frac{\xi_j^n}{\sqrt{n}},$$

and the closedness of the discrete Skorokhod integral concludes. □

Proof of Theorem 18. The $\mathbb{L}_n^{2,2}$-assumption guarantees that, for every $i\in\mathbb{N}$, $(D_i^n Z^n)\mathbf{1}_{\mathbb{N}\setminus\{i\}}\in\mathbb{L}_n^{1,2}$. As $(Z^n_{\lceil n\cdot\rceil})_{n\in\mathbb{N}}$ is norm bounded in $L^2(\Omega\times[0,\infty))$ by the assumed strong convergence to Z, we observe in view of Propositions 15 and 19 that (i) is equivalent to

(i') $\sup_{n\in\mathbb{N}}\mathbb{E}[|\delta^n(Z^n)|^2]<\infty$ and $\sup_{n\in\mathbb{N}}\frac{1}{n}\sum_{i=1}^{\infty}\mathbb{E}[|D_i^n\delta^n(Z^n)|^2]<\infty$.

Thanks to Theorems 8 and 12, assertion (i') is equivalent to

(ii') $Z\in D(\delta)$, $\delta(Z)\in\mathbb{D}^{1,2}$, $(\delta^n(Z^n))_{n\in\mathbb{N}}$ converges weakly to $\delta(Z)$ in $L^2(\Omega,\mathcal{F},P)$, and $(D^n\delta^n(Z^n))_{n\in\mathbb{N}}$ converges to $D\delta(Z)$ weakly in $L^2(\Omega\times[0,\infty))$.

Due to the strong convergence of $(Z^n_{\lceil n\cdot\rceil})_{n\in\mathbb{N}}$ to Z and the weak convergence of $(D^n_{\lceil n\cdot\rceil}\delta^n(Z^n))_{n\in\mathbb{N}}$ to $D(\delta(Z))$, the continuous time duality between Skorokhod integral and Malliavin derivative and its discrete time counterpart in Proposition 13 imply

$$\|\delta^n(Z^n)\|^2_{L^2(\Omega,\mathcal{F},P)}=\int_0^\infty\mathbb{E}[Z^n_{\lceil ns\rceil}D^n_{\lceil ns\rceil}\delta^n(Z^n)]ds\to\int_0^\infty\mathbb{E}[Z_sD_s\delta(Z)]ds=\|\delta(Z)\|^2_{L^2(\Omega,\mathcal{F},P)}.$$

Hence we obtain the convergence of $(\delta^n(Z^n))_{n\in\mathbb{N}}$ to $\delta(Z)$ in the strong topology, i.e., assertion (ii') is equivalent to assertion (ii). □

4. Strong and weak L^2-approximation of the Itô integral and the Clark-Ocone derivative

In this section, we first specialize the approximation result for the Skorokhod integral to predictable integrands. In this way, we obtain necessary and sufficient conditions for strong and weak L^2-convergence of discrete Itô integrals with respect to the noise $(\xi_i^n)_{i\in\mathbb{N}}$ to Itô integrals with respect to the Brownian motion B. Then, we discuss strong and weak L^2-approximations to the Clark-Ocone derivative, which provides the predictable integral representation of a random variable in $L^2(\Omega,\mathcal{F},P)$ with respect to the Brownian motion B.
Suppose $Z^n \in L^2_n(\Omega\times\mathbb{N})$ is predictable with respect to $(\mathcal{F}_i^n)_{i\in\mathbb{N}}$, i.e., for every $i\in\mathbb{N}$, Z_i^n is measurable with respect to $\mathcal{F}_{i-1}^n = \sigma(\xi_1^n,\dots,\xi_{i-1}^n)$. Then,

$$\delta^n(Z^n) = \sum_{i=1}^{\infty} Z_i^n \frac{\xi_i^n}{\sqrt{n}} =: \int Z^n dB^n,$$

which means that the discrete Skorokhod integral reduces to the discrete Itô integral. Analogously, the Skorokhod integral $\delta(Z)$ is well-known to coincide with the Itô integral $\int_0^\infty Z_s dB_s$, when $Z\in L^2(\Omega\times[0,\infty))$ is predictable with respect to the augmented Brownian filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$, see, e.g. [16, Theorem 7.41]. In this case of predictable integrands, the approximation theorem for Skorokhod integrals (Theorem 8) can be improved as follows.

Theorem 20. *Suppose $Z\in L^2(\Omega\times[0,\infty))$ is predictable with respect to the augmented Brownian filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$, and, for every $n\in\mathbb{N}$, $Z^n\in L^2_n(\Omega\times\mathbb{N})$ is predictable with respect to $(\mathcal{F}_i^n)_{i\in\mathbb{N}}$. Then, the following are equivalent:*

(i) *$(Z^n_{\lceil n\cdot\rceil})_{n\in\mathbb{N}}$ converges to Z strongly (resp. weakly) in $L^2(\Omega\times[0,\infty))$.*
(ii) *The sequence of discrete Itô integrals $\left(\int Z^n dB^n\right)_{n\in\mathbb{N}}$ converges strongly (resp. weakly) in $L^2(\Omega,\mathcal{F},P)$ to $\int_0^\infty Z_s dB_s$.*

Remark 21. *We note that, in order to study convergence of Itô integrals (with respect to different filtrations), techniques of convergence in distribution on the Skorokhod space of right-continuous functions with left limits are classically applied. E.g., the results by [19] immediately imply the following result in our setting: Suppose that Z is predictable with respect to the Brownian filtration and its paths are right-continuous with left limits. Moreover, assume that Z^n is predictable with respect to $(\mathcal{F}_i^n)_{i\in\mathbb{N}}$ and $(Z^n_{\lfloor 1+n(\cdot)\rfloor})$ converges to Z uniformly on compacts in probability. Then,*

$$\lim_{n\to\infty}\sum_{i=1}^{\lfloor n\cdot\rfloor} Z_i^n \frac{1}{\sqrt{n}}\xi_i^n = \int_0^{\cdot} Z_{s-}dB_s,$$

uniformly on compacts in probability. In contrast, our Theorem 20 provides an L^2-theory and, in particular, includes the converse implication, namely that convergence of the discrete Itô integrals implies convergence of the integrands.

The proof of Theorem 20 will make use of the following proposition.

Proposition 22. *Suppose $g,h\in\mathcal{E}$. Then, strongly in $L^2(\Omega\times[0,\infty))$,*

$$\lim_{n\to\infty}\exp^{\diamond_n}(I^n(\check{g}^n\mathbf{1}_{[1,\lceil n\cdot\rceil-1]}))\check{h}^n(\lceil n\cdot\rceil) = \exp^{\diamond}(I(g\mathbf{1}_{(0,\cdot]}))h(\cdot).$$

Proof. Recall that the support of h is contained in $[0,M]$ for some $M\in\mathbb{N}$. Hence, we can decompose,

$$\begin{aligned}&\int_0^\infty \mathbb{E}\left[\left(\exp^{\diamond_n}(I^n(\check{g}^n\mathbf{1}_{[1,\lceil nt\rceil-1]}))\check{h}^n(\lceil nt\rceil) - \exp^{\diamond}(I(g\mathbf{1}_{(0,t]}))h(t)\right)^2\right]dt\\ \le\ & 2\int_0^M \mathbb{E}\left[\left(\exp^{\diamond_n}(I^n(\check{g}^n\mathbf{1}_{[1,\lfloor nt\rfloor]})) - \exp^{\diamond}(I(g\mathbf{1}_{(0,t]}))\right)^2\right]h(t)^2dt\end{aligned}$$

$$+2\int_0^\infty \mathbb{E}\left[\left(\exp^{\diamond_n}(I^n(\check{g}^n \mathbf{1}_{[1,\lceil nt\rceil-1]}))\right)^2\right]|\check{h}^n(\lceil nt\rceil)-h(t)|^2dt,$$

since $\lceil nt\rceil - 1 = \lfloor nt\rfloor$ for Lebesgue almost every $t\geq 0$. As, by (2)

$$\sup_{n\in\mathbb{N},\, t\in[0,\infty)} \mathbb{E}\left[\left(\exp^{\diamond_n}(I^n(\check{g}^n \mathbf{1}_{[1,\lceil nt\rceil-1]}))\right)^2\right] \leq \sup_{n\in\mathbb{N}} e^{\|\check{g}^n(\lceil n\cdot\rceil)\|^2_{L^2([0,\infty))}} < \infty,$$

the second term goes to zero by (5). Moreover, by the boundedness of h, the first one tends to zero by the dominated convergence theorem, since, for every $t\in[0,\infty)$, by Proposition 2,

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\exp^{\diamond_n}(I^n((\check{g}^n \mathbf{1}_{[1,\lfloor nt\rfloor]}))) - \exp^{\diamond}(I(g\mathbf{1}_{(0,t]}))\right)^2\right]=0.$$

□

Proof of Theorem 20. '$(i)\Rightarrow(ii)$': By the isometry for discrete Itô integrals, we have

$$\mathbb{E}\left[\left(\int Z^n dB^n\right)^2\right] = \mathbb{E}\left[\left|\sum_{i=1}^\infty Z_i^n \frac{1}{\sqrt{n}}\xi_i^n\right|^2\right] = \int_0^\infty \mathbb{E}\left[|Z^n_{\lceil ns\rceil}|^2\right]ds. \tag{19}$$

Hence, if $(Z^n_{\lceil n\cdot\rceil})_{n\in\mathbb{N}}$ converges to Z weakly in $L^2(\Omega,\mathcal{F},P)$, then the left-hand side in (19) is bounded in $n\in\mathbb{N}$, and so Theorem 8 implies the asserted weak $L^2(\Omega,\mathcal{F},P)$ convergence of the sequence of discrete Itô integrals to $\int_0^\infty Z_s dB_s$. If $(Z^n_{\lceil n\cdot\rceil})_{n\in\mathbb{N}}$ converges to Z strongly in $L^2(\Omega,\mathcal{F},P)$, then, by (19) and the continuous time Itô isometry,

$$\lim_{n\to\infty}\mathbb{E}\left[\left(\int Z^n dB^n\right)^2\right] = \int_0^\infty \mathbb{E}\left[|Z_s|^2\right]ds = \mathbb{E}\left[\left(\int_0^\infty Z_s dB_s\right)^2\right],$$

which turns the weak $L^2(\Omega,\mathcal{F},P)$-convergence of the sequence of discrete Itô integrals into strong $L^2(\Omega,\mathcal{F},P)$-convergence.

'$(ii)\Rightarrow(i)$': We first assume that the sequence of discrete Itô integrals converges weakly in $L^2(\Omega,\mathcal{F},P)$ to the continuous time Itô integral. By the implication '$(i)\Rightarrow(ii)$' (which we have already proved) and Proposition 22, we obtain, for every $g,h\in\mathcal{E}$,

$$\lim_{n\to\infty}\sum_{i=1}^\infty \exp^{\diamond_n}(I^n(\check{g}^n\mathbf{1}_{[1,i-1]}))\check{h}^n(i)\frac{1}{\sqrt{n}}\xi_i^n = \int_0^\infty \exp^{\diamond}(I(g\mathbf{1}_{(0,s]}))\,h(s)\,dB_s \tag{20}$$

strongly in $L^2(\Omega,\mathcal{F},P)$. As Z^n is predictable, we get, for every $g,h\in\mathcal{E}$, by the discrete Itô isometry,

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^\infty (S^nZ_i^n)(\check{g}^n)\check{h}^n(i) &= \frac{1}{n}\sum_{i=1}^\infty \mathbb{E}[\mathbb{E}[Z_i^n|\mathcal{F}_{i-1}^n]\exp^{\diamond_n}(I^n(\check{g}^n))\check{h}^n(i)] \\
&= \frac{1}{n}\sum_{i=1}^\infty \mathbb{E}[Z_i^n \exp^{\diamond_n}(I^n(\check{g}^n\mathbf{1}_{[1,i-1]}))\check{h}^n(i)] \\
&= \mathbb{E}\left[\left(\sum_{i=1}^\infty Z_i^n\frac{1}{\sqrt{n}}\xi_i^n\right)\left(\sum_{i=1}^\infty \exp^{\diamond_n}(I^n(\check{g}^n\mathbf{1}_{[1,i-1]}))\check{h}^n(i)\frac{1}{\sqrt{n}}\xi_i^n\right)\right].
\end{aligned}$$

The assumed weak $L^2(\Omega,\mathcal{F},P)$-convergence of the sequence of discrete Itô integrals and the strong $L^2(\Omega,\mathcal{F},P)$-convergence in (20) now imply

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^\infty (S^nZ_i^n)(\check{g}^n)\check{h}^n(i) = \mathbb{E}\left[\left(\int_0^\infty Z_s dB_s\right)\left(\int_0^\infty e^{\diamond I(g\mathbf{1}_{(0,s]})}\,h(s)\,dB_s\right)\right].$$

As $(\exp^{\diamond}(I(g\mathbf{1}_{(0,s]})))_{s\in[0,\infty)}$ is a uniformly integrable martingale and Z is predictable, we obtain, by the Itô isometry and the definition of the S-transform,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{\infty}(S^n Z_i^n)(\breve{g}^n)\check{h}^n(i)=\int_0^{\infty}(SZ_s)(g)h(s)ds,\quad g,h\in\mathcal{E}.$$

We can now apply Theorem 10. As $\int_0^\infty \mathbb{E}\left[|Z^n_{\lceil ns\rceil}|^2\right]ds$ is bounded in n by (19) and by the assumed weak $L^2(\Omega,\mathcal{F},P)$-convergence of the discrete Itô integrals, the latter Theorem implies that $(Z^n_{\lceil n\cdot\rceil})_{n\in\mathbb{N}}$ converges to Z weakly in $L^2(\Omega\times[0,\infty))$. If we instead assume strong $L^2(\Omega,\mathcal{F},P)$-convergence of the sequence of the discrete Itô integrals, a straightforward application of the isometries for discrete and continuous-time Itô integrals turns the weak $L^2(\Omega\times[0,\infty))$-convergence again into strong convergence. □

We now turn to the Clark-Ocone derivative. Recall that a Brownian motion has the predictable representation property with respect to its natural filtration, i.e., for every $X\in L^2(\Omega,\mathcal{F},P)$ there is a unique $(\mathcal{F}_t)_{t\in[0,\infty)}$-predictable process $\nabla X\in L^2(\Omega\times[0,\infty))$ such that

$$X=\mathbb{E}[X]+\int_0^\infty \nabla_s X dB_s. \tag{21}$$

We refer to ∇X as the *generalized Clark-Ocone derivative* and recall that $(\nabla_t X)_{t\geq 0}$ is the predictable projection of the Malliavin derivative $(D_tX)_{t\geq0}$, if $X\in\mathbb{D}^{1,2}$. By Itô's isometry the operator $\nabla: L^2(\Omega,\mathcal{F},P)\to L^2(\Omega\times[0,\infty))$ is continuous with norm 1.
Except in the case of binary noise, the discrete time approximation $B^{(n)}$ of the Brownian motion B does not satisfy the discrete time predictable representation property with respect to $(\mathcal{F}_i^n)_{i\in\mathbb{N}}$. Nonetheless one can consider the discrete time predictable projection of the discretized Malliavin derivative

$$\nabla_i^n X:=\mathbb{E}[D_i^nX|\mathcal{F}_{i-1}^n]=\sqrt{n}\mathbb{E}[\xi_i^nX|\mathcal{F}_{i-1}^n],\quad X\in L^2(\Omega,\mathcal{F},P),\ i\in\mathbb{N},$$

as discretization of the generalized Clark-Ocone derivative. We refer to $(\nabla_i^nX)_{i\in\mathbb{N}}$ as *discretized Clark-Ocone derivative of* X and note that it has been extensively studied in the context of discretization of backward stochastic differential equations, see, e.g., [6, 10, 30].
The operator

$$\nabla^n: L^2(\Omega,\mathcal{F},P)\to L^2_n(\Omega\times\mathbb{N}),\quad X\mapsto(\nabla_i^nX)_{i\in\mathbb{N}}$$

is continuous with norm one. Indeed, introducing the shorthand notation $\mathbb{E}_{n,i}[\cdot]=\mathbb{E}[\cdot|\mathcal{F}_i^n]$ and noting that the martingale $(\mathbb{E}_{n,i}[X])_{i\in\mathbb{N}}$ is, for fixed $n\in\mathbb{N}$, uniformly integrable, and, thus, converges almost surely to $\mathbb{E}[X|\mathcal{F}^n]$, as i tends to infinity, one gets, by Hölder's and Jensen's inequality,

