Betti Numbers of Syzygies and Cohomology of Coherent Sheaves

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Theorem (Hilbert's Syzygy Theorem, 1890)

M has a finite free resolution F

$$(0 \leftarrow M \leftarrow) F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow \ldots \leftarrow F_{n+1} \leftarrow 0$$

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of length $\leq n+1$ (= the number of variables), where each $F_i = \oplus_j S(-j)^{\beta_{i,j}}$ is a direct sum of free modules generated in degree j.

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The polynomial nature of the Hilbert function of M

$$h_M : \mathbb{Z} \to \mathbb{Z},\ k \mapsto \dim_{\mathbb{K}} M_k.$$

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$$h_M(k) = \sum_{i=0}^{n+1} (-1)^i \dim_{\mathbb{K}} (F_i)_k = \sum_{i=0}^{n+1} (-1)^i \sum_j \beta_{i,j} \binom{k-j+n}{n}$$

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$$\begin{aligned} h_M(k) &= \sum_{i=0}^{n+1}(-1)^i \dim_{\mathbb{K}}(F_i)_k = \sum_{i=0}^{n+1}(-1)^i \sum_j \beta_{i,j}\binom{k-j+n}{n} \\ &= p_M(k) \text{ for } k \gg 0, \end{aligned}$$

for the Hilbert polynomial $p_M(t) \in \mathbb{Q}[t]$.

Syzygies und Hilbert series

$$h_M(k) = \sum_{i=0}^{n+1} (-1)^i \sum_j \beta_{i,j} \binom{n+k-j}{n}$$

implies also the rationality of the Hilbert series:

$$H_M(z) = \sum_k h_M(k) z^k = \frac{\sum_j (\sum_{i=0}^{n+1} (-1)^i \beta_{i,j}) z^j}{(1-z)^{n+1}} \quad ,$$

because

$$H_S(z) = \sum_{k=0}^{\infty} \binom{k+n}{n} z^k = \frac{1}{(1-z)^{n+1}} \quad .$$

Geometric interpretation of the Hilbert polynomial

Let $A = S/\langle f_1, \ldots f_\ell \rangle$ be an algebra with the f_j homogeneous polynomials and let $X = V(f_1, \ldots, f_\ell) \subset \mathbb{P}^n$ be the vanishing loci. Then:

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- $\deg X = r!($ lead coefficient of $p_A(t)) = mult(A) = Q_A(1)$, where $H_M(z) = \frac{Q_A(z)}{(1-z)^{r+1}}$ is the coprime rational expression for the Hilbert series. In particular, the numerator in the formula
$$H_M(z) = \frac{\sum_j (\sum_{i=0}^{n+1} (-1)^i \beta_{ij}) z^j}{(1-z)^{n+1}} \quad ,$$
vanishes at $z = 1$ to the order $c = codim\, X$.

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- $p_M(k) = \chi(\mathcal{F}(k)) = \sum_{i=0}^n (-1)^i h^i(\mathcal{F}(k))$ for all k.
- A family of sheaves $\mathcal{F}_\tau$ is flat, iff the coefficients of the Hilbert polynomials are constant as functions of τ.

Graded Betti numbers

The coefficients of the Hilbert polynomial are the fundamental numerical invariants of a graded S-module.

The graded Betti numbers $\beta_{i,j}$ of a minimal resolution

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Minimal means that at each step we choose a minimal homogeneous generating system. Then

$$image(F_{i+1}) \subset \langle x_0, \ldots, x_n \rangle F_i$$

and

$$\beta_{i,j} = \dim(F_i \otimes \mathbb{K})_j = \dim_{\mathbb{K}} Tor_i^S(M, \mathbb{K})_j.$$

Betti Tables

We abbreviate the numerical information of a minimal free resolution, say

$$S \leftarrow S(-2)^{10} \leftarrow \begin{array}{c} S^{16}(-3) \\ \oplus \\ S^{3}(-4) \end{array} \leftarrow \begin{array}{c} S^{3}(-4) \\ \oplus \\ S^{16}(-5) \end{array} \leftarrow S^{10}(-6) \leftarrow S(-8) \leftarrow 0$$

in a table

$\beta_{i,i+k}$	$i=0$	1	2	3	4	5
$k=0$	1	–	–	–	–	–
1	–	10	16	3	–	–
2	–	–	3	16	10	–
3	–	–	–	–	–	1

The traditional approach to the study of Betti numbers is the question, which Betti numbers are possible for a module with given Hilbert function or Hilbert polynomial.

