Prof. Dr. Christian Bender
M.Sc. Steffen Meyer

Saarland University
WiSe 2019/2010

27th November 2019

Stochastics II

7. Tutorial

Exercise 1 (4 **Points**) Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}, P)$ be a filtered probability space and $X = (X_t)_{t\in[0,\infty)}$ be a $\mathbb{F}$-adapted and integrable stochastic process. Show that the following statements are equivalent.

(i) X is a supermartingale.

(ii) For all $s, t \in [0, \infty)$ and every $A \in \mathcal{F}_s$ we have

$$P(A) > 0 \Rightarrow E^{P(.|A)}[X_{s+t}] \leq E^{P(.|A)}[X_s].$$

(iii) For all $s, t \in [0, \infty)$ and every $A \in \mathcal{F}_s$ we have

$$\int_A X_{s+t} dP \leq \int_A X_s dP.$$

(iv) For all $s, t \in [0, \infty)$ and every bounded, non-negative and $\mathcal{F}_s$-measurable function Y, we have

$$E[Y X_{s+t}] \leq E[Y X_s].$$

Exercise 2 (**6 Points**) For a martingale $X = (X_n)_{n\in\mathbb{N}_0}$ in $L^2(P)$ we define the process $(\langle X\rangle_n)_{n\in\mathbb{N}_0}$ as $\langle X\rangle_0 := 0$ and

$$\langle X\rangle_n := \langle X\rangle_{n-1} + E[(X_n - X_{n-1})^2|\mathcal{F}_{n-1}]$$

for every $n \in \mathbb{N}$. Then $\langle X\rangle$ is predictable, increasing, and $(X_n^2 - \langle X\rangle_n)_{n\in\mathbb{N}_0}$ is a martingale as well (see exercise 4 of the 6th tutorial).
Now let M be a martingale in $L^2(P)$ such that $\langle M\rangle_n \to \infty$ for $n \to \infty$. Show that

$$\lim_{n\to\infty} \frac{M_n}{\langle M\rangle_n} = 0 \quad P\text{-allmost surely.}$$

Hint: Consider the process

$$W_n := \sum_{1\leq k\leq n} \frac{M_k - M_{k-1}}{1 + \langle M\rangle_k}$$

and use exercise 4 of the 6th tutorial to show that W_n converges. Then use Kronecker's Lemma to conclude the statement.

Exercise 3 (5 Points)

(i) Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_n)_{n\in\mathbb{N}_0}, P)$ be a filtered probability space, $X = (X_n)_{n\in\mathbb{N}_0}$ be a $\mathbb{F}$-submartingale and σ be a $\mathbb{F}$-stopping time satisfying $E[\sigma] < \infty$. Furthermore assume there exists a constant $C > 0$ such that

$$|X_{n+1} - X_n| \leq C$$

P-a.s. for every $n \in \mathbb{N}_0$. Show that

$$E[X_\sigma] \geq E[X_0].$$

(ii) Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables with

$$P(\{\xi_n = -1\}) = P(\{\xi_n = 1\}) = \frac{1}{2}.$$

Moreover define $S_n := \sum_{k=1}^n \xi_k$ and $\tau := \inf\{n \in \mathbb{N} | S_n = 1\}$. Show that $E[\tau] = \infty$.