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Stochastics II

2. Tutorial

Exercise 1 **(3 Points)** Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence in $L^1(\mathcal{F},P)$ and $\mathcal{G}\subset\mathcal{F}$ a σ-field. Show that: If $\xi_n\geq 0$ and $\liminf_{n\to\infty}\xi_n\in L^1(\mathcal{F},P)$, then it follows

$$E[\liminf_{n\to\infty}\xi_n|\mathcal{G}]\leq\liminf_{n\to\infty}E[\xi|\mathcal{G}]\quad (P\text{-a.s.}).$$

Exercise 2 **(4 Points)** Let X and Y be independent random variables on the probability space $(\Omega,\mathcal{F},P)$ which map Ω into the measurable space $(\mathcal{X},\mathcal{A})$, respectively $(\mathcal{Y},\mathcal{G})$. Furthermore let $g:\mathcal{X}\times\mathcal{Y}\to\mathbb{R}$ be a measurable function, such that $g(X,Y)\in L^1(\mathcal{F},P)$. Show that:

$$E[g(X,Y)|X]=E[g(x,Y)]_{|x=X}\quad (P\text{-a.s.}).$$

Exercise 3 **(4 Points)** Let $X:=(X_t)_{t\in\mathcal{T}}$ be a stochastic process with state space $(E,\mathcal{E})$.

(i) Show that X is measurable in each of the following situations:
  (a) $\mathcal{T}$ is at most countable.
  (b) $\mathcal{T}=[0,\infty)$, $(E,\mathcal{E})=(\mathbb{R}^D,\mathcal{B}^D)$ and X has right-continuous paths.

(ii) Give an example for a stochastic process that is not measurable.

Exercise 4 **(5 Points)** Let $Y_k;=\sum_{j=1}^k Z_j$, $k\in\mathbb{N}$, where $(Z_j)_{1\leq j\leq k}$ is an independent family of exponential(λ)-distributed random variables with $\lambda>0$.

(i) Show that Y_k is a Γ-distributed random variable with parameters (k,λ), i.e. the density f_k of Y_k is given by

$$f_k(u)=\frac{(\lambda u)^{k-1}}{(k-1)!}\lambda e^{-\lambda u}\mathbb{1}_{(0,\infty)}(u)$$

for every $u\in\mathbb{R}$.

(ii) Show that

$$P(\{Y_k>\vartheta\})=\sum_{i=0}^{k-1}\frac{(\lambda\vartheta)^j}{j!}e^{-\lambda\vartheta}$$

for every $\vartheta>0$.