Prof. Dr. Christian Bender
M.Sc. Steffen Meyer

Saarland University
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Stochastics II

6. Tutorial

Exercise 1 **(3 Points)** Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function.

(i) Let X be an integrable random variable, such that $f(X) \in L^1(P)$ and let $\mathcal{G}$ be a sub σ-field of $\mathcal{F}$. Show that

$$E[f(X)|\mathcal{G}] \geq f(E[X|\mathcal{G}]) \quad P\text{-a.s.}$$

(ii) Show that $(f(X_t))_{t\in[0,\infty)}$ is a submartingale, if

a) $(X_t)_{t\in[0,\infty)}$ is a martingale and $f(X_t) \in L^1(P)$ for every $t \in [0, \infty)$.

b) $(X_t)_{t\in[0,\infty)}$ is a submartingale, $f(X_t) \in L^1(P)$ for every $t \in [0, \infty)$ and f is non-decreasing.

Hint: Every convex function $f : \mathbb{R} \to \mathbb{R}$ can be expressed as the supremum over all linear functions which lie completely below the graph of f.

Exercise 2 **(2 Points)** Suppose $M = (M_t)_{t\in[0,\infty)}$ is a martingale such that $E[M_t^2] < \infty$ for every $t \geq 0$. Show that $E[(M_t - M_s)(M_v - M_u)] = 0$ for every $u \leq v \leq s \leq t$.

Exercise 3 **(3 Points)** let J be an arbitrary index set and $X := (X_j)_{j\in J} \subset L^1(P)$ be a family of random variables. Show that X is uniformly integrable, if there exists a measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\lim_{t\to\infty} \frac{f(t)}{t} = \infty \text{ and } \sup_{j\in J} E[f(|X_j|)] < \infty.$$

Exercise 4 **(6 Points)** Let $M = (M_n)_{n\in\mathbb{N}_0}$ be a martingale in discrete time with $M_n \in L^2(P)$ for every $n \in \mathbb{N}_0$.

(i) Show that the following statements are equivalent:

a) M_n converges in $L^2(P)$ to some random variable M_∞ in $L^2(P)$..

b) $\sup_n E[M_n^2] < \infty$

(c) $\sum_{n\in\mathbb{N}} E[(M_n - M_{n-1})^2] < \infty$.

(ii) Show that M_n also converges P-almost surely to M_∞, if the statements in (i) hold true.

Exercise 5 **(4 Points)** Let $M = (M_n)_{n\in\mathbb{N}_0}$ be a martingale with $M_n \in L^2(P)$ for every $n \in \mathbb{N}_0$. Show that:

(i) There exists an increasing and predictable process $A = (A_n)_{n\in\mathbb{N}_0}$ with $A_0 = 0$, such that the process $N = (N_n)_{n\in\mathbb{N}_0}$ defined as $N_n := M_n^2 - A_n$ for every $n \in \mathbb{N}_0$ is a martingale.

(ii) If $E[A_\infty] < \infty$, then M_n converges P-almost surely and in $L^2(P)$ to a random variable M_∞ in $L^2(P)$, where $A_\infty := \lim_{n\to\infty} A_n$.