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Syzygies of k-Gonal Curves

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0. Introduction

The main focus in this thesis is on the coordinate rings of canonically embedded curves $C \subset \mathbb{P}^{g-1}$ of genus g especially on the minimal free resolutions of those. The shape of a minimal free resolution can be expressed in a Betti tables and in the case of canonical curves these tables look as follows

	0	1	2	...	$g-4$	$g-3$	$g-2$
0	1	-	-	...	-	-	-
1	-	$\beta_{1,2}$	$\beta_{2,3}$	...	$\beta_{g-4,g-3}$	$\beta_{g-3,g-2}$	-
2	-	$\beta_{g-3,g-2}$	$\beta_{g-4,g-3}$	...	$\beta_{2,3}$	$\beta_{1,2}$	-
3	—	-	-	...	-	-	1

Green's famous conjecture which was recently proved for general curves in [Voi05] states that the non-vanishing of certain Betti numbers is equivalent to the existence of special linear series on C.

A related problem is the question which Betti tables can occur for a canonical curve C. For canonical curves of genus $g \leq 8$ Schreyer shows in [Sch86] that the graded Betti numbers depend on and determine the existence of special linear series, by giving a complete list of all possible Betti tables. The idea behind Schreyer's result is the following:

A special pencil of degree k on a canonical curve sweeps out a $(k-1)$-dimensional rational normal scroll X. One can obtain a minimal resolution of $\mathscr{O}_C$ in terms of $\mathscr{O}_X$-modules and then resolve the $\mathscr{O}_X$-modules occurring in this resolution by $\mathscr{O}_{\mathbb{P}^{g-1}}$-modules. An iterated mapping cone construction then gives a possibly non-minimal resolution of $\mathscr{O}_C$ as an $\mathscr{O}_{\mathbb{P}^{g-1}}$-module. In particular, the 2-linear strand of the resolution of $X \subset \mathbb{P}^{g-1}$ is a summand of the 2-linear strand of resolution of $C \subset \mathbb{P}^{g-1}$.

The question arises which Betti numbers of C coincide with those of the scroll X. Under suitable hypothesis on C, which will be specified in Chapter 1, the results in [Sch86] lead to conjecture that $\beta_{m,m+1}(C) = \beta_{m,m+1}(X)$ holds for $m \geq \lceil \frac{g-1}{2} \rceil$. Under the same hypothesis on C, the purpose of this thesis is to prove the following theorem.

Theorem. *Let $C \subset \mathbb{P}^{g-1}$ be a general 5-gonal canonical curve of odd genus $g = 2n+1 \geq 13$ then $\beta_{m,m+1}(C) > \beta_{m,m+1}(X)$ for $m = \lceil \frac{g-1}{2} \rceil$.*

After summarizing the relevant theory in Chapter 1, we explain in Chapter 2 how the construction of rational g-nodal canonical curves that admit a g^1_k can be implemented in the computer algebra system *Macaulay2*. With the methods described in this chapter, we can in particular verify the above theorem for general rational g-nodal canonical curves of genus $g \leq 17$. In Chapter 3 we will finally prove the theorem and give a similar result for 5-gonal curves of even genus.

1. Background and Motivation

In this chapter we give an overview of the theory needed in the Chapters 2 and 3. Our primary focus in this thesis is on the coordinate ring of projective varieties, especially on the minimal free resolutions of the coordinate rings of canonically embedded curves.

1.1. Free Resolutions and Syzygies

For a field $\mathbb{k}$ let $\mathbb{P}^n = \mathbb{P}^n(\mathbb{k})$ be the projective n-space and let $S = \mathbb{k}[x_0, ..., x_n]$ be the coordinate ring of $\mathbb{P}^n$. If $X \subset \mathbb{P}^n$ is a projective variety, then we denote by $I_X \subset S$ the vanishing ideal and by $S_X := S/I_X$ the coordinate ring of X. Recall that the ideal I_X is finitely generated by the Noetherian property of S. The generators of I_X have relations, and these relations also can have relations and so on. Hilbert's Syzygy Theorem states that this process terminates after finitely many steps.

Theorem 1.1 (Hilbert's Syzygy Theorem). *A finitely generated graded S-module M has a finite graded free resolution*

$$0 \longrightarrow F_m \xrightarrow{\varphi_m} F_{m-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow M \longrightarrow 0.$$

Moreover we may take $m \leq n+1$, the number of variables of S.

Proof. See [Eis95, Corollary 15.11]. □

Definition 1.2. *A complex of graded S-modules*

$$\ldots \longrightarrow F_i \xrightarrow{\varphi_i} F_{i-1} \cdots$$

is called minimal *if the image of φ_i is contained in $\mathfrak{m}F_{i-1}$ for each i, where $\mathfrak{m} = \langle x_0, ..., x_n \rangle$ is the irrelevant ideal of S.*

By choosing a minimal set of generators in each step in the theorem above we obtain a minimal free resolution. On the other hand, by canceling trivial subcomplexes, a free resolution can be be minimized to obtain a minimal free resolution.

Theorem 1.3. *A minimal free resolution of a graded S-module M is unique up to an isomorphism of complexes that induces the identity map on M.*

Proof. See [Eis05, Theorem 1.6]. □

The theorem above states in particular that the number of generators of each degree j required to generate F_i is independent of the minimal free resolution. This leads to the definition of one of the main objects of interest in this thesis.

Definition 1.4 (Betti Numbers). *Let M be a finitely generated graded S-modul and*

$$0 \longrightarrow F_m \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

be a minimal free resolution of M. The free modules F_i are of the form $F_i = \bigoplus_j S(-j)^{\beta_{ij}}$. We call the numbers β_{ij} the graded Betti numbers *of M. If $X \subset \mathbb{P}^n$ is a projective variety, then we call the* Betti numbers *of X those of the coordinate ring S_X.*

The Betti numbers of a finitely generated graded module M are usually summarized in a *Betti table* of the form

	0	1	...	$m-1$	m
0	$\beta_{0,0}$	$\beta_{1,1}$	...	$\beta_{m-1,m-1}$	$\beta_{m,m}$
1	$\beta_{0,1}$	$\beta_{1,2}$	...	$\beta_{m-1,m}$	$\beta_{m,m+1}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
s	$\beta_{0,s}$	$\beta_{1,s+1}$	...	$\beta_{m-1,m+s-1}$	$\beta_{m,m+s}$

where the i^{th} column corresponds to the module F_i, in the minimal free resolution of M, and the Betti numbers in the i^{th} column specify the degrees of the generators of F_i.
If $M = S/I$ for some ideal $I \subset S$, then $\beta_{0,0} = 1$ and $\beta_{1,1} = \beta_{1,1}... = \beta_{m,m} = 0$. We define the 2-*linear strand* of S/I to be the subcomplex

$$S/I \longleftarrow S \longleftarrow S(-2)^{\beta_{1,2}} \longleftarrow S(-3)^{\beta_{2,3}} \longleftarrow \cdots \longleftarrow S(-m-1)^{\beta_{m,m+1}} \longleftarrow 0$$

of the minimal free resolution of S/I. The *length* of the 2-linear strand is the largest number n such that $\beta_{n,n+1} \neq 0$.

Lemma 1.5. *If $I \subset J$ are ideals containing no linear forms, then the 2-linear strand of the resolution of S/I is a summand of the minimal free resolution of S/J. In particular the length of the 2-linear strand of S/J is greater or equal than that of the 2-linear strand of S/I.*

Proof. See [Eis92, Lemma 1]. □

Proposition 1.6. *Let $\{\beta_{ij}\}$ be the graded Betti numbers of a finitely generated S-module. If for a given number i there is a number d such that $\beta_{ij} = 0$ for all $j < d$, then $\beta_{i+1,j+1} = 0$ for all $j < d$.*

Proof. See [Eis05, Proposition 1.9]. □

So writing the Betti numbers in a table with $(i,j)^{\text{th}}$ entry $\beta_{i,i+j}$ displays the Betti numbers in a more compact way because, if the i^{th} column of a Betti table has zeros above the j^{th} row, then the $(i+1)^{\text{th}}$ column has also zeros above the j^{th} row, by the theorem above.

Proposition 1.7. *The graded Betti numbers of a finitely generated graded S-module can be expressed as follows*

$$\beta_{ij} = \dim_{\Bbbk} \operatorname{Tor}_i^S(M, \Bbbk)_j \,.$$

Proof. See [Eis05, Proposition 1.7]. □

Example 1.8. The twisted cubic curve $C \subset \mathbb{P}^3$ has a minimal free resolution of the form

$$S(-3)^2 \longrightarrow S(-2)^3 \longrightarrow S$$

and we read off the Betti table

	0	1	2
0	1	-	-
1	-	3	2

.

1.2. Canonical Curves

If $C \subset \mathbb{P}^r$ is a curve embedded in some projective space by a complete linear series $|\mathscr{L}|$, then some properties of the coordinate ring S_C, such as the graded Betti numbers, depend on the curve and on the line bundle $\mathscr{L}$. In the case of non-hyperelliptic curves we can consider the linear series $|\omega_C|$ associated to the canonical bundle ω_C on a curve C. This linear series defines an embedding, and the properties of the coordinate ring S_C depend on C alone. We refer to such curves as *canonical curves,* and we refer to the embedding

$$j : C \hookrightarrow \mathbb{P}H^0(C, \omega_C) = \mathbb{P}^{g-1}$$

defined by $|\omega_C|$ as the canonical embedding. The following theorem is due to Noether.

Theorem 1.9. *Let $C \subset \mathbb{P}^{g-1}$ be a non-hyperelliptic canonical curve, then*

$$S_C = \sum_{n \geq 0} H^0(C, \omega_C^{\otimes n}).$$

In particular, $H^1(C, \mathscr{I}_C(m)) = 0$ for all $m \geq 0$.

Proof. See [ACGH85, §2 Chapter 3]. □

The main objects of interest in this thesis are the Betti numbers of $S_C = \sum_{n \geq 0} H^0(C, \omega_C^{\otimes n})$.

Proposition 1.10. *Let C be a non-hyperelliptic canonical curve of genus $g \geq 3$ then*

$$\omega_C = \mathscr{E}xt^{g-2}(\mathscr{O}_C, \mathscr{O}_{\mathbb{P}^{g-1}}) \cong \mathscr{O}_C(1).$$

The minimal free resolution of S_C is therefore, up to shift, self dual with

$$\beta_{i,j} = \beta_{g-2-i, g-1-j}$$

and has a Betti table of the form as indicated below

	0	1	2	...	$g-4$	$g-3$	$g-2$
0	1	-	-	...	-	-	-
1	-	$\beta_{1,2}$	$\beta_{2,3}$	...	$\beta_{g-4,g-3}$	$\beta_{g-3,g-2}$	-
2	-	$\beta_{g-3,g-2}$	$\beta_{g-4,g-3}$	...	$\beta_{2,3}$	$\beta_{1,2}$	-
3	–	-	-	...	-	-	1

where $\beta_{1,2} = \binom{g-2}{2}$.

Proof. See [Eis05, Proposition 9.6]. □

Definition 1.11. *A scheme X is called* arithmetically Gorenstein *if it has, up to shift, a self dual resolution.*

In particular, a canonically embedded curve is arithmetically Gorenstein.

Definition 1.12. *If $\mathscr{L}$ be a line bundle on a curve C, then the Clifford index of $\mathscr{L}$ is defined by*

$$\operatorname{Cliff}(\mathscr{L}) = \deg \mathscr{L} - 2(h^0(\mathscr{L}) - 1) = g + 1 - h^0(\mathscr{L}) - h^1(\mathscr{L}).$$

The Clifford index of a curve C is defined by taking the minimum of all "relevant" line bundles on C

$$\operatorname{Cliff}(C) = \min\{\operatorname{Cliff}(\mathscr{L}) \mid h^0(\mathscr{L}) \geq 2 \text{ and } h^1(\mathscr{L}) \geq 2\}.$$

Note that $\operatorname{Cliff}(\mathscr{L}) = \operatorname{Cliff}(\mathscr{L}^{-1} \otimes \omega_C)$ by Serre Duality.

Theorem 1.13 (Clifford's Theorem)**.** *If $\mathscr{L}$ is a special line bundle (that is $h^1(\mathscr{L}) \neq 0$), then $\operatorname{Cliff}(\mathscr{L}) \geq 0$ with equality if and only if $\mathscr{L} = \mathscr{O}_C$, or $\mathscr{L} = \omega_C$, or C is hyperelliptic and $\mathscr{L}$ is a power of the degree 2 line bundle with 2 independent sections.*

Proof. See [ACGH85, §1 Chapter 3]. □

For a canonical curve $C \subset \mathbb{P}^{g-1}$ it is conjectured in [Gre84] that there is a relation between the non-vanishing of certain Betti numbers and the existence of spacial linear series on C.

Notation 1.14. *Throughout the rest of this thesis g^r_k a denotes an r-dimensional linear series of degree k on a curve C.*

Conjecture 1.15 (Green's Conjecture)**.** *Let $C \subset \mathbb{P}^{g-1}$ be a canonical curve over a field $\Bbbk$ with $\operatorname{char}(\Bbbk) = 0$. Then the following is conjectured to be equivalent*

(1) $\beta_{i,i+1} \neq 0$

(2) there exists a linear series g^r_k on C with $r \geq 1$, $k \leq g-1$ and Clifford index $k - 2r = g - 2 - i$.

Remark 1.16. The direction "(2) ⇒ (1)" is proved in [Gre84] and the other direction was recently proved for general curves in [Voi05].

1.3. Line Bundles, Scrolls and Pencils

In this section, based on [Sch86], we consider the varieties swept out by g^1_k's on a canonical curve C. In [Sch86] Schreyer uses the approach presented in this section to classify all possible Betti tables of canonical curves of genus $g \leq 8$. He proceeds as follows: the variety swept out by a g^1_k on a canonical curve is a rational normal scroll X of dimension $d = k-1$. One can resolve the curve C as an $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-module, where $\mathbb{P}(\mathscr{E})$ is a $\mathbb{P}^{d-1}$-bundle associated to the scroll X. In the next step one can resolve the $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-module occurring in this resolution by $\mathscr{O}_{\mathbb{P}^{g-1}}$-modules and receive a possibly non-minimal resolution of C as an $\mathscr{O}_{\mathbb{P}^{g-1}}$-module by an iterated mapping cone construction.

Definition 1.17. *Let $e_1 \geq e_2 \geq ... \geq e_d \geq 0$ be integers and $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(e_1) \oplus ... \oplus \mathscr{O}_{\mathbb{P}^1}(e_d)$ be a locally free sheaf of rank d on $\mathbb{P}^1$. We denote by $\pi : \mathbb{P}(\mathscr{E}) \longrightarrow \mathbb{P}^1$ the corresponding $\mathbb{P}^{d-1}$-bundle.*
A rational normal scroll *X of type $S(e_1, ..., e_d)$ is the image of $\mathbb{P}(\mathscr{E})$ in $\mathbb{P}^r = H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}(1))$, where $r = f + d - 1$ with $f = e_1 + ... + e_d \geq 2$.*

Proposition 1.18. *The variety $X = S(e_1, ..., e_d)$ defined above is a non-degenerate variety of minimal degree*

$$\deg X = f = r - d + 1 = \operatorname{codim} X.$$

X is smooth if and only if $e_i > 0$ for all $i = 1, ..., d$. In this case the map $j : \mathbb{P}(\mathscr{E}) \to X$ is an isomorphism. If otherwise some of the $e_i = 0$, then $j : \mathbb{P}(\mathscr{E}) \to X \subset H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}(1)) = \mathbb{P}^r$ is a resolution of singularities.

Proof. See [Har81, Section 3] or [EH87, Section 1]. □

Remark 1.19. The singularities of X are rational , i.e.,

$$j_* \mathscr{O}_{\mathbb{P}(\mathscr{E})} = \mathscr{O}_X \text{ and } R^i j_* \mathscr{O}_{\mathbb{P}(\mathscr{E})} = 0 \text{ for } i > 0,$$

and we can therefore consider $\mathbb{P}(\mathscr{E})$ instead of X for most cohomological considerations.

