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Saarland University
WiSe 2019/2020

9th January 2020

Stochastics II

11. Tutorial

Definition A function $\varrho : \mathcal{T} \times \mathcal{T} \to \mathbb{R}$ is called positive semi-definite, if

$$\forall k \in \mathbb{N}\ \forall t_1, \ldots, t_k \in \mathcal{T}\ \forall \lambda_1, \ldots, \lambda_k \in \mathbb{R} : \sum_{i=1}^{k} \sum_{j=1}^{k} \lambda_i \lambda_j \varrho(t_i, t_j) \geq 0.$$

Exercise 1 (4 **Points**) Let $\mathcal{T} := [0, T]$, $m : \mathcal{T} \to \mathbb{R}$ be arbitrary and $\varrho : \mathcal{T} \times \mathcal{T} \to \mathbb{R}$ be symmetric and positive semi-definite. Show that there exists a probability space $(\Omega, \mathcal{F}, P)$ and a Gaussian process $X = (X_t)_{t \in \mathcal{T}}$ on $(\Omega, \mathcal{F}, P)$ such that $E[X_t] = m(t)$ and $\mathrm{Cov}(X_s, X_t) = \varrho(s, t)$ for all $s, t \in \mathcal{T}$.

Exercise 2 (**10 Points**) Let $T \in \mathbb{R}$ and $(X_t)_{t \in [0,T)}$ be the stochastic process defined by

$$X_t := \frac{T - t}{T} W_{\frac{tT}{T-t}}$$

for every $t \in [0, T)$, where $(W_t)_{t \in [0,\infty)}$ is a Brownian motion.

(i) Show that $X_T := \lim_{t \nearrow T} X_t$ P-a.s. exists and derive the distribution of X_T.

(ii) Show that $(X_t)_{t \in [0,T]}$ is a Gaussian process and calculate $E[X_t]$ and $\mathrm{Cov}(X_s, X_t)$ for $s, t \in [0, T]$.

(iii) Show that the random variables

$$Z_i := \frac{X_{t_i}}{T - t_i} - \frac{X_{t_{i-1}}}{T - t_{i-1}}$$

are independent for $k \in \mathbb{N}$, $t_1, \ldots, t_k \in [0, T)$.

(iv) Show that

$$P\left(\bigcap_{i=1}^{k} \{X_{t_i} \leq x_i\}\right) = \lim_{\epsilon \searrow 0} P\left(\bigcap_{i=1}^{k} \{W_{t_i} \leq x_i\} \middle| \{W_T \in (-\epsilon, \epsilon)\}\right)$$

for $i = 1, \ldots, k \in \mathbb{N}$, $t_1, \ldots, t_k \in [0, T)$ and $x_1, \ldots, x_k \in \mathbb{R}$.

Exercise 3 (**5 Points**) *Preliminaries for the proof of theorem 11.7:*

(i) Show that

$$\frac{x}{1 + x^2} e^{-x^2/2} \leq \int_x^\infty e^{-t^2/2} dt \leq \frac{1}{x} e^{-x^2/2}$$

for every $x > 0$ and conclude that

$$P(\{\xi > x\}) \geq \frac{1}{\sqrt{2}^3 \sqrt{\pi}} \frac{1}{x} e^{-x^2/2}$$

for every $\mathcal{N}(0,1)$-distributed random variable ξ and every $x \geq 1$.

(ii) We define for $n \in \mathbb{N}$ and $q > 1$

$$a_n := \frac{e^{-\log\log(q^n - q^{n-1})}}{\sqrt{\log\log(q^n - q^{n-1})}}.$$

Show that $\sum_{n \in \mathbb{N}} a_{2n} = \infty$.