Example: Canonical curves of genus 7 [S 1986]

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trigonal $\qquad \exists\, g^2_6$

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4-gonal $\qquad$ general case, $char(\mathbb{K}) \neq 2$

Boij-Söderberg Cone

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Boij and Söderberg conjectured a complete description of this cone.

Pure resolution

A pure resolution is the resolution of a CM-module, which has shape

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The Betti numbers $\beta_i = \beta_{i,d_i}$ of a pure resolution are determined by the degree sequence

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Corollary

The Betti table of a pure resolution spans an extremal ray of the Boij-Söderberg cone.

Rays of the Boij-Söderberg Cone

Theorem (Eisenbud-S, Boij-Söderberg, 2008)

Existence. For every degree sequence there exists a CM-module with a pure resolution.
Spanning and Decomposition. Each Betti table is a unique positive rational linear combination of pure Betti tables in a unique chain of degree sequences.

Here ”chain” refers to the natural partial order on degree sequences

$$(d_0, d_1, \ldots, d_c) \leq (e_0, e_1, \ldots, e_{c'}) :\Leftrightarrow c \geq c' \text{ and } d_i \leq e_i \, \forall i \leq c'.$$

1^{st} Application: Decomposition and Bounds

Let B_x denote the Betti table $B_x = \begin{matrix} 1 & - & - & - & - & - \\ - & 10 & 16 & x & - & - \\ - & - & x & 16 & 10 & - \\ - & - & - & - & - & 1 \end{matrix}.$

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According to the theorem, B_x is a linear combination of

$$A = \begin{matrix} 5 & - & - & - & - & - \\ - & 60 & 128 & 90 & - & - \\ - & - & - & - & 20 & - \\ - & - & - & - & - & 3 \end{matrix} \quad A^* = \begin{matrix} 3 & - & - & - & - & - \\ - & 20 & - & - & - & - \\ - & - & 90 & 128 & 60 & - \\ - & - & - & - & - & 5 \end{matrix}$$

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$$B_{11} = \frac{1}{45} B_0 + \frac{11}{90} A + \frac{11}{90} A^*.$$

Boij-Söderberg monoid

We do not think that B_{11} can be realized by an algebra. Only an integral multiple actually occurs. One can see the same phenomenon already with pure sequences.

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$$\begin{matrix} 2 & 4 & - & - \\ - & - & 4 & 2 \end{matrix}$$

is realized.

Dependence on the characteristic [Kunte 2008]

The monoid of actual Betti tables depends on the characteristic of the ground field.

$$\begin{array}{cccccc} 1 & - & - & - & - & - \\ - & 10 & 16 & - & - & - \\ - & - & - & 16 & 10 & - \\ - & - & - & - & - & 1 \end{array}$$

occurs for all fields of $char(\mathbb{K}) \neq 2$, while in $char(\mathbb{K}) = 2$ an algebra which this Hilbert function has Betti number at least

$$\begin{array}{cccccc} 1 & - & - & - & - & - \\ - & 10 & 16 & 1 & - & - \\ - & - & 1 & 16 & 10 & - \\ - & - & - & - & - & 1 \end{array}$$

Fan Structure

In a bounded range, say

$$\beta_{i,j} \neq 0 \text{ only for } j \text{ with } \underline{d}_i \leq j \leq \overline{d}_i$$

with bounds $\underline{d} = (\underline{d}_0, \ldots, \underline{d}_c) \leq \overline{d} = (\overline{d}_0, \ldots, \overline{d}_c)$, every maximal chain has the same number of elements $b = \sum_{i=0}^{c}(\overline{d}_i - \underline{d}_i)$.