Proposition 1.20. *The Picard group $\operatorname{Pic}(\mathbb{P}(\mathscr{E}))$ is generated by the ruling $R = [\pi^* \mathscr{O}_{\mathbb{P}^1}(1)]$ and the hyperplane class $H = [j^* \mathscr{O}_{\mathbb{P}^r}(1)]$. The intersection products are given by*

$$H^d = f, \quad H^{d-1} \cdot R = 1 \text{ and } R^2 = 0$$

Proof. See [Har81, Section 3] or [EH87, Section 1]. □

Remark 1.21. Following [Sch86, (1.3)], the cohomology groups $H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH + bR))$ can be computed explicitly. For $a \geq 0$ we have

$$H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH + bR)) \cong H^0(\mathbb{P}^1, (S_a(\mathscr{E}))(b)),$$

where $S_a(\mathscr{E})$ denotes the a^{th} symmetric power of the sheaf $\mathscr{E}$. To be more precise let $\Bbbk[s, t]$ be the coordinate ring of $\mathbb{P}^1$ and let

$$\varphi_i \in H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H - e_i R)), \quad i = 1, ..., d$$

be the basic sections obtained from the inclusion of the i^{th} summand

$$\mathscr{O}_{\mathbb{P}^1} \longrightarrow \mathscr{E}(-e_i) \cong \pi_* \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H - e_i R).$$

We can identify the sections $\psi \in H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH+bR))$ with homogeneous polynomials

$$\psi = \sum_{\alpha} P_\alpha(s,t) \varphi_1^{\alpha_1} \dots \varphi_d^{\alpha_d}$$

of degree $a = \alpha_1 + \cdots + \alpha_d$ in φ_i's and with polynomial coefficients $P_\alpha \in \Bbbk[s,t]$ of degree $\deg P_\alpha = \alpha_1 e_1 + \cdots + \alpha_d e_d + b$. We say that b is *sufficiently large* if $\alpha_1 e_1 + \cdots + \alpha_d e_d + b \geq -1$ for all partitions $a = \alpha_1 + \cdots + \alpha_d$. For b sufficiently large we get the dimension

$$h^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH + bR)) = f \binom{a+d-1}{d} + (b+1) \binom{a+d-1}{d-1}.$$

If we choose a basis

$$x_{i,j} = s^j t^{e_i - j} \varphi_i \quad \text{with } j = 0, ..., e_i \text{ and } i = 1, ..., d \tag{1.1}$$

for $H^0 \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H) \cong H^0 \mathscr{O}_{\mathbb{P}^r}(1)$, we see that the defining equations of X are determinantal. Consider the matrix

$$\Phi = \begin{pmatrix} x_{1,0} & \cdots & x_{1,e_1-1} & x_{2,0} & \cdots & \cdots & x_{d,e_d-1} \\ x_{1,1} & \cdots & x_{1,e_1} & x_{2,1} & \cdots & \cdots & x_{d,e_d} \end{pmatrix}, \tag{1.2}$$

then the 2×2 minors of Φ generate the homogeneous ideal of X by [EH87, Section 1]. In more intrinsic terms Φ can be obtained from the multiplication map

$$H^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(R)) \otimes (H^0 \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H - R)) \longrightarrow H^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(H)).$$

In the following we want to explain how to resolve the $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-modules $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH + bR)$ as $\mathscr{O}_{\mathbb{P}^r}$-modules. To this end let

$$\Phi : F \longrightarrow G$$

be a map of locally free sheaves of rank f and g with $f \geq g$ on a smooth variety V. Under suitable hypothesis on Φ we get a complex $\mathscr{C}^b$ that resolves the b^{th} symmetric power of the cokernel of Φ, for b sufficiently large (see [BE75]). The j^{th} term of the complex $\mathscr{C}^b$ is defined by

$$\mathscr{C}^b_j = \begin{cases} \bigwedge^j F \otimes S_{b-j} G \otimes \mathscr{O}_{\mathbb{P}^r}(-j), & \text{for } 0 \leq j \leq b \\ \bigwedge^{j+g-1} F \otimes D_{j-b-1} G^* \otimes \bigwedge^g G^* \otimes \mathscr{O}_{\mathbb{P}^r}(-j-1), & \text{for } j \geq b+1 \end{cases}$$

where $S_j G$ denotes the j^{th} symmetric power algebra and $D_j G^*$ the j^{th} divided power algebra (see [Eis95, Appendix 2] for a short survey on symmetric and divided power algebras). The differentials $\delta_j : \mathscr{C}^b_j \to \mathscr{C}^b_{j-1}$ are defined by the multiplication with $\Phi \in H^0(V, F^* \otimes G)$ for $j \neq b+1$ and by $\bigwedge^g \Phi \in H^0(V, \bigwedge^g F^* \otimes \bigwedge^g G)$ for $j = b+1$.

Theorem 1.22. *If we regard the map $\Phi : F(-1) \longrightarrow G$ from (1.2) as a map between bundles $F = H^0\mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-R) \otimes \mathscr{O}_{\mathbb{P}^r} = \mathscr{O}^f_{\mathbb{P}^r}$ and $G = H^0\mathscr{O}_{\mathbb{P}(\mathscr{E})}(R) \otimes \mathscr{O}_{\mathbb{P}^r} = \mathscr{O}^2_{\mathbb{P}^r}$, then the Eagon-Northcott type complex $\mathscr{C}^b(a) := \mathscr{C}^b \otimes \mathscr{O}_{\mathbb{P}^r}(a)$, defined above, gives a minimal free resolution of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH + bR)$ as an $\mathscr{O}_{\mathbb{P}^r}$-module.*

Proof. See [Sch86, section 1]. □

Remark 1.23. The complex $\mathscr{C}^0$ is the well known Eagon-Northcott complex associated to the matrix Φ, i.e., $\mathscr{C}^0$ resolves the ideal sheaf $I_g(\Phi)$ generated by the $(g \times g)$ minors of the matrix Φ.

Now let $C \subset \mathbb{P}^{g-1}$ be a canonically embedded curve of genus g. Let further

$$\{D_\lambda\}_{\lambda\in\mathbb{P}^1} \subset |D|$$

be a pencil of divisors on the curve C satisfying $h^0(C, \omega_C \otimes \mathscr{O}_C(D)^{-1}) = f \geq 2$ and let $G \subset H^0(C, \mathscr{O}_C(D))$ be the 2-dimensional subspace defining the pencil. The variety

$$X = \bigcup_{\lambda\in\mathbb{P}^1} \overline{D_\lambda}\ ,$$

where $\overline{D_\lambda} \subset \mathbb{P}^{g-1}$ denotes the linear span of the divisor D_λ, is a rational normal scroll of degree f by [EH87, Theorem 2]. In a more algebraic way, X can be defined by the vanishing of the 2×2 minors of the $2 \times f$ multiplication matrix

$$\Phi : G \otimes H^0(C, \omega_C \otimes \mathscr{O}_C(D)^{-1}) \longrightarrow H^0(C, \omega_C).$$

The type of the scroll is uniquely determined by the pencil $\{D_\lambda\}_{\lambda\in\mathbb{P}^1}$ and can explicitly be calculated by [Sch86, (2.4)].
Conversely, let X be a rational normal scroll of degree f containing a canonical curve C. Then the ruling R on X cuts out a pencil of divisors $\{D_\lambda\}_{\lambda\in\mathbb{P}^1} \subset |D|$ on C (possibly with base points), such that $h^0(C, \omega_C \otimes \mathscr{O}_C(D)^{-1}) = f$.

Remark 1.24. By the geometric version of the Riemann-Roch Theorem, the following holds for any effective divisor D of degree k on C :

$$\dim \overline{D} = \deg D - \dim |D| - 1.$$

So if

$$g^1_k = \{D_\lambda\}_{\lambda\in\mathbb{P}^1}$$

is a base point free complete pencil of divisors of degree $k \leq g-1$ on C, then the geometric version of the Riemann-Roch Theorem implies that $\dim \overline{D_\lambda} = k - 2$. It follows that

$$X = \bigcup_{\lambda\in\mathbb{P}^1} \overline{D_\lambda} \subset \mathbb{P}^{g-1}$$

is a $(k-1)$-dimensional rational normal scroll of degree $f = g - k + 1$.

Notation 1.25. *During the rest of this section $C \subset \mathbb{P}^{g-1}$ will denote a canonical curve with a basepoint free g^1_k. The variety X is the scroll of degree f and dimension $d = k-1$ defined by this pencil and $\mathbb{P}(\mathscr{E})$ will denote the $\mathbb{P}^{d-1}$-bundle corresponding to X. If we regard C as an $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-module, we will write $C \subset \mathbb{P}(\mathscr{E})$.*

Theorem 1.26. *i) $C \subset \mathbb{P}(\mathscr{E})$ has a resolution $F_\bullet$ of type*

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-kH + (f-2)R) \longrightarrow \sum_{j=1}^{\beta_{k-3}} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-(k-2)H + a_j^{(k-3)}R) \longrightarrow$$

$$\ldots \longrightarrow \sum_{j=1}^{\beta_1} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H + a_j^{(1)}R) \longrightarrow \mathscr{O}_{\mathbb{P}(\mathscr{E})} \longrightarrow \mathscr{O}_C \longrightarrow 0$$

with $\beta_i = \frac{i(k-2-i)}{k-1}\binom{k}{i+1}$.

ii) The complex $F_\bullet$ is self dual, i.e.,

$$\mathscr{H}om(F_\bullet, \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-kH + (f-2)R)) \cong F_\bullet$$

iii) If all $b_k \geq -1$, then an iterated mapping cone construction

$$\left[\left[\ldots\left[\mathscr{C}^{(f-2)}(-k) \longrightarrow \sum_{j=1}^{\beta_{k-3}} \mathscr{C}^{(a_j^{(k-3)})}(-k-2)\right] \longrightarrow \ldots\right] \longrightarrow \mathscr{C}^0\right]$$

gives a, not necessarily minimal, resolution of C as an $\mathscr{O}_{\mathbb{P}^{g-1}}$-module.

Proof. See [Sch86, Corollary (4.4)] and [Sch86, Lemma (4.2)]. □

Remark 1.27. The $a_i^{(k)}$'s in part *i)* of the theorem above satisfy certain linear equations obtained from the Euler characteristic of the complex $F_\bullet$ (see [Sch86, (3.3)] for details). Schreyer shows in [Sch86, Section 6], that for example in the case of 5-gonal curves C we have

- $\sum_i^5 a_i = 2g - 12$
- $a_i + b_i = f - 2$, where $f = g - 4$ is the degree of the scroll swept out by the g^1_5 on C, and $a_i := a_i^{(1)}, b_i := a_i^{(2)}$.

For a section $\Psi : \mathscr{O}_X(-H+bR) \longrightarrow \mathscr{O}_X(aR)$ in $H^0(\mathscr{O}_X(H-(b-a)R))$ we obtain comparison maps $\rho_\bullet : \mathscr{C}^b_\bullet(-1) \longrightarrow \mathscr{C}^a_\bullet$ which are determined by Ψ up to homotopy. By degree reasons

$$\mathrm{Hom}(\mathscr{C}^b_{a+1}(-1), \mathscr{C}^a_{a+2}) = \mathrm{Hom}(\mathscr{C}^b_a(-1), \mathscr{C}^a_{a+1}) = 0$$

and therefore the $(a+1)^{st}-$comparison map ρ_{a+1} is uniquely determined by Ψ (not only up to homotopy). We have the following lemma.

Lemma 1.28. *For $\Psi : \mathscr{O}_X(-H+bR) \longrightarrow \mathscr{O}_X(aR)$, i.e., $\Psi \in H^0\mathscr{O}_X(H-(b-a)R)$ and $a \leq b \leq f-1$ sufficiently large, the $(a+1)^{st}$-comparison map coincides, up to a scalar factor, with the map induced by the composition*

$$\rho : \bigwedge^{a+1} F \otimes S_{b-a-1}G \cong \bigwedge^{a+1} F \otimes H^0\mathscr{O}_X((b-a+1)R) \xrightarrow{id\otimes\Psi} \bigwedge^{a+1} F \otimes F \xrightarrow{\wedge} \bigwedge^{a+2} F.$$

Proof. See appendix in [MS86]. □

Remark 1.29. Similar results as the lemma above apply to the j^{th}-comparism map if $\operatorname{Hom}(\mathscr{C}^b_j(-1), \mathscr{C}^a_{j+1}) = \operatorname{Hom}(\mathscr{C}^b_{j-1}(-1), \mathscr{C}^a_j) = 0$.

The following example is due to Schreyer (see [Sch86, Section 7]).

Example 1.30. Let $C \subset \mathbb{P}^6$ be a canonical curve of genus 7 that admits a g^1_5 and has invariants $(a_1, ..., a_5) = (0,0,0,1,1)$ (by [Sch86, Section 7] this is in fact the generic case). Let further $X = \bigcup_{D\in g^1_5} \overline{D} \subset \mathbb{P}^6$ be the 4-dimensional rational normal scroll of degree $f = g-5+1 = 3$ swept out by the g^1_5 and let $\mathbb{P}(\mathscr{E})$ be the corresponding $\mathbb{P}^3$-bundle associated to X. By Theorem 1.26 and Remark 1.27, the minimal resolution of C as an $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-module is of the form

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-5H+R) \longrightarrow \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+R)^{\oplus 3} \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H)^{\oplus 2} \end{matrix} \xrightarrow{\Psi} \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H)^{\oplus 3} \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+R)^{\oplus 2} \end{matrix} \longrightarrow \mathscr{O}_{\mathbb{P}(\mathscr{E})} \to \mathscr{O}_C \longrightarrow 0,$$

and by the Structure Theorem for Gorenstein ideals in codimension 3 (see [BE77]), the 5×5 matrix Ψ is skew-symmetric and its 5 Pfaffians generate the ideal of C. As stated in Theorem 1.22, we can resolve the $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-modules in the resolution above by the Eagon-Northcott type complexes $\mathscr{C}$ and obtain:

$$\begin{tikzcd}
0 & 0 & 0 & 0 & \\
\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-5H+R) \arrow[r] \arrow[u] & \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+R)^{\oplus 3} \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H)^{\oplus 2} \end{matrix} \arrow[r, "\Psi"] \arrow[u] & \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H)^{\oplus 3} \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+R)^{\oplus 2} \end{matrix} \arrow[r] \arrow[u] & \mathscr{O}_{\mathbb{P}(\mathscr{E})} \arrow[r] \arrow[u] & \mathscr{O}_C \\
G(-5) \arrow[r] \arrow[u] & \begin{matrix} G(-3)^{\oplus 3} \\ \oplus \\ \mathscr{O}_{\mathbb{P}^6}(-3)^{\oplus 2} \end{matrix} \arrow[r] \arrow[u] & \begin{matrix} \mathscr{O}_{\mathbb{P}^6}(-2)^{\oplus 3} \\ \oplus \\ G(-2)^{\oplus 2} \end{matrix} \arrow[r] \arrow[u] & \mathscr{O}_{\mathbb{P}^6} \arrow[u] & \\
F(-6) \arrow[r] \arrow[u] & \begin{matrix} F(-4)^{\oplus 3} \\ \oplus \\ \bigwedge^2 F(-5)^{\oplus 2} \end{matrix} \arrow[r] \arrow[u] & \begin{matrix} \bigwedge^2 F(-4)^{\oplus 3} \\ \oplus \\ F(-3)^{\oplus 2} \end{matrix} \arrow[r] \arrow[u] & \bigwedge^2 F(-2) \arrow[u] & \\
\bigwedge^3 F(-8) \arrow[r] \arrow[u] & \begin{matrix} \bigwedge^3 F(-6)^{\oplus 3} \\ \oplus \\ \bigwedge^3 F\otimes D_1G^*(-6)^{\oplus 2} \end{matrix} \arrow[r] \arrow[u] & \begin{matrix} \bigwedge^3 F\otimes D_1G^*(-5)^{\oplus 3} \\ \oplus \\ \bigwedge^3 F(-5)^{\oplus 2} \end{matrix} \arrow[r] \arrow[u] & \bigwedge^3 F \otimes D_1G^*(-3) \arrow[u] &
\end{tikzcd}$$

By Theorem 1.26, the iterated mapping cone

$$\left[\left[\left[\mathscr{C}^1(-5) \longrightarrow \begin{matrix}\mathscr{C}^1(-3)^{\oplus 3}\\ \oplus \\ \mathscr{C}^0(-3)^{\oplus 2}\end{matrix}\right] \longrightarrow \begin{matrix}\mathscr{C}^0(-2)^{\oplus 3}\\ \oplus \\ \mathscr{C}^1(-2)^{\oplus 2}\end{matrix}\right] \longrightarrow \mathscr{C}^0\right]$$

gives a not necessarily minimal resolution of C as an $\mathscr{O}_{\mathbb{P}^6}$-module. By degree reasons the only non-minimal part that can occur in the iterated mapping cone arises from the 3×3 submatrix

$$\widetilde{\Psi} = \begin{pmatrix} 0 & \Psi_{12} & \Psi_{13} \\ -\Psi_{12} & 0 & \Psi_{23} \\ -\Psi_{13} & -\Psi_{23} & 0 \end{pmatrix}$$

of Ψ, where Ψ_{ij} can be identified with sections in $H^0\mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-R)$:

$$\begin{tikzcd}
\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+R)^{\oplus 3} \arrow[r, "\widetilde{\Psi}"] & \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H)^{\oplus 3} \\
G(-3)^{\oplus 3} \arrow[r] \arrow[u] & \mathscr{O}_{\mathbb{P}^6)}(-2) \arrow[u] \\
F(-4)^{\oplus 3} \arrow[r, "\alpha"] \arrow[u] & \bigwedge^2 F(-4)^{\oplus 3} \arrow[u] \\
\bigwedge^3 F(-6)^{\oplus 3} \arrow[r] \arrow[u] & \bigwedge^3 \otimes D_1 G^*(-5)^{\oplus 3} \arrow[u]
\end{tikzcd}$$

To be more precise, the graded Betti numbers of the resolution of C as an $\mathscr{O}_{\mathbb{P}}^6$- module depend on, and determine, the rank of the map $\alpha : F(-4)^{\oplus 3} \to \bigwedge^2 F(-4)^{\oplus 3}$ that is given by the wedge product with $\widetilde{\Psi}$. Thus α has full rank, i.e., $\mathrm{rank}(\alpha) = 9$ if the characteristic of the ground field is not equal to 2. We can now sum up the ranks of the modules in the iterated mapping cone construction and write down the Betti table of the curve C if we take the rank of α into account:

	0	1	2	3	4	5
0	1	-	-	-	-	-
1	-	10	16	9−rank(α)	-	-
2	-	-	9−rank(α)	16	10	-
3	-	-	-	-	-	1

In [Sch86] Schreyer uses the approach presented in the example above to give a complete list of all possible Betti tables of canonical curves up to genus 8. In [Sag58] Sagraloff classifies all possible Betti tables of canonical curves of genus $g = 9$. If we exclude the hyperelliptic case and assume that the invariants $e_1, ..., e_d$ of the scroll $X = S(e_1, .., e_d)$, on which the canonical curve C lies, as well as the numbers $a_j^{(i)}$ in the resolution of C as a $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-module (see Theorem 1.26) satisfy the balancing condition below, then the Betti tables of canonical curves carrying a g_k^1 look as in Table 1.1.