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^\infty \mathbb{E}\left[\left(\sqrt{n}\,\mathbb{E}_{n,i-1}\left[\xi_i^nX\right]\right)^2\right]&=\sum_{i=1}^\infty\mathbb{E}\left[\left(\mathbb{E}_{n,i-1}\left[\xi_i^n\left(\mathbb{E}_{n,i}[X]-\mathbb{E}_{n,i-1}[X]\right)\right]\right)^2\right]\\&\leq\sum_{i=1}^\infty\mathbb{E}\left[\mathbb{E}_{n,i-1}\left[(\xi_i^n)^2\right]\mathbb{E}_{n,i-1}\left[\left(\mathbb{E}_{n,i}[X]-\mathbb{E}_{n,i-1}[X]\right)^2\right]\right]\\&=\mathbb{E}\left[\sum_{i=1}^\infty\left((\mathbb{E}_{n,i}[X])^2-(\mathbb{E}_{n,i-1}[X])^2\right)\right]=\mathbb{E}\left[(\mathbb{E}[X|\mathcal{F}^n])^2\right]-\mathbb{E}\left[X\right]^2\\&\leq\mathbb{E}\left[(X)^2\right]-\mathbb{E}\left[X\right]^2.\end{aligned}$$

We now denote by

$$\mathcal{P}^n:=\left\{a+\int Z^ndB^n;\ a\in\mathbb{R},\ Z^n\in L^2_n(\Omega\times\mathbb{N})\text{ predictable}\right\}$$

the closed subspace in $L^2(\Omega,\mathcal{F},P)$, which admits a discrete time predictable integral representation. Note that, for every $X\in L^2(\Omega,\mathcal{F},P)$, $a\in\mathbb{R}$, and $(\mathcal{F}^n_i)_{i\in\mathbb{N}}$-predictable $Z^n\in L^2_n(\Omega\times\mathbb{N})$, by the discrete Itô isometry,

$$\begin{aligned}\mathbb{E}\left[X\left(a+\int Z^n dB^n\right)\right] &= a\mathbb{E}[X]+\frac{1}{\sqrt{n}}\sum_{i=1}^\infty \mathbb{E}[X\xi^n_i\mathbb{E}[Z^n_i|\mathcal{F}^n_{i-1}]] \\ &= a\mathbb{E}[X]+\frac{1}{n}\sum_{i=1}^\infty \mathbb{E}\left[Z^n_i\sqrt{n}\mathbb{E}[X\xi^n_i|\mathcal{F}^n_{i-1}]\right] \\ &= \mathbb{E}\left[\left(\mathbb{E}[X]+\int\nabla^n X dB^n\right)\left(a+\int Z^n dB^n\right)\right].\end{aligned}$$

Hence,

$$\pi_{\mathcal{P}^n}X=\mathbb{E}[X]+\int\nabla^n X dB^n, \tag{22}$$

where, for any closed subspace $\mathcal{A}$ in $L^2(\Omega,\mathcal{F},P)$, $\pi_{\mathcal{A}}$ denotes the orthogonal projection on $\mathcal{A}$. Our first approximation result for the Clark-Ocone derivative now reads as follows:

Theorem 23. *Suppose $(X^n)_{n\in\mathbb{N}}$ is a sequence in $L^2(\Omega,\mathcal{F},P)$ and $X\in L^2(\Omega,\mathcal{F},P)$. Then, the following are equivalent, as n tends to infinity:*

(i) *$(\pi_{\mathcal{P}^n}X^n-\mathbb{E}[X^n])_{n\in\mathbb{N}}$ converges to $X-\mathbb{E}[X]$ strongly (weakly) in $L^2(\Omega,\mathcal{F},P)$.*
(ii) *$(\nabla^n_{\lceil n\cdot\rceil}X^n)_{n\in\mathbb{N}}$ converges to ∇X strongly (weakly) in $L^2(\Omega\times[0,\infty))$.*

A sufficient condition for (i), (ii) is that $(X^n)_{n\in\mathbb{N}}$ converges to X strongly (weakly) in $L^2(\Omega,\mathcal{F},P)$.

Proof. Recall that by (21) and (22)

$$\begin{aligned}X-\mathbb{E}[X] &= \int_0^\infty \nabla X_s dB_s, \\ \pi_{\mathcal{P}^n}X^n-\mathbb{E}[X^n] &= \int\nabla^n X^n dB^n.\end{aligned}$$

Hence, Theorem 20 provides the equivalence of (i) and (ii). As, for every $g\in\mathcal{E}$, $\exp^{\diamond_n}(I^n(\check{g}^n))\in\mathcal{P}^n$ by (3), the sufficient condition is a consequence of the following lemma. □

Lemma 24. *Suppose that $\mathcal{A}^n$, $n\in\mathbb{N}$, are closed subspaces of $L^2(\Omega,\mathcal{F},P)$ such that for every $n\in\mathbb{N}$,*

$$\{\exp^{\diamond_n}(I^n(\check{g}^n)),\ g\in\mathcal{E}\}\subset\mathcal{A}_n.$$

Then, strong (weak) $L^2(\Omega,\mathcal{F},P)$-convergence of $(X^n)_{n\in\mathbb{N}}$ to X implies that $(\pi_{\mathcal{A}^n}X^n)_{n\in\mathbb{N}}$ converges to X strongly (weakly) in $L^2(\Omega,\mathcal{F},P)$ as well.

Proof. As, for every $g\in\mathcal{E}$,

$$\mathbb{E}[(\pi_{\mathcal{A}^n}X^n)\exp^{\diamond_n}(I^n(\check{g}^n))]=\mathbb{E}[X^n\,\pi_{\mathcal{A}^n}(\exp^{\diamond_n}(I^n(\check{g}^n)))]=\mathbb{E}[X^n\exp^{\diamond_n}(I^n(\check{g}^n))],$$

we obtain that $(S^nX^n)(\check{g}^n)=(S^n(\pi_{\mathcal{A}^n}X^n))(\check{g}^n)$. In the case of weak convergence, Theorem 1 now immediately applies, because

$$\mathbb{E}\left[(\pi_{\mathcal{A}^n}X^n)^2\right]\le\mathbb{E}\left[(X^n)^2\right].$$

In the case of strong convergence, we also make use of Theorem 1, and note that by the already established weak convergence of $(\pi_{\mathcal{A}^n}X^n)_{n\in\mathbb{N}}$ and Hölder's inequality, as n tends to infinity,

$$\mathbb{E}\left[(\pi_{\mathcal{A}^n}X^n)^2\right]=\mathbb{E}\left[X(\pi_{\mathcal{A}^n}X^n)\right]+\mathbb{E}\left[(X^n-X)(\pi_{\mathcal{A}^n}X^n)\right]\to\mathbb{E}\left[X^2\right].$$

□

We shall finally discuss an alternative approximation of the generalized Clark-Ocone derivative, which involves orthogonal projections on appropriate finite-dimensional subspaces. To this end, we denote by $\mathcal{H}^n$ the strong closure in $L^2(\Omega, \mathcal{F}, P)$ of the linear span of

$$\Xi^n := \left\{ \Xi^n_A := \prod_{i \in A} \xi^n_i, \quad A \subseteq \mathbb{N}, |A| < \infty \right\},$$

and emphasize that $\mathcal{H}^n = L^2(\Omega, \mathcal{F}^n, P)$, if and only if the noise distribution of ξ is binary. As Ξ^n consists of an orthonormal basis of $\mathcal{H}^n$, every $X^n \in \mathcal{H}^n$ has a unique expansion in terms of this Hilbert space basis, which is called the *Walsh decomposition* of X^n,

$$X^n = \sum_{|A|<\infty} X^n_A \Xi^n_A, \tag{23}$$

where $X^n_A = \mathbb{E}[X^n \Xi^n_A]$ satisfies $\sum_{|A|<\infty} (X^n_A)^2 < \infty$. The expectation and $L^2(\Omega, \mathcal{F}, P)$-inner product can be computed in terms of the Walsh decomposition via $\mathbb{E}[X^n] = X^n_\emptyset$ and

$$\mathbb{E}[X^n Y^n] = \sum_{|A|<\infty} X^n_A Y^n_A, \quad X^n, Y^n \in \mathcal{H}^n,$$

cp. [12]. A direct computation shows that the Walsh decomposition of a discrete Wick exponential is given by

$$\exp^{\diamond_n}(I^n(f^n)) = \sum_{|A|<\infty} \left(n^{-|A|/2} \prod_{i \in A} f^n(i) \right) \Xi^n_A. \tag{24}$$

In view of the Möbius inversion formula [1, Theorem 5.5], we obtain, for every finite subset B of $\mathbb{N}$,

$$\Xi^n_B = n^{|B|/2} \sum_{C \subseteq B} (-1)^{|B|-|C|} \exp^{\diamond_n}(I^n(\mathbf{1}_C)).$$

Hence, the set $\{\exp^{\diamond_n}(I^n(\check{g}^n)),\ g \in \mathcal{E}\}$ is total in $\mathcal{H}^n$.

We now consider the finite-dimensional subspaces

$$\mathcal{H}^n_i := \mathrm{span}\{\Xi^n_A,\ A \subset \{1, \ldots, i\}\},$$

and introduce, as a second approximation of the generalized Clark-Ocone derivative, the operator

$$\bar{\nabla}^n : L^2(\Omega, \mathcal{F}, P) \to L^2_n(\Omega \times \mathbb{N}), \quad X \mapsto (\pi_{\mathcal{H}^n_{i-1}}(\nabla^n_i X))_{i \in \mathbb{N}}.$$

Notice that

$$\bar{\nabla}^n_i X = \sqrt{n}\, \pi_{\mathcal{H}^n_{i-1}}(\xi^n_i X),$$

if $\xi^n_i X \in L^2(\Omega, \mathcal{F}, P)$.

We are now going to show the following variant of Theorem 23.

Theorem 25. *Suppose $(X^n)_{n\in\mathbb{N}}$ is a sequence in $L^2(\Omega, \mathcal{F}, P)$ and $X \in L^2(\Omega, \mathcal{F}, P)$. Then, the following are equivalent, as n tends to infinity:*

(i) $(\pi_{\mathcal{H}^n} X^n - \mathbb{E}[X^n])_{n\in\mathbb{N}}$ *converges to* $X - \mathbb{E}[X]$ *strongly (weakly) in* $L^2(\Omega, \mathcal{F}, P)$.

(ii) $(\bar{\nabla}^n_{\lceil n\cdot\rceil} X^n)_{n\in\mathbb{N}}$ *converges to* ∇X *strongly (weakly) in* $L^2(\Omega \times [0, \infty))$.

A sufficient condition for (i), (ii) is that $(X^n)_{n\in\mathbb{N}}$ converges to X strongly (weakly) in $L^2(\Omega, \mathcal{F}, P)$.

The proof is based on the simple observation that $\mathcal{H}^n \subset \mathcal{P}^n$, i.e., for every $X^n \in \mathcal{H}^n$,

$$X^n = \mathbb{E}[X^n] + \sum_{i=1}^{\infty} \nabla^n_i X^n \frac{1}{\sqrt{n}} \xi^n_i. \tag{25}$$

In order to show this, we recall that $\{\exp^{\diamond_n}(I^n(\check{g}^n)),\ g \in \mathcal{E}\}$ is total in $\mathcal{H}^n$. Thus, by continuity of the discretized Clark-Ocone derivative and by the discrete Itô isometry, it suffices to show (25) in the case $X^n = \exp^{\diamond_n}(I^n(\check{f}^n))$ for $f \in \mathcal{E}$. A direct computation shows,

$$\nabla^n_i \exp^{\diamond_n}(I^n(f^n)) = f^n(i) \exp^{\diamond_n}(I^n(f^n \mathbf{1}_{[1,i-1]})), \tag{26}$$

which in view of (3) completes the proof of (25).

Proof of Theorem 25. We first note that, for every $X \in L^2(\Omega, \mathcal{F}, P)$,

$$\begin{aligned}\mathbb{E}[\pi_{\mathcal{H}^n} X] &= \mathbb{E}[X], && (27)\\ \bar{\nabla}_i^n X &= \nabla_i^n(\pi_{\mathcal{H}^n} X). && (28)\end{aligned}$$

Indeed, as

$$\pi_{\mathcal{H}^n} X = \mathbb{E}[X] + \sum_{1\le |A| <\infty} \mathbb{E}[X\Xi_A^n]\Xi_A^n,$$

Eq. (27) is obvious. In order to prove (28), we recall first that $\nabla_i^n(\pi_{\mathcal{H}^n} X) \in \mathcal{H}_{i-1}^n$ (by (26) and continuity of the discretized Clark-Ocone derivative) and then note that, for every $A \subset \{1, \dots, i-1\}$,

$$\begin{aligned}&\mathbb{E}\left[\Xi_A^n \mathbb{E}\left[\xi_i^n X | \, \mathcal{F}_{i-1}^n\right]\right] = \mathbb{E}[\Xi_{A\cup\{i\}}^n X] = \mathbb{E}[\Xi_{A\cup\{i\}}^n \pi_{\mathcal{H}^n}(X)]\\ &= \mathbb{E}\left[\Xi_A^n \mathbb{E}\left[\xi_i^n \pi_{\mathcal{H}^n}(X) | \, \mathcal{F}_{i-1}^n\right]\right] = \mathbb{E}\left[\Xi_A^n \frac{1}{\sqrt{n}} \nabla_i^n(\pi_{\mathcal{H}^n} X)\right].\end{aligned}$$

In particular, by (25), (27), and (28)

$$\pi_{\mathcal{H}^n} X = \mathbb{E}[X] + \int \bar{\nabla}^n X dB^n, \qquad (29)$$

which is the analogue of (22). The proof of Theorem 23 can now be repeated verbatim with $\mathcal{P}^n$ replaced by $\mathcal{H}^n$. □

We close this section with two remarks.

Remark 26. *In view of Lemma 24 and the inclusion $\mathcal{H}^n \subset \mathcal{P}^n$ we observe that, for any sequence $(X^n)_{n\in\mathbb{N}}$ in $L^2(\Omega, \mathcal{F}, P)$,*

$$\begin{aligned}&\lim_{n\to\infty} X_n = X \quad \text{strongly (weakly) in } L^2(\Omega, \mathcal{F}, P)\\ \Rightarrow\ &\lim_{n\to\infty} \pi_{\mathcal{P}^n} X_n = X \quad \text{strongly (weakly) in } L^2(\Omega, \mathcal{F}, P)\\ \Rightarrow\ &\lim_{n\to\infty} \pi_{\mathcal{H}^n} X_n = X \quad \text{strongly (weakly) in } L^2(\Omega, \mathcal{F}, P).\end{aligned}$$

In particular, by Theorems 23 and 25, if the sequence of discretized Clark-Ocone derivatives $(\nabla_{\lceil n\cdot\rceil}^n X^n)_{n\in\mathbb{N}}$ converges to ∇X strongly (weakly) in $L^2(\Omega \times [0, \infty))$, then so does the sequence of modified discretized Clark-Ocone derivatives $(\bar{\nabla}_{\lceil n\cdot\rceil}^n X^n)_{n\in\mathbb{N}}$.

