Addendum to: A-posteriori estimates for backward SDEs *

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Abstract

In this addendum we make the constants in Theorem 3.1 explicit.

Throughout this addendum, the assumptions and notation of Theorem 3.1 in Bender and Steiner, A-posteriori estimates for backward SDEs, are in force without further notice. This reference will be abbreviated by [BS] from now on.

Lemma 0.1.

$$\begin{aligned}
&\sum_{i=1}^{N} \sup_{s\in[t_i,t_{i+1}]} \mathrm{E}\left[|Y_s - Y_{t_i}|^2\right] \\
\leq\; & \left[4(1+K^2T)e^{T(K+1)^2} + 8K^2 e^{T(K+1)^2}(T+2+2K^2T)\,|\pi|\right] \mathrm{E}\left[|\xi|^2\right] \\
&+ \left[4(1+K^2T)e^{T(K+1)^2} + \left(8K^2 e^{T(K+1)^2}(3+2K^2T)+4\right)|\pi|\right] E\left[\int_0^T |f(r,0,0)|^2\,dr\right].
\end{aligned}$$

Proof. Define

$$\tilde{C}_i = \sup_{s\in[t_i,t_{i+1}]} \mathrm{E}\left[|Y_s - Y_{t_i}|^2\right].$$

Due to the defintion of Y, Itô's isometry, an several applications of Young's inequality we have

$$\begin{aligned}
\tilde{C}_i &\leq 2 \sup_{s\in[t_i,t_{i+1}]} \mathrm{E}\left[\left|\int_{t_i}^{s} f(r,Y_r,Z_r)dr\right|^2 + \int_{t_i}^{s} |Z_r|^2\,dr\right] \\
&\leq 4\,|\pi|\,\mathrm{E}\left[\int_{t_i}^{t_{i+1}} \left(|f(r,Y_r,Z_r)-f(r,0,0)|^2 + |f(r,0,0)|^2\right)dr\right] + 2\mathrm{E}\left[\int_{t_i}^{t_{i+1}} |Z_r|^2\,dr\right] \\
&\leq 8K^2\,|\pi|\,\mathrm{E}\left[\int_{t_i}^{t_{i+1}} \left(|Y_r|^2 + |Z_r|^2\right)dr\right] + 4\,|\pi|\,\mathrm{E}\left[\int_{t_i}^{t_{i+1}} |f(r,0,0)|^2\,dr\right] + 2\mathrm{E}\left[\int_{t_i}^{t_{i+1}} |Z_r|^2\,dr\right].
\end{aligned}$$

Hence,

$$\sum_{i=0}^{N-1} \tilde{C}_i \leq 8K^2\,|\pi|\,\mathrm{E}\left[\int_0^T |Y_r|^2\,dr\right] + (8K^2\,|\pi|+2)\mathrm{E}\left[\int_0^T |Z_r|^2\,dr\right] + 4\,|\pi|\,\mathrm{E}\left[\int_0^T |f(r,0,0)|^2\,dr\right].$$

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By Proposition 2.2 in El Karoui (1997) with $\beta = (K+1)^2$ we get

$$\mathrm{E}\left[\int_0^T |Y_t|^2\,dt\right] \le e^{T(K+1)^2}\left(T\mathrm{E}\left[|\xi|^2\right] + \mathrm{E}\left[\int_0^T |f(t,0,0)|^2\,dt\right]\right),$$
$$\mathrm{E}\left[\int_0^T |Z_t|^2\,dt\right] \le 2(1+K^2T)e^{T(K+1)^2}\left(\mathrm{E}\left[|\xi|^2\right] + \mathrm{E}\left[\int_0^T |f(t,0,0)|^2\,dt\right]\right).$$

Thus,

$$\sum_{i=0}^{N-1}\tilde{C}_i \le \left[4(1+K^2T)e^{T(K+1)^2} + 8K^2e^{T(K+1)^2}(T+2+2K^2T)\,|\pi|\right] E\,|\xi|^2$$
$$+\left[4(1+K^2T)e^{T(K+1)^2} + \left(8K^2e^{T(K+1)^2}(3+2K^2T)+4\right)|\pi|\right] E\left[\int_0^T |f(r,0,0)|^2\,dr\right].$$

□

The following lemma is a variant of Lemma 3.3 in [BS] with explicit constants.