Definition 1.31. *Let $(a_1, ..., a_d)$ be a partition of $a = \sum_i a_i$. We call the partition $(a_1, .., a_d)$* balanced *or* generic *if $a_1 = ... = a_r = q+1$ and $a_{r+1} = ... = a_d = q$ where $a = q \cdot d + r$ and $0 \leq r < d$.*
A rational normal scroll $X = S(e_1, ..., e_d)$ of dimension d and degree $f = \sum_i e_i$ is said to be of generic type *if $(e_1, ..., e_d)$ satisfies the balancing condition above.*
We say that a k-gonal canonical curve satisfies the balancing conditions *if*

- *the scroll X of dimension d swept out by the g^1_k on C is of generic type and*
- *the partitions $(a_1^{(i)}, ..., a_{\beta_{k-3}}^{(i)})$ that occur in Theorem 1.26 are balanced for all $i = 1, ..., k-3$.*

genus	case	Betti table
$g=3$	generic	$\begin{array}{cc} 1 & - \\ - & - \\ - & - \\ - & 1 \end{array}$
$g=4$	generic	$\begin{array}{ccc} 1 & - & - \\ - & 1 & - \\ - & 1 & - \\ - & - & 1 \end{array}$
$g=5$	generic	$\begin{array}{cccc} 1 & - & - & - \\ - & 3 & - & - \\ - & - & 3 & - \\ - & - & - & 1 \end{array}$
	g^1_3	$\begin{array}{cccc} 1 & - & - & - \\ - & 3 & 2 & - \\ - & 2 & 3 & - \\ - & - & - & 1 \end{array}$
$g=6$	generic	$\begin{array}{ccccc} 1 & - & - & - & - \\ - & 6 & 5 & - & - \\ - & - & 5 & 6 & - \\ - & - & - & - & 1 \end{array}$
	g^1_3	$\begin{array}{ccccc} 1 & - & - & - & - \\ - & 6 & 8 & 3 & - \\ - & 3 & 8 & 6 & - \\ - & - & - & - & 1 \end{array}$
$g=7$	generic	$\begin{array}{cccccc} 1 & - & - & - & - & - \\ - & 10 & 16 & - & - & - \\ - & - & - & 16 & 10 & - \\ - & - & - & - & - & 1 \end{array}$
	g^1_4	$\begin{array}{cccccc} 1 & - & - & - & - & - \\ - & 10 & 16 & 3 & - & - \\ - & - & 3 & 16 & 10 & - \\ - & - & - & - & - & 1 \end{array}$
$g=7$	g^1_3	$\begin{array}{cccccc} 1 & - & - & - & - & - \\ - & 10 & 20 & 15 & 4 & - \\ - & 4 & 15 & 20 & 10 & - \\ - & - & - & - & - & 1 \end{array}$
$g=8$	generic	$\begin{array}{ccccccc} 1 & - & - & - & - & - & - \\ - & 15 & 35 & 21 & - & - & - \\ - & - & - & 21 & 35 & 15 & - \\ - & - & - & - & - & - & 1 \end{array}$
	g^1_4	$\begin{array}{ccccccc} 1 & - & - & - & - & - & - \\ - & 15 & 35 & 25 & 4 & - & - \\ - & - & 4 & 25 & 35 & 15 & - \\ - & - & - & - & - & - & 1 \end{array}$
	g^1_3	$\begin{array}{ccccccc} 1 & - & - & - & - & - & - \\ - & 15 & 40 & 45 & 24 & 5 & - \\ - & 5 & 24 & 45 & 40 & 15 & - \\ - & - & - & - & - & - & 1 \end{array}$
$g=9$	generic	$\begin{array}{cccccccc} 1 & - & - & - & - & - & - & - \\ - & 21 & 64 & 70 & - & - & - & - \\ - & - & - & - & 70 & 64 & 21 & - \\ - & - & - & - & - & - & - & 1 \end{array}$
	g^1_5	$\begin{array}{cccccccc} 1 & - & - & - & - & - & - & - \\ - & 21 & 64 & 70 & 4 & - & - & - \\ - & - & - & 4 & 70 & 64 & 21 & - \\ - & - & - & - & - & - & - & 1 \end{array}$
	g^1_4	$\begin{array}{cccccccc} 1 & - & - & - & - & - & - & - \\ - & 21 & 64 & 75 & 24 & 5 & - & - \\ - & - & 5 & 24 & 75 & 64 & 21 & - \\ - & - & - & - & - & - & - & 1 \end{array}$
	g^1_3	$\begin{array}{cccccccc} 1 & - & - & - & - & - & - & - \\ - & 21 & 70 & 105 & 84 & 35 & 6 & - \\ - & 6 & 35 & 84 & 105 & 70 & 21 & - \\ - & - & - & - & - & - & - & 1 \end{array}$

Table 1.1.: Betti tables of k-gonal curves of genus $g \leq 9$ satisfying the balancing conditions.

Lemma 1.32. *Let $C \subset \mathbb{P}^{g-1}$ be a general k-gonal canonical curve and let X be the scroll swept out by the special pencil. Then X is of generic type.*

Proof. The proof of this lemma uses *Ballico's Theorem* (see [Bal89]) and can be found in detail in [Gei13, Section 4]. □

Remark 1.33. Having generic Betti numbers is an open condition. We can therefore prove that the generic k-gonal curve of genus g satisfies the balancing conditions by computing one example for each g. In Appendix B this is done exemplary for case of a 5-gonal curve of genus 13.

By Lemma 1.5 we know that the 2-linear strand of a k-gonal canonical curve is a summand of the 2-linear strand of the scroll X swept out by the special pencil. If we set $m = \lceil \frac{g-1}{2} \rceil$ and compare the Betti numbers $\beta_{m,m+1}, ..., \beta_{g-3,g-2}$ in the 2-linear strand of the curves in the table above with those of the 2-linear strand of the scrolls X, then we see that these numbers coincide.
It is natural to ask whether this is the case for larger genus.

Conjecture 1.34 (Schreyer)**.** *Let C be a general non-hyperelliptic k-gonal canonical curve of genus g and let X be the scroll swept out by the g^1_k on C. Suppose further that C satisfies the balancing conditions. If we set $m = \lceil \frac{g-1}{2} \rceil$, then the following is conjectured to be true*

$$\beta_{m,m+1}(C) = \beta_{m,m+1}(X), ..., \beta_{g-3,g-2}(C) = \beta_{g-3,g-2}(X).$$

Remark 1.35. With the notation as in the conjecture above, Lemma 1.5 states in particular that if $\beta_{m,m+1}(C) = \beta_{m,m+1}(X)$, then

$$\beta_{m+1,m+2}(C) = \beta_{m+1,m+2}, ..., \beta_{g-3,g-2}(C) = \beta_{g-3,g-2}(X).$$

We therefore refer to $\beta_{\text{crit}}(C) = \beta_{m,m+1}(C)$ as the *critical Betti number* of the canonical curve C. Furthermore we refer to $\beta_{\text{exp}}(C) = \beta_{m,m+1}(X)$ as the *expected critical Betti number*. By Theorem 1.22, we can compute $\beta_{\text{exp}}(C)$ as

$$\beta_{\text{exp}}(C) = \text{rank}(\bigwedge^{m+1} F \otimes D_{m-1}G^*) = \binom{f}{m+1} \cdot m,$$

where $F = \mathscr{O}^f_{\mathbb{P}^{g-1}}$, $G = \mathscr{O}^2_{\mathbb{P}^{g-1}}$ and $f = g - k + 1$ is the degree of the scroll X swept out by the g^1_k on C.

Remark 1.36. Any curve of genus g has a g^1_k for $k \geq \frac{1}{2}g + 1$ (see [Har77, §5 Chapter 4]). On the other hand the generic curve of genus g has no g^1_k for $k < \frac{1}{2}g + 1$. By Voisin's proof of Green's Conjecture for generic curves, see [Voi05], it follows in particular that $\beta_{\text{crit}}(C) = \beta_{\text{exp}}(C) = 0$ in the generic case.

In the following chapter we will describe how to construct k-gonal g-nodal canonical curves using the computer algebra system *Macaulay2* (c.f. [GS]), and see that the conjecture above has several exceptional cases. In Chapter 3 we will take a closer look at one of the exceptional cases and show that the conjecture in the form as above fails for 5-gonal curves with odd genus $g \geq 13$ and even genus $g \geq 28$.

2. Construction of Nodal Curves with Special Pencils

In this chapter we summarize the theory needed for the construction of k-gonal canonical nodal curves and the effective computation of single Betti numbers. In the last section of this chapter we give an example of how this can be implemented in *Macaulay2*.

2.1. Construction of Nodal Curves

We start by summarizing a construction method of g-nodal curves described in [CEFS61, Section 4]. Following [CEFS61], we begin by recalling some basic facts.
Suppose that C is a rational g-nodal curve with a normalisation $\nu : \mathbb{P}^1 \longrightarrow C$ and let further $\{P_i, Q_i\}_{i=1,\dots,g}$ be the preimages of the g nodes. By [ACG11, Chapter 10 §2], a line bundle $\mathscr{L} \in \operatorname{Pic}^d(C)$ is given by an isomorphism $\nu^*(\mathscr{L}) \cong \mathscr{O}_{\mathbb{P}^1}(d)$ and gluing data between the residue class fields

$$\frac{b_j}{a_j} : \mathscr{O}_{\mathbb{P}^1}(d) \otimes \kappa(P_j) \cong \mathscr{O}_{\mathbb{P}^1}(d) \otimes \kappa(Q_j).$$

If we denote by $S = \Bbbk[s,t]$ the homogeneous coordinate ring of $\mathbb{P}^1$, then the cohomology group $H^0(C, \mathscr{L})$ is given by

$$H^0(C, \mathscr{L}) \cong \{f \in S_d \mid b_j f(P_j) = a_j f(Q_j) \ \text{ for } \ j = 1, ..., g\}.$$

In other words, the space $H^0(C, \mathscr{L}) \subset H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(d))$ is the solution space of a system of homogeneous equations.
If we assume that all points $P_i,\ Q_i$ lie in an affine chart $\mathfrak{U}_0$, then a basis of sections of the dualizing sheaf ω_C is given by

$$\left\{\omega_j = \frac{\mathrm{d}z}{(z-p_j)(z-q_j)}\right\}_{j=1,\dots,g} ,$$

where $z = \frac{s}{t}$, $P_j = (1 : p_j)$ and $Q_j = (1 : q_j)$. In this case the canonical map j_{ω_C} is induced by

$$j_{\omega_C} : \mathbb{P}^1 \to \mathbb{P}^{g-1} \ , \quad z \mapsto (\prod_{i \neq 1}(z-p_i)(z-q_i) : ... : \prod_{i \neq g}(z-p_i)(z-q_i)).$$

We are now able to write down a construction method for nodal curves which can be implemented in a computer algebra system, as it is done in the *Macaulay2*-package $NodalCurves$, available at [Sch12b].

Construction 1.
Step 1. We choose $2g$ distinct points $\{P_i, Q_i\}_{i=1,...,g}$ in $\mathbb{P}^1$.
Step 2. We define g quadrics $\mathfrak{q}_1, ..., \mathfrak{q}_g$ by

$$\mathfrak{q}_i := \begin{vmatrix} P_i^{(0)} & s \\ P_i^{(1)} & t \end{vmatrix} \cdot \begin{vmatrix} Q_i^{(0)} & s \\ Q_i^{(1)} & t \end{vmatrix},$$

where $P_j = (P_j^{(0)} : P_j^{(1)})$, $Q_j = (Q_j^{(0)} : Q_j^{(1)})$ and $S = \mathbb{k}[s,t]$ denotes homogeneous the coordinate ring of $\mathbb{P}^1$.
A basis of $H^0(C, \omega_C)$ is now given by

$$\left\{ s_i := \prod_{\substack{j \neq i \\ j=1}}^{g} \mathfrak{q}_i \in S_{2g-2} \;\middle|\; i = 1, ..., g \right\}.$$

The matrix $s = (s_1, ..., s_g)$ defines the canonical embedding $j_{\omega_C} : \mathbb{P}^1 \to C \subset \mathbb{P}^{g-1}$, and the ideal of the g-nodal curve C is given by the kernel of the map $s : T = \mathbb{k}[t_0, ..., t_{g-1}] \to S$.
Step 3. We can read off the canonical multipliers

$$\alpha_i := s_i(P_i^{(0)}, P_i^{(1)}) \text{ and } \beta_i := s_i(Q_i^{(0)}, Q_i^{(1)})) \text{ for } i = 1, ..., g.$$

It follows by construction, that $\beta_i s_j(P_i) = \alpha_i s_j(Q_i)$ for all $i, j \in \{1, ..., g\}$.

Remark 2.1. The basis $\{s_i\}_{i=1,...,g}$ computed in Step 2 is not sufficiently general so that the variables $t_{g-1}, t_{g-2} \in T = \mathbb{k}[t_0, ..., t_{g-1}]$ do not form a regular sequence for T/I_C. This might cause computational problems since we want to make the computation feasible by Artinian reduction (see Section 2.3). In this case we can construct a matrix $\widetilde{s}$ defining the canonical embedding by taking general linear combination of the $\{s_i\}$ for each entry of $\widetilde{s}$. The curve $\widetilde{C}$ obtained from the matrix $\widetilde{s}$ has the same geometric invariants as the curve C, but one has a better chance that $t_{g-1}, t_{g-2} \in T$ form a regular sequence for $T/I_{\widetilde{C}}$.

2.2. Nodal Curves with Special Pencils

Our purpose in this section is the following: Given two integers $g, k \leq \lceil \frac{g}{2} \rceil$, we want to construct a rational g-nodal canonical curve that carries a g_k^1. Therefore, we have to find two polynomials $f, g \in S_k = \mathbb{k}[s,t]_k$ such that

$$\begin{matrix} \beta_i f(P_i) = \alpha_i f(Q_i) \\ \beta_i g(P_i) = \alpha_i g(Q_i) \end{matrix} \quad \text{for } i = 1, ..., g,$$

for some multipliers $\{\alpha_i, \beta_i\}_{i=1,...,g}$ and $2g$ distinct points $\{P_i, Q_i\}_{i=1,...,g}$. If we do the construction described in the last section with these points we obtain g-nodal k-gonal canonical curve.

Suppose that

$$f = f_1 s^k + f_1 s^{k-1} t + \cdots + f_m t^k, \text{ with } \ m = \dim_{\Bbbk} S_k = \binom{k+1}{k}$$

with relations

$$\beta_i f(P_i) - \alpha_i f(Q_i) = 0 \text{ for } \ i = 1, ..., g.$$

This yields a system of linear equations in the coefficients of f.
If we fix a basis $\mathcal{B} = \{s^k, s^{k-1}t, ..., t^k\}$ of S_k and denote by $\mathcal{B}_i$ the i^{th} basis element viewed as a function

$$\mathcal{B}_i : \mathbb{P}^1 \to \Bbbk, \quad P = (p_0 : p_1) \mapsto \mathcal{B}_i(P) = p_0^{k-i} p_1^i,$$

then this system of equations can be written as follows:

$$A \cdot \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} = 0 \text{ with a } g \times m \text{ matrix } A \text{ with } (A)_{i,j} = \beta_i \mathcal{B}_j(P_i) - \alpha_i \mathcal{B}_j(Q_i). \tag{2.1}$$

Finding $f, g \in S$ that build a basis for a g_k^1 is now equivalent to $\dim_{\Bbbk} \ker A \geq 2$. The matrix A depends on k, the multipliers and the $2g$ points $\{P_i, Q_i\}_{i=1,...,g}$, and we have very little chances to find the desired f, g if we pick the $2g$ points randomly and do the construction described in the previous section.

Construction 2. We construct the $2g$ points $\{P_i, Q_i\}$ in a good way, such that the construction from the previous section works out. We proceed as follows:
Step 1. We construct a morphism

$$\varphi : \mathbb{P}^1 \to \mathbb{P}^1, \quad (p_0 : p_1) \mapsto (f(p_0, p_1) : g(p_0, p_1))$$

with $f, g \in S_k$ and points $\{R_i = (R_i^{(0)}, R_i^{(1)})\}_{i=1,...,g} \in \mathbb{P}^1$ such that

$$\begin{vmatrix} f & R_i^{(0)} \\ g & R_i^{(1)} \end{vmatrix} = R_i^{(0)} f - R_i^{(1)} g$$

has at least two linear factors for all $i \in \{1, ..., g\}$. Such morphism φ and points R_i can be found in a few seconds using a computer algebra system as it is done in [Bop13].
Step 2. We can now define $2g$ linear forms $\{l_1^{(i)}, l_2^{(i)}\}_{i=1,...,g}$:

$$\begin{vmatrix} f & R_i^{(0)} \\ g & R_i^{(1)} \end{vmatrix} = \underbrace{(as - bt)}_{=:l_1^{(i)}} \underbrace{(cs - dt)}_{=:l_2^{(i)}} r_i, \text{ with } r_i \in S_{k-2}$$

and compute the desired $2g$ points $P_i = V(l_1^{(i)}), \ Q_i = V(l_2^{(i)})$ as the vanishing loci of the linear forms $l_1^{(i)}$ and $l_2^{(i)}$.

