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Stochastics II

12. Tutorial

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}, P)$ be the underlying filtered probability space for the whole assignment.

Exercise 1 (5 **Points)** Let $X := (X_t)_{t\in[0,\infty)}$ be a Brownian motion with respect to $\mathbb{F}$ and let τ be a $\mathbb{F}$-stopping time. Show that $E[X_\tau] = 0$ if:

(i) $(|X_{t\wedge\tau}|)_{(t\in[0,\infty)}$ is bounded by a P-integrable random variable.

(ii) $E[\tau] < \infty$.

Hint: To solve (ii), take a look at the random variable

$$\sum_{k=1}^{\lceil\tau\rceil} \max_{t\in[0,1]} |X_{k+t} - X_k|.$$

Exercise 2 **(3 Points)** Let τ be a $\mathbb{F}$-stopping time. Show that

(i) $\sigma - \tau$ is a $(\mathcal{F}_{t+\tau})_{t\in[0,\infty)}$-stopping time, if σ is a $\mathbb{F}$ stopping time.

(ii) $\varrho + \tau$ is a $\mathbb{F}$-stopping time, if ϱ is a $(\mathcal{F}_{t+\tau})_{t\in[0,\infty)}$-stopping time.

Exercise 3 **(7 Points)** Let $W := (X_t)_{t\in[0,\infty)}$ be a Brownian motion with respect to $\mathbb{F}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a function with continuous and bounded first derivative. Calculate the quadratic variation of $(f(W_t))_{t\in[0,\infty)}$.
Hint: Show that

$$\begin{aligned}\big(f(W_{t_i}) - f(W_{t_{i-1}})\big)^2 &= f'(W_{t_{i-1}})^2(t_i - t_{i-1}) \\ &\quad + f'(W_{t_{i-1}})^2 \left((W_{t_i} - W_{t_{i-1}})^2 - (t_i - t_{i-1})\right) \\ &\quad + \left(f'(W_{u_i})^2 - f'(W_{t_i})^2\right)(W_{t_i} - W_{t_{i-1}})^2\end{aligned}$$

for $t_i > t_{i-1}$, where $u_i = u_i(\omega) \in [t_{i-1}, t_i]$.

Exercise 4 **(3 Points)** Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[0,\infty)$, such that $\frac{\gamma_n}{n} \to 1$ for $n \to \infty$. Show that

$$\lim_{n\to\infty} \sup_{0\le k\le n} \left|\frac{\gamma_k - k}{n}\right| = 0.$$