Lemma 0.2.

$$\max_{0\le i\le N}\mathrm{E}\left[\left|Y_{t_i} - Y^\pi_{t_i}\right|^2\right] + \sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\mathrm{E}\left[\left|Z_t - Z^\pi_{t_i}\right|^2\right]dt$$
$$\le d_1\mathcal{E}_\pi(\hat{Y}^\pi,\hat{Z}^\pi) + K_y\,|\pi|\left(d_2\mathrm{E}\left[|\xi|^2\right] + d_3E\left[\int_0^T |f(r,0,0)|^2\,dr\right]\right)$$
$$+d_4\mathrm{E}\left[|\xi-\xi^\pi|^2\right] + d_5\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\mathrm{E}\left[|f(t,Y_t,Z_t) - f^\pi(t_i,Y_t,Z_t)|^2\right]dt$$

holds for $|\pi| \le \min\{1, (8K(1+4K))^{-1}, (4K^2(T+1)(D\vee 2) + 16TK^4(1+T)^2(D\vee 2)^2)^{-1}\}$*, where*

$$\begin{aligned}
d_1 &= (\frac{d_4}{4}+2)(2+D)\left(2+C_1(1+K^2T(T+D))\right)\\
d_2 &= \frac{d_4}{4K}\left((4+4K^2)e^{(K+1)^2T} + 8K^2e^{T(K+1)^2}(T+2+2K^2T)\,|\pi|\right)\\
d_3 &= \frac{d_4}{4K}\left(4(1+K^2T)e^{T(K+1)^2} + \left(8K^2e^{T(K+1)^2}(3+2K^2T)+4\right)|\pi|\right)\\
d_4 &= (1+2(1+2\beta_\pi)T)e^{\alpha_\pi T} + 4 + 4\beta_\pi\,|\pi|\\
d_5 &= \frac{d_4}{4K}\\
\beta_\pi &= e^{(4K+16K^2)|\pi|}(4K+16K^2)\\
\alpha_\pi &= \beta_\pi\left(1+\frac{1}{2(1-|\pi|/2)^2}\,|\pi|\right) + \frac{1}{2(1-|\pi|/2)^2},
\end{aligned}$$

and C_1 is the constant of Theorem 2.1 in [BS].

Proof. The proof is similar to Theorem 3.1 in Bouchard and Touzi (2004). We first define the process $\check{Y}^\pi$ on $[t_i, t_{i+1})$, $i = 0, \dots, N-1$, by

$$\check{Y}_t^\pi = \mathrm{E}\left[Y_{t_{i+1}}^\pi | \mathcal{F}_t\right] - f^\pi(t_i, Y_{t_i}^\pi, Z_{t_i}^\pi)(t_{i+1} - t)$$

with $\check{Y}_T^\pi = \xi^\pi$. Then, $\check{Y}_{t_i}^\pi = Y_{t_i}^\pi$ for $t_i \in \pi$ by (3.3) in [BS]. Moreover, thanks to (3.5) in [BS], the pairs (Y, Z) and $(\check{Y}_t^\pi, \check{Z}_t^\pi)$ solve on $t \in [t_i, t_{i+1})$ the following differential equations

$$\begin{aligned} Y_t &= Y_{t_{i+1}} - \int_t^{t_{i+1}} f(s, Y_s, Z_s) ds - \int_t^{t_{i+1}} Z_s dW_s, \\ \check{Y}_t^\pi &= Y_{t_{i+1}}^\pi - \int_t^{t_{i+1}} f^\pi(t_i, Y_{t_i}^\pi, Z_{t_i}^\pi) dt - \int_t^{t_{i+1}} \check{Z}_s^\pi dW_s, \end{aligned}$$