If we do the construction from Section 2.1 with this set of points, we get a canonical curve C, with the property that the following diagram commutes

$$\begin{tikzcd} \mathbb{P}^1 \arrow[rr, "\varphi=(f,g)"] \arrow[dr, "\nu"'] & & \mathbb{P}^1 \\ & C \arrow[ur, "d:1"'] & \end{tikzcd}$$

Step 3. In this step we compute multipliers $\{\alpha_i, \beta_i\}_{i=1,..,g}$ such that

$$\begin{matrix} \beta_i f(P_i) = \alpha_i f(Q_i) \\ \beta_i g(P_i) = \alpha_i g(Q_i) \end{matrix} \quad \text{for} \quad i = 1, ..., g.$$

We have

$$\begin{vmatrix} f(P_i) & g(P_i) \\ f(Q_i) & g(Q_i) \end{vmatrix} = f(P_i)g(Q_i) - f(Q_i)g(P_i) = 0$$

since $R_i^{(1)} f(P_i) - R_i^{(2)} g(P_i) = R_i^{(1)} f(Q_i) - R_i^{(2)} g(Q_i) = 0$. It follows that

$$\frac{f(P_i)}{g(P_i)} = \frac{R_i^{(1)}}{R_i^{(2)}} = \frac{f(Q_i)}{g(Q_i)},$$

and we can therefore define the multipliers $\{\alpha_i, \beta_i\}_{i=1,..,g}$ to be

$$\beta_i := 1 \text{ and } \alpha_i := \frac{f(P_i)}{f(Q_i)} = \frac{g(P_i)}{g(Q_i)} \ \forall i = 1, .., g.$$

Remark 2.2. For the computation of the sections that define the canonical embedding, as in Section 2.1, it is not necessary to compute the multipliers. It is sufficient that

$$\begin{vmatrix} f & R_i^{(0)} \\ g & R_i^{(1)} \end{vmatrix} \text{ has two linear or one quadratic factor } \forall i = 1, ..., g.$$

Indeed, let $q \in S_2$ be the quadratic factor and let $l_i^{(1)},\ l_i^{(2)}$ be the linear factors of q in some extension field $\mathbb{K}$. Let further $P_i,\ Q_i$ be the points corresponding to the two linear factors in $\mathbb{P}^1(\mathbb{K})$. Constructing the quadric $\mathfrak{q}_i$ as in Section 2.1 we get

$$\mathfrak{q}_i = \begin{vmatrix} P_i^{(0)} & s \\ P_i^{(1)} & t \end{vmatrix} \cdot \begin{vmatrix} Q_i^{(0)} & s \\ Q_i^{(1)} & t \end{vmatrix} = (P_i^{(0)} t - P_i^{(1)} s)(Q_i^{(0)} t - Q_i^{(1)} s) = -q.$$

It follows that it is not even necessary to compute the points $\{P_i, Q_i\}_{i=1,...,g}$ that will glue together to a node in order to construct a canonically embedded g-nodal curve with a g_k^1. This can be useful if we want to construct nodal curves with special pencils over a field $\Bbbk$ which has very small characteristic, because in this case it might happen that we actually do not find enough points R_i, so that $\begin{vmatrix} f & R_i^{(0)} \\ g & R_i^{(1)} \end{vmatrix}$ has two linear factors.

2.3. Speeding Up the Computation

In this section we develop techniques, namely *Artinian reduction* and *Koszul cohomology*, that will speed up the computation of the critical Betti numbers $\beta_{\text{crit}}(C) = \beta_{m,m+1}(C)$ of canonical curves C, where $m = \lceil \frac{g-1}{2} \rceil$.

Artinian Reduction Since it is our intent to compute Betti tables or single Betti numbers of canonically embedded curves, we have to make this computation feasible. The effort of the computation of a minimal free resolution depends crucially on the numbers of variables of the homogeneous coordinate ring $T = \Bbbk[t_0, ..., t_{g-1}]$ of the canonical space $\mathbb{P}(H^0(C, \omega_C))$. In the case of canonically embedded curves *Artinian reduction* allows us to reduce the number of variables by two.

Lemma 2.3. *Let S be a ring, M be an S-module and $\underline{x} = \langle x_1, ..., x_d \rangle$ be an M-sequence. Let further*

$$N_2 \xrightarrow{\varphi_2} N_1 \xrightarrow{\varphi_1} N_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

be an exact sequence of S-modules. Then the induced sequence

$$N_2/\underline{x}N_2 \xrightarrow{\bar{\varphi_2}} N_1/\underline{x}N_1 \xrightarrow{\bar{\varphi_1}} N_0/\underline{x}N_0 \xrightarrow{\bar{\varphi_0}} M/\underline{x}M \longrightarrow 0$$

is exact.

Proof. The proof is based on [Ver05, Section 1]. It is sufficient to prove the result for one M-regular element x, since we have the following diagram

$$\begin{tikzcd}
N_2 \arrow[r, "\varphi_2"] \arrow[d, "\cdot x"] & N_1 \arrow[r, "\varphi_1"] \arrow[d, "\cdot x"] & N_0 \arrow[r, "\varphi_0"] \arrow[d, "\cdot x"] & M \arrow[r] \arrow[d, "\cdot x"] & 0 \\
N_2 \arrow[r, "\varphi_2"] \arrow[d] & N_1 \arrow[r, "\varphi_1"] \arrow[d] & N_0 \arrow[r, "\varphi_0"] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
N_2/xN_2 \arrow[r, "\bar{\varphi_2}"] & N_1/xN_1 \arrow[r, "\bar{\varphi_1}"] & N_0/xN_0 \arrow[r, "\bar{\varphi_0}"] & M/xM \arrow[r] & 0
\end{tikzcd}$$

and can repeat this step for every M-regular element in the sequence $\underline{x} = \langle x_1, ..., x_d \rangle$ if we consider $M/\underline{x}M$ as an $S/\langle x_1, ..., x_{d-1} \rangle S$-module.
The induced sequence is obtained from the original sequence by tensoring with S/xS. It follows that it is enough to show that $\ker \overline{\varphi}_1 \subset \operatorname{im} \overline{\varphi}_2$, since tensoring is right exact.
We denote by $\overline{y}$ the image of y modulo x. If $\overline{\varphi}_1(\overline{y}) = 0$, for some $y \in N_1$, then $\varphi_1(y) = xz$ for some $z \in N_0$. Therefore we have $\varphi_0(\varphi_1)(y) = \varphi_0(xz) = x\varphi_0(z) = 0$ and it follows that $\varphi_0(z) = 0$ since x is M-regular. By the exactness of the original sequence $z = \varphi_1(y')$ for some $y' \in N_1$. Hence we have $\varphi_1(y - xy') = 0$ and therefore $(y - xy') \in \varphi_2(N_2)$ and $\overline{y} \in \overline{\varphi}_2(N_2)$. $\square$

Corollary 2.4. *Let S be a ring and let*

$$N_\bullet : \quad \dots \longrightarrow N_m \xrightarrow{\varphi_m} N_{m-1} \longrightarrow \dots \longrightarrow N_0 \xrightarrow{\varphi_0} N_{-1} \longrightarrow 0$$

be an exact sequence of S-modules. If $\underline{x} = \langle x_1, ..., x_d \rangle$ is N_i-regular for all i then $N_\bullet \otimes S/\underline{x}S$ is exact.

Proof. The proof is based on [Ver05, Section 1]. By the same argument as in the last lemma it is sufficient to prove the statement for one element x, which is N_i-regular for all i. The image of φ_{i+1} is a submodule of N_i and it follows in particular that x is $(\mathrm{im}\varphi_{i+1})$-regular. We can now split the sequence above in sequences of the form

$$N_{i+3} \longrightarrow N_{i+2} \longrightarrow N_{i+1} \longrightarrow \mathrm{im}\varphi_{i+1} \longrightarrow 0$$

and conclude that $N_\bullet \otimes S/\underline{x}S$ is exact by the previous lemma. $\square$

Remark 2.5. Let M be a graded Cohen-Macaulay module of dimension d over the standard graded polynomial ring S and let further $\underline{x} = \langle x_1, ..., x_d \rangle$ be an M-sequence. We denote by $T = S/\underline{x}S$ and by $A = M/\underline{x}M$, the Artinian quotient of S and M, respectively. By the last corollary we deduce that a minimal free resolution of A as a T-module is obtained from a minimal free resolution of M as an S-module by tensoring with $T = S/\underline{x}S$. In particular, it follows for the graded Betti numbers of M that

$$\beta_{ij}(M) = \dim_{\Bbbk} \mathrm{Tor}_i^S(M, \Bbbk)_j = \dim_{\Bbbk} \mathrm{Tor}_i^T(A, \Bbbk)_j = \beta_{ij}(A).$$

By [Eis95, 21.15], we get the following if we denote by $\omega_S \cong S(-\dim S)$ the dualizing module of S and by $\omega_M := \mathrm{Ext}_S^M(M, \omega_S)$ the dualizing module of M. Starting with a minimal free resolution of M, we get a minimal free resolution of ω_M by applying $\mathrm{Hom}(\_, \omega_S)$. Therefore we have

$$\beta_{ij}(M) = \dim_{\Bbbk} \mathrm{Tor}_i^S(M, \Bbbk)_j = \dim_{\Bbbk} \mathrm{Tor}^S_{\mathrm{codim} M - i}(\omega_M, \Bbbk)_{\dim S - j}\,.$$

If the last $\dim M$ variables of S form an M-regular sequence, then the *Macaulay2*-package *extrasForTheKernel.m2*, available at [Sch12a], provides a function that computes the Artinian reduction of a module M.

Koszul Cohomology Koszul cohomology allows us to compute single Betti numbers of canonical curves by computing the rank of two matrices.
Let V be a $\Bbbk$-vector space of dimension n, $S := \bigoplus_{n \geq 0} S_n V$ be the symmetric algebra and B be a graded S-module $B = \bigoplus_{n \geq 0} B_n$. We define a complex

$$\mathscr{K}_{p,q}(B, V) := \bigwedge^p V \otimes B_q$$

with coboundary maps

$$\delta : \mathscr{K}_{p,q}(B, V) \longrightarrow \mathscr{K}_{p-1,q+1}(B, V)$$

defined as the composition

$$\begin{array}{ccc} \bigwedge^p V \otimes B_q & & v_1 \wedge \cdots \wedge v_p \otimes b \\ \downarrow & & \downarrow \\ \bigwedge^{p-1} V \otimes V \otimes B_q & & \sum_i (-1)^i v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_p \otimes v_i \otimes b \\ \downarrow & & \downarrow \\ \bigwedge^{p-1} V \otimes B_{q+1} & & \sum_i (-1)^i v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_p \otimes v_i b \end{array} .$$

Proposition 2.6. *We have $\delta \circ \delta = 0$ for the coboundary maps defined above.*

Proof. See [Gre84, Section 1]. □

Definition 2.7. *The group*

$$K_{p,q}(B, V) = H^0(\mathscr{K}_{p-*,p+*}(B, V), \delta)$$

is called the Koszul cohomology group.

Proposition 2.8. *Let V be a $\Bbbk$-vector space of dimension n, $S = \bigoplus_{n \geq 0} S_n V$ be the symmetric algebra and $B = \bigoplus_{n \geq 0} B_n$ be a graded S-module. The dimension of the Koszul cohomology group can be expressed as follows*

$$\dim_{\Bbbk} K_{p,q}(B, V) = \dim_{\Bbbk} \operatorname{Tor}_p^S(B, \Bbbk)_{p+q} = \beta_{p,p+q}(B).$$

Proof. See [Gre84, Section 1]. □

Notation 2.9. *Let X be a projective variety and let $\mathscr{L}$ be a line bundle on X. We write*

$$K_{p,q}(X, V) := K_{p,q}(X, \mathscr{L}) := K_{p,q}(B, V),$$

where $V = H^0(X, \mathscr{L})$ and $B_n = H^0(X, \mathscr{L}^n)$.

With the notation as above we are now able to formulate Green's Conjecture in terms of Koszul cohomology:

Conjecture 2.10 (Green's Conjecture in terms of Koszul cohomology)**.** *Let $C \subset \mathbb{P}^{g-1}$ be a canonical curve of genus g over a field $\Bbbk$ with $\operatorname{char}(\Bbbk) = 0$. The following is conjectured to be true*

$$K_{l,2}(C, \omega_C) = 0, l \leq a \iff \operatorname{Cliff}(C) > a.$$

For our intents in this thesis we want to compute $\beta_{\text{crit}}(C) := \beta_{m,m+1}(C)$ where C is a k-gonal canonical curve of genus g and $m = \left\lceil \frac{g-1}{2} \right\rceil$. Koszul cohomology reduces this computation to the computation of the ranks of two matrices, namely the two coboundary maps

$$d_1 := \delta_{m+1,0} : \bigwedge^{m+1} V \otimes B_0 \longrightarrow \bigwedge^{m} V \otimes B_1$$

and

$$d_2 := \delta_{m,1} : \bigwedge^m V \otimes B_1 \longrightarrow \bigwedge^{m-1} V \otimes B_2 \,,$$

where $B = T/I_C$, $V = \mathrm{span}(t_0, ..., t_{g-1})$ and $T = \bigoplus_{n\geq 0} S_n V = \Bbbk[t_0, ..., t_{g-1}]$. In this case the critical Betti number is given by $\beta_{\mathrm{crit}}(C) = \dim_{\Bbbk}(\ker(d_2)/\mathrm{im}(d_1))$. We can compute the sizes of the matrices d_1 and d_2 with the following consideration:
Recall from Proposition 1.10, that the ideal of a non-hyperelliptic canonical curve has no linear and $\binom{g-2}{2}$ quadratic generators. By Theorem 1.9, the sequence

$$0 \longrightarrow H^0(\mathscr{I}_C(k)) \longrightarrow H^0(\mathscr{O}_{\mathbb{P}^{g-1}}(k)) \longrightarrow H^0(\mathscr{O}_C(k)) \longrightarrow 0$$

is exact and we therefore compute

- $\dim_{\Bbbk}(\bigwedge^{m+1} V \otimes B_0) = \binom{g}{m+1}$
- $\dim_{\Bbbk}(\bigwedge^{m} V \otimes B_1) = \binom{g}{m} \cdot \dim_{\Bbbk} B_1 = \binom{g}{m} \cdot g$
- $\dim_{\Bbbk}(\bigwedge^{m-1} V \otimes B_2) = \binom{g}{m-1} \cdot \dim_{\Bbbk} B_2 = \binom{g}{m-1} \cdot [\binom{g+1}{2} - \binom{g-2}{2}] = \binom{g}{m-1} \cdot (3g-3)$.

The matrix d_1 only depends on the genus g, since I_C has no linear generators. The rank of d_1 is given by $\mathrm{rank}(d_1) = \binom{g}{m+1}$ and for the computation of the critical Betti number $\beta_{\mathrm{crit}}(C)$ it is now sufficient to determine the rank of the matrix d_2.
The effort of the computation of $\mathrm{rank}(d_2)$ drops remarkably if we consider the Artinian reduced case. To this end let $T_{\mathrm{red}} = \Bbbk[t_0, ..., t_{g-3}] = \bigoplus_{n\geq 0} S_n \widetilde{V}$ be the symmetric algebra with $\widetilde{V} = \mathrm{span}(t_0, ..., t_{g-3})$, and $\widetilde{B} = T_{\mathrm{red}}/I_{\mathrm{red}}$, where I_{red} denotes the Artinian reduction of I_C. By Remark 2.5, we have $\beta_{i,j}(I_C) = \beta_{i,j}(I_{\mathrm{red}})$ and it follows in particular that I_{red} has no linear and $\binom{g-2}{2}$ quadratic generators as well. Consequently we have

- $\dim_{\Bbbk}(\bigwedge^{m+1} \widetilde{V} \otimes \widetilde{B}_0) = \binom{g-2}{m+1}$
- $\dim_{\Bbbk}(\bigwedge^{m} \widetilde{V} \otimes \widetilde{B}_1) = \binom{g-2}{m} \cdot (g-2)$
- $\dim_{\Bbbk}(\bigwedge^{m-1} \widetilde{V} \otimes \widetilde{B}_2) = \binom{g-2}{m-1} \cdot [\binom{g-1}{2} - \binom{g-2}{2}] = \binom{g-2}{m-1} \cdot (g-2)$,

and $\mathrm{rank}(d_1) = \binom{g-2}{m+1}$ in the Artinian reduced case.