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Theorem (Erman, 2009)

The Boij-Söderberg monoid of actual Betti tables in a bounded range is a finitely generated monoid.

The index of actual Betti tables along a ray may be abitrarily large.

2^{nd}: Multiplicity Conjecture [Huneke-Srinivasan, 1998]

Theorem (Eisenbud-S, 08)

Let $A = S/I$ be a CM-algebra. If the resolution

$$0 \leftarrow A \leftarrow S \leftarrow F_1 \leftarrow \ldots \leftarrow F_c \leftarrow 0$$

has nonzero terms $\beta_{i,j} \neq 0$ only in the range $\underline{d}_i \leq j \leq \overline{d}_i$, then

$$\frac{1}{c!}\prod_{i=1}^{c}\underline{d}_i \leq mult(A) \leq \frac{1}{c!}\prod_{i=1}^{c}\overline{d}_i$$

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with equality on either side iff A has a pure resolution.

Proof [Boij-Söderberg]. Write the Betti table of A as a convex combination of pure Betti tables in a chain. $\square$

3^{rd} App: Betti numbers over regular local rings

Let R be a regular local ring and M a finitely generated R-module of projective dimension c. The minimal finite free resolution of M has shape

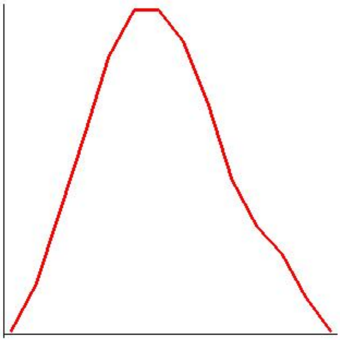
$$0 \leftarrow M \leftarrow R^{\beta_0} \leftarrow R^{\beta_1} \leftarrow \ldots \leftarrow R^{\beta_c} \leftarrow 0$$

Theorem (Erman, 09)

The cone of Betti tables of R-modules of projective dimension $= c$ is the cone over interior of the simplex spanned by

$$(1,1,0,\ldots,0),\ (0,1,1,\ldots 0), \ldots, (0,\ldots,0,1,1) \in \mathbb{Q}^{c+1}$$

Plot of possible Betti numbers over a regular local ring



naive expectation

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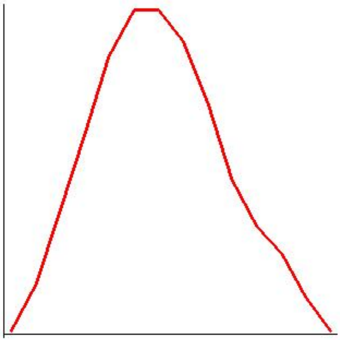

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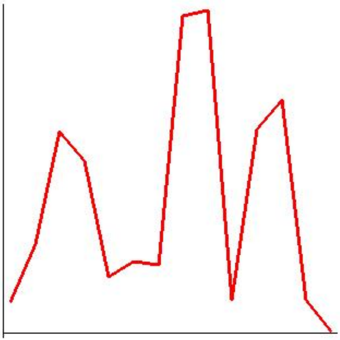

also possible

Facet equation

The simplices of the Boij-Söderberg cone correspond to chains of degree sequence. A facet of a simplex is obtained by dropping a vertex. The following chain corresponds to a typical outer facet of the simplicial fan.

$(1,2,3,4) > (0,2,3,4) > (0,1,3,4) > (0,1,2,4) > (0,1,2,3) > \ldots > (0,1,2,3,6) > (0,1,2,3,5) > \ldots$

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The simplices of the Boij-Söderberg cone correspond to chains of degree sequence. A facet of a simplex is obtained by dropping a vertex. The following chain corresponds to a typical outer facet of the simplicial fan. The equation $\langle \beta, \delta \rangle = 0$ of this facet has coefficients, which can be computed recursively.