By Itô's formula we then obtain

$$\begin{aligned} &\mathrm{E}\left[\left|Y_t - \check{Y}_t^\pi\right|^2\right] + \int_t^{t_{i+1}} \mathrm{E}\left[\left|Z_s - \check{Z}_s^\pi\right|^2\right] ds \\ &\quad\le \mathrm{E}\left[\left|Y_{t_{i+1}} - Y_{t_{i+1}}^\pi\right|^2\right] + 2\int_t^{t_{i+1}} \mathrm{E}\left[(Y_s - \check{Y}_s^\pi)\left(f(s, Y_s, Z_s) - f^\pi(t_i, Y_{t_i}^\pi, Z_{t_i}^\pi)\right)\right] ds \\ &\quad= (I) + (II). \end{aligned}$$

Young's inequality and the Lipschitz condition on f^π yield for some $\gamma > 0$ (to be fixed later),

$$\begin{aligned} (II) &\le \gamma \int_t^{t_{i+1}} \mathrm{E}\left[\left|Y_s - \check{Y}_s^\pi\right|^2\right] ds + \frac{1 + (4K)^{-1}}{\gamma} \int_{t_i}^{t_{i+1}} \mathrm{E}\left[\left|f^\pi(t_i, Y_s, Z_s) - f^\pi(t_i, Y_{t_i}^\pi, Z_{t_i}^\pi)\right|^2\right] ds \\ &\quad + \frac{1 + 4K}{\gamma} \int_{t_i}^{t_{i+1}} \mathrm{E}\left[\left|f(s, Y_s, Z_s) - f^\pi(t_i, Y_s, Z_s)\right|^2\right] ds \\ &\le \gamma \int_t^{t_{i+1}} \mathrm{E}\left[\left|Y_s - \check{Y}_s^\pi\right|^2\right] ds + 2\frac{1 + (4K)^{-1}}{\gamma} \int_{t_i}^{t_{i+1}} K_y^2 \,\mathrm{E}\left[\left|Y_s - Y_{t_i}^\pi\right|^2\right] + K^2 \,\mathrm{E}\left[\left|Z_s - Z_{t_i}^\pi\right|^2\right] ds \\ &\quad + \frac{1 + 4K}{\gamma} \int_{t_i}^{t_{i+1}} \mathrm{E}\left[\left|f(s, Y_s, Z_s) - f^\pi(t_i, Y_s, Z_s)\right|^2\right] ds. \end{aligned}$$

Define

$$\tilde{C}_i = \sup_{s \in [t_i, t_{i+1}]} E\left|Y_s - Y_{t_i}\right|^2.$$

Then,

$$E\left|Y_s - Y_{t_i}^\pi\right|^2 \le 2\tilde{C}_i + 2E\left|Y_{t_i} - Y_{t_i}^\pi\right|^2.$$

Hence,

$$
\begin{aligned}
(II) \leq\ & \gamma \int_t^{t_{i+1}} \mathrm{E}\left[\left|Y_s - \check{Y}_s^\pi\right|^2\right] ds + \frac{1+4K}{\gamma} K \left(\Delta_i \mathrm{E}\left[\left|Y_{t_i} - Y_{t_i}^\pi\right|^2\right] + \int_{t_i}^{t_{i+1}} \mathrm{E}\left[\left|Z_s - \check{Z}_s^\pi\right|^2\right] ds\right) \\
& + \frac{1+4K_y}{\gamma} K_y \tilde{C}_i \Delta_i + \frac{1+4K}{\gamma} K \mathrm{E}\left[\left(\int_{t_i}^{t_{i+1}} \check{Z}_s^\pi dW_s - Z_{t_i}^\pi \Delta W_i\right)^2\right] \\
& + \frac{1+4K}{\gamma} \int_{t_i}^{t_{i+1}} \mathrm{E}\left[\left|f(s, Y_s, Z_s) - f^\pi(t_i, Y_s, Z_s)\right|^2\right] ds \\
=:\ & \gamma \int_t^{t_{i+1}} \mathrm{E}\left[\left|Y_s - \check{Y}_s^\pi\right|^2\right] ds + \frac{1+4K}{\gamma} K A_i + B_i.
\end{aligned}
$$