2.4. Exemplary Implementation

In this section we show how the construction of rational g-nodal k-gonal canonical curves, as described in Section 2.2, can be implemented in the computer algebra system *Macaulay2*, available at [GS].
We begin by defining the two functions *getFactors* and *getCoordinates*, which we use in the code below. The function *getFactors* computes two lists L_1 and L_2 containing the linear and quadratic factors of a polynomial $f \in S = \Bbbk[s,t]$,

```
i1 : getFactors:=(f)->(
       S:=ring f;
       (L1,L2):=({},{});
       facs:=apply(toList(factor f), fac->fac#0) ;
       if (f==0_S) then (L1,L2)=({},{}) else
       L1=select(facs,fac->degree fac=={1});
       L2=select(facs, fac->degree fac=={2});
        L1,L2);
```

and the function *getCoordinates* computes the vanishing loci $V(l)$ of a linear polynomial in two variables.

```
i2 : getCoordinates:=(l)->(
       S:=ring l;
       if rank source (coefficients(l))#0==2 then
       matrix{{-(coefficients l)_1_(1,0), (coefficients l)_1_(0,0)}}
       else if
         sub((coefficients(l))#0,S) - map(S^1,S^1,(entries vars S)_0#0)==0
            then matrix{{0,1}}
       else if
         sub((coefficients(l))#0,S) - map(S^1,S^1,(entries vars S)_0#1)==0
            then matrix {{1,0}}   );
```

Taking Remark 2.2 into account, we begin with the construction from Section 2.2.
We define the field $\Bbbk$, the polynomial ring $S = \Bbbk[x_0, x_1]$, and the desired gonality and genus.

```
i3 : kk=ZZ/(p=1009)
```

```
i4 : S=kk[s,t]
```

```
i5 : (k,g)=(5,11)
```

We begin with Step 1 and search a generic morphism $f = \binom{f_0}{f_1} : S(-k) \to S^2$, so that

$$h_i = \begin{vmatrix} f_0 & R_i^{(0)} \\ f_1 & R_i^{(1)} \end{vmatrix}$$

has at least two linear or one quadratic factor for g distinct points $(R_i^{(0)} : R_i^{(1)}) \in \mathbb{P}^1(\Bbbk)$, and save these factors in a list L.

```
i6 : L={};
```

```
i7 : pts:=random(apply(p,i-> matrix{{i,1}})|{matrix{{1,0}}});
```

```
i8 :    while (#L<g or #(unique flatten L)< #(flatten L))
                         do (  (L,L1,L2)=({},{},{});
                         f:=random(S^1, S^{2:-k});
                         j:=0;
       while (#L<g and j<p)
       do    ( if #(L1=(getFactors(sub(det(f||sub(pts_j,S)),S)))#0)>=2
             then L= L|{L1} --adding at least two linear factors to L
             else if #(L2=(getFactors(sub(det(f||sub(pts_j,S)),S)))#1)>=1
             then L=L|{L2}; --adding at least one quadratic factor to L
           j=j+1; );););
```

Note that if we want to compute the $2g$ points that glue together to the g-nodes, then we have to exclude that quadratic factors are saved in the list L.
Now we construct the quadrics $\mathfrak{q}_i$ which are needed to compute the sections $\{s_1, ..., s_g\}$ that define the canonical embedding as in Step 2 of Construction 1. If h_i already has a quadratic factor we can define $\mathfrak{q}_i$ to be this factor by Remark 2.2. Otherwise we use the following function that computes two points P_i, Q_i from two of the linear factors of h_i, using the function *getCoordinates*, and construct the quadric $\mathfrak{q}_i$ from these points.

```
i9 : linFactorsToQuadric:=(l)->(
       lh:=random(l);
       (P,Q):=(sub(getCoordinates(lh_0),S),sub(getCoordinates(lh_1),S));
       quad:=(det(P||(vars S)))*(det(Q||(vars S)));
       quad );

i10 : quadrics:={};

i11 : for i from 0 to (#L-1)
        do ( a:=(degree((L_i)_0))_0;
             if a==1
               then quadrics=quadrics|{linFactorsToQuadric(L_i)}
             else      quadrics=quadrics|{(L_i)_(random(#L_i))} );
```

We construct the matrix $s = (s_1, ..., s_g)$ and compute the ideal I_C of the canonically embedded curve as the kernel of the map $s : T = \Bbbk[t_0, ..., t_{g-1}] \longrightarrow S$. But at first we multiply the matrix s with a general matrix $M \in \mathrm{GL}(g, \Bbbk)$ to obtain more general sections, and have a better chance that $\{t_{g-2}, t_{g-1}\}$ forms a regular sequence for T/I_C. If this is the case, then the *Macaulay2*-package *extrasForTheKernel.m2* (see [Sch12a]) provides a function that computes the Artinian reduction of the ideal I_C .

```
i12 : sections :=matrix {(apply(g, i -> product(g, j->
                          if i == j then 1_S else sub(quadrics_j,S))))};


              1       11
o12 : Matrix S  <--- S
```

```
i13 : while (M=random(S^{g:0},S^{g:0}); det M==0) do ();

i14 : sections2:=sections*M;

i15 : ideal sections==ideal sections2

o15 = true

i16 : Tcan=kk[T_0..T_(g-1)]

i17 : IC=ideal mingens ker map(S,Tcan,sections2);
```

Finally, we compute the Betti table of the 2-linear strand of I_C in the Artinian reduced and non-reduced case and compare the times needed for the computations. After that we compute $\beta_{\text{exp}}(C)$ and check whether the critical and expected Betti number coincide.

```
i18 : time betti res (IC,DegreeLimit=>1)
     -- used 157.961 seconds

             0  1   2   3   4  5 6
o18 = total: 1 36 160 315 294 35 6
          0: 1  .   .   .   .  . .
          1: . 36 160 315 294 35 6

o18 : BettiTally

i19 : loadPackage("extrasForTheKernel")

i20 : time (Tred,Ired)=artinianReduction(IC);
     -- used 0.162335 seconds

i21 : time betti res (Ired,DegreeLimit=>1)
     -- used 27.4634 seconds

             0  1   2   3   4  5 6
o21 = total: 1 36 160 315 294 35 6
          0: 1  .   .   .   .  . .
          1: . 36 160 315 294 35 6

i22 : f=g-k+1;

i23 : m=ceiling((g-1)/2);

i24 : betaExp=binomial(f,m+1)*m

o24 = 35
```

The package *extrasForTheKernel.m2* also provides a function that computes Koszul cohomology maps. We can use this function to compute the critical Betti number via Koszul cohomology as pointed out in Section 2.3.

```
i25 : time Mred=koszulMap(Ired,m,1);
     -- used 0.971997 seconds

                1134        1134
o25 : Matrix kk     <--- kk

i26 : time binomial(g-2,m)*(g-2)-rank(Mred)-binomial(g-2,m+1)
     -- used 29.246 seconds

o26 = 35
```

The *Macaulay2*-package *kGonalNodalCurves.m2*, see [Bop13], provides several functions concerning the construction of k-gonal nodal curves and the computation of the critical Betti number. In particular, all the functions needed for the construction above are implemented in this package.

Remark 2.11. The Koszul cohomology maps can be represented as sparse matrices, but in the computation above we saw that the computation of the 2-linear strand of the resolution of C and the calculation of the single critical Betti number almost took the same time. This is mainly because *Macaulay2* does not support fast linear algebra for sparse matrices yet.
The package *kGonalNodalCurves.m2* provides a function $sparseKoszulMatrix$ that computes a list $\mathcal{L}$ containing the size of the desired Koszul map $\delta_{m,1}$, the characteristic of the ground field and a list containing the non-zero entries and the position of those. Using the computer algebra system *Sage*, see [S^{+}12], we can reconstruct the Koszul map in the class for sparse matrices from the list $\mathcal{L}$.
The computation of the critical Betti number with this method is much faster than the direct computation in *Macaulay2*. In the case of a 5-gonal curve of genus 11, as in the construction above, *Sage* needs about 2 seconds to reconstruct the matrix $\delta_{m,1}$ and 0.35 seconds to compute its rank.

For k-gonal curves of genus $g \leq 17$ and $k \leq \lceil \frac{g}{2} \rceil$ we find lists of the critical Betti numbers computed with the methods of this chapter in Appendix A. In particular, among the computed examples, we have $\beta_{\text{crit}}(C) > \beta_{\text{exp}}(C)$ for $k = 5$ and $g = 13, 15, 17$.

3. A Theoretical Approach

In this chapter we give a theoretical explanation of the extra syzygies occurring in the minimal free resolution of 5-gonal canonical curves.
Throughout this chapter we denote by $C \subset \mathbb{P}^{g-1}$ a 5-gonal canonical curve of genus g. Furthermore, we denote by X the 4-dimensional scroll of degree $f = g - 4$, swept out by the g^1_5 on C, and by $\mathbb{P}(\mathscr{E})$ the corresponding $\mathbb{P}^3$-bundle. We assume that all curves C satisfy the balancing conditions and denote by $\beta_{\text{crit}}(C) = \beta_{m,m+1}(C)$ the critical Betti number and by $\beta_{\exp}(C) = \beta_{m,m+1}(X)$ the expected Betti number, where $m = \lceil \frac{g-1}{2} \rceil$.
Recall from Theorem 1.26 and Remark 1.27 that a 5-gonal curve $C \subset \mathbb{P}(\mathscr{E})$ has a resolution of the form

$$\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-5H+(f-2)R) \to \sum_{i=1}^{5} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+b_iR) \xrightarrow{\Psi} \sum_{i=1}^{5} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+a_iR) \to \mathscr{O}_{\mathbb{P}(\mathscr{E})} \to \mathscr{O}_C$$

where $\sum_{i=1}^{5} a_i = 2g-12$, $a_i + b_i = f - 2$. By the Structure Theorem for Gorenstein ideals in codimension 3 (see [BE77]), it follows that the 5×5 matrix Ψ is skew-symmetric and its 5 Pfaffians generate the ideal I_C.
With the notation as in section 1.3 we denote by $F = H^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-R)) \otimes \mathscr{O}_{\mathbb{P}^{g-1}}$ and by $G = H^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(R)) \otimes \mathscr{O}_{\mathbb{P}^{g-1}}$. For the computation of the rank of the comparison maps $\mathscr{C}^b_j \longrightarrow \mathscr{C}^a_j$ we will consider the $\Bbbk$-vector spaces $H^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-R))$ and $H^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(R))$ most of the time instead of the free sheaves. By abuse of notation we will also denote by F and G the corresponding $\Bbbk$-vector spaces.

3.1. 5-Gonal Curves of Genus 13

By Appendix A, we have $\beta_{\text{crit}}(C) = \beta_{\exp}(C) + 6$ in the case of a 5-gonal 13-nodal canonical curve of genus 13 and by Computation 1 in Appendix B it follows, that the 5-gonal canonical curves of genus 13 constructed as described in Chapter 2 satisfy the balancing conditions. We give an explanation of the 6 extra syzygies by a direct computation.
In this section $C \subset \mathbb{P}^{12}$ denotes a 5-gonal canonical of genus 13 satisfying the balancing conditions. The scroll X swept out by the g^1_5 on C is therefore a 4-dimensional scroll of type $S(3,2,2,2)$ and degree $f = 9$, and the numbers a_i and b_i are given by

$$(a_1, a_2, a_3, a_4, a_5) = (2,3,3,3,3) \text{ and } (b_1, b_2, b_3, b_4, b_5) = (5,4,4,4,4).$$

By Theorem 1.26, $C \subset \mathbb{P}(\mathscr{E})$ has a resolution of the form

$$0 \to \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-5H+7R) \to \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+5R) \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+4R)^{\oplus 4} \end{matrix} \xrightarrow{\Psi} \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+2R) \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+3R)^{\oplus 4} \end{matrix} \to \mathscr{O}_{\mathbb{P}(\mathscr{E})} \to \mathscr{O}_C \to 0$$

where Ψ is a skew-symmetric matrix with entries as indicated below

$$(\Psi) \sim \begin{pmatrix} 0 & (H-2R) & (H-2R) & (H-2R) & (H-2R) \\ (H-2R) & 0 & (H-R) & (H-R) & (H-R) \\ (H-2R) & (H-R) & 0 & (H-R) & (H-R) \\ (H-2R) & (H-R) & (H-R) & 0 & (H-R) \\ (H-2R) & (H-R) & (H-R) & (H-R) & 0 \end{pmatrix} \tag{3.1}$$

According to Remark 1.21, we can write down bases of the relevant cohomology groups. Let $\{s,t\}$ be a basis $H^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(R))$ and $\{\varphi_0\}$ be a basis of $H^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-3R))$ then a basis of $H^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-2R))$ is given by $\{s\varphi_0, t\varphi_0, \varphi_1, f\varphi_2, \varphi_3\}$ and basis of $H^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-R))$ by $\{s^2\varphi_0, st\varphi_0, t^2\varphi_0, s\varphi_1, s\varphi_2, s\varphi_3, t\varphi_1, t\varphi_2, t\varphi_3\}$.
As in Example 1.30, we resolve the $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-modules in the resolution of C by $\mathscr{O}_{\mathbb{P}^{12}}$-modules and determine the ranks of those maps induced by Ψ which give non-minimal parts in the iterated mapping cone construction:

$$\begin{tikzcd}
\cdots \arrow[r] & \begin{matrix}\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+5R)\\ \bigoplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+4R)^{\oplus 4}\end{matrix} \arrow[r, "\Psi"] & \begin{matrix}\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+2R)\\ \bigoplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+3R)^{\oplus 4}\end{matrix} \arrow[r] & \cdots \\
& \vdots \arrow[u] & \vdots \arrow[u] & \\
\cdots \arrow[r] & \sum_{i=1}^5 \mathscr{C}_3^{b_i}(-3) = \begin{matrix}\bigwedge^3 F\otimes S_2G(-6)\\ \bigoplus \\ (\bigwedge^3 F\otimes S_1G(-6))^4\end{matrix} \arrow[u] \arrow[r] & \sum_{i=1}^5 \mathscr{C}_3^{a_i}(-2) = \begin{matrix}\bigwedge^4 F(-6)\\ \bigoplus \\ (\bigwedge^4 F(-5))^4\end{matrix} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & \sum_{i=1}^5 \mathscr{C}_4^{b_i}(-3) = \begin{matrix}\bigwedge^4 F\otimes S_1G(-7)\\ \bigoplus \\ (\bigwedge^4 F(-7))^4\end{matrix} \arrow[u] \arrow[r] & \sum_{i=1}^5 \mathscr{C}_4^{a_i}(-2) = \begin{matrix}\bigwedge^5 F\otimes D_1G^*(-7)\\ \bigoplus \\ (\bigwedge^5 F(-7))^4\end{matrix} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & \sum_{i=1}^5 \mathscr{C}_5^{b_i}(-3) = \begin{matrix}\bigwedge^5 F(-8)\\ \bigoplus \\ (\bigwedge^5 F(-9))^4\end{matrix} \arrow[u] \arrow[r] & \sum_{i=1}^5 \mathscr{C}_5^{a_i}(-2) = \begin{matrix}\bigwedge^6 F\otimes D_2G^*(-8)\\ \bigoplus \\ (\bigwedge^6 F\otimes D_1G^*(-8))^4\end{matrix} \arrow[u] \arrow[r] & \cdots \\
& \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

The maps indicated in the diagram above are the only ones which give non-minimal parts in the iterated mapping cone construction.
At first we take a look at those maps determining the Betti number $\beta_{m+1,m+2}(C)$. By the skew-symmetry of Ψ, the induced maps

$$\bigwedge^3 F \otimes S_2G(-6) \longrightarrow \bigwedge^4 F(-6) \quad \text{and} \quad \bigwedge^5 F(-8) \longrightarrow \bigwedge^6 F \otimes D_2G^*(-8)$$

are both zero. The maps

$$\psi_3 : (\bigwedge^3 F \otimes S_1G(-6))^4 \longrightarrow \bigwedge^4 F(-6) \quad \text{and} \quad \psi_5 : \bigwedge^5 F(-8) \longrightarrow (\bigwedge^6 F \otimes D_1G^*(-8))^4$$

are dual to each other by the Gorenstein property of canonical curves, and the rank of one of those maps determines the Betti number $\beta_{m+1,m+2}(C)$ in the minimal free resolution of $C \subset \mathbb{P}^{12}$.

Proposition 3.1. *The maps ψ_3 and ψ_5 above have full rank if Ψ is sufficiently general.*

Proof. The maps ψ_3 and ψ_4 are dual to each other and thus have the same rank. We show the surjectivity of ψ_3. Recall that $S_jG \cong H^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(jR))$ by Remark 1.21.
The map ψ_3 is induced by the 1×4 submatrix $\Psi_{(14)} = (\Psi_{12}, \Psi_{13}, \Psi_{14}, \Psi_{15})$ of Ψ where $\Psi_{1,j} \in H^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-2R))$, and as in Lemma 1.28 the map is given as the composition

$$\begin{array}{c} (\bigwedge^3 F \otimes S_1G)^4 \cong \bigwedge^3 F \otimes H^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(R))^4 \\ \Big\downarrow {\scriptstyle id\otimes\Psi_{(14)}} \\ \bigwedge^3 F \otimes H^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-R)) \cong \bigwedge^3 F \otimes F \xrightarrow{\wedge} \bigwedge^4 F \end{array}$$

We assume that $\Psi_{(14)}$ is sufficiently general, such that the 4 entries are independent. After a suitable base change, we can furthermore assume that the entries of $\Psi_{(14)}$ are among the basis elements of $H^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-2R))$. With this considerations we can immediately choose preimages for every basis element in $\bigwedge^4 F$. Therefore $\Psi_{(14)}$ is surjective and thus ψ_3 and ψ_5 have full rank. $\square$

In particular, it follows that $\beta_{m+1,m+2}(C) = \beta_{m+1,m+2}(X)$ and it remains to compute the rank of the map

$$\psi_4 : \begin{array}{c} \bigwedge^4 F\otimes S_1G(-7) \\ \oplus \\ (\bigwedge^4 F(-7))^4 \end{array} \longrightarrow \begin{array}{c} \bigwedge^5 F\otimes D_1G^*(-7) \\ \oplus \\ (\bigwedge^5 F(-7))^4 \end{array}$$

which determines the critical Betti number $\beta_{m,m+1}(C)$.

Proposition 3.2. *If Ψ is sufficiently general, then the induced map ψ_4 above has a 6-dimensional kernel.*

Proof. We first consider the part of the map ψ_4, that defines the map

$$\psi_{(41)} : \bigwedge^4 F \otimes S_1G(-7) \longrightarrow (\bigwedge^5 F(-7))^4 .$$

As in Lemma 1.28, this map is given as the composition

$$\begin{array}{c} \bigwedge^4 F \otimes S_1G \cong \bigwedge^4 F \otimes H^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(R)) \\ \Big\downarrow {\scriptstyle id\otimes\Psi_{(41)}} \\ \bigwedge^4 F \otimes H^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-R))^4 \cong \bigwedge^4 F \otimes F^4 \xrightarrow{\wedge} (\bigwedge^5 F)^4 \end{array}$$

where $\Psi_{(41)} = -(\Psi_{12}, \Psi_{13}, \Psi_{14}, \Psi_{15})^t$. We assume again, that the entries of $\Psi_{(41)}$ are given by independent basis elements of $H^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-2R))$, say $\Psi_{(41)} = (s\varphi_0, \varphi_1, \varphi_2, \varphi_3)$. We see that elements of the form

$$(\lambda s + \mu t)s\varphi_0 \wedge (\lambda s + \mu t)\varphi_1 \wedge (\lambda s + \mu t)\varphi_2 \wedge (\lambda s + \mu t)\varphi_3 \otimes (\lambda s + \mu t), \text{with } \lambda, \mu \in \Bbbk$$

lie in the kernel of the composition. Expanding one of those elements we get

$$\lambda^5 s^2\varphi_0 \wedge s\varphi_1 \wedge s\varphi_2 \wedge s\varphi_3 \otimes s + \cdots + \mu^5 st\varphi_0 \wedge t\varphi_1 \wedge t\varphi_2 \wedge t\varphi_3 \otimes t$$

and conclude that a rational normal curve of degree 5 lies in $\mathbb{P}(\mathrm{Syz})$ where

$$\mathrm{Syz} = \mathrm{Tor}_m^T(T/I_C, \Bbbk)_{m+1}$$

is the linear space spanned by the m^{th} linear syzygies and $I_C \subset T$ denotes the ideal of the canonical curve. In particular, it follows that $\dim_{\Bbbk}(\mathrm{Syz}) = 6$. Therefore, ψ_4 has at least a 6-dimensional kernel.
It is now sufficient to give one example, so that the map ψ_4 has a 6-dimensional kernel. This example is given by the computation that led to Appendix A and we conclude that the 6-dimensional kernel can always be described as above, if Ψ is sufficiently general. $\square$

In Computation 2 in Appendix B we also verify computationally that the map $\psi_{(41)}$ in the proposition above has a 6-dimensional kernel.
By the proposition above, it follows in particular, that $\beta_{\text{crit}}(C) = \beta_{\exp}(C) + 6$ and we conclude that the general 5-gonal canonical curve of genus 13 has a Betti table of the following form

	0	1	2	3	4	5	6	7	8	9	10	11
0	1	-	-	-	-	-	-	-	-	-	-	-
1	-	55	320	891	1416	1218	222	63	8	-	-	-
2	-	-	-	8	63	222	1218	1416	891	320	55	-
3	-	-	-	-	-	-	-	-	-	-	-	1

Remark 3.3. By Computation 3 in Appendix B, it follows that for a general skew-symmetric matrix Ψ of the type as in (3.1) none of the entries can be made zero by suitable row and column operations. Schreyer shows in [Sch86, Section 5] that the vanishing of one of the entries of Ψ by suitable row and column operations is equivalent to the existence of an additional linear series on C. In the case of a 5-gonal curve of genus 13, this would imply the existence of an additional g^1_7 on C which would give a geometric interpretation of the extra syzygies.