(1,2,3,4)>(0,2,3,4)>(0,1,3,4)>(0,1,2,4)>(0,1,2,3)>...>(0,1,2,3,6)>(0,1,2,3,5)>...

$$\delta = (\delta_{i,j}) = \begin{array}{ccccc} \vdots & \vdots & \vdots & \vdots & \vdots \\ 21 & -12 & 5 & 0 & -3 \\ 12 & -5 & 0 & 3 & -4 \\ 5 & 0 & -3 & 4 & -3 \\ 0 & 3 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 12 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

Facet equation

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$$\beta = (\beta_{i,j}) = \begin{matrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ 3 & 8 & 6 & - & - \\ - & - & - & 1 & - \\ - & - & - & - & - \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{matrix} \qquad \delta = (\delta_{i,j}) = \begin{matrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ 21 & -12 & 5 & 0 & -3 \\ 12 & -5 & 0 & 3 & -4 \\ 5 & 0 & -3 & 4 & -3 \\ \mathbf{0} & 3 & -4 & 3 & 0 \\ 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & 5 \\ 0 & 0 & 0 & 0 & 12 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{matrix}$$

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$$\beta = (\beta_{i,j}) = \begin{matrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ 10 & 36 & 45 & 20 & - \\ - & - & - & - & - \\ - & - & - & - & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{matrix} \qquad \delta = (\delta_{i,j}) = \begin{matrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ 21 & -12 & 5 & 0 & -3 \\ 12 & -5 & 0 & 3 & -4 \\ 5 & 0 & -3 & 4 & -3 \\ 0 & 3 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 12 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{matrix}$$

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$$\beta = (\beta_{i,j}) = \begin{matrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ 1 & 3 & 3 & 1 & - \\ - & - & - & - & - \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{matrix} \qquad \delta = (\delta_{i,j}) = \begin{matrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ 21 & -12 & 5 & 0 & -3 \\ 12 & -5 & 0 & 3 & -4 \\ 5 & 0 & -3 & 4 & -3 \\ 0 & 3 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 12 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{matrix}$$

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Our key discovery was that such $\delta_{i,j}$'s are the dimensions of cohomology groups of certain coherent sheaves.

Cohomology Tables

Let $\mathcal{E}$ be a coherent sheaf on $\mathbb{P}^n$, for example a vector bundle. We have the dimensions of the cohomology groups

$$\gamma_{i,j} = h^i(\mathbb{P}^n, \mathcal{E}(j)).$$

We identify the cohomology table $\gamma(\mathcal{E}) = (\gamma_{i,j})$ with an element of

$$\prod_{j \in \mathbb{Z}} \mathbb{Q}^{n+1}.$$

Supernatural Bundles

A vector bundle $\mathcal{E}$ on $\mathbb{P}^n$ has natural cohomology, if for each twist k at most one group $H^i(\mathcal{E}(k)) \neq 0$. It is supernatural, if in addition the Hilbert polynomial

$$\chi(\mathcal{E}(t)) = \frac{rank\ \mathcal{E}}{n!} \prod_{k=1}^{n} (t - z_k)$$

has pairwise distinct integral roots $z = (z_1 > \ldots > z_n)$.

Boij-Söderberg Analog for Vector Bundles

Theorem (E-S, 08)

The cohomology table of an arbitrary vector bundle on $\mathbb{P}^n$ is a unique positive rational linear combination of cohomology tables of supernatural bundles, whose degree sequences form a unique chain.

Here chain refers to the natural partial order

$$z = (z_1 < \ldots < z_n) \geq z' = (z'_1 < \ldots, < z'_n) :\Leftrightarrow z_i \geq z'_i$$

on zero sequences.