Summarizing, we have

$$
\begin{aligned}
\mathrm{E}\left[\left|Y_t - \check{Y}_t^\pi\right|^2\right] &\leq \mathrm{E}\left[\left|Y_t - \check{Y}_t^\pi\right|^2\right] + \int_t^{t_{i+1}} \mathrm{E}\left[\left|Z_s - \check{Z}_s^\pi\right|^2\right] ds \\
&\leq \mathrm{E}\left[\left|Y_{t_{i+1}} - Y_{t_{i+1}}^\pi\right|^2\right] + \gamma \int_t^{t_{i+1}} \mathrm{E}\left[\left|Y_s - \check{Y}_s^\pi\right|^2\right] ds + \frac{1+4K}{\gamma} K A_i + B_i.
\end{aligned}
\tag{1}
$$

By Gronwall's lemma it follows that $\mathrm{E}\left[|Y_t - \check{Y}_t^\pi|^2\right] \leq e^{\gamma \Delta_i}(\mathrm{E}\left[|Y_{t_{i+1}} - Y_{t_{i+1}}^\pi|^2\right] + (1+4K)KA_i/\gamma + B_i)$. Inserting this result into the second inequality of (1) yields

$$
\begin{aligned}
&\mathrm{E}\left[\left|Y_{t_i} - Y_{t_i}^\pi\right|^2\right] + \int_{t_i}^{t_{i+1}} \mathrm{E}\left[\left|Z_t - \check{Z}_t^\pi\right|^2\right] dt \\
&\quad \leq (1 + \gamma \Delta_i e^{\gamma \Delta_i})(\mathrm{E}\left[\left|Y_{t_{i+1}} - Y_{t_{i+1}}^\pi\right|^2\right] + \frac{1+4K}{\gamma} K A_i + B_i)
\end{aligned}
$$

Choosing $\gamma = 4K(1+4K)$ and $|\pi| \leq 1/(2\gamma)$ leads to

$$
\begin{aligned}
\mathrm{E}\left[\left|Y_{t_i} - Y_{t_i}^\pi\right|^2\right] + \int_{t_i}^{t_{i+1}} \mathrm{E}\left[\left|Z_t - \check{Z}_t^\pi\right|^2\right] dt &\leq (1 + e^{\gamma \Delta_i} \gamma \Delta_i)\left(\mathrm{E}\left[\left|Y_{t_{i+1}} - Y_{t_{i+1}}^\pi\right|^2\right] + B_i\right) \\
&+ \frac{1}{2} \Delta_i \mathrm{E}\left[\left|Y_{t_i} - Y_{t_i}^\pi\right|^2\right] + \frac{1}{2} \int_{t_i}^{t_{i+1}} \mathrm{E}\left[\left|Z_t - \check{Z}_t^\pi\right|^2\right] dt.
\end{aligned}
$$

Hence, for $|\pi| \leq 1/(2\gamma)$,

$$
\begin{aligned}
&(1 - \frac{1}{2}\Delta_i) \mathrm{E}\left[\left|Y_{t_i} - Y_{t_i}^\pi\right|^2\right] + \frac{1}{2} \int_{t_i}^{t_{i+1}} \mathrm{E}\left[\left|Z_t - \check{Z}_t^\pi\right|^2\right] dt \\
&\quad \leq (1 + \beta_\pi \Delta_i) \Bigg\{ \mathrm{E}\left[\left|Y_{t_{i+1}} - Y_{t_{i+1}}^\pi\right|^2\right] \\
&\quad + \frac{K_y}{4K} \tilde{C}_i \Delta_i + \frac{1}{4} \mathrm{E}\left[\left(\int_{t_i}^{t_{i+1}} \check{Z}_s^\pi dW_s - Z_{t_i}^\pi \Delta W_i\right)^2\right] \\
&\quad + \frac{1}{4K} \int_{t_i}^{t_{i+1}} \mathrm{E}\left[\left|f(s, Y_s, Z_s) - f^\pi(t_i, Y_s, Z_s)\right|^2\right] ds \Bigg\}.
\end{aligned}
\tag{2}
$$