Remark 27. *The following result can be derived from [6, Theorem 5 and the examples in Section 5] under the additional assumption that $\mathbb{E}[|\xi|^{2+\epsilon}] < \infty$ for some $\epsilon > 0$ and on a finite time horizon: Strong convergence of $(X^n)_{n\in\mathbb{N}}$ to X in $L^2(\Omega, \mathcal{F}, P)$ implies convergence of the sequence of discretized Clark-Ocone derivatives as stated in (ii) of Theorem 23. Our Theorem 25 additionally shows that the conditional expectations $\mathbb{E}[\cdot|\mathcal{F}_{i-1}^n]$ in the definition of the discretized Clark-Ocone derivative can be replaced by the projection on the finite dimensional subspace $\mathcal{H}_i^n$, i.e., if $(X^n)_{n\in\mathbb{N}}$ converges to X strongly in $L^2(\Omega, \mathcal{F}, P)$, then*

$$\left(\sqrt{n}\, \pi_{\mathcal{H}_{\lceil nt\rceil -1}^n}(\xi_{\lceil nt\rceil}^n\, (\tau_n(X^n))\right)_{t\in[0,\infty)} \to \nabla X$$

strongly in $L^2(\Omega \times [0, \infty))$, where τ_n denotes the truncation at $\pm n$.
We also note that, in view of (29),

$$\bar{\nabla}_i X = \frac{(\pi_{\mathcal{H}_i^n} X) - (\pi_{\mathcal{H}_{i-1}^n} X)}{B_i^n - B_{i-1}^n}$$

can be rewritten as difference operator (where we apply the convention $\frac{\xi_i^n}{\xi_i^n} = 1$ when ξ_i^n vanishes). This representation shows the close relation to the weak $L^2(\Omega \times [0, \infty))$-approximation result for *the generalized Clark-Ocone derivative which is derived in [20, Corollary 4.1] for the case of binary noise.*

5. Strong L^2-approximation of the chaos decomposition

In this section, we apply Theorem 1 in order to characterize strong $L^2(\Omega, \mathcal{F}, P)$-convergence of a sequence (X^n) (where X^n can be represented via multiple Wiener integrals with respect to the discrete time noise $(\xi_i^n)_{i\in\mathbb{N}})$ via convergence of the coefficient functions of such a discrete chaos decomposition.
Recall first, that every $X \in L^2(\Omega, \mathcal{F}, P)$ has a unique Wiener chaos decomposition in terms of multiple Wiener integrals

$$X = \sum_{k=0}^{\infty} I^k(f_X^k), \tag{30}$$

where $f_X^k \in \widetilde{L^2}([0,\infty)^k)$, see e.g. [23, Theorem 1.1.2]. Here, we denote by $L^2([0,\infty)^k)$ the Hilbert space of square-integrable functions with respect to the k-dimensional Lebesgue measure and by $\widetilde{L^2}([0,\infty)^k)$ the subspace of functions in $L^2([0,\infty)^k)$ which are symmetric in the k variables. We apply the standard convention $\widetilde{L^2}([0,\infty)^0) = L^2([0,\infty)^0) = \mathbb{R}$, $I^0(f^0) = f^0$, and recall that, for $k \geq 1$ and $f^k \in \widetilde{L^2}([0,\infty)^k)$, the multiple Wiener integral can be defined as iterated Itô integral:

$$I^k(f^k) = k! \int_0^\infty \int_0^{t_k} \cdots \int_0^{t_2} f^k(t_1, \ldots, t_k) dB_{t_1} \cdots dB_{t_{k-1}} dB_{t_k}.$$

The Itô isometry therefore immediately implies the following well-konwn Wiener-Itô isometry for multiple Wiener integrals,

$$\mathbb{E}[I^k(f^k)\, I^{k'}(g^{k'})] = \delta_{k,k'}\ k!\ \langle f^k, g^k \rangle_{L^2([0,\infty)^k)} \tag{31}$$

for functions $f^k \in \widetilde{L^2}([0,\infty)^k)$ and $g^{k'} \in \widetilde{L^2}([0,\infty)^{k'})$.
The main theorem of this section now reads as follows:

Theorem 28 (Wiener chaos limit theorem)**.** *Suppose $(X^n)_{n\in\mathbb{N}}$ is a sequence in $L^2(\Omega, \mathcal{F}, P)$. Then the following assertions are equivalent as n tends to infinity:*

(i) *The sequence $(\pi_{\mathcal{H}^n} X^n)$ converges strongly in $L^2(\Omega, \mathcal{F}, P)$.*

(ii) *For every $k \in \mathbb{N}_0$, the sequence $(\widehat{f_{X^n}^{n,k}})_{n\in\mathbb{N}}$, defined via*

$$\widehat{f_{X^n}^{n,k}}(u_1, \ldots, u_k) := \mathbb{E}\left[X^n \frac{n^{k/2}}{k!} \Xi^n_{\{\lceil nu_1 \rceil, \ldots, \lceil nu_k \rceil\}} \right] \mathbf{1}_{\{|\{\lceil nu_1 \rceil, \ldots, \lceil nu_k \rceil\} \cap \mathbb{N}| = k\}}. \tag{32}$$

is strongly convergent in $L^2([0,\infty)^k)$ and

$$\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{k=m}^{\infty} k! \|\widehat{f_{X^n}^{n,k}}\|^2_{L^2([0,\infty)^k)} = 0. \tag{33}$$

In this case, the limit X of $(\pi_{\mathcal{H}^n} X^n)_{n\in\mathbb{N}}$ has the Wiener chaos decomposition $X = \sum_{k=0}^{\infty} I^k(f_X^k)$ with $f_X^k = \lim_{n\to\infty} \widehat{f_{X^n}^{n,k}}$ in $L^2([0,\infty)^k)$.

We recall that, by Remark 26, the strong $L^2(\Omega, \mathcal{F}, P)$-convergence of (X^n) to X is a sufficient condition for the strong approximation of the chaos coefficients of X as stated in (ii) of the above theorem.
Before proving Theorem 28, we briefly discuss this result. To this end, we first recall the relation between Walsh decomposition and discrete chaos decomposition. The discrete multiple Wiener

integrals are defined analogously to the continuous setting, see e.g. [25, Section 1.3]. For all $k, n \in \mathbb{N}$ we consider the Hilbert space

$$L^2_n(\mathbb{N}^k) := \left\{ f^{n,k} : \mathbb{N}^k \to \mathbb{R} : \sum_{(i_1,\dots,i_k)\in\mathbb{N}^k} \Big(f^{n,k}(i_1,\dots,i_k)\Big)^2 < \infty \right\}$$

endowed with the inner product

$$\langle f^{n,k}, g^{n,k}\rangle_{L^2_n(\mathbb{N}^k)} := n^{-k} \sum_{(i_1,\dots,i_k)\in\mathbb{N}^k} f^{n,k}(i_1,\dots,i_k) g^{n,k}(i_1,\dots,i_k).$$

The closed subspace of symmetric functions in $L^2_n(\mathbb{N}^k)$ which vanish on the diagonal part

$$\partial_k := \Big\{ (i_1,\dots,i_k) \in \mathbb{N}^k : \; |\{i_1,\dots,i_k\}| < k \Big\}$$

is denoted by $\widetilde{L^2_n}(\mathbb{N}^k)$.

Then, for $k \in \mathbb{N}$, the discrete multiple Wiener integral of $f^{n,k} \in \widetilde{L^2_n}(\mathbb{N}^k)$ with respect to the random walk B^n is defined as

$$I^{n,k}(f^{n,k}) := n^{-k/2} k! \sum_{(i_1,\dots,i_k)\in\mathbb{N}^k,\; i_1<\dots<i_k} f^{n,k}(i_1,\dots,i_k)\Xi^n_{\{i_1,\dots,i_k\}}.$$

We notice that $I^{n,k}$ is linear on $\widetilde{L^2_n}(\mathbb{N}^k)$ and fulfills $\mathbb{E}[I^{n,k}(f^{n,k})] = 0$ as well as the isometry

$$\mathbb{E}[I^{n,k}(f^{n,k}) I^{n,k'}(g^{n,k'})] = \delta_{k,k'} \; k! \; \langle f^{n,k}, g^{n,k}\rangle_{L^2_n(\mathbb{N}^k)} \tag{34}$$

for $f^{n,k} \in \widetilde{L^2_n}(\mathbb{N}^k)$, $g^{n,k'} \in \widetilde{L^2_n}(\mathbb{N}^{k'})$ and possibly different orders $k, k' \in \mathbb{N}$. As in the continuous time setting, we apply the convention that $I^{n,0}$ is the identity on $\widetilde{L^2_n}(\mathbb{N}^0) := \mathbb{R}$, and refer to [25, Section 1.3] for further properties of such discrete multiple Wiener integrals.

We now fix $X^n \in \mathcal{H}^n$. The Walsh decomposition $X^n = \sum_{|A|<\infty} \mathbb{E}[X^n \Xi^n_A]\Xi^n_A$ implies that the discrete analog of the Wiener chaos decomposition

$$X^n = \sum_{k=0}^{\infty} n^{-k/2} k! \sum_{(i_1,\dots,i_k)\in\mathbb{N}^k, i_1<\dots<i_k} \frac{n^{k/2}}{k!} X^n_{\{i_1,\dots,i_k\}} \Xi^n_{\{i_1,\dots,i_k\}} = \sum_{k=0}^{\infty} I^{n,k}(f^{n,k}_{X^n}), \tag{35}$$

holds for the integrands $f^{n,k}_{X^n} \in \widetilde{L^2_n}(\mathbb{N}^k)$ given by

$$f^{n,k}_{X^n}(i_1,\dots,i_k) := \begin{cases} \mathbb{E}\left[\frac{n^{k/2}}{k!} X^n \Xi^n_{\{i_1,\dots,i_k\}}\right], & |\{i_1,\dots,i_k\} \cap \mathbb{N}| = k \\ 0, & \text{otherwise.} \end{cases} \tag{36}$$

Hence, this discrete analog of the Wiener chaos decomposition (30) for random variables in $\mathcal{H}^n$ is a reformulation of the Walsh decomposition (23).

Given a general element $f^{n,k} \in \widetilde{L^2_n}(\mathbb{N}^k)$ we define its embedding into simple continuous time functions in k variables as

$$\begin{aligned} \widehat{f^{n,k}}(u_1,\dots,u_k) &:= f^{n,k}\left(\lceil nu_1\rceil, \dots, \lceil nu_k\rceil\right) \\ &= \sum_{i_1,\dots,i_k=1}^{\infty} f^{n,k}(i_1,\dots,i_k) \mathbf{1}_{(\frac{i_1-1}{n},\frac{i_1}{n}]\times\dots\times(\frac{i_k-1}{n},\frac{i_k}{n}]}(u_1,\dots,u_k), \end{aligned}$$

which is consistent with the notation already applied in (32) and (36). Here and in what follows, we apply the convention that $f^{n,k}$ vanishes when one of its arguments is set to zero.

We can now rephrase Theorem 28 in the following way:

The sequence (X^n), with $X^n \in \mathcal{H}^n$, converges to X strongly in $L^2(\Omega, \mathcal{F}, P)$, if and only if, for all orders $k \in \mathbb{N}_0$, the sequence of coefficient functions of the discrete chaos decomposition of X^n converge (after the natural embedding into continuous time) to the coefficient functions of the Wiener chaos of X strongly in $L^2([0,\infty)^k)$ and the tail condition (33) is satisfied.

Remark 29. *Convergence of discrete multiple Wiener integrals to continuous multiple Wiener integrals was studied in [28] as a main tool for proving noncentral limit theorems. The results in Section 4 of the latter reference imply that, for every $k \in \mathbb{N}_0$, the sequence of discrete multiple Wiener integrals $(I^{n,k}(f^{n,k}))_{n\in\mathbb{N}}$ converges in distribution to the multiple Wiener integral $I^k(f^k)$, if $(\widehat{f^{n,k}})_{n\in\mathbb{N}}$ converges to f^k strongly in $L^2([0,\infty)^k)$. Our result lifts this convergence in distribution to strong $L^2(\Omega,\mathcal{F},P)$-convergence and, more importantly, adds the converse:*

$$L^2(\Omega,\mathcal{F},P)\text{-}\lim_{n\to\infty} I^{n,k}(f^{n,k}) = I^k(f^k) \;\Leftrightarrow\; L^2([0,\infty)^k)\text{-}\lim_{n\to\infty} \widehat{f^{n,k}} = f^k.$$

We note that the $L^2(\Omega,\mathcal{F},P)$-convergence of the sequence $(I^{n,k}(f^{n,k}))$ even implies convergence in $L^p(\Omega,\mathcal{F},P)$ for $p>2$, if $\mathbb{E}[|\xi|^r]<\infty$ for some $r>p$. Indeed, in this case, the sequence $(|I^{n,k}(f^{n,k})|^p)$ is uniformly integrable by the hypercontractivity inequality of [17] in the variant of [4, Proposition 5.2].

The following elementary corollary of Theorem 28 generalizes Proposition 2. It makes use of the fact that the chaos decompositions of (discrete) Wick exponentials are given, for all $f \in L^2([0,\infty))$, $f^n \in L^2_n(\mathbb{N})$, by

$$e^{\diamond I(f)} = \sum_{k=0}^{\infty} I^k(\frac{1}{k!} f^{\otimes k}),\;\; \exp^{\diamond_n}(I^n(f^n)) = \sum_{k=0}^{\infty} I^{n,k}(\frac{1}{k!}(f^n)^{\otimes k}\mathbf{1}_{\partial_k^c}). \tag{37}$$

For a proof of the continuous case see e.g. [16, Theorem 3.21, Theorem 7.26]. The statement of the discrete case is a direct consequence of (24).

Corollary 30. *Suppose $f \in L^2([0,\infty))$ and (f^n) is a sequence with $f^n \in L^2_n(\mathbb{N})$ for every $n \in \mathbb{N}$. Then, as n tends to infinity (in the sense of strong convergence),*

$$\begin{aligned}\widehat{f^n} \to f \text{ in } L^2([0,\infty)) &\Leftrightarrow I^n(f^n) \to I(f) \text{ in } L^2(\Omega,\mathcal{F},P)\\ &\Leftrightarrow \exp^{\diamond_n}(I^n(f^n)) \to \exp^{\diamond}(I(f)) \text{ in } L^2(\Omega,\mathcal{F},P).\end{aligned}$$

Proof. In view of Theorem 28 and (37), we only have to show that $\widehat{f^n} \to f$ strongly in $L^2([0,\infty))$ implies that $\widehat{(f^n)^{\otimes k}\mathbf{1}_{\partial_k^c}} \to f^{\otimes k}$ strongly in $L^2([0,\infty)^k)$, for every $k \geq 2$. This is a consequence of the following lemma. □

Lemma 31. *(i) Fix $k \in \mathbb{N}_0$. Suppose $(f^{n,k})_{n\in\mathbb{N}}$ is a sequence such that $f^{n,k} \in L^2_n(\mathbb{N}^k)$ for every $n \in \mathbb{N}$ and $(\widehat{f^{n,k}})$ converges to some f^k strongly in $L^2([0,\infty)^k)$. Then, the sequence $(\widehat{f^{n,k}\mathbf{1}_{\partial_k^c}})$ converges to f^k strongly in $L^2([0,\infty)^k)$ as well.*

(ii) Suppose $(f^n)_{n\in\mathbb{N}}$ is a sequence such that $f^n \in L^2_n(\mathbb{N})$ for every $n \in \mathbb{N}$ and $(\widehat{f^n})$ converges to some f strongly in $L^2([0,\infty))$. Then, for every $k \geq 2$, the sequences $(\widehat{(f^n)^{\otimes k}})$ and $(\widehat{(f^n)^{\otimes k}\mathbf{1}_{\partial_k^c}})$ converge to $f^{\otimes k}$ strongly in $L^2([0,\infty)^k)$.