The Pairing

The crucial new concept is the following pairing between Betti tables of modules and cohomology tables of coherent sheaves. We define $\langle \beta, \gamma \rangle$ for a Betti table $\beta = (\beta_{i,k})$ and a cohomology table $\gamma = (\gamma_{j,k})$ by

$$\langle \beta, \gamma \rangle = \sum_{i \geq j} (-1)^{i-j} \sum_k \beta_{i,k} \gamma_{j,-k}$$

Note that if $\widetilde{F}_i = \oplus_{k \in \mathbb{Z}} \mathcal{O}(-k)^{\beta_{i,k}}$ then

$$\langle \beta(F), \gamma(\mathcal{E}) \rangle = \sum_{i \geq j} (-1)^{i-j} h^j(\widetilde{F}_i \otimes \mathcal{E})$$

Positivity 1

Theorem (E-S, 2008/09)

For F any free resolution of a finitely generated graded $\mathbb{K}[x_0, \ldots, x_n]$-module M and $\mathcal{E}$ any coherent sheaf on $\mathbb{P}^n$, we have

$$\langle \beta(F), \gamma(\mathcal{E}) \rangle \geq 0.$$

Moreover, if M has finite length and $H^{i+1}(\widetilde{F}_i \otimes \mathcal{E}) = 0$ for all $i \geq 0$, then

$$\langle \beta(F), \gamma(\mathcal{E}) \rangle = 0.$$

Sketch of Proof. We treat the case where $\mathcal{E}$ is a vector bundle. In this case we have an exact complex

$$0 \leftarrow \mathcal{M}_0 \leftarrow \widetilde{F}_0 \otimes \mathcal{E} \rightarrow \widetilde{F}_1 \otimes \mathcal{E} \leftarrow \ldots \leftarrow \widetilde{F}_r \otimes \mathcal{E} \leftarrow 0$$

with $\mathcal{M}_0 = \widetilde{M} \otimes \mathcal{E}$. Breaking it up in short exact sequences

$$\begin{array}{ccccccccc} 0 & \leftarrow & \mathcal{M}_0 & \leftarrow & \widetilde{F}_0 \otimes \mathcal{E} & \leftarrow & \mathcal{M}_1 & \leftarrow & 0 \\ 0 & \leftarrow & \mathcal{M}_1 & \leftarrow & \widetilde{F}_1 \otimes \mathcal{E} & \leftarrow & \mathcal{M}_2 & \leftarrow & 0 \\ 0 & \leftarrow & \mathcal{M}_2 & \leftarrow & \widetilde{F}_2 \otimes \mathcal{E} & \leftarrow & \mathcal{M}_3 & \leftarrow & 0 \\ & & & & \vdots & & & & \end{array}$$

we get the desired functional by taking the alternating sum of the Euler characteristics of initial parts of the corresponding long exact sequences in cohomology:

$$\begin{array}{ccccccc}
 & & H^0(\widetilde{F}_0 \otimes \mathcal{E}) & \leftarrow & H^0(\mathcal{M}_1) & \leftarrow & 0 \\
 & & & & & & \\
 & & H^1(\widetilde{F}_1 \otimes \mathcal{E}) & \leftarrow & H^1(\mathcal{M}_2) & \leftarrow & \\
H^0(\mathcal{M}_1) & \leftarrow & H^0(\widetilde{F}_1 \otimes \mathcal{E}) & \leftarrow & H^0(\mathcal{M}_2) & \leftarrow & 0 \\
 & & & & & & \\
 & & H^2(\widetilde{F}_2 \otimes \mathcal{E}) & \leftarrow & H^2(\mathcal{M}_3) & \leftarrow & \\
H^1(\mathcal{M}_2) & \leftarrow & H^1(\widetilde{F}_2 \otimes \mathcal{E}) & \leftarrow & H^1(\mathcal{M}_3) & \leftarrow & \\
H^0(\mathcal{M}_2) & \leftarrow & H^0(\widetilde{F}_2 \otimes \mathcal{E}) & \leftarrow & H^0(\mathcal{M}_3) & \leftarrow & 0 \\
 & & \vdots & & & &
\end{array}$$