In particular, for $|\pi| \le \min\{1, 1/(2\gamma)\}$,

$$\begin{aligned}
&\mathrm{E}\left[\left|Y_{t_i} - Y^\pi_{t_i}\right|^2\right] \\
&\quad \le (1+\alpha_\pi \Delta_i)\left\{\mathrm{E}\left[\left|Y_{t_{i+1}} - Y^\pi_{t_{i+1}}\right|^2\right] + \frac{K_y}{4K}\tilde{C}_i\Delta_i + \frac{1}{4}\mathrm{E}\left[\left(\int_{t_i}^{t_{i+1}} \check{Z}^\pi_s dW_s - Z^\pi_{t_i}\Delta W_i\right)^2\right]\right. \\
&\qquad \left. + \frac{1}{4K}\int_{t_i}^{t_{i+1}} \mathrm{E}\left[|f(s,Y_s,Z_s) - f^\pi(t_i,Y_s,Z_s)|^2\right] ds\right\}.
\end{aligned}$$

Thanks to the discrete Gronwall lemma we get for $|\pi| \le \min\{\frac{1}{2\gamma}, 1\}$

$$\begin{aligned}
\mathrm{E}\left[\left|Y_{t_i} - Y^\pi_{t_i}\right|^2\right] \quad \le \quad & \exp\{\alpha_\pi T\}\left\{\mathrm{E}\left[|\xi - \xi^\pi|^2\right]\right. \\
& + \frac{K_y}{4K}|\pi|\sum_{j=i}^{N-1}\tilde{C}_j + \frac{1}{4}\sum_{j=i}^{N-1}\mathrm{E}\left[\left(\int_{t_j}^{t_{j+1}} \check{Z}^\pi_s dW_s - Z^\pi_{t_j}\Delta W_j\right)^2\right] \\
& \left. + \frac{1}{4K}\sum_{j=i}^{N-1}\int_{t_j}^{t_{j+1}} \mathrm{E}\left[|f(t,Y_t,Z_t) - f^\pi(t_j,Y_t,Z_t)|^2\right] dt\right\}. \qquad (3)
\end{aligned}$$

Next we sum (2) up from $i = 0$ to $N-1$ and obtain

$$\begin{aligned}
& \sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} \mathrm{E}\left[\left|Z_t - \check{Z}^\pi_t\right|^2\right] dt \\
\le \quad & 2(1+\beta_\pi|\pi|)\left\{\mathrm{E}\left[|\xi - \xi^\pi|^2\right] + \frac{K_y}{4K}|\pi|\sum_{j=0}^{N-1}\tilde{C}_j\right. \\
& + \frac{1}{4}\sum_{j=0}^{N-1}\mathrm{E}\left[\left(\int_{t_j}^{t_{j+1}} \check{Z}^\pi_s dW_s - Z^\pi_{t_j}\Delta W_j\right)^2\right] \\
& \left. + \frac{1}{4K}\sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}} \mathrm{E}\left[|f(t,Y_t,Z_t) - f^\pi(t_j,Y_t,Z_t)|^2\right] dt\right\} \\
& + 2T\left(\beta_\pi + \frac{1}{2}\right)\max_{t_i\in\pi}\mathrm{E}\left[\left|Y_{t_i} - Y^\pi_{t_i}\right|^2\right] \qquad (4)
\end{aligned}$$

By (3), (4), Young's inequality and Itô's isometry,

$$\begin{aligned}
& \max_{t_i\in\pi}\mathrm{E}\left[\left|Y_{t_i} - Y^\pi_{t_i}\right|^2\right] + \sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} \mathrm{E}\left[\left|Z_t - Z^\pi_{t_i}\right|^2\right] dt \\
\le \quad & \max_{t_i\in\pi}\mathrm{E}\left[\left|Y_{t_i} - Y^\pi_{t_i}\right|^2\right] + 2\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} \mathrm{E}\left[\left|Z_t - \check{Z}^\pi_t\right|^2\right] dt + 2\sum_{i=0}^{N-1}\mathrm{E}\left[\left(\int_{t_i}^{t_{i+1}} \check{Z}^\pi_s dW_s - Z^\pi_{t_i}\Delta W_i\right)^2\right] \\
\le \quad & d_4\left(\mathrm{E}\left[|\xi - \xi^\pi|^2\right] + \frac{K_y}{4K}|\pi|\sum_{j=0}^{N-1}\tilde{C}_j + \frac{1}{4K}\sum_{j=i}^{N-1}\int_{t_j}^{t_{j+1}} \mathrm{E}\left[|f(t,Y_t,Z_t) - f^\pi(t_j,Y_t,Z_t)|^2\right] dt\right) \\
& + (\frac{d_4}{4} + 2)\sum_{j=i}^{N-1}\mathrm{E}\left[\left(\int_{t_i}^{t_{i+1}} \check{Z}^\pi_s dW_s - Z^\pi_{t_i}\Delta W_i\right)^2\right]
\end{aligned}$$