Proof. (i) We decompose,

$$\|\widehat{f^{n,k}\mathbf{1}_{\partial_k^c}} - f^k\|_{L^2([0,\infty)^k)} \leq \|\widehat{f^{n,k}\mathbf{1}_{\partial_k^c}} - \widehat{f^{n,k}}\|_{L^2([0,\infty)^k)} + \|\widehat{f^{n,k}} - f^k\|_{L^2([0,\infty)^k)}.$$

The second term goes to zero by assumption. The first one equals

$$\left(\int_{[0,\infty)^k} |f^{n,k}(\lceil nu_1\rceil,\ldots,\lceil nu_k\rceil)|^2 \mathbf{1}_{\{|\{\lceil nu_1\rceil,\ldots,\lceil nu_k\rceil\}|<k\}}\right)^{1/2}.$$

The sequence of integrands tends to 0 almost everywhere, because

$$\lim_{n\to\infty} \mathbf{1}_{\{|\{\lceil nu_1\rceil,\ldots,\lceil nu_k\rceil\}|<k\}} = \mathbf{1}_{\{\,u_l=u_p,\ \text{for some } l\neq p\}}.$$

Moreover, the sequence of integrands inherits uniform integrability from the $L^2([0,\infty)^k)$-convergent series $(\widehat{f^{n,k}})$. Therefore, the first term goes to zero by interchanging limit and integration.

(ii) As tensor powers commute with discretization and embedding, i.e.

$$(\check{g}^n)^{\otimes k} = ((g)\check{^{\otimes k}})^n, \qquad \widehat{h^n}^{\otimes k} = \widehat{(h^n)^{\otimes k}}$$

for all $k \in \mathbb{N}$, $g \in \mathcal{E}$, $h^n \in L^2_n(\mathbb{N})$, and as the tensor product is continuous, we observe inductively that $\widehat{(f^n)^{\otimes k}} \to f^{\otimes k}$ strongly in $L^2([0,\infty)^k)$. Then, for the second sequence, part (i) applies. □

We now start to prepare the proof of Theorem 28.

Proposition 32. *Let $k \in \mathbb{N}_0$. Then, for all $g \in \mathcal{E}$ and sequences $(f^{n,k})_{n\in\mathbb{N}}$ such that $f^{n,k} \in \widetilde{L^2_n}(\mathbb{N}^k)$ and $\sup_{n\in\mathbb{N}} \|f^{n,k}\|_{L^2_n(\mathbb{N})} < \infty$,*

$$\lim_{n\to\infty} \left|(S^n I^{n,k}(f^{n,k}))(\check{g}^n) - (SI^k(\widehat{f^{n,k}}))(g)\right| = 0.$$

Proof. First note that, by (34), (37), and as $f^{n,k}$ vanishes on the diagonal ∂_k,

$$\begin{aligned}(S^n\ I^{n,k}(f^{n,k}))(\check{g}^n) &= \mathbb{E}\left[I^{n,k}(f^{n,k})\exp^{\diamond_n}(I^n(\check{g}^n))\right] = \langle f^{n,k}, (\check{g}^n)^{\otimes k}\rangle_{L^2_n(\mathbb{N}^k)} \\ &= \int_{[0,\infty)^k} \widehat{f^{n,k}}(x_1,\ldots,x_k)\widehat{(\check{g}^n)^{\otimes k}}(x_1,\ldots,x_k)dx_1\cdots dx_k.\end{aligned}$$

Analogously, making use of the Wiener-Itô isometry for the continuous chaos decomposition (31) instead of (34), we get

$$(SI^k(\widehat{f^{n,k}}))(g) = \int_{[0,\infty)^k} \widehat{f^{n,k}}(x_1,\ldots,x_k)g^{\otimes k}(x_1,\ldots,x_k)dx_1\cdots dx_k.$$

Hence, by the Cauchy-Schwarz inequality, we conclude

$$\begin{aligned}&\left|(S^n I^{n,k}(f^{n,k}))(\check{g}^n) - (SI^k(\widehat{f^{n,k}}))(g)\right| \\ &= \left|\int_{[0,\infty)^k} \widehat{f^{n,k}}(x_1,\ldots,x_k)\left(\widehat{(\check{g}^n)^{\otimes k}} - g^{\otimes k}\right)(x_1,\ldots,x_k)dx_1\cdots dx_k\right| \\ &\le \left(\sup_{m\in\mathbb{N}} \|f^{m,k}\|_{L^2_n(\mathbb{N})}\right)^{1/2} \|g^{\otimes k} - \widehat{(\check{g}^n)^{\otimes k}}\|_{L^2([0,\infty)^k)},\end{aligned}$$

which tends to zero for $n \to \infty$ by Lemma 31. □

Corollary 33. *Suppose $g \in \mathcal{E}$. Then, for every $k \in \mathbb{N}$,*

$$I^{n,k}((\check{g}^n)^{\otimes k}\mathbf{1}_{\partial_k^c}) \to I^k(g^{\otimes k})$$

strongly in $L^2(\Omega, \mathcal{F}, P)$.

Proof. We check item (ii) in Theorem 1. To this end, we decompose, for every $g, h \in \mathcal{E}$,

$$\begin{aligned}&\left|(S^n I^{n,k}((\check{g}^n)^{\otimes k}\mathbf{1}_{\partial_k^c}))(\check{h}^n) - (SI^k(g^{\otimes k}))(h)\right| \\ \le\ &\left|(S^n I^{n,k}((\check{g}^n)^{\otimes k}\mathbf{1}_{\partial_k^c}))(\check{h}^n) - (SI^k(\widehat{(\check{g}^n)^{\otimes k}\mathbf{1}_{\partial_k^c}}))(h)\right| \\ &+ \left|(SI^k(\widehat{(\check{g}^n)^{\otimes k}\mathbf{1}_{\partial_k^c}}))(h) - (SI^k(g^{\otimes k}))(h)\right|.\end{aligned}$$

The first term on the righthand side tends to zero by Proposition 32. The second one equals, by the isometry for multiple Wiener integrals,

$$\int_{[0,\infty)^k} h^{\otimes k}(x)\left(\widehat{(\check{g}^n)^{\otimes k}\mathbf{1}_{\partial_k^c}} - g^{\otimes k}\right)(x)dx$$

and goes to zero by Lemma 31. Consequently,

$$\lim_{n\to\infty} (S^n I^{n,k}((\check{g}^n)^{\otimes k}\mathbf{1}_{\partial_k^c}))(\check{h}^n) = (SI^k(g^{\otimes k}))(h)$$

for all $k \in \mathbb{N}_0$ and $g, h \in \mathcal{E}$. For $h = g$, this implies $\mathbb{E}[I^{n,k}((\breve{g}^n)^{\otimes k}\mathbf{1}_{\partial_k^c})^2] \to \mathbb{E}[I^k(g^{\otimes k})^2]$ by the orthogonality of (discrete) multiple Wiener integrals of different orders. Thus, Theorem 1 applies. $\square$

We are now in the position to give the proof of Theorem 28.

Proof of Theorem 28. '*(i)* $\Rightarrow$ *(ii)*': We denote the limit of $(\pi_{\mathcal{H}^n} X^n)_{n\in\mathbb{N}}$ in $L^2(\Omega, \mathcal{F}, P)$ by X and recall that

$$\pi_{\mathcal{H}^n} X^n = \sum_{|A|<\infty} \mathbb{E}[X^n \Xi_A^n]\Xi_A^n = \sum_{k=0}^{\infty} I^{n,k}(f_{X^n}^{n,k}),$$

with $f_{X^n}^{n,k}$ as defined in (36). Throughout the proof we omit the subscripts from the coefficients and write $\pi_{\mathcal{H}^n} X^n = \sum_{k=0}^{\infty} I^{n,k}(f^{n,k})$ and $X = \sum_{k=0}^{\infty} I^k(f^k)$. Thanks to Corollary 33 and the orthogonality of (discrete) multiple Wiener integrals of different orders, we obtain, for every $k \in \mathbb{N}_0$,

$$(S^n I^{n,k}(f^{n,k}))(\breve{g}^n) = \frac{1}{k!}\mathbb{E}[\pi_{\mathcal{H}^n}(X^n) I^{n,k}((\breve{g}^n)^{\otimes k}\mathbf{1}_{\partial_k^c})] \to \frac{1}{k!}\mathbb{E}[X I^k(g^{\otimes k})](SI^k(f^k))(g).$$

The estimate $\sup_{n\in\mathbb{N}} \mathbb{E}[(I^{n,k}(f^{n,k}))^2] \le \sup_{n\in\mathbb{N}} \mathbb{E}[(\pi_{\mathcal{H}^n} X^n)^2] < \infty$ now yields, in view of Theorem 1, weak $L^2(\Omega, \mathcal{F}, P)$-convergence of $(I^{n,k}(f^{n,k}))_{n\in\mathbb{N}}$ towards $I^k(f^k)$. As $\pi_{\mathcal{H}^n} X^n \to X$ strongly in $L^2(\Omega, \mathcal{F}, P)$, we thus obtain

$$\mathbb{E}[(I^{n,k}(f^{n,k}))^2] = \mathbb{E}[I^{n,k}(f^{n,k})\pi_{\mathcal{H}^n} X^n] \to \mathbb{E}[I^k(f^k)X] = \mathbb{E}[(I^k(f^k))^2]. \tag{38}$$

Hence, $I^{n,k}(f^{n,k}) \to I^k(f^k)$ strongly in $L^2(\Omega, \mathcal{F}, P)$ for all $k \in \mathbb{N}_0$ by Theorem 1. Moreover, for every $g \in \mathcal{E}$, we obtain

$$\begin{aligned}
&\langle g^{\otimes k}, \widehat{f^{n,k}} - f^k\rangle_{L^2([0,\infty)^k)} = (SI^k(\widehat{f^{n,k}}))(g) - (SI^k(f^k))(g) \\
&= \Big((SI^k(\widehat{f^{n,k}}))(g) - (S^n I^{n,k}(f^{n,k}))(\breve{g}^n)\Big) \\
&\quad + \mathbb{E}\left[I^{n,k}(f^{n,k}) \exp^{\diamond_n}(I^n(\breve{g}^n)) - I^k(f^k)\exp^{\diamond}(I(g))\right] \to 0,
\end{aligned}$$

by Propositions 2 and 32, and the $L^2(\Omega, \mathcal{F}, P)$-convergence of $(I^{n,k}(f^{n,k}))_{n\in\mathbb{N}}$ to $I^k(f^k)$. Since the set $\{g^{\otimes k}, g \in \mathcal{E}\}$ is total in $\widetilde{L^2}([0,\infty)^k)$, we may conclude that $(\widehat{f^{n,k}})$ converges weakly in $\widetilde{L^2}([0,\infty)^k)$ to f^k by [29, Theorem V.1.3]. Finally, (38) and the isometry for discrete multiple Wiener integrals turn this weak convergence into strong $L^2([0,\infty)^k)$-convergence. In particular, the kth coefficient in the chaos decomposition of the limiting random variable X is the strong $L^2([0,\infty)^k)$-limit of $(\widehat{f^{n,k}})$, as asserted. It remains to show (33). However, by (38) and the isometries for (discrete) multiple Wiener integrals,

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{k=m}^{\infty} k! \|\widehat{f^{n,k}}\|^2_{L^2([0,\infty)^k)} = \lim_{n\to\infty} \sum_{k=m}^{\infty} \|I^{n,k}(f^{n,k})\|^2_{L^2(\Omega,\mathcal{F},P)} \\
&= \lim_{n\to\infty} \left(\|\pi_{\mathcal{H}^n} X^n\|^2_{L^2(\Omega,\mathcal{F},P)} - \sum_{k=0}^{m-1} \|I^{n,k}(f^{n,k})\|^2_{L^2(\Omega,\mathcal{F},P)} \right) \\
&= \|X\|^2_{L^2(\Omega,\mathcal{F},P)} - \sum_{k=0}^{m-1} \|I^k(f^k)\|^2_{L^2(\Omega,\mathcal{F},P)} \to 0
\end{aligned}$$

as m tends to infinity.

'*(ii)* $\Rightarrow$ *(i)*': In order to lighten the notation, we again denote the function $f_{X^n}^{n,k}$ from (36) by $f^{n,k}$. Assuming (ii), the strong $L^2([0,\infty)^k)$-limit of $\widehat{f^{n,k}}$ exists and will be denoted f^k. We first show that $(I^{n,k}(f^{n,k}))$ converges to $I^k(f^k)$ strongly in $L^2(\Omega,\mathcal{F},P)$ for all $k\in\mathbb{N}_0$ by means of Theorem 1. To this end, we observe that, for every $g\in\mathcal{E}$,

$$(S^n\, I^{n,k}(f^{n,k}))(\check{g}^n) = \Big((S^n\, I^{n,k}(f^{n,k}))(\check{g}^n) - (S\, I^k(\widehat{f^{n,k}}))(g)\Big) + (S\, I^k(\widehat{f^{n,k}}))(g) \to (S\, I^k(f^k))(g)$$

by Proposition 32 and the isometry for continuous multiple Wiener integrals. Moreover, again, by the isometries for discrete and continuous multiple Wiener integrals,

$$\mathbb{E}\left[(I^{n,k}(f^{n,k}))^2\right] = k!\|f^{n,k}\|^2_{L^2_n(\mathbb{N}^k)} = k!\|\widehat{f^{n,k}}\|^2_{L^2([0,\infty)^k)} \to k!\|f^k\|^2_{L^2([0,\infty)^k)} = \mathbb{E}\left[(I^k(f^k))^2\right].$$

So, Theorem 1 applies indeed. With the $L^2(\Omega,\mathcal{F},P)$-convergence of $I^{n,k}(f^{n,k})$ to $I^k(f^k)$ at hand, we can now decompose, for every $m\in\mathbb{N}$,

$$\begin{aligned}
&\limsup_{n\to\infty}\mathbb{E}\left[\left|\pi_{\mathcal{H}^n}X^n - \sum_{k=0}^{\infty} I^k(f^k)\right|^2\right]\\
&\le 3\limsup_{n\to\infty}\Bigg(\|\sum_{k=0}^{m-1} I^k(f^k) - \sum_{k=0}^{m-1} I^{n,k}(f^{n,k})\|^2_{L^2(\Omega,\mathcal{F},P)}\\
&\qquad + \sum_{k=m}^{\infty}\|I^k(f^k)\|^2_{L^2(\Omega,\mathcal{F},P)} + \sum_{k=m}^{\infty}\|I^{n,k}(f^{n,k})\|^2_{L^2(\Omega,\mathcal{F},P)}\Bigg)\\
&= 3\sum_{k=m}^{\infty} k!\|f^k\|^2_{L^2([0,\infty)^k)} + 3\limsup_{n\to\infty}\sum_{k=m}^{\infty} k!\|\widehat{f^{n,k}}\|^2_{L^2([0,\infty)^k)}.
\end{aligned}\tag{39}$$

By Fatou's lemma and the strong convergence of $(\widehat{f^{n,k}})_{n\in\mathbb{N}}$ to f^k,

$$\sum_{k=m}^{\infty} k!\|f^k\|^2_{L^2([0,\infty)^k)} \le \liminf_{n\to\infty}\sum_{k=m}^{\infty} k!\|\widehat{f^{n,k}}\|^2_{L^2([0,\infty)^k)}.$$

Hence, letting m tend to infinity in (39), we observe, thanks to (33), that $(\pi_{\mathcal{H}^n}X^n)$ converges strongly in $L^2(\Omega,\mathcal{F},P)$. □

We close this section with an example.

