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Stochastics II

13. Tutorial

Definition: A $\mathbb{R}^D$-valued stochastic process $W = (W_t)_{t\in[0,\infty)}$ is called a D-dimensional Brownian motion with respect to a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}$, if the following conditions hold true:

- W has continuous paths, $W_0 = 0$ and W is $\mathbb{F}$-adapted.
- $W_t - W_s$ is for every $0 \le s < t$ independent of $\mathcal{F}_s$ and Gaussian distributed with mean vector 0 and covariance matrix $(t-s)I_D$, where I_D is the unit matrix.

Exercise 1 **(5 Points)** Show that there exits a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}, P)$ and a $\mathbb{R}^D$ valued process $W = (W_t)_{t\in[0,\infty)}$, such that W is a D-dimensional Brownian motion with respect to $\mathbb{F}$.

Exercise 2 **(5 Points)** Let $W = (W_t)_{t\in[0,\infty)}$ be a D-dimensional Brownian motion with respect to a filtration $\mathbb{F}$. Show that the components $W^{(1)}, \dots, W^{(D)}$ are independent one-dimensional Brownian motions with respect to $\mathbb{F}$.

Exercise 3 **(6 Points)** Consider the finite difference scheme

$$
\begin{aligned}
-\frac{V^{(n)}_{i+1,j} - V^{(n)}_{i,j}}{\tau} &= \frac{1}{2}\left(\frac{V^{(n)}_{i+1,j-1} - 2V^{(n)}_{i+1,j} + V^{(n)}_{i+1,j+1}}{h^2}\right) + g(jh) \\
V^{(n)}_{n,j} &= 0,
\end{aligned}
$$

where $n \in \mathbb{N}$, $i = 1, \dots, n$, $j = -i, \dots, i$, $\tau = \frac{1}{n}$, $h = \sqrt{\frac{3}{n}}$ and g is a bounded Lipschitz continuous function. We define the random walk

$$
S_k := \sum_{i=1}^{k} \xi_i
$$

where the random variables ξ_i, $i = 1, \dots k$ are independent indentical distributed with

$$
P(\{\xi_i = a\}) = \begin{cases} \frac{1}{6}, & a = -\sqrt{3} \\ \frac{2}{3}, & a = 0 \\ \frac{1}{6}, & a = \sqrt{3} \end{cases}.
$$

We also define for $n \in \mathbb{N}$ the discrete time process

$$
\begin{aligned}
Y^{(n)}_i &= \sum_{j=-i}^{i} V^{(n)}_{i,j} \mathbb{1}_{\{S_i = j\sqrt{3}\}}, \quad i = 0, \dots n-1 \\
Y^{(n)}_n &= 0
\end{aligned}
$$

Show that

(i) $Y_i^{(n)} = E[Y_{i+1}^{(n)} | \sigma(\xi_1, \dots, \xi_i)] + g(\frac{1}{\sqrt{n}} S_i)\tau$ for $i = 0, \dots, n-1$.

(ii) $V_{0,0}^{(n)} \to E[\int_0^1 g(W_s) ds]$ for $n \to \infty$, where W is a Brownian motion.