3.2. 5-Gonal Curves of Odd Genus

The main aim of this section is the proof of the following theorem.

Theorem 3.4. *Let C be a general 5-gonal canonical curve of odd genus $g = 2n + 1 \geq 13$ satisfying the balancing conditions, then $\beta_{crit}(C) > \beta_{exp}(C)$.*

Throughout this section we assume that all curves are of odd genus $g = 2n + 1 \geq 13$ and satisfy the balancing conditions. We can therefore distinguish the following 5 cases to which we will refer in the rest of this section.

Case 1. $(a_1, ..., a_5) = (a, a, a, a, a), (b_1, ..., b_5) = (b, b, b, b, b) \Leftrightarrow a = 4r + 2$ and $n = 5r + 5$ for some $r \geq 1$.

Case 2. $(a_1, ..., a_5) = (a - 1, a, a, a, a), (b_1, ..., b_5) = (b + 1, b, b, b, b) \Leftrightarrow a = 4r - 1$ and $n = 5r + 1$ for some $r \geq 1$.

Case 3. $(a_1, ..., a_5) = (a - 1, a - 1, a, a, a), (b_1, ..., b_5) = (b + 1, b + 1, b, b, b) \Leftrightarrow a = 4r$ and $n = 5r + 2$ for some $r \geq 1$.

Case 4. $(a_1, ..., a_5) = (a + 1, a + 1, a, a, a), (b_1, ..., b_5) = (b - 1, b - 1, b, b, b) \Leftrightarrow a = 4r$ and $n = 5r + 3$ for some $r \geq 1$.

Case 5. $(a_1, ..., a_5) = (a + 1, a, a, a, a), (b_1, ..., b_5) = (b - 1, b, b, b, b) \Leftrightarrow a = 4r + 1$ and $n = 5r + 4$ for some $r \geq 1$.

Note that the degree of the scroll X is given by $f = g - k + 1 = 2n - 3$, the number b above is given by $b = f - 2 - a = 2n - a - 5$ and $n = m = \lceil \frac{g-1}{2} \rceil$.

Remark 3.5. The rank of the map

$$\psi := \psi_{n-2} : \sum_{i=1}^{5} \mathscr{C}_{n-2}^{b_i}(-3) \longrightarrow \sum_{i=1}^{5} \mathscr{C}_{n-2}^{a_i}(-2)$$

in the iterated mapping cone construction determines the critical Betti number. A simple computation shows that $\operatorname{rank}(\sum_{i=1}^{5} \mathscr{C}_{n-2}^{b_i}) = \operatorname{rank}(\sum_{i=1}^{5} \mathscr{C}_{n-2}^{a_i})$ for all cases listed above. Furthermore we compute that among the cases above we always have

$$\min\{b_i\} \geq n - 2 \geq \max\{a_i\}$$

and therefore

$$\mathscr{C}_{n-2}^{b_i} = \bigwedge^{n-2} F \otimes S_{b_i - n+2} G \quad \text{and} \quad \mathscr{C}_{n-2}^{a_i} = \bigwedge^{n-1} F \otimes D_{n-a_i-3} G^*$$

which means in particular, that the map ψ has no submatrices which give rise to minimal parts in the free resolution of $C \subset \mathbb{P}^{g-1}$ obtained by an iterated mapping cone construction.

Proof of Theorem 3.4.
Case 1: We have $n = 5r+5,\ a = 4r+2,\ (a_1, ..., a_5) = (a, ..., a)$ and therefore $(b_1, ..., b_5) = (b, ..., b)$ with $b = f - 2 - a = 2n - a - 5$. As in Lemma 1.28, the map

$$\psi : \sum_{i=1}^{5} \mathscr{C}^b_{n-2} = \bigwedge^{n-2} F \otimes (S_{b-(n-2)}G)^5 \longrightarrow \sum_{i=1}^{5} \mathscr{C}^a_{n-2} = \bigwedge^{n-1} F \otimes (D_{n-a-3}G^*)^5$$

is given as the composition

$$\begin{aligned} &\bigwedge^{n-2} F \otimes (S_{b-(n-2)}G)^5 \cong \bigwedge^{n-2} F \otimes (S_{n-a-3}G)^5 \otimes S_{n-a-3}G \otimes D_{n-a-3}G^* \\ &\longrightarrow \bigwedge^{n-2} F \otimes (S_{2n-2a-6}G)^5 \otimes D_{n-a-3}G^* \xrightarrow{id \otimes \Psi \otimes id} \bigwedge^{n-2} F^5 \otimes D_{n-a-3}G^* \\ &\xrightarrow{\wedge \otimes id} (\bigwedge^{n-1} F)^5 \otimes D_{n-a-3}G^* \end{aligned}$$

where Ψ is the skew-symmetric 5×5 matrix with entries in $H^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(H - (b-a)R))$. We show that the composition ψ has a non-trivial decomposable element in the kernel. Since the multiplication map $(S_{n-a-3}G)^5 \otimes S_{n-a-3}G \longrightarrow (S_{2n-2a-6}G)^5$ is not injective, we have to show that there exists an $f \in \bigwedge^{n-2} F$ and an element $g \in (S_{n-a-3}G)^5$ such that $\Psi(f \otimes (g \cdot g')) = 0\ \forall g' \in S_{n-a-3}G$. We argue by calculating the relevant dimensions.
Let $g \in (S_{n-a-3}G)^5$ be a basis element, say $g = (g_1, 0, \dots, 0)^t$ and let $\{g'_1, ..., g'_{n-a-2}\}$ be a basis of $S_{n-a-3}G$. By the skew-symmetry of Ψ, we have $\Psi(gg'_i) = (0, f^{(i)}_1, \dots, f^{(i)}_4)^t$ for all $i = 1, ..., n-a-2$. We compute that

$$\begin{aligned} n - 2 = 5r + 3 \geq \dim_{\Bbbk}(\Psi(g \cdot S_{n-a-3}G)) &= 4 \cdot \dim_{\Bbbk} S_{n-a-3}G = 4(n-a-2) \\ &= 4(5r + 5 - (4r+2) - 2) = 4r + 4 \end{aligned}$$

holds for every $r \geq 1$ and we can choose an element of the form

$$f^{(1)}_1 \wedge \dots \wedge f^{(1)}_4 \wedge f^{(2)}_1 \wedge \dots \wedge f^{(n-a-2)}_4 \wedge \tilde{f} \otimes g \in \ker \psi$$

for some $\tilde{f} \in \bigwedge^{r-1} F$.
The $f^{(i)}_j$ above are independent if Ψ is sufficiently general which means that the first row of Ψ has no syzygies in $S_{2n-2a-6}G$. This is a closed condition and it is now sufficient to give one example where this is the case. The entries of Ψ are elements of $H^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(H - (b-a)R))$ and with notation as in Remark 1.21 the basis elements of $H^0(H - (b-a)R)$ can be represented by elements of the form $p\varphi_i$ for $i = 1, ..., 4$ and some $p \in \Bbbk[s,t] = \bigoplus_{k \geq 0} S_k G$. If the first row of Ψ is of the form $(0, p_1\varphi_1, ..., p_4\varphi_4)$ for $p_j \in \Bbbk[s,t]$ then we immediately see that the elements $f^{(i)}_j = g \cdot g'_i \cdot p_j \varphi_j$ are independent. It follows that the $f^{(i)}_j$ are independent in the general case which finishes the proof for Case 1.
Next we take a look at Case 3 since the approach for Case 3 can be applied to all the other cases.

Case 3: We have $n = 5r + 2,\ a = 4r,\ (a_1, ..., a_5) = (a-1, a-1, a, a, a)$ and $(b_1, ..., b_5) = (b+1, b+1, b, b, b)$ with $b = 2n - a - 5$. We show that the map

$$\psi : \sum_{i=1}^{2} \mathscr{C}_{n-2}^{(b+1)} \oplus \sum_{i=1}^{3} \mathscr{C}_{n-2}^{b} \longrightarrow \sum_{i=1}^{2} \mathscr{C}_{n-2}^{(a-1)} \oplus \sum_{i=1}^{3} \mathscr{C}_{n-2}^{a} ,$$

induced by skew-symmetric matrix

$$\Psi : \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+(b+1)R)^2 \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+bR)^3 \end{matrix} \longrightarrow \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+(a-1)R)^2 \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+aR)^3 \end{matrix} .$$

has a non-trivial decomposable element in the kernel. We refer by

$$\Psi_{(22)} : \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H + (b+1)R)^2 \longrightarrow \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H + (a-1)R)^2$$

to the (2×2)-block of Ψ, by

$$\Psi_{(32)} : \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H + (b+1)R)^2 \longrightarrow \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H + (a)R)^3$$

to the (3×2)-block and so on. We first consider the maps

$$\psi_{(22)} : \textstyle\sum_{i=1}^{2} \mathscr{C}_{n-2}^{(b+1)} \longrightarrow \sum_{i=1}^{2} \mathscr{C}_{n-2}^{(a-1)} \text{ and } \psi_{(32)} : \sum_{i=1}^{2} \mathscr{C}_{n-2}^{(b+1)} \longrightarrow \sum_{i=1}^{3} \mathscr{C}_{n-2}^{a} ,$$

induced by the blocks $\Psi_{(22)}$ and $\Psi_{(32)}$. As in Lemma 1.28, the map $\psi_{(22)}$ is given as the composition

$$\sum_{i=1}^{2} \mathscr{C}_{n-2}^{(b+1)} = \bigwedge^{n-2} F \otimes (S_{n-a-2}G)^2 \cong \bigwedge^{n-2} F \otimes (S_{n-a-2}G)^2 \otimes S_{n-a-2}G \otimes D_{n-a-2}G^*$$

$$\longrightarrow \bigwedge^{n-2} F \otimes (S_{2n-2a-4}G)^2 \otimes D_{n-a-2}G^* \xrightarrow{id \otimes \Psi_{(22)} \otimes id} \bigwedge^{n-2} F \otimes F^2 \otimes D_{n-a-2}G^*$$

$$\xrightarrow{\wedge \otimes id} (\bigwedge^{n-1} F)^2 \otimes D_{n-a-2}G^* = \sum_{i=1}^{2} \mathscr{C}_{n-2}^{a-1} ,$$

and the map $\psi_{(32)}$ is given as the composition

$$\sum_{i=1}^{2} \mathscr{C}_{n-2}^{(b+1)} = \bigwedge^{n-2} F \otimes (S_{n-a-2}G)^2 \cong \bigwedge^{n-2} F \otimes (S_{n-a-2}G)^2 \otimes S_{n-a-3}G \otimes D_{n-a-3}G^*$$

$$\longrightarrow \bigwedge^{n-2} F \otimes (S_{2n-2a-5}G)^2 \otimes D_{n-a-3}G^* \xrightarrow{id \otimes \Psi_{(32)} \otimes id} \bigwedge^{n-2} F \otimes F^3 \otimes D_{n-a-3}G^*$$

$$\xrightarrow{\wedge \otimes id} (\bigwedge^{n-1} F)^3 \otimes D_{n-a-3}G^* = \sum_{i=1}^{3} \mathscr{C}_{n-2}^{a} .$$

Finding a decomposable element in the kernel of the map

$$\sum_{i=1}^{2} \mathscr{C}_{n-2}^{(b+1)} \longrightarrow \begin{matrix} \sum_{i=1}^{2} \mathscr{C}_{n-2}^{(a-1)} \\ \oplus \\ \sum_{i=1}^{3} \mathscr{C}_{n-2}^{a} \end{matrix}$$

means that we have to find an $f \in \bigwedge^{n-2} F$ and an element $g \in (S_{n-a-2}G)^2$ such that

$$\Psi_{(22)}((f \otimes (g \cdot g'))) = 0, \ \forall g' \in S_{n-a-2}G \text{ and } \Psi_{(32)}(f \otimes (g \cdot g'')) = 0, \ \forall g'' \in S_{n-a-3}G\,.$$

Again we choose a basis element of the form $g = \binom{g_1}{0} \in (S_{b-n+2}G)^2$ and compute the relevant dimensions. We denote by $\{g_1', ..., g_{n-a-1}'\}$ a basis of $S_{n-a-2}G$ and by $\{g_1'', ..., g_{n-a-2}''\}$ a basis of $S_{n-a-3}G$. By the skew-symmetry of $\Psi_{(22)}$, we have

$$\Psi_{(22)}(g \cdot g_i') = \begin{pmatrix} 0 \\ f_1^{(i)} \end{pmatrix} \text{ and } \Psi_{(32)}(g \cdot g_i'') = \begin{pmatrix} \tilde{f}_1^{(i)} \\ \tilde{f}_2^{(i)} \\ \tilde{f}_3^{(i)} \end{pmatrix} \quad \forall i$$

and compute that

$$\begin{aligned} n-2 = 5r &\geq \dim_{\Bbbk}(\Psi_{(22)}(g \cdot S_{n-a-2}G)) + \dim_{\Bbbk}(\Psi_{(32)}(g \cdot S_{n-a-3}G)) \\ &= 1 \cdot \dim_{\Bbbk}(S_{n-a-2}G) + 3 \cdot \dim_{\Bbbk}(S_{n-a-3}G) \\ &= 1(n-a-1) + 3(n-a-2) = 4r+1 \end{aligned}$$

holds for $r \geq 1$. Thus we can choose an element of the form

$$f_1^{(1)} \wedge \cdots \wedge f_1^{(n-a-1)} \wedge \tilde{f}_1^{(1)} \wedge \cdots \wedge \tilde{f}_3^{(n-a-2)} \wedge \tilde{f} \otimes \begin{pmatrix} g \\ 0 \end{pmatrix} \in \ker \psi$$

for some $\tilde{f} \in \bigwedge^{r-1} F$. By the same argument as in Case 1 we conclude that the elements $\{f_1^{(1)}, ..., \tilde{f}_3^{(n-a-2)}\}$ are independent if Ψ is sufficiently general. This finishes the proof of the third case. Note that if we would consider the map

$$\begin{pmatrix} \psi_{(23)} \\ \psi_{(33)} \end{pmatrix} : \sum_{i=1}^{3} \mathscr{C}_{n-2}^{(b)} \longrightarrow \sum_{i=1}^{2} \mathscr{C}_{n-2}^{(a-1)} \oplus \sum_{i=1}^{3} \mathscr{C}_{n-2}^{a}$$

instead of the map $\binom{\psi_{(22)}}{\psi_{(32)}}$, then doing the above would give the following inequality

$$n-2 = 5r \geq 2 \cdot \dim_{\Bbbk}(S_{n-a-2}G) + 2 \cdot \dim_{\Bbbk}(S_{n-a-3}G) = 2(n-a-1) + 2(n-a-2) = 4r+2$$

and we could at first read off an element in the kernel for $r \geq 2$. We do the same for the other cases.

Case 2: We have $n = 5r + 1,\ a = 4r - 1,\ (a_1, ..., a_5) = (a - 1, a, a, a, a),\ (b_1, ..., b_5) = (b + 1, b, b, b, b)$ with $b = f - 2 - a = 2n - a - 5$. We do the computation as above for the map

$$\psi : \mathscr{C}_{n-2}^{(b+1)} \oplus \sum_{i=1}^{4} \mathscr{C}_{n-2}^{b} \longrightarrow \mathscr{C}_{n-2}^{(a-1)} \oplus \sum_{i=1}^{4} \mathscr{C}_{n-2}^{a}$$

induced by the skew-symmetric matrix

$$\Psi : \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+(b+1)R) \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+bR)^4 \end{matrix} \longrightarrow \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+(a-1)R) \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+aR)^4 \end{matrix} .$$

Note that the map $\Psi_{(11)} : \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H + (b + 1)R) \longrightarrow \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H + (a - 1)R)$ is zero by the skew-symmetry of Ψ. We find an element in the kernel of ψ by considering the map $\psi_{(41)} : \mathscr{C}_{n-2}^{(b+1)} = \bigwedge^{n-2} F \otimes S_{n-a-2}G \longrightarrow (\bigwedge^{n-1} F)^4 \otimes D_{n-a-2}G^* = \sum_{i=1}^{4} \mathscr{C}_{n-2}^{(a-1)}$. We compute that

$$n - 2 = 5r - 1 \geq 4 \cdot \dim_{\Bbbk}(S_{n-a-2}G) = 4(n - a - 1) = 4r$$

holds for $r \geq 1$. By the same argument as in Case 1 and Case 3 we find a non-trivial element in the kernel of ψ.