The assertion now follows by Lemma 0.1 above (taking the definition of $\tilde{C}_i$ into account) and (3.6) in [BS].

□

We finally present a version of Theorem 3.1 in [BS] in terms of the explicit constants defined in Lemma 0.2 above.

Theorem 0.3. *Suppose (H) and ($H^{\mathcal{F}}_{\pi}$) and that $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\Xi)$, where Ξ is independent of $\mathcal{F}_T$. Then, for every pair of $\mathcal{G}_{t_i}$-adapted square integrable processes $(\hat{Y}^{\pi}_{t_i}, \hat{Z}^{\pi}_{t_i})_{t_i\in\pi}$*

$$\begin{aligned} & \max_{t_i\in\pi} \mathrm{E}_0\left[|Y_{t_i} - \hat{Y}^{\pi}_{t_i}|^2\right] + \sum_{i=0}^{N-1} \mathrm{E}_0\left[\int_{t_i}^{t_{i+1}} \left(|Y_t - \hat{Y}^{\pi}_{t_i}|^2 + |Z_t - \hat{Z}^{\pi}_{t_i}|^2\right) dt\right] \\ \le\ & 2(2T+1)(d_1 + DC_1)\mathcal{E}_{\pi}(\hat{Y}^{\pi}, \hat{Z}^{\pi}) \\ & +2(2T+1)d_4 \mathrm{E}\left[|\xi - \xi^{\pi}|^2\right] + 2(2T+1)d_5 \sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} \mathrm{E}\left[|f(t, Y_t, Z_t) - f^{\pi}(t_i, Y_t, Z_t)|^2\right] dt \\ & + |\pi|\left(\tilde{d}_2 \mathrm{E}\left[|\xi|^2\right] + \tilde{d}_3 E\left[\int_0^T |f(r,0,0)|^2\, dr\right]\right) \end{aligned}$$

for $|\pi| \le \min\{1, (8K(1+4K))^{-1}, (4K^2(T+1)(D\vee 2) + 16TK^4(1+T)^2(D\vee 2)^2)^{-1}\}$, *where*

$$\begin{aligned} \tilde{d}_2 &= 2K(2T+1)d_2 + 8(1+K^2T)e^{T(K+1)^2} + 16K^2 e^{T(K+1)^2}(T+2+2K^2T)\,|\pi| \\ \tilde{d}_3 &= 2K(2T+1)d_3 + 8(1+K^2T)e^{T(K+1)^2} + \left(16K^2 e^{T(K+1)^2}(3+2K^2T) + 8\right)|\pi| \end{aligned}$$

Moreover,

$$\begin{aligned} \mathcal{E}_{\pi}(\hat{Y}^{\pi}, \hat{Z}^{\pi}) \le\ & (10+16K^2T)\Bigg(\max_{t_i\in\pi} \mathrm{E}_0\left[|Y_{t_i} - \hat{Y}^{\pi}_{t_i}|^2\right] + \sum_{i=0}^{N-1} \mathrm{E}_0\left[\int_{t_i}^{t_{i+1}} \left(|Y_t - \hat{Y}^{\pi}_{t_i}|^2 + |Z_t - \hat{Z}^{\pi}_{t_i}|^2\right) dt\right] \\ & +\mathrm{E}\left[|\xi - \xi^{\pi}|^2\right] + \sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} \mathrm{E}\left[|f(t, Y_t, Z_t) - f^{\pi}(t_i, Y_t, Z_t)|^2\right] dt\Bigg). \end{aligned}$$