Hence,

$$\langle \beta(F), \gamma(\mathcal{E}) \rangle = \sum_{j=0}^{n} \dim \operatorname{coker} \left(H^j(\mathcal{M}_{j+1}) \to H^j(\widetilde{F}_j \otimes \mathcal{E}) \right) \geq 0.$$

$\square$

Facet equation 2

The facet equation in the example above is obtained from the vector bundle $\mathcal{E}$ on $\mathbb{P}^2 \xhookrightarrow{\iota} \mathbb{P}^3$, that is the kernel of a general map $\mathcal{O}_{\mathbb{P}^2}^5(-1) \to \mathcal{O}_{\mathbb{P}^2}^3$.

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$$
\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \vdots \\
21 & -12 & 5 & 0 & -3 \\
12 & -5 & 0 & 3 & -4 \\
5 & 0 & -3 & 4 & -3 \\
0 & 3 & -4 & 3 & 0 \\
0 & 4 & -3 & 0 & 5 \\
0 & 3 & 0 & -5 & 12 \\
0 & 0 & 5 & -12 & 21 \\
0 & 0 & 12 & -21 & 32 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

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$$
\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \vdots \\
21 & -12 & 5 & 0 & -3 \\
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5 & 0 & -3 & 4 & -3 \\
0 & 3 & -4 & 3 & 0 \\
0 & 4 & -3 & 0 & 5 \\
0 & 3 & 0 & -5 & 12 \\
0 & 0 & 5 & -12 & 21 \\
0 & 0 & 12 & -21 & 32 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

This is not quite the functional we wanted, which had zeros in places of some of the nonzero values.

Positivity 2

We define “truncated” functionals $\langle -, \gamma \rangle_{\tau,\kappa}$ by

$$\begin{aligned}\langle \beta, \gamma \rangle_{\tau,\kappa} \quad = \quad & \textstyle\sum_{k \le \kappa} \beta_{\tau,k}\gamma_{\tau,-k} + \sum_{j<\tau} \sum_k \beta_{j,k}\gamma_{j,-k} \\ & - \textstyle\sum_{k \le \kappa+1} \beta_{\tau+1,k}\gamma_{\tau,-k} - \sum_{j<\tau} \sum_k \beta_{j+1,k}\gamma_{j,-k} \\ & + \textstyle\sum_{i>j+1} (-1)^{i-j} \sum_k \beta_{i,k}\gamma_{j,-k}\end{aligned}$$

Theorem

For F the minimal free resolution of a finitely generated graded $\mathbb{K}[x_0, \ldots, x_n]$-module and $\mathcal{E}$ any coherent sheaf on $\mathbb{P}^n$, we have

$$\langle \beta(F), \gamma(\mathcal{E}) \rangle_{\tau,\kappa} \geq 0.$$

Existence

With these functionals, the proof of both Main Theorems reduce to proof of the existence of supernatural vector bundles and CM-modules for arbitrary zero or degree sequences.

Theorem (Eisenbud-S, 08)

1. *There exists a CM-module with pure resolution for any given degree sequence* $(d_0, \ldots, d_c)$.
2. *There exists supernatural vector bundle for any given zero sequence* $z = (z_1, \ldots, z_n)$.

In case $char(\mathbb{K}) = 0$, Eisenbud-Fløystad-Weyman [2007] gave different construction.

Supernatural sheaves

A coherent (possibly torsion) sheaf $\mathcal{F}$ on $\mathbb{P}^n$ with *supernatural cohomology* has the Hilbert polynomial

$$\chi(\mathcal{F}(d)) = \frac{\deg \mathcal{F}}{s!} \prod_{i=1}^{s} (d - z_i)$$

with distinct integral roots. It will be convenient to put $z_{s+1} = z_{s+2} = \ldots = -\infty$, and to define a partial order on all root sequences by $z \geq z'$ by

$$z_1 \geq z_1', \ldots, z_n \geq z_n'.$$

Let γ^z denote the cohomology table of a supernatural sheaf with root sequence z and degree $= s!$.