Case 4: We have $n = 5r + 3,\ a = 4r,\ (a_1, ..., a_5) = (a + 1, a + 1, a, a, a)$ and $(b_1, ..., b_5) = (b - 1, b - 1, b, b, b)$. We consider the map

$$\begin{pmatrix} \psi_{(23)} \\ \psi_{(33)} \end{pmatrix} : \sum_{i=1}^{3} \mathscr{C}_{n-2}^{b} = \bigwedge^{n-2} F \otimes (S_{n-a-3}G)^3 \longrightarrow \begin{matrix} \sum_{i=1}^{2} \mathscr{C}_{n-2}^{(a+1)} \\ \oplus \\ \sum_{i=1}^{3} \mathscr{C}_{n-2}^{a} \end{matrix} = \begin{matrix} (\bigwedge^{n-1} F)^2 \otimes D_{n-a-4}G^* \\ \oplus \\ (\bigwedge^{n-1} F)^3 \otimes D_{n-a-3}G^* \end{matrix}$$

induced by the (2×3)- and the (3×3)-block of Ψ. With the same considerations as in the other cases we compute that

$$n-2 = 5r+1 \geq 2 \cdot \dim_{\Bbbk}(S_{n-a-4}G) + 2 \cdot \dim_{\Bbbk}(S_{n-a-3}G) = 2(n-a-3) + 2(n-a-2) = 4r+2$$

holds for $r \geq 1$ and we therefore find a non-trivial element in the kernel of the map

$$\psi : \sum_{i=1}^{2} \mathscr{C}_{n-2}^{(b-1)} \oplus \sum_{i=1}^{3} \mathscr{C}_{n-2}^{b} \longrightarrow \sum_{i=1}^{2} \mathscr{C}_{n-2}^{(a+1)} \oplus \sum_{i=1}^{3} \mathscr{C}_{n-2}^{a} .$$

Case 5: We have $n = 5r + 4,\ a = 4r + 1, (a_1, ..., a_5) = (a + 1, a, a, a, a)$ and $(b_1, ..., b_5) = (b - 1, a, a, a, a)$ with $b = 2n - a - 5$. We consider the map

$$\begin{pmatrix} \psi_{(14)} \\ \psi_{(44)} \end{pmatrix} : \sum_{i=1}^{4} \mathscr{C}_{n-2}^{b} = \bigwedge^{n-2} F \otimes (S_{n-a-3}G)^4 \longrightarrow \begin{matrix} \mathscr{C}_{n-2}^{(a+1)} \\ \oplus \\ \sum_{i=1}^{4} \mathscr{C}_{n-2}^{a} \end{matrix} = \begin{matrix} \bigwedge^{n-1} F \otimes S_{n-a-4}G \\ \oplus \\ (\bigwedge^{n-1} F)^4 \otimes D_{n-a-3}G^* \end{matrix}$$

induced by the (1×4)- and the 3×3-block of the skew-symmetric matrix Ψ. With the same arguments as before we compute that

$$n-2 = 5r+2 \geq 1 \cdot \dim_{\Bbbk}(S_{n-a-4}G) + 3 \cdot \dim_{\Bbbk}(S_{n-a-3}G) = (n-a-3) + 3(n-a-2) = 4r+3$$

holds for $r \geq 1$ and we can read off an element in the kernel of the map

$$\psi : \mathscr{C}_{n-2}^{(b-1)} \oplus \sum_{i=1}^{4} \mathscr{C}_{n-2}^{b} \longrightarrow \mathscr{C}_{n-2}^{(a+1)} \oplus \sum_{i=1}^{4} \mathscr{C}_{n-2}^{a} .$$

This finishes the proof of Theorem 3.4. $\square$

We can immediately generalize Theorem 3.4 and get the following.

Theorem 3.6. *Let C be a general 5-gonal canonical curve of odd genus $g = 2n+1$ satisfying the balancing conditions. Then*

$$\beta_{m+c,m+c+1}(C) > \beta_{m+c,m+c+1}(X) \text{ for } g \geq 30c + 13 .$$

Remark 3.7. The idea of the proof is the same as in the proof of Theorem 3.4. We distinguish again the cases 1-5. The rank of the map

$$\psi := \psi_{n-2+c} : \sum_{i=1}^{5} \mathscr{C}_{n-2+c}^{b_i} \longrightarrow \sum_{i=1}^{5} \mathscr{C}_{n-2+c}^{a_i}$$

determines the Betti number $\beta_{m+c,m+c+1}(C)$, and for $c \geq 0$ we have

$$\operatorname{rank}(\textstyle\sum_{i=1}^{5} \mathscr{C}_{n-2+c}^{b_i}) \leq \operatorname{rank}(\textstyle\sum_{i=1}^{5} \mathscr{C}_{n-2+c}^{a_i}) .$$

We note that for $g \geq 30c+13$ we always have $\min\{b_i\} \geq n-2+c \geq \max\{a_i\}$ which means that

$$\mathscr{C}_{n-2+c}^{b_i} = \bigwedge^{n-2+c} F \otimes S_{b_i-(n-2+c)}G \text{ and } \mathscr{C}_{n-2+c}^{a_i} = \bigwedge^{n-1+c} F \otimes D_{n+c-a_i-3}G^* \quad \forall i .$$

In particular ψ gives no minimal parts in the resolution of $C \subset \mathbb{P}^{g-1}$ by an iterated mapping cone construction.

Proof of Theorem 3.6. We repeat the argument in detail for Case 1. Throughout the proof Ψ denotes again the 5×5 skew-symmetric matrix in the resolution of C as an $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-module.

Case 1: We have $n = 5r + 5$, $a = 4r + 2$ and $(a_1, ..., a_5) = (a, a, a, a, a)$. The map $\sum_{i=1}^{5} \mathscr{C}_{n-2+c}^{b} = \bigwedge^{n-2+c} F \otimes (S_{b-n-c+2}G)^5 \xrightarrow{\psi} \sum_{1=0}^{5} \mathscr{C}_{n-2+c}^{a} = (\bigwedge^{n-1+c} F)^5 \otimes D_{n+c-a-3}G^*$ is given as the composition

$$\bigwedge^{n-2+c} F \otimes (S_{b-n-c+2}G)^5 \cong \bigwedge^{n-2+c} F \otimes (S_{n-a-c-3}G)^5 \otimes S_{n+c-a-3}G \otimes D_{n+c-a-3}G^*$$

$$\longrightarrow \bigwedge^{n-2+c} F \otimes S_{2n-2a-6}G \otimes D_{n+c-a-3}G^* \xrightarrow{id \otimes \Psi \otimes id} \bigwedge^{n-2+c} F \otimes F^5 \otimes D_{n+c-a-3}G^*$$

$$\xrightarrow{\wedge \otimes id} (\bigwedge^{n-1+c} F)^5 \otimes D_{n+c-a-3}G^* .$$

We choose a basis element of the form $g = (g_1, 0, \ldots, 0)^t \in (S_{b-n-c+2}G)^5$ and show that there exists an $f \in \bigwedge^{n-2+c} F$ such that $f \wedge \Psi(g \cdot g') = 0 \ \forall g' \in S_{n+c-a-3}G$. By the skew-symmetry of Ψ it follows that $\Psi(g \cdot g')$ is of the form $(0, f_1, \ldots, f_4)^t$ for some element $g' \in S_{n+c-a-3}G$. We compute that

$$n - 2 + c = 5r + c - 1 \geq 4 \cdot \dim_{\Bbbk}(S_{n+c-a-3}G) = 4(n + c - a - 2) = 4r + 4c + 4$$

holds for $r \geq 3c+1$. This means that if $r \geq 3c+1$, then we can directly read of a non-trivial element in the kernel of ψ as in the proof of Theorem 3.4.

For the other cases we only compute the relevant dimensions, since the argument is exactly the same as above or in the proof of Theorem 3.4. The blocks of Ψ that induce the maps considered below are the same as in the proof of Theorem 3.4 and doing the computation for the other blocks would give larger bounds on r.

Case 2: $n = 5r + 1,\ a = 4r - 1$ and $(a_1, ..., a_5) = (a - 1, a, a, a, a)$. We can directly read off an element in the kernel if the following inequality is satisfied

$$n - 2 + c = 5r + c - 1 \geq 4 \cdot \dim_{\Bbbk}(S_{n+c-a-3}G) = 4(n + c - a - 2) = 4r + 4c\,.$$

This is the case for $r \geq 3c + 1$.

Case 3: $n = 5r + 2,\ a = 4r$ and $(a_1, .., a_5) = (a - 1, a - 1, a, a, a)$. We can directly read off an element in the kernel if

$$n - 2 + c = 5r + c \geq 1 \cdot \dim_{\Bbbk}(S_{n+c-a-2}G) + 3 \cdot \dim_{\Bbbk}(S_{n+c-a-3}G) = 4r + 4c + 1$$

which holds for $r \geq 3c + 1$.

Case 4: $n = 5r + 3,\ a = 4r$ and $(a_1, ..., a_5) = (a + 1, a + 1, a, a, a)$. We can directly read off an element in the kernel if

$$n - 2 + c = 5r + c \geq 2 \cdot \dim_{\Bbbk}(S_{n+c-a-4}G) + 2 \cdot \dim_{\Bbbk}(S_{n+c-a-3}G) = 4r + 4c + 2$$

which holds for $r \geq 3c + 1$.

Case 5: $n = 5r + 4,\ a = 4r + 1$ and $(a_1, ..., a_5) = (a + 1, a, a, a, a)$ We can directly read off an element in the kernel if

$$n - 2 + c = 5r + c + 2 \geq 1 \cdot \dim_{\Bbbk}(S_{n+c-a-4}G) + 3 \cdot \dim_{\Bbbk}(S_{n+c-a-3}G) = 4r + 4c + 3$$

which holds for $r \geq 3c + 1$.

Altogether we see that the smallest $g = 2n+1$, so that we can directly read off a non-trivial kernel of the map

$$\psi := \psi_{n-2+c} : \sum_{i=1}^{5} \mathscr{C}^{b_i}_{n-2+c} \longrightarrow \sum_{i=1}^{5} \mathscr{C}^{a_i}_{n-2+c}\,,$$

is given by $g = 2 \cdot (5r + 1) + 1 = 2 \cdot (5 \cdot (3c + 1) + 1) + 1 = 30c + 13.$ □

3.3. 5-Gonal Curves of Even Genus

We use the approach of the previous section and prove a similar result for 5-gonal canonical curves of even genus. Throughout this section we assume that all curves are of even genus $g = 2n \geq 8$ and satisfy the balancing conditions. We distinguish the following 5 cases:

Case 1. $(a_1, ..., a_5) = (a-1, a, a, a, a)$, $(b_1, ..., b_5) = (b+1, b, b, b, b) \Leftrightarrow a = 4r-3$ and $n = 5r-1$ for some $r \geq 1$.

Case 2. $(a_1, ..., a_5) = (a-1, a-1, a, a, a)$, $(b_1, ..., b_5) = (b+1, b+1, b, b, b) \Leftrightarrow a = 4r-2$ and $n = 5r$ for some $r \geq 1$.

Case 3. $(a_1, ..., a_5) = (a+1, a+1, a, a, a)$, $(b_1, ..., b_5) = (b-1, b-1, b, b, b) \Leftrightarrow a = 4r-2$ and $n = 5r+1$ for some $r \geq 1$.

Case 4. $(a_1, ..., a_5) = (a+1, a, a, a, a)$, $(b_1, ..., b_5) = (b-1, b, b, b, b) \Leftrightarrow a = 4r-1$ and $n = 5r+2$ for some $r \geq 1$.

Case 5. $(a_1, ..., a_5) = (a, a, a, a, a)$, $(b, ..., b_5) = (b-1, b, b, b, b) \Leftrightarrow a = 4r$ and $n = 5r+3$ for some $r \geq 1$.

The aim of this section is to prove the following theorem.

Theorem 3.8. *Let $C \subset \mathbb{P}^{g-1}$ be a general 5-gonal canonical curve of even genus $g = 2n$ satisfying the balancing conditions. Then*

$$\beta_{m+c,m+c+1}(C) > \beta_{m+c,m+c+1}(X) \text{ for } g \geq 30c + 28\,.$$

The degree of the scroll X is given by $f = 2n-4$ and the number b above is given by $b = f - 2 - a = 2n - a - 6$.
Note that the statements of Remark 3.7 also apply in the situation of Theorem 3.8.

Proof of Theorem 3.8. The idea of the proof is similar to the proof of Theorem 3.4. We repeat the argument in detail for Case 1.
Case 1: $n = 5-1$, $a = 4r-3$ and $(a_1, ..., a_5) = (a-1, a, a, a, a)$. We show that the map

$$\psi : \begin{matrix} \mathscr{C}^{(b+1)}_{n-2+c} \\ \oplus \\ \mathscr{C}^{b}_{n-2+c} \end{matrix} = \begin{matrix} \bigwedge^{n-2+c} F \otimes S_{b-n-c+3} G \\ \oplus \\ \bigwedge^{n-2+c} F \otimes (S_{b-n-c+2} G)^4 \end{matrix} \longrightarrow \begin{matrix} \mathscr{C}^{(a-1)}_{n-2+c} \\ \oplus \\ \mathscr{C}^{a}_{n-2+c} \end{matrix} = \begin{matrix} \bigwedge^{n-1+c} F \otimes D_{n+c-a-4} G^* \\ \oplus \\ \bigwedge^{n-1+c} F \otimes (D_{n+c-a-3} G^*)^4 \end{matrix}$$

induced by the skew-symmetric matrix

$$\Psi : \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+(b+1)R) \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+bR)^4 \end{matrix} \longrightarrow \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+(a-1)R) \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+aR)^4 \end{matrix}$$

has a non-trivial kernel for $r \geq 3c+3$. By the skew-symmetry of Ψ it is sufficient to show that the map $\psi_{(41)}$ induced by the (4×1)-block

$$\Psi_{(41)} : \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H + (b+1)R) \longrightarrow \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H + aR)^4$$

has a non-trivial decomposable element in the kernel. As in Lemma 1.28, the map $\psi_{(41)}$ is given as the composition

$$\bigwedge^{n-2+c} F \otimes S_{b-n-c+3}G \cong \bigwedge^{n-2+c} F \otimes S_{n-a-c-3}G \otimes S_{n+c-a-3}G \otimes D_{n+c-a-3}G^*$$

$$\longrightarrow \bigwedge^{n-2+c} F \otimes S_{2n-2a-6}G \otimes D_{n+c-a-3}G^* \xrightarrow{id\otimes\Psi_{(41)}\otimes id} \bigwedge^{n-2+c} F \otimes F^4 \otimes D_{n+c-a-3}G^*$$

$$\xrightarrow{\wedge\otimes id} (\bigwedge^{n-1+c} F)^4 \otimes D_{n+c-a-3}G^*\,,$$

and we thus have to find an $f \in \bigwedge^{n-2+c} F$ and an element $g \in S_{n-a-c-3}G$ such that $f \wedge \Psi_{(41)}(gg') = 0$ for all $g' \in S_{n+c-a-3}G$. As in the proof of Theorem 3.4, we compute

$$\begin{aligned} n-2+c = 5r+c-3 \geq \dim_{\Bbbk}(\Psi_{(41)}(g \cdot S_{n+c-a-3}G)) &= 4(n+c-a-2) \\ &= 4r+4c \end{aligned}$$

holds for $r \geq 3c+3$, which means that we can directly read off a non-trivial element in the kernel if $r \geq 3c+3$.
Note that doing the above for the map

$$\psi_{(54)} : \bigwedge^{n-2+c} F \otimes (S_{b-n-c+2}G)^4 \longrightarrow \bigwedge^{n-1+c} F \otimes (S_{n+c-a-4}G) \oplus \bigwedge^{n-1+c} F \otimes (S_{n+c-a-3}G)^4$$

would give a larger bound on r.
For the cases 2-5 we proceed as above or in the proof of Theorem 3.4. In all cases we compute that we can at first directly read off a non-trivial element in the kernel of the map

$$\psi : \sum_{i=1}^{5} \mathscr{C}^{b_i}_{n-2+c} \longrightarrow \sum_{i=1}^{5} \mathscr{C}^{a_i}_{n-2+c}$$

if $r \geq 3c+3$. Altogether, we see that

$$\beta_{m+c,m+c+1}(C) > \beta_{m+c,m+c+1}(X) \text{ for } g \geq 2(5(3c+3)-1) = 30c+28.$$

$\square$

3.4. Final Remarks

Remark 3.9. It arises the question whether the bounds given by Theorem 3.6 and 3.8 are sharp. For $c = 1$, it follows by Theorem 3.6 that $\beta_{m+c,m+c+1}(C) > \beta_{m+c,m+c+1}(X)$ holds for $g = 43$. This is the smallest bound on g given by Theorem 3.6 and by Computation 4 in Appendix B it follows that ψ_{n-2+c} has at least no decomposable elements in the kernel for $g = 41$.
Note that the sharpness of Theorem 3.6 and 3.8 would imply the following equivalence:

$$c > \begin{cases} \frac{g-13}{30} & \text{for } g = 2n+1 \\ \frac{g-28}{30} & \text{for } g = 2n \end{cases} \text{for some } n \geq 2 \iff \beta_{n+c,n+c+1}(C) = \beta_{n+c,n+c+1}(X)\,.$$

Remark 3.10. By Appendix A, we also have $\beta_{\text{crit}}(C) > \beta_{\exp}(C)$ in the case of a 7-gonal curve of genus 17. It would be interesting to obtain similar results as Theorem 3.6 and 3.8 for 7-gonal curves, or more generally for curves of arbitrary gonality.