If, additionally, f and f^{π} do not depend on y, then

$$\begin{aligned} & \max_{t_i\in\pi} \mathrm{E}_0\left[|Y_{t_i} - \hat{Y}^{\pi}_{t_i}|^2\right] + \sum_{i=0}^{N-1} \mathrm{E}_0\left[\int_{t_i}^{t_{i+1}} |Z_t - \hat{Z}^{\pi}_{t_i}|^2 dt\right] \\ \le\ & 2(d_1 + DC_1)\mathcal{E}_{\pi}(\hat{Y}^{\pi}, \hat{Z}^{\pi}) + 2d_4 \mathrm{E}\left[|\xi - \xi^{\pi}|^2\right] + 2d_5 \sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} \mathrm{E}\left[|f(t, Y_t, Z_t) - f^{\pi}(t_i, Y_t, Z_t)|^2\right] dt \end{aligned}$$

for $|\pi| \le \min\{1, (8K(1+4K))^{-1}, (4K^2(T+1)(D\vee 2) + 16TK^4(1+T)^2(D\vee 2)^2)^{-1}\}$.

Moreover,

$$\begin{aligned} \mathcal{E}_{\pi}(\hat{Y}^{\pi}, \hat{Z}^{\pi}) \le\ & (10+16K^2T)\Bigg(\max_{t_i\in\pi} \mathrm{E}_0\left[|Y_{t_i} - \hat{Y}^{\pi}_{t_i}|^2\right] + \sum_{i=0}^{N-1} \mathrm{E}_0\left[\int_{t_i}^{t_{i+1}} |Z_t - \hat{Z}^{\pi}_{t_i}|^2\ dt\right] \\ & +\mathrm{E}\left[|\xi - \xi^{\pi}|^2\right] + \sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} \mathrm{E}\left[|f(t, Z_t) - f^{\pi}(t_i, Z_t)|^2\right] dt\Bigg). \end{aligned}$$

Proof. We follow the lines of the proof of Theorem 3.1 in [BS]. Inserting the result of Lemma 0.2 instead of Lemma 3.3 in [BS], eq. (3.7) in [BS] becomes

$$\begin{aligned}
&\max_{t_i\in\pi} \mathrm{E}_0\left[|Y_{t_i}-\hat{Y}^\pi_{t_i}|^2\right] + \sum_{i=0}^{N-1} \mathrm{E}_0\left[\int_{t_i}^{t_{i+1}} |Z_t-\hat{Z}^\pi_{t_i}|^2 dt\right] \\
\leq\ & 2(d_1+DC_1)\mathcal{E}_\pi(\hat{Y}^\pi,\hat{Z}^\pi) + K_y\,|\pi|\left(2d_2\mathrm{E}\left[|\xi|^2\right] + 2d_3 E\left[\int_0^T |f(r,0,0)|^2\,dr\right]\right) \\
&+2d_4\mathrm{E}\left[|\xi-\xi^\pi|^2\right] + 2d_5\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} \mathrm{E}\left[|f(t,Y_t,Z_t)-f^\pi(t_i,Y_t,Z_t)|^2\right] dt. \qquad (5)
\end{aligned}$$

The third inequality follows by setting $K_y = 0$.

In order to complete the proof of the first inequality we estimate

$$\sum_{i=0}^{N-1} \mathrm{E}_0\left[\int_{t_i}^{t_{i+1}} |Y_t-\hat{Y}^\pi_{t_i}|^2 dt\right] \leq 2T\max_{t_i\in\pi}\mathrm{E}_0\left[|Y_{t_i}-\hat{Y}^\pi_{t_i}|^2\right] + 2\,|\pi|\sum_{i=1}^{N}\sup_{s\in[t_i,t_{i+1}]}\mathrm{E}\left[|Y_s-Y_{t_i}|^2\right].$$

Combining this ineqaulity with (5), we obtain the first inequality after applying Lemma 0.1.

The second and fourth inequality are shown in the proof of Theorem 3.1 in [BS].

□

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