Example 34. *Fix $X\in L^2(\Omega,\mathcal{F},P)$. Theorem 28 with $X^n = X$ for every $n\in\mathbb{N}$, implies that the chaos coefficients f^k_X, $k\in\mathbb{N}_0$, of X are given as the strong $L^2([0,\infty)^k)$-limit of*

$$\widehat{f^{n,k}}(u_1,\dots,u_k) := \frac{1}{k!}\,\mathbb{E}\left[X\left(\prod_{l=1}^{k}\frac{B^n_{\lceil nu_l\rceil} - B^n_{(\lceil nu_l\rceil-1)}}{1/n}\right)\right]\mathbf{1}_{\{|\{\lceil nu_1\rceil,\dots,\lceil nu_k\rceil\}\cap\mathbb{N}|=k\}}.$$

This formula can be further simplified when X is $\mathcal{F}_T$-measurable. Then, one can show, analogously to Example 4 (ii), that the sequence $(\pi_{\mathcal{H}^n_{\lfloor nT\rfloor}}X)$ converges to X strongly in $L^2(\Omega,\mathcal{F},P)$. Applying Theorem 28 with the latter sequence, shows that the chaos coefficients f^k_X, $k\in\mathbb{N}_0$, are the strong $L^2([0,\infty)^k)$-limit of

$$\widehat{f^{n,k}}(u_1,\dots,u_k) := \frac{1}{k!}\,\mathbb{E}\left[X\left(\prod_{l=1}^{k}\frac{B^n_{\lceil nu_l\rceil} - B^n_{(\lceil nu_l\rceil-1)}}{1/n}\right)\right]\mathbf{1}_{\{|\{\lceil nu_1\rceil,\dots,\lceil nu_k\rceil\}\cap\{1,\dots,\lfloor nT\rfloor\}|=k\}}.$$

In this case, for each fixed $n\in\mathbb{N}$, only finitely many of the functions $\widehat{f^{n,k}}$, $k\in\mathbb{N}_0$, are not constant zero, and these are simple functions with finitely many steps sizes only.
These two approximation formulas for the chaos coefficients of X are one way to give a rigorous meaning of the heuristic formula

$$f^k_X(u_1,\dots,u_k) = \frac{1}{k!}\mathbb{E}\left[X\left(\prod_{l=1}^{k}\dot{B}_{u_l}\right)\right],$$

where $\dot{B}$ is white noise, which is called Wiener's intuitive recipe in [9]. The latter paper provides another rigorous meaning to Wiener's recipe via nonstandard analysis, which is closely related to our approximation formulas in the special case of symmetric Bernoulli noise. The authors show that

$$f_X^k({}^\circ t_1, \dots, {}^\circ t_k) = \frac{1}{k!}\, {}^\circ\mathbb{E}\left[x(b)\left(\frac{\Delta b_{t_1}}{\Delta t}\cdots\frac{\Delta b_{t_k}}{\Delta t}\right)\right],$$

$t_l \in T = \{j\Delta t,\ 0 \le j < N^2\}$, where N is infinite, $\Delta t = 1/N$, $b_t(\omega) = \sqrt{\Delta t}\sum_{s<t}\omega(s)$, $t \in T$, $\omega \in \Omega := \{-1,1\}^T$, which is equipped with the internal counting measure, $x(b)$ is a lifting of X, $\mathbb{E}$ is the expectation operator with respect to the internal counting measure, and the circle denotes the standard part.

6. Strong L^2-approximation of the Skorokhod integral and the Malliavin derivative

In this section, we apply the Wiener chaos limit theorem (Theorem 28) in order to prove strong L^2-approximation results for the Skorokhod integral and the Malliavin derivative. For the construction of the approximating sequences we compose the discrete Skorokhod integral and the discretized Malliavin derivative with the orthogonal projection on $\mathcal{H}^n$, i.e. on the subspace of random variables which admit a discrete chaos decomposition in terms of multiple integrals with respect to the discrete time noise $(\xi_i^n)_{i\in\mathbb{N}}$.
We first treat the Malliavin derivative and aim at proving the following result.

Theorem 35. *Suppose $(X^n)_{n\in\mathbb{N}}$ converges strongly in $L^2(\Omega, \mathcal{F}, P)$ to X and, for every $n \in \mathbb{N}$, $\pi_{\mathcal{H}^n} X^n \in \mathbb{D}_n^{1,2}$. Then the following are equivalent:*

(i) $\lim_{m\to\infty}\limsup_{n\to\infty}\sum_{k=m}^{\infty} kk!\|\widehat{f_{X_n}^{n,k}}\|^2_{L^2([0,\infty)^k)} = 0$ *(with $\widehat{f_{X_n}^{n,k}}$ as defined in (32)).*

(ii) *$X \in \mathbb{D}^{1,2}$ and the sequence $(D^n_{\lceil n\cdot\rceil}(\pi_{\mathcal{H}^n}X^n))_{n\in\mathbb{N}}$ converges to DX strongly in $L^2(\Omega \times [0,\infty))$ as n tends to infinity.*

Note first, that by continuity of D_i^n for a fixed time $i \in \mathbb{N}$, we get

$$\begin{aligned} D_i^n(\pi_{\mathcal{H}^n}X^n) &= \sum_{|A|<\infty} \mathbb{E}[X^n \Xi_A^n] D_i^n \Xi_A^n = \sqrt{n}\sum_{|A|<\infty;\, i\in A} \mathbb{E}[X^n\Xi_A^n]\Xi^n_{A\setminus\{i\}} \\ &= \sqrt{n}\sum_{|B|<\infty;\, i\notin B}\mathbb{E}[X^n\Xi^n_{B\cup\{i\}}]\Xi_B^n. \end{aligned}$$

By the relation (35)–(36) between Walsh decomposition and discrete chaos decomposition, this identity can be reformulated as

$$D_i^n(\pi_{\mathcal{H}^n}X^n) = \sum_{k=1}^{\infty} kI^{n,k-1}(f^{n,k}_{X^n}(\cdot, i)).$$

Hence, the isometry for discrete multiple Wiener integrals (34) implies

$$\frac{1}{n}\sum_{i=1}^{\infty}\mathbb{E}\left[|D_i^n(\pi_{\mathcal{H}^n}X^n)|^2\right] = \sum_{k=1}^{\infty} kk!\|\widehat{f_{X^n}^{n,k}}\|^2_{L^2([0,\infty)^k)}, \tag{40}$$

i.e.,

$$\pi_{\mathcal{H}^n}X^n \in \mathbb{D}_n^{1,2} \;\Leftrightarrow\; \sum_{k=1}^{\infty} kk!\|\widehat{f_{X^n}^{n,k}}\|^2_{L^2([0,\infty)^k)} < \infty.$$

This is in line with the characterization of the continuous Malliavin derivative in terms of the chaos decomposition, see e.g. [23], which we show to be equivalent to Definition 11 in the Appendix:

$$X \in \mathbb{D}^{1,2} \;\Leftrightarrow\; \sum_{k=1}^{\infty} kk!\|f_X^k\|^2_{L^2([0,\infty)^k)} < \infty, \tag{41}$$

and, if this is the case,

$$D_t X = \sum_{k=1}^{\infty} k I^{n,k-1}(f_X^k(\cdot, t)), \text{ a.e. } t \geq 0, \tag{42}$$

$$\int_0^\infty \mathbb{E}[(D_t X)^2]dt = \sum_{k=1}^{\infty} kk! \|f_X^k\|^2_{L^2([0,\infty)^k)}.$$

After these considerations on the connection between (discretized) Malliavin derivative and (discrete) chaos decomposition, the proof of Theorem 35 turns out to be rather straightforward.

Proof of Theorem 35. By Theorem 28 (in conjunction with Remark 26), we observe that, for every $k \in \mathbb{N}_0$, $(\widehat{f^{n,k}_{X_n}})_{n\in\mathbb{N}}$ converges to f_X^k strongly in $L^2([0,\infty)^k)$. Hence, by (40), (41), and (42),

$$\begin{aligned}
(i) \quad &\Leftrightarrow \quad \lim_{n\to\infty} \sum_{k=1}^{\infty} kk! \|\widehat{f^{n,k}_{X^n}}\|^2_{L^2([0,\infty)^k)} = \sum_{k=1}^{\infty} kk! \|f_X^k\|^2_{L^2([0,\infty)^k)} < \infty \\
&\Leftrightarrow \quad X \in \mathbb{D}^{1,2} \text{ and } \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{\infty} \mathbb{E}\left[|D_i^n(\pi_{\mathcal{H}^n} X^n)|^2\right] = \int_0^\infty \mathbb{E}[(D_t X)^2]dt.
\end{aligned}$$

Hence, the asserted equivalence is a direct consequence of Theorem 12. □

We now wish to derive an analogous strong approximation result for the Skorokhod integral, which requires some additional notation. For every $Z^n \in L^2_n(\Omega \times \mathbb{N})$ and $k \in \mathbb{N}_0$, we denote

$$\mathfrak{f}^{n,k}_{Z^n}(i_1, \dots, i_k, i) := f^{n,k}_{Z^n_i}(i_1, \dots, i_k) = \begin{cases} \mathbb{E}\left[\frac{n^{k/2}}{k!} Z_i^n \Xi^n_{\{i_1,\dots,i_k\}}\right], & |\{i_1, \dots, i_k\} \cap \mathbb{N}| = k \\ 0, & \text{otherwise.} \end{cases}$$

Then, with $\pi_{\mathcal{H}^n} Z^n := (\pi_{\mathcal{H}^n} Z_i^n)_{i\in\mathbb{N}}$,

$$\sum_{k=0}^{\infty} k! \|\mathfrak{f}^{n,k}_{Z^n}\|^2_{L^2_n(\mathbb{N}^{k+1})} = \|\pi_{\mathcal{H}^n} Z^n\|^2_{L^2_n(\Omega\times\mathbb{N})} < \infty,$$

but $\mathfrak{f}^{n,k}_{Z^n}$ is symmetric in the first k variables only and does not, in general, vanish on the diagonal. For a function F in k variables, we denote its symmetrization by

$$\widetilde{F}(y_1, \dots, y_k) = \frac{1}{k!} \sum_{\pi} F(y_{\pi(1)}, \dots, y_{\pi(k)}),$$

where the sum runs over the group of permutations of $\{1, \dots, k\}$. With this notation, $\widetilde{\mathfrak{f}}^{n,k}_{Z^n} \mathbf{1}_{\partial^c_{k+1}}$ is an element of $\widetilde{L^2_n}(\mathbb{N}^{k+1})$.
We can now state:

Theorem 36. *Suppose that, for every $n \in \mathbb{N}$, $Z^n \in L^2_n(\Omega \times \mathbb{N})$ and $\pi_{\mathcal{H}^n} Z^n \in D(\delta^n)$. Moreover, assume that $(Z^n_{\lceil n\cdot\rceil})_{n\in\mathbb{N}}$ converges to Z strongly in $L^2(\Omega \times [0,\infty))$. Then, the following assertions are equivalent:*

(i) $\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{k=m}^{\infty} k! \|\widetilde{\mathfrak{f}}^{n,k-1}_{Z^n} \mathbf{1}_{\partial^c_k}\|^2_{L^2_n(\mathbb{N}^k)} = 0.$
(ii) *$Z \in D(\delta)$ and $(\delta^n(\pi_{\mathcal{H}^n} Z^n))$ converges to $\delta(Z)$ strongly in $L^2(\Omega, \mathcal{F}, P)$ as n tends to infinity.*

As a preparation of the proof we note that, for every $M \in \mathbb{N}$,

$$\sum_{i=1}^{M} \mathbb{E}[\pi_{\mathcal{H}^n} Z_i^n | \mathcal{F}^n_{M,-i}] \frac{\xi_i^n}{\sqrt{n}} = \sum_{i=1}^{M} \sum_{|A|<\infty} \mathbb{E}[Z_i^n \Xi_A^n]\, \mathbb{E}[\Xi_A^n | \mathcal{F}^n_{M,-i}] \frac{\xi_i^n}{\sqrt{n}}$$

$$
\begin{aligned}
&= n^{-1/2} \sum_{i=1}^{M} \sum_{A \subset \{1,\dots,M\}} \mathbf{1}_{\{i \notin A\}} \, \mathbb{E}[Z_i^n \Xi_A^n] \Xi_{A \cup \{i\}}^n \\
&= n^{-1/2} \sum_{k=1}^{M} \sum_{B \subset \{1\dots,M\}, |B|=k} \sum_{i \in B} \mathbb{E}[Z_i^n \Xi_{B \setminus \{i\}}^n] \Xi_B^n \\
&= n^{-1/2} \sum_{k=1}^{M} k \sum_{\substack{(i_1,\dots,i_k) \in \mathbb{N}^k, \\ i_1 < \dots < i_k}} \mathbf{1}_{[1,M]}^{\otimes k}(i_1, \dots, i_k) \frac{1}{k} \sum_{j=1}^{k} \mathbb{E}[Z_{i_j}^n \Xi_{\{i_1,\dots,i_k\} \setminus \{i_j\}}^n] \Xi_{\{i_1,\dots,i_k\}}^n \\
&= \sum_{k=1}^{M} I^{n,k}(\widetilde{\mathfrak{f}}_{Z^n}^{n,k-1} \mathbf{1}_{[1,M]}^{\otimes k} \mathbf{1}_{\partial_k^c}).
\end{aligned}
$$

Hence, by the isometry for discrete multiple Wiener integrals,

$$
\pi_{\mathcal{H}^n} Z^n \in D(\delta^n) \;\Leftrightarrow\; \sum_{k=1}^{\infty} k! \|\widetilde{\mathfrak{f}}_{Z^n}^{n,k-1} \mathbf{1}_{\partial_k^c}\|_{L_n^2(\mathbb{N}^k)}^2 < \infty, \tag{43}
$$

and, if this is the case,

$$
\delta^n(\pi_{\mathcal{H}^n} Z^n) = \sum_{k=1}^{\infty} I^{n,k}(\widetilde{\mathfrak{f}}_{Z^n}^{n,k-1} \mathbf{1}_{\partial_k^c}), \tag{44}
$$

i.e., $f_{\delta^n(\pi_{\mathcal{H}^n} Z^n)}^{n,0} = 0$ and, for every $k \in \mathbb{N}$, $f_{\delta^n(\pi_{\mathcal{H}^n} Z^n)}^{n,k} = \widetilde{\mathfrak{f}}_{Z^n}^{n,k-1} \mathbf{1}_{\partial_k^c}$.
For the proof of Theorem 36, we also provide the following variant of Theorem 28, '$(i) \Rightarrow (ii)$', for stochastic processes.