Remark 3.11. Throughout this chapter we assumed that all curves satisfy the balancing conditions. It is an open problem whether this is the case for general k-gonal curves of genus g. In the case of 5-gonal canonical curves one can use the Structure Theorem for Gorenstein ideals in codimension 3 (see [BE77]) to produce 5-gonal curves of arbitrary genus satisfying the balancing conditions. Indeed, if we take a general skew-symmetric matrix

$$\Psi : \textstyle\sum_{i=1}^{5} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H + b_i R) \longrightarrow \textstyle\sum_{i=1}^{5} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H + a_i R),$$

where $(a_1, ..., a_5)$ satisfies the balancing condition, then we can obtain the ideal of a 5-gonal canonical curve as the ideal generated by the Pfaffians of Ψ. In this case it remains to show that the curves produced this way are absolutely irreducible.

A. Critical Betti Numbers

We summarize the expected critical Betti numbers and those computed with the *Macaulay2*-package *kGonalNodalCurves.m2* (see [Bop13]). Recall that

$$\beta = \beta_{\text{crit}}(C) = \beta_{m,m+1}(C) \text{ with } m = \left\lceil \frac{g-1}{2} \right\rceil$$

and

$$\beta_{\exp}(C) = \beta_{m,m+1}(X)\,,$$

where X denotes the scroll swept out by the special pencil on C.

$g = 5:$

gonality	expected β	computed β
3	2	2
4	0	0

$g = 6:$

gonality	expected β	computed β
3	3	3
4	0	0

$g = 7:$

gonality	expected β	computed β
3	15	15
4	3	3
5	0	0

$g = 8:$

gonality	expected β	computed β
3	24	24
4	4	4
5	0	0

$g = 9:$

gonality	expected β	computed β
3	84	84
4	24	24
5	4	4
6	0	0

$g = 10$

gonality	expected β	computed β
3	140	140
4	35	35
5	5	5
6	0	0

$g = 11:$

gonality	expected β	computed β
3	420	420
4	140	140
5	35	35
6	5	5
7	0	0

$g = 12:$

gonality	expected β	computed β
3	720	720
4	216	216
5	48	48
6	6	6
7	0	0

$g = 13$:

gonality	expected β	computed β
3	1980	1980
4	720	720
5	216	222
6	48	48
7	6	6
8	0	0

$g = 14$

gonality	expected β	computed β
3	3465	3465
4	1155	1155
5	315	315
6	63	63
7	7	7
8	0	0

$g = 15$:

gonality	expected β	computed β
3	9009	9009
4	3465	3465
5	1155	1190
6	315	315
7	63	63
8	7	7
9	0	0

$g = 16$

gonality	expected β	computed β
3	16016	16016
4	5720	5720
5	1760	1760
6	440	440
7	80	80
8	8	8
9	0	0

$g = 17$:

gonality	expected β	computed β
3	40040	40040
4	16016	16016
5	5720	5904
6	1760	1760
7	440	456
8	80	80
9	8	16
10	0	0

B. Computations

Computation 1 We verify that a general 5-gonal 13-nodal canonical curve satisfies the balancing conditions. In the computation we will use the functions *getFactors* and *getCoordinates* from Section 2.4, and the function *canonicalMultipliers* from the *Macaulay2*-package *NodalCurves.m2* (see [Sch12b]), that computes the canonical multipliers as described in Section 2.1.
In the first step we compute g pairs of points (P_i, Q_i), the multipliers and the canonical multipliers in a normalized form, i.e., $\beta_i = 1$ for all pairs of multipliers (α_i, β_i).

```
i1 : p=10007, k=5,g=13;
     kk=ZZ/p;
     S=kk[x_0,x_1];
     f=random(S^1,S^{2:-k});

i2 : L={};
     while #unique L < g do (
          if #(q:=(getFactors(det (f||matrix{{random p-1,1}})))_0)>=2
          then L= L|{q});
     P:=transpose matrix apply(L,l->flatten entries getCoordinates l_0);
     Q:=transpose matrix apply(L,l->flatten entries getCoordinates l_1);

i3 : multL=apply(g,i->sub(sub(f_(0,0),transpose(P_{i})),kk)/
                      sub(sub(f_(0,0),transpose(Q_{i})),kk));
     multK0=sub(canonicalMultipliers(P,Q),kk);
     multK=apply(g,i->multK0_(0,i)/multK0_(1,i)); --normalized multipliers
```

For the computation of the type of the scroll X as described in [Sch86, (2.4)] it is necessary to compute $h^0(\omega_C \otimes \mathscr{O}_C(i \cdot D)^{-1})$, where $|D|$ is the special pencil on the curve C. To this end we define the function *getGlobalSections* that returns a basis of sections of a line bundle of given degree from the g pairs of points (P_i, Q_i) and the corresponding multipliers as described in (2.1).

```
i4 : getGlobalSections=(mults,P,Q,k)->(
            B:=flatten entries basis(k,S);
            M:=matrix apply(#mults,i->apply(k+1,j->sub(B_j,transpose(P_{i}))-
                                 mults_i*sub(B_j,transpose(Q_{i}))));
            basis(k,S)*syz M)
```

We define a function *H0KiD* that computes $H^0(\omega_C \otimes \mathscr{O}_C(i \cdot D)^{-1})$, and calculate the type of the scroll.

```
i5 :        H0KiD=n->(
            multKiD=apply(g,i->multK_i/(multL_i^n));
            getGlobalSections(multKiD,P,Q,2*g-2-n*k));

i6 : eDual={}; -- the dual partition
            for i from 0 to 3 do (
            Dual0:=rank source H0KiD(i) - rank source H0KiD(i+1);
            if Dual0<=0 then break else (eDual=eDual|{Dual0}));

i7 : e=apply(4,i->#select(eDual,e0->e0>=i+1)-1)--the type of the scroll

o7 = {3, 2, 2, 2}
```

The rest of this computation is based on the *Macaulay2* code in [Gei13, Section 4]. It remains to determine a minimal free resolution of $C \subset \mathbb{P}(\mathscr{E})$. To this end we compute sections defining the canonical embedding in such a way that we can easily find defining equations for the scroll X.

```
i8 : PHI0=apply(e,n->(
          H0KnD1:=H0KnD(n);
          phi:=H0KnD1*random(kk^(rank source H0KnD1),kk^1);
               s=apply(n+1,i->(
            (phi*f_(0,0)^(n-i)*f_(0,1)^i))_(0,0))));
     PHI=matrix{flatten PHI0};

i9 : T=kk[t_0..t_(g-1)];
     phi=map(S,T,PHI);
     Ican = saturate ker phi; --computation of the ideal of C in PP^12

i10 : (dim Ican, degree Ican, genus Ican)

o10 = (2, 24, 13)
```

In the next step we compute the scroll X. By Remark 1.21, the Cox-Ring

$$R_{\text{cox}} = \bigoplus_{a,b\in\mathbb{Z}} H^0(\mathscr{O}_X(aH + bR))$$

is a subring of $R = \Bbbk[\varphi_1, ...\varphi_4, s, t]$, where $\deg(s) = \deg(t) = (0, 1)$ and $\deg(\varphi_i) = (1, e_1 - e_i)$. We compute the image $J \subset R$ of the ideal I_C under the map $T/I_X \longrightarrow R$.

```
i11 : degs={apply(0..#e-1, i-> {1,max e -e_i}), 2:{0,1}};
      R=kk[pp_0..pp_3,s,t,Degrees=>degs];
      PSI=matrix{flatten apply(#e,i->flatten entries (basis({0,e_i},R)*pp_i))};
      psi=map(R,T,PSI);
      Scroll = ideal mingens ker psi;

i12 : betti res Scroll
```

```
             0  1   2   3   4   5   6  7 8
o12 = total: 1 36 168 378 504 420 216 63 8
          0: 1  .   .   .   .   .   .  . .
          1: . 36 168 378 504 420 216 63 8

i13 : T'=T/Scroll;
      psi'=map(R,T',PSI);
      J=psi'(ideal mingens sub(Ican,T'));
      J1=saturate(J,ideal basis({0,1},R)); --ideal of Ican in X
```

Recall that $\mathscr{I}_C$ has a resolution of the form
$\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-5H+7R) \longrightarrow \sum_{i=1}^{5} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+b_iR) \longrightarrow \sum_{i=1}^{5} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+a_iR) \longrightarrow \mathscr{I}_C \longrightarrow 0$,
and as in [Gei13, Section 4] we can now obtain the Betti numbers of the minimal free resolution of $I_C \subset X$ by picking the right degrees.

```
i14 : J2=ideal select(flatten entries gens J1,f->(degree f)_0==2);
      resX={gens J2};
      degsH={3,5};
      scan(#degsH,i->(
         M0=syz(resX_i);
         cols:=toList(0..rank source M0-1);
         M=M0_(select(cols,j->((degrees source M0)_j)_0==degsH_i));
         resX=resX|{M}));

i15 : betti chainComplex resX--the resolution of the curve as a PP(E)-module

             0 1 2 3
o15 = total: 1 5 5 1
          0: 1 . . .
          1: . . . .
          2: . . . .
          3: . . . .
          4: . 4 . .
          5: . 1 1 .
          6: . . 4 .
          7: . . . .
          8: . . . .
          9: . . . .
         10: . . . 1
```

In particular we see that $(a_1, ..., a_5)$ and $(b_1, ..., b_5)$ satisfy the balancing condition since $a_1 + ... + a_5 = 2g - 12$ and $a_i + b_i = f - 2$.

Computation 2 We verify that $\psi_{(41)}$ induced by the 4×1 submatrix $\Psi_{(41)}$ of Ψ has a 6-dimensional kernel. The map $\psi_{(41)}$ is given as the composition

$$\bigwedge^4 F \otimes G \xrightarrow{id \otimes \Psi_{(41)}} \bigwedge^4 F \otimes F^4 \xrightarrow{\wedge} (\bigwedge^5 F)^4 .$$

In the following we denote by `sv`$:= \Psi_{(41)}(s)$ and by `tv`$:= \Psi_{(41)}(t)$, where $\{s,t\}$ is a basis of $G \cong H^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(R))$.

```
i1 : R=QQ[sp_1,sp_2,sp_3,tp_1,tp_2,tp_3,s2p,stp,t2p,SkewCommutative=>true];
     sv=transpose matrix{{sp_1,sp_2,sp_3,stp}};
     tv= transpose matrix{{tp_1,tp_2,tp_3,t2p}};
     stv=sv|tv;

i2 : rels= syz(basis(4,R)**stv);
     tally degrees source rels

o2 = Tally{{4} => 6   }
          {5} => 1988
```

We conclude that $\psi_{(41)}$ has a 6-dimensional kernel.

Computation 3 We consider the case of a 5-gonal canonical curve of genus 13. For a general choice of Ψ, we show that none of the entries of the skew-symmetric matrix Ψ can be made zero, by suitable row and column operations that respect the skew-symmetric structure of Ψ. Without loss of generality we try to make the entry Ψ_{45} zero, i.e., we have to find a matrix A of full rank, with entries as indicated below

$$A \sim \left(\begin{array}{c|c} \mathrm{GL}(1,\Bbbk) & H^0(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(R)) \\ \hline 0 & \mathrm{GL}(4,\Bbbk) \end{array} \right)$$

such that $(A^t \Psi A)_{45} = 0$.
The entry $(A^t \Psi A)_{45}$ is given by $\mu \Psi \lambda^t$, where $\mu = (\mu_0 s + \mu_1 t + \mu_2 + ... + \mu_5)$ is the 4^{th} row of A^t and $\lambda^t = (\lambda_0 s + \lambda_1 t + \lambda_2 + ... + \lambda_5)^t$ is the 5^{th} column of A.

```
i1 : kk=ZZ/12347;
     e={3,2,2,2};--the type of the scroll
     degs={2:{0,1},apply(0..#e-1, i->{1,max e-e_i})};
     T=kk[s,t,p_0..p_3,Degrees=>degs];--the cox ring
     Z=kk[l_0..l_5,m_0..m_5]; --coefficientring
     ZT=Z[s,t,p_0..p_3,Degrees=>degs];

i2 : psi'=random(T^{1:{1,1},4:{1,2}},T^5);
     Psi=sub(psi'-transpose psi',ZT);
     lambda=matrix{{l_0*s+l_1*t,l_2..l_5}};
     mu=matrix{{m_0*s+m_1*t,m_2..m_5}};
```

```
i3 : --this entry has to be made zero:
     Psi41= (mu*Psi*(transpose lambda))_(0,0);
```

In the next step we compute the ideal I_{rels} generated by relations on the μ_i and λ_j. Since the matrix A must have full rank we have to ensure that

$$((\mu_0 : ... : \mu_5), (\lambda_0 : ... : \lambda_5)) \notin \Delta_{\mathbb{P}^5 \times \mathbb{P}^5} \ .$$

On the other hand we exclude solutions induced by the skew-symmetry of Ψ by claiming that rank $\begin{pmatrix} \mu_2 & \cdots & \mu_5 \\ \lambda_2 & \cdots & \lambda_5 \end{pmatrix} = 2$.

```
i4 : H0HR=basis({1,2},T);--basis of H^0(H-R)
     use Z
     Irels=sub(ideal flatten  {apply(9,i->coefficient(H0HR_(0,i),Psi41))},Z);
     -- ideal generated by the relations in lambda and mu
     Idiag=minors(2, matrix apply(6, i->{l_i,m_i}));--projective diagonal

i5 : Isols=saturate(Irels, Idiag)

o5 = ideal (m , m , m , m , l , l , l , l )
             5   4   3   2   5   4   3   2
```

It follows that rank $\begin{pmatrix} \mu_2 & \cdots & \mu_5 \\ \lambda_2 & \cdots & \lambda_5 \end{pmatrix} < 2$ and therefore none of the entries of Ψ can be made zero.

Computation 4 We consider the case of a general 5-gonal curve of genus $g = 41$ satisfying the balancing conditions. With notation as in Chapter 3 we have $(a_1, ..., a_5) = (14, ..., 14)$ and $(b_1, ..., b_5) = (21, ..., 21)$.
The map $\psi : (\mathscr{C}_{19}^{21})^5 = \bigwedge^{19} F \otimes (S_2G)^5 \longrightarrow (\mathscr{C}_{19}^{14})^5 = \bigwedge^{20} F \otimes (D_4G^*)^5$, induced by the skew-symmetric matrix Ψ, determines the Betti number $\beta_{m+c,m+c+1}(C)$. We show that there are no decomposable elements in the kernel of ψ. The map ψ is given as the composition

$$\bigwedge^{19} F \otimes (S_2G)^5 \cong \bigwedge^{19} F \otimes (S_2G)^5 \otimes S_4G \otimes D_4G^* \longrightarrow \bigwedge^{19} F \otimes (S_6G)^5 \otimes D_4G^*$$

$$\xrightarrow{id \otimes \Psi \otimes} \bigwedge^{19} F \otimes F^5 \otimes D_4G^* \xrightarrow{\wedge \otimes id} (\bigwedge^{20} F)^5 \otimes D_4G^*$$

Recall that having a decomposable element in the kernel of this composition means that there exists an $f \in \bigwedge^{19} F$ and an element $g \in (S_2G)^5$ such that $f \wedge \Psi(gg') = 0$ for all elements $g' \in S_4G$.
Let $\{g_1', ..., g_5'\}$ be a basis of S_4G. For the computation we use a brute force approach over

the field $\Bbbk = \mathbb{Z}/3$ to show that the linear space $F_g \subset F$ spanned by the entries of the 5×5 matrix $(\Psi(gg_1')|\cdots|\Psi(gg_5'))$ has at least rank 20 for all $g \in S_2G$. This means in particular, that the composition has no decomposable elements in the kernel.
We first build a list L_1 containing all elements of S_2G.

```
i1 : kk=ZZ/3;
     e=(e1,e2,e3,e4)=(10,9,9,9);--the type of the scroll X
     degs={apply(0..#e-1, i->{1,max e-e_i}),2:{0,1}};
     T=kk[p_0..p_3,s,t,Degrees=>degs];

i2 : comb2Lists:=(L1,L2)->(--builds all combinations from lists L1 and L2
        L:=unique flatten    apply(#L1, j->apply(#L2, k->L1_j+L2_k));
        L':=L|L1|L2;
        unique L'         );

i3 : N1=apply(char(kk),i->i*(flatten entries basis({0,2},T))_0);
     N2=apply(char(kk),i->i*(flatten entries basis({0,2},T))_1);
     N3=apply(char(kk),i->i*(flatten entries basis({0,2},T))_2);

i4 : L1=unique comb2Lists(comb2Lists(N1,N2),N3);
```

In the next step we compute $\dim_{\Bbbk} F_g$ for all $g \in (S_2G)^5$ and save the dimensions to a list.

```
i5 : S4G=flatten entries basis({0,4},T);--basis H^0(4R)
     Psi'=random(T^{5:{1,3}},T^{5:{0,0}});
     Psi=Psi'-transpose Psi';--the generic skew-symmetric matrix Psi

i6 : vec1=apply(#L1,i->transpose matrix{{L1_i,0,0,0,0}});
     vec2=apply(#L1,i->transpose matrix{{0,L1_i,0,0,0}});
     vec3=apply(#L1,i->transpose matrix{{0,0,L1_i,0,0}});
     vec4=apply(#L1,i->transpose matrix{{0,0,0,L1_i,0}});
     vec5=apply(#L1,i->transpose matrix{{0,0,0,0,L1_i}});

i7 : L={};
     time for index1 from 0 to 5 do (      -- used 43546.8 seconds
             for index2 from 0 to 5 do (
         for index3 from 0 to 5 do (
              for index4 from 0 to 5 do (
           for index5 from 0 to 5 do (
               vec=vec1_index1+vec2_index2+vec3_index3+
                   vec4_index4+vec5_index5;
              A=matrix {apply(5,j-> Psi*S4G_j*vec)};
                     L=L|{rank source mingens minors(1,A)} ;
                     L=unique L  ); ); ); ); )
i9 : L

o9 : {0, 20, 21, 22}
```

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