Proposition 37. *Suppose $Z^n \in L_n^2(\Omega \times \mathbb{N})$ for every $n \in \mathbb{N}$ and $(Z_{\lceil n \cdot \rceil}^n)$ converges strongly in $L^2(\Omega \times [0,\infty))$ to Z as n tends to infinity. Define the functions $\mathfrak{f}_Z^k \in L^2([0,\infty)^{k+1})$ via $\mathfrak{f}_Z^k(t_1, \dots, t_{k+1}) := f_{Z_{t_{k+1}}}^k(t_1, \dots, t_k)$. Then, for every $k \in \mathbb{N}_0$, as n tends to infinity, $\widehat{\mathfrak{f}_{Z^n}^{n,k}} \to \mathfrak{f}_Z^k$ strongly in $L^2([0,\infty)^{k+1})$.*

Proof. The proof largely follows the arguments in the proof of Theorem 28. We spell it out for sake of completeness. Let $g, h \in \mathcal{E}$. Then, by the isometry for (discrete) multiple Wiener integrals, Corollary 33, and (5),

$$
\begin{aligned}
\left\langle \widehat{\mathfrak{f}_{Z^n}^{n,k}}, \widehat{(\check{g}^n)}^{\otimes k} \otimes \widehat{\check{h}^n} \right\rangle_{L^2([0,\infty)^{k+1})} &= \frac{1}{n} \sum_{i=1}^{\infty} \left\langle f_{Z_i^n}^{n,k}, (\check{g}^n)^{\otimes k} \mathbf{1}_{\partial_k^c} \right\rangle_{L_n^2(\mathbb{N}^k)} \check{h}^n(i) \\
&= \frac{1}{n} \sum_{i=1}^{\infty} \mathbb{E}[(\pi_{\mathcal{H}^n} Z_i^n) I^{n,k}((\check{g}^n)^{\otimes k} \mathbf{1}_{\partial_k^c})] \check{h}^n(i) \\
&= \frac{1}{n} \sum_{i=1}^{\infty} \mathbb{E}[Z_i^n I^{n,k}((\check{g}^n)^{\otimes k} \mathbf{1}_{\partial_k^c})] \check{h}^n(i) \to \int_0^\infty \mathbb{E}[Z_s I^k(g^{\otimes k})] h(s) ds \\
&= \left\langle \mathfrak{f}_Z^k, g^{\otimes k} \otimes h \right\rangle_{L^2([0,\infty)^{k+1})}.
\end{aligned} \tag{45}
$$

As

$$
\sup_{n \in \mathbb{N}} \left\| \widehat{\mathfrak{f}_{Z^n}^{n,k}} \right\|_{L^2([0,\infty)^{k+1})}^2 = \sup_{n \in \mathbb{N}} \int_0^\infty \mathbb{E}\left[\left| I^{n,k}(f_{Z_{\lceil ns \rceil}^n}^{n,k}) \right|^2 \right] ds \le \sup_{n \in \mathbb{N}} \left\| Z_{\lceil n \cdot \rceil}^n \right\|_{L^2(\Omega \times [0,\infty))}^2 < \infty, \tag{46}
$$

$\widehat{(\check{g}^n)}^{\otimes k} \otimes \widehat{\check{h}^n} \to g^{\otimes k} \otimes h$ strongly in $L^2([0,\infty)^{k+1})$ by (5), and the set $\{g^{\otimes k} \otimes h : g, h \in \mathcal{E}\}$ is total in the closed subspace of functions in $L^2([0,\infty)^{k+1})$, which are symmetric in the first k

variables, we conclude again that $\widehat{\mathfrak{f}^{n,k}_{Z^n}}$ converges weakly to $\mathfrak{f}^k_Z$ in this subspace. Hence, it only remains to argue that

$$\left\|\widehat{\mathfrak{f}^{n,k}_{Z^n}}\right\|^2_{L^2([0,\infty)^{k+1})} \to \left\|\mathfrak{f}^k_Z\right\|^2_{L^2([0,\infty)^{k+1})}, \quad n \to \infty.$$

As

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{\infty} \mathbb{E}[Z_i^n I^{n,k}((\check{g}^n)^{\otimes k}\mathbf{1}_{\partial_k^c})]\check{h}^n(i) &= \frac{1}{n}\sum_{i=1}^{\infty}(S^n I^{n,k}(f^{n,k}_{Z_i^n}))(\check{g}^n)\check{h}^n(i), \\ \int_0^\infty \mathbb{E}[Z_s I^k(g^{\otimes k})]h(s)ds &= \int_0^\infty (SI^k(f^k_{Z_s}))(g)h(s)ds, \end{aligned}$$

we may derive from (45)–(46) and Theorem 10, that $I^{n,k}(f^{n,k}_{Z^n_{\lceil n\cdot\rceil}})$ converges to $I^k(f^k_{Z_\cdot})$ weakly in $L^2(\Omega\times[0,\infty))$. Thus,

$$\begin{aligned} \left\|\widehat{\mathfrak{f}^{n,k}_{Z^n}}\right\|^2_{L^2([0,\infty)^{k+1})} &= \int_0^\infty \mathbb{E}\left[I^{n,k}(f^{n,k}_{Z^n_{\lceil ns\rceil}})Z_s\right]ds + \int_0^\infty \mathbb{E}\left[I^{n,k}(f^{n,k}_{Z^n_{\lceil ns\rceil}})(Z^n_{\lceil ns\rceil}-Z_s)\right]ds \\ &\to \int_0^\infty \mathbb{E}\left[I^k(f^k_{Z_s})Z_s\right]ds = \left\|\mathfrak{f}^k_Z\right\|^2_{L^2([0,\infty)^{k+1})}. \end{aligned}$$

□

Proof of Theorem 36. By the linearity of the embedding operator $\widehat{(\cdot)}$, Minkowski inequality, Proposition 37, and Lemma 31, we obtain, for every $k\in\mathbb{N}_0$,

$$\left\|\widehat{\widetilde{\mathfrak{f}}^{n,k}_{Z^n}\mathbf{1}_{\partial^c_{k+1}}} - \widetilde{\mathfrak{f}}^k_Z\right\|_{L^2([0,\infty)^{k+1})} = \left\|\widetilde{\widehat{\mathfrak{f}^{n,k}_{Z^n}\mathbf{1}_{\partial^c_{k+1}}}} - \widetilde{\mathfrak{f}}^k_Z\right\|_{L^2([0,\infty)^{k+1})} \le \left\|\widehat{\mathfrak{f}^{n,k}_{Z^n}\mathbf{1}_{\partial^c_{k+1}}} - \mathfrak{f}^k_Z\right\|_{L^2([0,\infty)^{k+1})} \to 0$$

as n tends to infinity. Thus, due to Theorem 28 and (44),

$$(i) \Leftrightarrow (\delta^n(\pi_{\mathcal{H}^n}Z^n))_{n\in\mathbb{N}} \text{ converges strongly in } L^2(\Omega,\mathcal{F},P).$$

Now, the implication '$(ii)\Rightarrow(i)$' is obvious, while the converse implication is a consequence of Theorem 8. □

Remark 38. *As a by-product of the proof of Theorem 36, we recover, thanks to Theorem 28, the well-known chaos decomposition of the Skorokhod integral as*

$$\delta(Z) = \sum_{k=1}^{\infty} I^k(\widetilde{\mathfrak{f}}^{k-1}_Z).$$

7. Examples

In this section, we first specialize the previous results to the case of binary noise in discrete time. Then, we explain how to translate the main results of this paper into the language of convergence in distribution without imposing the condition that the discrete-time noise is embedded into the driving Brownian motion.

7.1. Binary noise.

In this subsection, we specialize to the case of binary noise, i.e., we suppose that, for some constant $b>0$,

$$P(\{\xi=-1/b\}) = \frac{b^2}{b^2+1}, \quad P(\{\xi=b\}) = \frac{1}{b^2+1}.$$

We illustrate, that in this binary case, our approximation formulas for the Malliavin derivative and the Skorokhod integral give rise to a straightforward numerical implementation.
We recall first that Malliavin calculus on the Bernoulli space is well-studied, see, e.g. [12], [21], [25], and the references therein, usually with the aim to explain the main ideas of Malliavin calculus by discussing the analogous operators in the simple toy setting. Note first that $L^2(\Omega,\mathcal{F}^n_i,P)$

equals $\mathcal{H}_i^n$ in the binary case (and in this case only) by observing that both spaces have dimension 2^i in this case. Hence, $L^2(\Omega, \mathcal{F}^n, P)$ coincides with $\mathcal{H}^n$ for binary noise, and we can drop the orthogonal projections $\pi_{\mathcal{H}^n}$ on $\mathcal{H}^n$ in the statement of all previous results. In particular, every random variable $X^n \in L^2(\Omega, \mathcal{F}^n, P)$ then admits a chaos decomposition in terms of discrete multiple Wiener integrals, and the representations of the discretized Malliavin derivative and the discrete Skorokhod integral in terms of the discrete chaos in Section 6 show that these operators coincide with the Malliavin derivative and the Skorokhod integral on the Bernoulli space, see [25].

In the binary case, the representations for the discrete Malliavin derivative and the discrete Skorokhod integral can be simplified considerably. Suppose $X^n \in L^2(\Omega, \mathcal{F}^n, P)$. Then, there is a measurable map $F_{X^n} : \mathbb{R}^\infty \to \mathbb{R}$ such that $X^n = F_{X^n}(\xi_1^n, \xi_2^n, \ldots)$. A direct computation shows that, for every $i \in \mathbb{N}$,

$$\begin{aligned} D_i^n X &= \sqrt{n}\mathbb{E}[\xi_i^n F_{X^n}(\xi_1^n, \xi_2^n, \ldots)|\mathcal{F}_{-i}^n] \\ &= \frac{\sqrt{n}b}{b^2+1}\left(F_{X^n}(\xi_1^n, \ldots, \xi_{i-1}^n, b, \xi_{i+1}^n, \ldots) - F_{X^n}(\xi_1^n, \ldots, \xi_{i-1}^n, -1/b, \xi_{i+1}^n, \ldots)\right), \end{aligned}$$

hence, the Malliavin derivative becomes a difference operator. Moreover, for $Z^n \in L_n^2(\Omega \times \mathbb{N})$ and $N \in \mathbb{N}$, the discrete Skorokhod integral can be rewritten as

$$\delta^n(Z^n \mathbf{1}_{[1,N]}) = \sum_{i=1}^N Z_i^n \frac{\xi_i^n}{\sqrt{n}} - \frac{1}{n}\sum_{i=1}^N (\xi_i^n)^2 D_i^n Z_i^n,$$

which can either be derived from [25, Proposition 1.8.3] or by expanding Z_i^n in its Walsh decomposition and noting that, for every finite subset $A \subset \mathbb{N}$,

$$\left(\Xi_A^n - \mathbb{E}[\Xi_A^n|\mathcal{F}_{-i}^n]\right)\sqrt{n}\xi_i^n = \left\{\begin{array}{cc} \sqrt{n}\Xi_{A\setminus\{i\}}^n(\xi_i^n)^2, & i \in A \\ 0, & i \notin A \end{array}\right\} = (\xi_i^n)^2 D_i^n \Xi_A^n.$$

Hence, for $Z^n \in L_n^2(\Omega \times \mathbb{N})$ and $N \in \mathbb{N}$,

$$\begin{aligned} \delta^n(Z^n\mathbf{1}_{[1,N]}) &= \sum_{i=1}^N F_{Z_i^n}(\xi_1^n, \xi_2^n, \ldots)\frac{\xi_i^n}{\sqrt{n}} \\ &\quad - \sum_{i=1}^N \frac{(\xi_i^n)^2 b}{\sqrt{n}(b^2+1)}\left(F_{Z_i^n}(\xi_1^n, \ldots, \xi_{i-1}^n, b, \xi_{i+1}^n, \ldots) F_{Z_i^n}(\xi_1^n, \ldots, \xi_{i-1}^n, -1/b, \xi_{i+1}^n, \ldots)\right). \qquad (47) \end{aligned}$$

Recall that the discrete noise $(\xi_i^n)_{i\in\mathbb{N}}$, can be constructed from the underlying Brownian motion $(B_t)_{t\in[0,\infty)}$ via a Skorokhod embedding as

$$\xi_i^n = \sqrt{n}\left(B_{\tau_i^n} - B_{\tau_{i-1}^n}\right),$$

where, in the binary case,

$$\tau_0^n := 0\ ,\ \tau_i^n := \inf\left\{s \geq \tau_{i-1}^n : B_s - B_{\tau_{i-1}^n} \in \left\{\frac{b}{\sqrt{n}}, \frac{-1}{b\sqrt{n}}\right\}\right\}, \qquad (48)$$

and the Brownian motion at the first-passage times τ_i^n can be simulated by the acceptance-rejection algorithm of Burq&Jones [7].

We close this paper by a toy example which illustrates how to numerically compute Skorokhod integrals by our approximation results.

Example 39. *In this example, we approximate the Skorokhod integral $\delta(Z)$ for the process*

$$Z_t = \operatorname{sign}(1/2 - t)(B_1 B_{1-t} - (1-t)))\mathbf{1}_{[0,1]}(t), \quad t \geq 0,$$

where we choose the sign-function to be rightcontinuous at 0. For the discrete time approximation we consider

$$Z_i^n = \operatorname{sign}(1/2 - i/n)\left(B_n^n B_{n-i}^n - (1-i/n)\right)\mathbf{1}_{[1,n-1]}(i), \quad i \in \mathbb{N},$$

and note that $(Z^n_{\lceil nt \rceil})$ *converges to* Z_t *for almost every* $t \geq 0$ *in probability by (1). Hence, by uniform integrability and dominated convergence, it is easy to check that* $(Z^n_{\lceil n \cdot \rceil})_{n\in\mathbb{N}}$ *converges to* Z *strongly in* $L^2(\Omega \times [0,\infty))$*. We next observe that in the discrete chaos decomposition of* $\delta^n(Z^n)$*, all the coefficient functions* $f^{n,k}_{\delta^n(Z^n)}$ *for* $k \geq 4$ *vanish, because* Z^n_i *is a polynomial of degree 2 in* B^n*. Hence, the tail condition in Theorem 36 is trivially satisfied and, consequently,* $(\delta^n(Z^n))_{n\in\mathbb{N}}$ *converges to* $\delta(Z)$ *strongly in* $L^2(\Omega, \mathcal{F}, P)$*. We now suppose that* B^n *is constructed via the Skorokhod embedding (48) and simulate, for* $n = 4, 8, \ldots, 2^{15}$*, 10000 independent copies* $(B^{n,l})_{l=1,\ldots,10000}$ *of* B^n *by the Burq&Jones algorithm. The corresponding realizations of* $\delta^n(Z^n)$

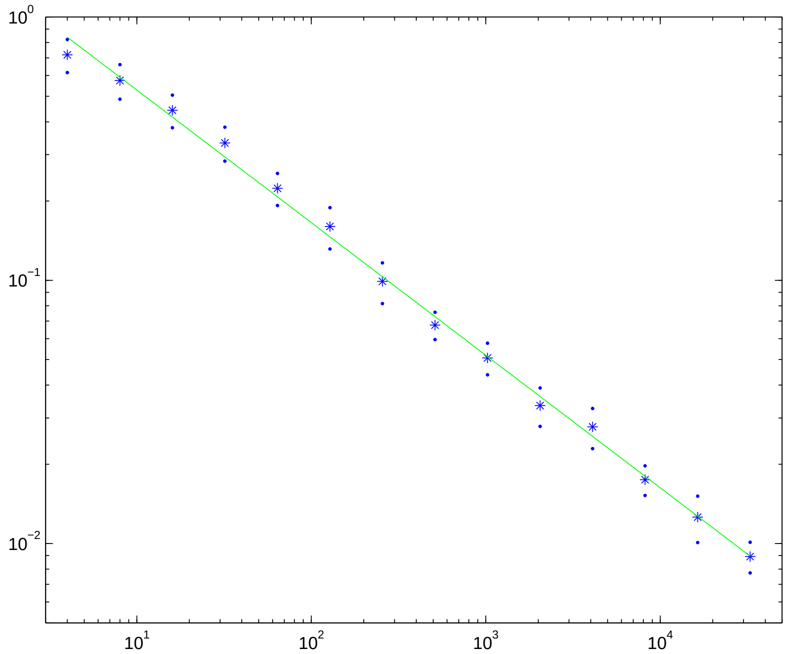

FIGURE 1. Log-log plot of the simulated strong $L^2(\Omega, \mathcal{F}, P)$-approximation as the number of time steps increases.

and $\delta(Z)$ *along the* l*th trajectory of the underlying Brownian motion are denoted* $\delta^n_l(Z^n)$ *and* $\delta_l(Z)$*,* $l = 1, \ldots 10000$*, respectively. For the discrete Skorokhod integral we implement formula (47) with* $N = n$*, while for the continuous Skorokhod integral we exploit that it can be computed analytically and equals*

$$\delta(Z) = B_1 B_{1/2}^2 - \frac{B_1}{2} - B_{1/2}.$$

Figure 1 shows, in the case of symmetric binary noise ($b = 1$*), a log-log-plot of the empirical mean (indicated by crosses) of* $|\delta^n_l(Z^n) - \delta_l(Z)|^2$*,* $l = 1, \ldots, 10000$*, and the corresponding (asymptotical) 95%-confidence bounds (indicated by dots) as the number of time steps* n *increases. A linear regression (solid line) exhibits a slope of* -0.5036 *and, thus, indicates that strong* $L^2(\Omega, \mathcal{F}, P)$*-convergence takes place at the expected rate of* $1/2$*.*

7.2. **Donsker type theorems.** In this subsection, we explain how the main results of the paper can be translated into Donsker type results on convergence in distribution, without assuming that the discrete time noise is embedded into the driving Brownian motion B. To this end, we construct, without loss of generality, the discrete time noise as coordinate mapping on the canonical space $(\mathbb{R}^\infty, \mathcal{B}^\infty, P_\xi^\infty)$, where P_ξ is the distribution of ξ and P_ξ^∞ its countable product measure. We denote expectation with respect to P_ξ^∞ by $E[\cdot]$, while $\mathbb{E}[\cdot]$ still denotes expectation on $(\Omega, \mathcal{F}, P)$, i.e. on the probability space which carries the driving Brownian motion B. We write $L^2(\mathbb{R}^\infty \times \mathbb{N})$ for the set of mappings $z : \mathbb{N} \to L^2(\mathbb{R}^\infty, \mathcal{B}^\infty, P_\xi^\infty)$ such that $\sum_{i=1}^\infty E[z_i^2] < \infty$.

We also note that, for every $n \in \mathbb{N}$, the map $F \mapsto F((\xi_i^n)_{i\in\mathbb{N}})$ provides a one-to-one correspondence between $L^2(\mathbb{R}^\infty, \mathcal{B}^\infty, P_\xi^\infty)$ and $L^2(\Omega, \mathcal{F}^n, P)$ thanks to the Doob-Dynkin lemma. The rescaled random walk approximation to B on $(\mathbb{R}^\infty, \mathcal{B}^\infty, P_\xi^\infty)$ is given by $b_t^n(x) = \frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor nt \rfloor} x_i$, and, hence, $B_t^n = b_t^n((\xi_i^n)_{i\in\mathbb{N}})$.
The translation of the strong L^2-approximation results derived in the previous sections into results on convergence in distribution relies on the following key lemma. Here, $\mathbb{D}([0,\infty))$ denotes the space of rightcontinuous functions with left limits from $[0,\infty)$ to $\mathbb{R}$ endowed with the Skorokhod topology.

Lemma 40. *a) Suppose $(F^n)_{n\in\mathbb{N}}$ is a sequence in $L^2(\mathbb{R}^\infty, \mathcal{B}^\infty, P_\xi^\infty)$ and $X \in L^2(\Omega, \mathcal{F}, P)$. Then, the following are equivalent:*

- (i) *The sequence $(F^n((\xi_i^n)_{i\in\mathbb{N}}))_{n\in\mathbb{N}}$ converges to X strongly in $L^2(\Omega, \mathcal{F}, P)$.*
- (ii) *The sequence $(F^n, b^n)_{n\in\mathbb{N}}$ converges in distribution to (X, B) in $\mathbb{R} \times \mathbb{D}([0,\infty))$ and the sequence $(|F^n|^2)_{n\in\mathbb{N}}$ is uniformly integrable (with respect to P_ξ^∞).*

b) Suppose $(z^n)_{n\in\mathbb{N}}$ is a sequence in $L^2(\mathbb{R}^\infty \times \mathbb{N})$ and $Z \in L^2(\Omega \times [0,\infty))$. Then, the following are equivalent:

- (i) *The sequence $(z_{\lceil n\cdot \rceil}^n((\xi_i^n)_{i\in\mathbb{N}}))_{n\in\mathbb{N}}$ converges to Z strongly in $L^2(\Omega \times [0,\infty))$.*
- (ii) *The sequence $(z_{\lceil n\cdot \rceil}^n, b^n)_{n\in\mathbb{N}}$ converges in distribution to (Z, B) in $L^2([0,\infty)) \times \mathbb{D}([0,\infty))$ and the sequence $(n^{-1}\sum_{i=1}^n (z_i^n)^2)_{n\in\mathbb{N}}$ is uniformly integrable (with respect to P_ξ^∞).*

Proof. We first prove b). We wish to apply Theorem 44 in Appendix B with $E = \mathbb{D}([0,\infty))$, $\mathcal{H} = L^2([0,\infty))$, $L = B$, and $L^n = B^n$. To this end, we first note that the sequence $(B_{\lfloor nt \rfloor/n} - B^n)$ converges to zero in finite-dimensional distributions by (1). It is easy to check by standard criteria such as [5, Theorem 15.6] that this sequence is tight in $\mathbb{D}([0,\infty))$. Hence, this sequence converges to zero in distribution in $\mathbb{D}([0,\infty))$. Thus, by the (generalized) continuous mapping theorem [5, Theorem 5.5], $\sup_{t\in[0,K]} |B_{\lfloor nt \rfloor/n} - B_t^n|$ converges to zero in distribution (and consequently in probability) for every $K \in \mathbb{N}$. Then, by continuity of B, (B^n) converges to B uniformly on compacts in probability. In particular, the Skorokhod distance between B^n and B converges to zero in probability. Moreover, $L^2(\Omega \times [0,\infty)) = L^2(\Omega, \mathcal{F}, P; L^2([0,\infty)))$ by [16, Appendix C]. Therefore, Theorem 44 is indeed applicable and yields the equivalence of (i) and

- (ii') The sequence $(z_{\lceil n\cdot \rceil}^n((\xi_j^n)_{j\in\mathbb{N}}), B^n)_{n\in\mathbb{N}}$ converges in distribution to (Z, B) in $L^2([0,\infty)) \times \mathbb{D}([0,\infty))$ and the sequence $(n^{-1}\sum_{i=1}^n |z_i^n((\xi_j^n)_{j\in\mathbb{N}})|^2)_{n\in\mathbb{N}}$ is uniformly integrable (with respect to P).

Since $(z_{\lceil n\cdot \rceil}^n((\xi_j^n)_{j\in\mathbb{N}}), B^n)$ has the same distribution under P as $(z_{\lceil n\cdot \rceil}^n, b^n)$ has under P_ξ^∞, the equivalence of (ii) and (ii') is immediate. The proof of a) is the very same, but with $\mathcal{H} = \mathbb{R}$. $\square$

As an example, we now explain how to reformulate two of the main convergence results for the Malliavin derivative (Theorems 12 and 16) in the present setting. We leave it to the reader to rewrite Theorem 35 and the corresponding theorems for the Skorokhod integral in the obvious way. For $F \in L^2(\mathbb{R}^\infty, \mathcal{B}^\infty, P_\xi^\infty)$ and $i \in \mathbb{N}$, we denote

$$\mathcal{D}_i^n F := \sqrt{n} \int_{\mathbb{R}} x_i F(x_1, \ldots, x_i, \ldots) P_\xi(dx_i),$$

whence, $(\mathcal{D}_i^n F)((\xi_j^n)_{j\in\mathbb{N}}) = D_i^n F((\xi_j^n)_{j\in\mathbb{N}})$.

Theorem 41. *Suppose $(F^n)_{n\in\mathbb{N}}$ is a sequence in $L^2(\mathbb{R}^\infty, \mathcal{B}^\infty, P_\xi^\infty)$ such that $(|F^n|^2)_{n\in\mathbb{N}}$ is uniformly integrable and $(F^n, b^n)_{n\in\mathbb{N}}$ converges in distribution to (X, B) in $\mathbb{R} \times \mathbb{D}([0,\infty))$ for some $X \in L^2(\Omega, \mathcal{F}, P)$. Then:*

a) If the sequence $(n^{-1}\sum_{i=1}^\infty (\mathcal{D}_i^n F^n)^2)_{n\in\mathbb{N}}$ is uniformly integrable, then $X \in \mathbb{D}^{1,2}$ and the following are equivalent as n tends to infinity:

- (i) *$(n^{-1}\sum_{i=1}^\infty E[(\mathcal{D}_i^n F^n)^2])_{n\in\mathbb{N}}$ converges to $\int_0^\infty \mathbb{E}[(D_s X)^2] ds$.*

(ii) *There is a* $Z \in L^2(\Omega \times [0,\infty))$ *such that* $(\mathcal{D}^n_{\lceil n\cdot\rceil} F^n, b^n)_{n\in\mathbb{N}}$ *converges in distribution to* (Z, B) *in* $L^2([0,\infty)) \times \mathbb{D}([0,\infty))$.

(iii) $(\mathcal{D}^n_{\lceil n\cdot\rceil} F^n, b^n)_{n\in\mathbb{N}}$ *converges in distribution to* (DX, B) *in* $L^2([0,\infty)) \times \mathbb{D}([0,\infty))$.

b) If

$$\sup_{n\in\mathbb{N}} \left(\frac{1}{n} \sum_{i=1}^{\infty} E\left[(\mathcal{D}^n_i F^n)^2\right] + \frac{1}{n^2} \sum_{i,j=1, i\neq j}^{\infty} E\left[(\mathcal{D}^n_j \mathcal{D}^n_i F^n)^2\right] \right) < \infty,$$

then $(n^{-1}\sum_{i=1}^{\infty}(\mathcal{D}^n_i F^n)^2)_{n\in\mathbb{N}}$ *is uniformly integrable and* $(\mathcal{D}^n_{\lceil n\cdot\rceil} F^n, b^n)_{n\in\mathbb{N}}$ *converges in distribution to* (DX, B) *in* $L^2([0,\infty)) \times \mathbb{D}([0,\infty))$.

Remark 42. *The general assumptions of the previous theorem are, in particular, satisfied, when* $X = g(B)$ *and* $F^n = g(b^n)$, $n \in \mathbb{N}$, *for some bounded and continuous function* $g : \mathbb{D}([0,\infty)) \to \mathbb{R}$.

Proof of Theorem 41. We write $X^n := F^n((\xi^n_i)_{i\in\mathbb{N}})$. Note first, that the assumptions guarantee, in view of Lemma 40 a), that $(X^n)_{n\in\mathbb{N}}$ converges to X strongly in $L^2(\Omega, \mathcal{F}, P)$.

a) Uniform integrability implies norm-boundedness. Hence,

$$\sup_{n\in\mathbb{N}} \frac{1}{n} \sum_{i=1}^{\infty} \mathbb{E}\left[(D^n_i X^n)^2\right] = \sup_{n\in\mathbb{N}} \frac{1}{n} \sum_{i=1}^{\infty} E\left[(\mathcal{D}^n_i F^n)^2\right] < \infty.$$

Thus, by Theorem 12, $X \in \mathbb{D}^{1,2}$ and $(D^n_{\lceil n\cdot\rceil} X^n)$ converges to DX weakly in $L^2(\Omega \times [0,\infty))$.

'(i)⇒ (iii)': As

$$\|D^n_{\lceil n\cdot\rceil} X^n\|^2_{L^2(\Omega\times[0,\infty))} = \frac{1}{n} \sum_{i=1}^{\infty} E[(\mathcal{D}^n_i F^n)^2], \tag{49}$$

condition (i) turns weak $L^2(\Omega\times[0,\infty))$- convergence of $(D^n_{\lceil n\cdot\rceil} X^n)$ to DX into strong convergence. Hence, Lemma 40 b) concludes.

'(iii)⇒ (i)': Assuming (iii), we obtain from Lemma 40 b), that $(D^n_{\lceil n\cdot\rceil} X^n)$ converges to DX strongly in $L^2(\Omega \times [0,\infty))$. Hence, the sequence of $L^2(\Omega \times [0,\infty))$-norms converges as well, which thanks to (49) implies (i).

'(ii)⇒ (iii)': If (ii) holds, $(D^n_{\lceil n\cdot\rceil} X^n)$ converges to Z strongly in $L^2(\Omega \times [0,\infty))$ by Lemma 40 b). The strong $L^2(\Omega \times [0,\infty))$-limit Z must, of course, coincide with the weak $L^2(\Omega \times [0,\infty))$-limit DX, which establishes (iii).

'(iii)⇒ (ii)': obvious.

b) This assertion is an immediate consequence of Lemma 40 and Theorem 16, '(i) ⇒ (ii)'. □

Appendix A. S-transform characterization of the Malliavin derivative

In this appendix, we prove the equivalence between the definition of the Malliavin derivative in terms of the S-transform (Definition 11) and the more classical characterization in terms of the chaos decomposition, see (41)–(42).

Proposition 43. *Suppose* $X = \sum_k I^k(f^k_X) \in L^2(\Omega, \mathcal{F}, P)$. *Then, the following are equivalent:*

(i) *There is a stochastic process* $Z \in L^2(\Omega \times [0,\infty))$ *such that, for every* $g, h \in \mathcal{E}$,

$$\int_0^{\infty} (SZ_s)(g)h(s)ds = \mathbb{E}\left[X \exp^{\diamond}(I(g)) \left(I(h) - \int_0^{\infty} g(s)h(s)ds\right)\right].$$

(ii) $\sum_{k=1}^{\infty} kk! \|f^k_X\|^2_{L^2([0,\infty)^k)} < \infty$.

If this is the case, then $Z_t = \sum_{k=1}^{\infty} kI^{n,k-1}(f^k_X(\cdot,t))$ *for almost every* $t \geq 0$.

Proof. We first note that, for every $f, g \in \mathcal{E}$,

$$e^{\diamond I(g)}\left(I(h) - \int_0^\infty g(s)h(s)ds\right) = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} I^k((\widetilde{g^{\otimes(k-1)} \otimes h})), \tag{50}$$

which can be verified by computing the S-transform of both sides. By the Cauchy-Schwarz inequality, we obtain for every $f, g \in \mathcal{E}$,

$$\begin{aligned}&\sum_{k=1}^{\infty} \int_{[0,\infty)^k} \left| k f_X^k(x)(g^{\otimes k-1} \otimes h)(x)\right| dx \\ &\leq \left(\sum_{k=1}^{\infty} k! \|f_X^k\|^2_{L^2([0,\infty)^k)}\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} \frac{k}{(k-1)!} \|g\|^{2(k-1)}_{L^2([0,\infty))} \|h\|^2_{L^2([0,\infty))}\right)^{\frac{1}{2}} < \infty.\end{aligned}$$

Hence, Fubini's theorem implies

$$\sum_{k=1}^{\infty} \int_{[0,\infty)^k} k f_X^k(x)(g^{\otimes k-1} \otimes h)(x)dx = \int_0^\infty \left(\sum_{k=1}^{\infty} \int_{[0,\infty)^{k-1}} k f_X^k(x,t) g^{\otimes k-1}(x)\right) h(t)dt,$$

i.e., by (50) and the isometry for multiple Wiener integrals,

$$\mathbb{E}\left[X \exp^{\diamond}(I(g)) \left(I(h) - \int_0^\infty g(s)h(s)ds\right)\right] = \int_0^\infty \left(\sum_{k=1}^{\infty} \int_{[0,\infty)^{k-1}} k f_X^k(x,t) g^{\otimes k-1}(x)\right) h(t)dt \tag{51}$$

for every $g, h \in \mathcal{E}$.

'$(i) \Rightarrow (ii)$': Assuming (i) and noting that (51) holds for every $g, h \in \mathcal{E}$, we observe that for every $g \in \mathcal{E}$, $\alpha \in \mathbb{R}$, and Lebesgue-almost every $s \in [0,\infty)$,

$$\sum_{k=1}^{\infty} \alpha^{k-1} \langle f_{Z_s}^{k-1}(\cdot), g^{\otimes(k-1)}\rangle_{L^2([0,\infty)^{k-1})} = (SZ_s)(\alpha g) = \sum_{k=1}^{\infty} \alpha^{k-1} \langle k f_X^k(\cdot, s), g^{\otimes(k-1)}\rangle_{L^2([0,\infty)^{k-1}}.$$

(Note, that the Lebesgue null set can be chosen independent of g, α. Indeed, one can first take $\alpha \in \mathbb{Q}$ and step functions g with rational step sizes and interval limits, and then pass to the limit). Comparing the coefficients in the power series and noting that $\{g^{\otimes k}, g \in \mathcal{E}\}$ is total in $\widetilde{L^2}([0,\infty)^k)$, we obtain, for every $k \geq 1$ and almost every $s \in [0,\infty)$,

$$k f_X^k(\cdot, s) = f_{Z_s}^{k-1}.$$

Therefore, the isometry for multiple Wiener-Itô integrals implies

$$\sum_{k=1}^{\infty} k k! \|f_X^k\|^2_{L^2([0,\infty)^k)} = \int_0^\infty \mathbb{E}[|Z_s|^2]ds < \infty. \tag{52}$$

'$(ii) \Rightarrow (i)$': Define $Z_t = \sum_{k=1}^{\infty} k I^{n,k-1}(f_X^k(\cdot, t))$. Assuming (ii), we observe by the first identity in (52) that Z belongs to $L^2(\Omega \times [0,\infty))$. By the isometry for multiple Wiener integrals and the chaos decomposition of a Wick exponential we get, for every $g, h \in \mathcal{E}$.

$$\int_0^\infty (SZ_s)(g)h(s)ds = \int_0^\infty \left(\sum_{k=1}^{\infty} \int_{[0,\infty)^{k-1}} k f_X^k(x,t) g^{\otimes k-1}(x)dx\right) h(t)dt.$$

Hence, (51) concludes. □

Appendix B. On the connection between strong L^2-convergence and convergence in distribution

In this appendix, we prove, that, under suitable conditions, strong L^2-convergence is completely specified by convergence in distribution of appropriate joint distributions in conjunction with uniform integrability. This is the essential tool for the results in Section 7.2.

We make use of the following notations and assumptions:

- $(\Omega, \mathcal{F}, P)$ is a probability space, (E, d_E) is a separable metric space, $\mathcal{B}_E$ denotes the Borel σ-field on E and $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is a separable real Hilbert space.
- $L : (\Omega, \mathcal{F}) \to (E, \mathcal{B}_E)$ is a measurable map.
- $L_n : (\Omega, \mathcal{F}) \to (E, \mathcal{B}_E)$ for all $n \in \mathbb{N}$, such that L_n is $\sigma(L)$-measurable and the sequence $(L_n)_{n\in\mathbb{N}}$ satisfies $d_E(L, L_n) \to 0$ in probability as n tends to infinity.
- $L^2(\Omega, \sigma(L), P; \mathcal{H})$ is the space of $\mathcal{H}$-valued, $\sigma(L)$-measurable and square-integrable random variables (i.e $X : (\Omega, \sigma(L)) \to \mathcal{H}$ measurable, where $\mathcal{H}$ is endowed with its Borel σ-field, and satisfies $\mathbb{E}[\|X\|^2_{\mathcal{H}}] < \infty$).

The convergence in distribution is denoted by $\Rightarrow$.

Theorem 44. *Suppose the notations and assumptions above, $X \in L^2(\Omega, \sigma(L), P; \mathcal{H})$ and $(X_n)_{n\in\mathbb{N}}$ satisfies $X_n \in L^2(\Omega, \sigma(L), P; \mathcal{H})$ for all $n \in \mathbb{N}$. Then the following assertions are equivalent as n tends to infinity:*

(i) *$X_n \to X$ strongly in $L^2(\Omega, \sigma(L), P; \mathcal{H})$.*
(ii) *$(X_n, L_n) \Rightarrow (X, L)$ in $\mathcal{H} \times E$ and $(\|X_n\|^2_{\mathcal{H}})_{n\in\mathbb{N}}$ is uniformly integrable.*

Proof. '$(i) \Rightarrow (ii)$': Due to the assumptions on the sequence $(L_n)_{n\in\mathbb{N}}$ and (i), we have

$$\|X_n - X\|^2_{\mathcal{H}} + d_E(L_n, L) \to 0 \text{ in probability}$$

as n tends to infinity. This immediately implies the convergence in distribution in (ii). Moreover, by the Cauchy-Schwarz inequality, we observe for $n \to \infty$,

$$\mathbb{E}[|\|X_n\|^2_{\mathcal{H}} - \|X\|^2_{\mathcal{H}}|] = \mathbb{E}[|\langle X_n - X, X_n + X\rangle|] \leq \mathbb{E}[|\|X_n - X\|^2_{\mathcal{H}}]^{1/2}\mathbb{E}[|\|X_n + X\|^2_{\mathcal{H}}]^{1/2} \to 0.$$

This gives the uniform integrability of the sequence $(\|X_n\|^2_{\mathcal{H}})_{n\in\mathbb{N}}$.

'$(ii) \Rightarrow (i)$': We firstly observe by the continuous mapping theorem, as n tends to infinity,

$$\|X_n\|^2_{\mathcal{H}} \Rightarrow \|X\|^2_{\mathcal{H}}.$$

Hence, the uniform integrability implies

$$\mathbb{E}[\|X_n\|^2_{\mathcal{H}}] \to \mathbb{E}[\|X\|^2_{\mathcal{H}}]. \tag{53}$$

Now we fix a uniformly continuous and bounded (u.c.b., for shorthand) map $f : E \to \mathbb{R}$ and $h \in \mathcal{H}$. The continuous mapping theorem gives that

$$f(L_n)\langle X_n, h\rangle_{\mathcal{H}} \Rightarrow f(L)\langle X, h\rangle_{\mathcal{H}}.$$

Moreover, by the de la Vallée-Poussin criterion, the inequality

$$\sup_{n\in\mathbb{N}} \mathbb{E}[|f(L_n)\langle X_n, h\rangle_{\mathcal{H}}|^2] \leq (\sup_{e\in E} |f(e)|^2)\|h\|^2_{\mathcal{H}} \sup_{n\in\mathbb{N}} \mathbb{E}[\|X_n\|^2_{\mathcal{H}}] < \infty$$

implies the uniform integrability of $(\langle X_n, f(L_n)h\rangle_{\mathcal{H}})_{n\in\mathbb{N}}$. Thus we conclude

$$\mathbb{E}[\langle X_n, f(L_n)h\rangle_{\mathcal{H}}] \to \mathbb{E}[\langle X, f(L)h\rangle_{\mathcal{H}}]. \tag{54}$$

Hence, (53) and (54) equal the expressions in Lemma 3 (ii) for the Hilbert space $L^2(\Omega, \sigma(L), P; \mathcal{H})$ and it suffices to verify the assumptions in Lemma 3, i.e. that

(iii) The set $\{f(L)h : f \text{ is u.c.b.}, h \in \mathcal{H}\}$ is total in $L^2(\Omega, \sigma(L), P; \mathcal{H})$.
(iv) For all u.c.b. f and $h \in \mathcal{H}$: $f(L_n)h \to f(L)h$ strongly in $L^2(\Omega, \sigma(L), P; \mathcal{H})$ as n tends to infinity.

'(iii)': As $L^2(\Omega, \sigma(L), P; \mathcal{H})$ is (a realization of) $L^2(\Omega, \sigma(L), P) \otimes \mathcal{H}$ (tensor product in the sense of Hilbert spaces, cf. [16, Appendix E]), it suffices to show that the set $\{f(L) : f \text{ is u.c.b.}\}$ is dense in $L^2(\Omega, \sigma(L), P)$. Let $X = F(L) \in L^2(\Omega, \sigma(L), P)$ and assume that $\mathbb{E}[Xf(L)] = 0$ for all u.c.b. f. Then

$$\int_E F(x)f(x)P_L(dx) = 0,$$

and, as in [5, Theorem 1.3], this implies

$$\int_A F(x)P_L(dx) = 0, \tag{55}$$

for every closed set $A \subset E$. Thanks to the regularity of probability measures on metric spaces, this implies (55) for all $A \in \mathcal{B}_E$. Hence, $F = 0$ P_L-a.s., and therefore $X = F(L) = 0$ P-a.s. Thus we conclude (iii).

'(iv)': Due to $d_E(L_n, L) \to 0$ in probability and the continuous mapping theorem, we obtain $f(L_n) \to f(L)$ in probability. Hence, the dominated convergence theorem implies

$$\mathbb{E}[\|f(L_n)h - f(L)h\|_{\mathcal{H}}^2] = \mathbb{E}[|f(L_n) - f(L)|^2]\|h\|_{\mathcal{H}}^2 \to 0$$

as n tends to infinity. □

References

[1] Aigner, M. (2007). *A course in Enumeration*. Springer, Berlin. MR 2339282

[2] Avram, F. (1988). Weak convergence of the variations, iterated integrals and Doléans-Dade exponentials of sequences of semimartingales. *Ann. Probab.* **16** 246–250. MR 0920268

[3] Avram, F. and Taqqu, M. (1986). Symmetric polynomials of random variables attracted to an infinitely divisible law. *Probab. Theory Relat. Fields* **71** 491–500. MR 0833266

[4] Bai, S. and Taqqu, M. (2014). Generalized Hermite processes, discrete chaos and limit theorems. *Stochastic Processes Appl.* **124** 1710–1739. MR 3163219

[5] Billingsley, P. (1968). *Convergence of Probability Measures*. John Wiley and Sons, Inc., New York-London-Sydney-Toronto.

[6] Briand, P., Delyon, B. and Mémin, J. (2002). On the robustness of backward stochastic differential equations. *Stochastic Processes Appl.* **97** 229–253. MR 1875334

[7] Burq, Z. A. and Jones, O. D. (2008). Simulation of Brownian motion at first-passage times. *Math. Comput. Simulation* **77** 64–71. MR 2388251

[8] Coquet, F., Mémin, J. and Slominski, L. (2001). On weak convergence of filtrations. *Séminaire de probabilités XXXV. Lecture Notes in Math.* **1755** Springer, Berlin. 306–328. MR 1837294

[9] Cutland, N. and Ng, S. (1991). On homogeneous chaos. *Math. Proc. Cambridge Philos. Soc.* **110** 353–363. MR 1113433

[10] Geiss, C., Geiss, S. and Gobet, E. (2012). Generalized fractional smoothness and L^p-variation of BSDEs with non-Lipschitz terminal condition. *Stochastic Process. Appl.* **122** 2078–2116. MR 2921973

[11] Gzyl, H. (2006). An exposé on discrete Wiener chaos expansions. *Bol. Asoc. Mat. Venez.* **13** 3–27. MR 2267626

[12] Holden, H., Lindstrøm, T., Øksendal, B. and Ubøe, J. (1993). Discrete Wick products. Stochastic analysis and related topics. *Stochastics Monogr.* **8**, Gordon and Breach, Montreux, 123–148. MR 1268009

[13] Holden H., Øksendal, B., Ubøe, J. and Zhang, T. (2010). *Stochastic Partial Differential Equations. A Modeling, White Noise Functional Approach*, 2nd ed. Springer, New York. MR 2571742

[14] Jacod, J., Méléard, S. and Protter, P. (2000). Explicit form and robustness of martingale representations. *Ann. Probab.* **28** 1747–1780. MR 1813842

[15] Jakubowski, A., Mémin, J. and Pagès, G. (1989). Convergence en loi des suites d'intégrales stochastiques sur l'espace D^1 de Skorokhod. *Probab. Theory Related Fields* **81** 111–137. MR 0981569

[16] Janson, S. (1997). *Gaussian Hilbert Spaces*. Cambridge University Press, Cambridge. MR 1474726

[17] Krakowiak, W. and Szulga, J. (1986). Random multilinear forms. *Ann. Probab.* **14** 955–973. MR 0841596

[18] Kuo, H.-H. (1996). *White Noise Distribution Theory*. Probability and Stochastics Series. CRC Press, Boca Raton. MR 1387829

[19] Kurtz, T. G. and Protter, P. (1991). Weak limit theorems for stochastic integrals and stochastic differential equations. *Ann. Probab.* **19** 1035–1080. MR 1112406

[20] Leão, D. and Ohashi, A. (2013). Weak approximation for Wiener functionals. *Ann. Appl. Probab.* **23** 1660–1691. MR 3098445

[21] Leitz-Martini, M. (2000). A discrete Clark-Ocone formula. *Maphysto Research Report* No. 29.

[22] Mörters, P. and Peres, Y. (2010). *Brownian motion.* Cambridge University Press, Cambridge. MR 2604525

[23] Nualart, D. (2006). *The Malliavin Calculus and Related Topics*, 2nd ed. Springer, New York. MR 2200233

[24] Nualart, D. and Pardoux, E. (1988). Stochastic calculus with anticipating integrands. *Probab. Theory Related Fields.* **78** 535–581. MR 0950346

[25] Privault, N. (2009). *Stochastic Analysis in Discrete and Continuous Settings. Lecture Notes in Mathematics* **1982**. Springer, Berlin. MR 2531026

[26] Shiryaev, A. N. (1996). *Probability*, 2nd ed. Springer, New York. MR 1368405

[27] Sottinen, T. (2001). Fractional Brownian motion, random walks and binary market models. *Finance Stoch.* **5** 343–355. MR 1849425

[28] Surgailis, D. (1982). Domains of attraction of self-similar multiple integrals. *Lithuanian Math. J.* **22** 185–201. MR 0684472

[29] Yosida, K. (1995). *Functional analysis.* Reprint of the sixth (1980) edition. Classics in Mathematics. Springer, Berlin. MR 1336382

[30] Zhang, J. (2004). A numerical scheme for BSDEs. *Ann. Appl. Probab.* **14** 459–488. MR 2023027

Saarland University, Department of Mathematics PO Box 151150, D-66041 Saarbrücken, Germany,
University of Mannheim, Institute of Mathematics A5,6, D-68131 Mannheim, Germany.
E-mail address: bender@math.uni-sb.de, parczewski@math.uni-mannheim.de