Boij-Söderberg analog for coherent sheaf

If Z is an infinite set of zero sequences, $(q_z)_{z\in Z}$ a sequence of numbers, and γ is a cohomology table, we write $\gamma = \sum_{z\in Z} q_z\gamma^z$, to mean that each entry $\sum_{z\in Z} q_z\gamma^z_{i,d}$ converges to $\gamma_{i,d}$.

Theorem (Eisenbud-S, 2009)

Let $\gamma(\mathcal{F})$ be the cohomology table of a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$. There is a unique chain of zero-sequences Z and a unique expression

$$\gamma(\mathcal{F}) = \sum_{z\in Z} q_z\gamma^z,$$

where the q_z are positive numbers.

Example

The ideal sheaf $\mathcal{I}_p$ of a point in $\mathbb{P}^2$ has the following cohomology table $(h^i\mathcal{I}_p(d-i))$

$\cdots$	10	6	3	1							2
$\cdots$	1	1	1	1	1						1
						2	5	9	14	$\cdots$	0
$\cdots$	-4	-3	-2	-1	0	1	2	3	4	$\cdots$	$d\backslash i$

where we dropped zero entries for the better visibility of the shape. Then

$$\gamma(\mathcal{I}_p) = \sum_{k=2}^{\infty} q_{(0,-k)}\gamma^{(0,-k)}$$

with

$$q_{(0,-k)} = \frac{2}{(k-1)k(k+1)}.$$

Idea of proof

Look at the supernatural sheaf with largest zero-sequence with the same upper shape as the given sheaf,

$$
\begin{array}{ccccccccccc|c}
\cdots & 10 & 6 & 3 & 1 & & & & & & & 2 \\
\cdots & 1 & 1 & 1 & 1 & 1 & & & & & & 1 \\
 & & & & & & 2 & 5 & 9 & 14 & \cdots & 0 \\
\hline
\end{array}
$$

Idea of proof

Look at the supernatural sheaf with largest zero-sequence with the same upper shape as the given sheaf,

$$
\begin{array}{cccccccccc|c}
\cdots & 10 & 6 & 3 & 1 & & & & & & 2 \\
\cdots & 1 & 1 & 1 & 1 & 1 & & & & & 1 \\
 & & & & & & 2 & 5 & 9 & 14 & \cdots & 0 \\
\hline
\end{array}
$$

in our case $\gamma^{(0,-2)}$,

$$
\begin{array}{ccccccccccc|c}
\cdots & 24 & 15 & 8 & 3 & & & & & & & 2 \\
 & & & & & 1 & & & & & & 1 \\
 & & & & & & 3 & 8 & 15 & 24 & \cdots & 0 \\
\hline
\end{array}
$$

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Look at the supernatural sheaf with largest zero-sequence with the same upper shape as the given sheaf,

$$\begin{array}{ccccccccccc|c}
\cdots & 10 & 6 & 3 & 1 & & & & & & & 2 \\
\cdots & 1 & 1 & 1 & 1 & 1 & & & & & & 1 \\
 & & & & & & 2 & 5 & 9 & 14 & \cdots & 0 \\
\hline
\end{array}$$

in our case $\gamma^{(0,-2)}$, and subtract as much as possible,

$$\begin{array}{ccccccccccc|c}
\cdots & 24 & 15 & 8 & 3 & & & & & & & 2 \\
 & & & & & 1 & & & & & & 1 \\
 & & & & & & 3 & 8 & 15 & 24 & \cdots & 0 \\
\hline
\end{array}$$

such that corners stay non-negative:

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$$
\begin{array}{ccccccccccc|c}
\cdots & 10 & 6 & 3 & 1 & & & & & & & 2 \\
\cdots & 1 & 1 & 1 & 1 & 1 & & & & & & 1 \\
 & & & & & & 2 & 5 & 9 & 14 & \cdots & 0 \\
\hline
\end{array}
$$

in our case $\gamma^{(0,-2)}$, and subtract as much as possible,

$$
\begin{array}{ccccccccccc|c}
\cdots & 24 & 15 & 8 & 3 & & & & & & & 2 \\
 & & & & & 1 & & & & & & 1 \\
 & & & & & & 3 & 8 & 15 & 24 & \cdots & 0 \\
\hline
\end{array}
$$

such that corners stay non-negative: $\gamma - \frac{1}{3}\gamma^{(0,-2)}$

$$
\begin{array}{ccccccccccc|c}
\cdots & 2 & 1 & \frac{1}{3} & 0 & & & & & & & 2 \\
\cdots & 1 & 1 & 1 & 1 & \frac{2}{3} & & & & & & 1 \\
 & & & & & & 1 & \frac{7}{3} & 4 & 6 & \cdots & 0 \\
\hline
\end{array}
$$

Idea of proof, 2nd step

Now look at

$$
\begin{array}{ccccccccccc|c}
\cdots & 2 & 1 & \frac{1}{3} & & & & & & & & 2 \\
\cdots & 1 & 1 & 1 & 1 & \frac{2}{3} & & & & & & 1 \\
 & & & & & & 1 & \frac{7}{3} & 4 & 6 & \cdots & 0 \\
\hline
\end{array}
$$

subtract a multiple of $\gamma^{(0,-3)}$:

$$
\begin{array}{ccccccccccc|c}
\cdots & 18 & 10 & 4 & & & & & & & & 2 \\
\cdots & & & & 2 & 2 & & & & & & 1 \\
 & & & & & & 4 & 10 & 18 & 28 & \cdots & 0 \\
\hline
\end{array}
$$

Idea of proof, 2nd step

Now look at

$$\begin{array}{cccccccccc|c}
\cdots & 2 & 1 & \frac{1}{3} & & & & & & & 2 \\
\cdots & 1 & 1 & 1 & 1 & \frac{2}{3} & & & & & 1 \\
 & & & & & & 1 & \frac{7}{3} & 4 & 6 & \cdots & 0 \\
\hline
\end{array}$$

subtract a multiple of $\gamma^{(0,-3)}$:

$$\begin{array}{cccccccccccc|c}
\cdots & 18 & 10 & 4 & & & & & & & & 2 \\
\cdots & & & & 2 & 2 & & & & & & 1 \\
 & & & & & & 4 & 10 & 18 & 28 & \cdots & 0 \\
\hline
\end{array}$$

We get $\gamma - \frac{1}{3}\gamma^{(0,-2)} - \frac{1}{12}\gamma^{(0,-3)}$.

$$\begin{array}{cccccccccc|c}
\cdots & \frac{1}{2} & \frac{1}{6} & & & & & & & & 2 \\
\cdots & 1 & 1 & 1 & \frac{5}{6} & \frac{1}{2} & & & & & 1 \\
 & & & & & & \frac{2}{3} & \frac{3}{2} & \frac{5}{2} & \frac{11}{3} & \cdots & 0 \\
\hline
\end{array}$$

Idea of proof, 2nd step

Now look at

$$
\begin{array}{ccccccccccc|c}
\cdots & 2 & 1 & \frac{1}{3} & & & & & & & & 2 \\
\cdots & 1 & 1 & 1 & 1 & \frac{2}{3} & & & & & & 1 \\
 & & & & & & 1 & \frac{7}{3} & 4 & 6 & \cdots & 0 \\
\hline
\end{array}
$$

subtract a multiple of $\gamma^{(0,-3)}$:

$$
\begin{array}{ccccccccccc|c}
\cdots & 18 & 10 & 4 & & & & & & & & 2 \\
\cdots & & & & 2 & 2 & & & & & & 1 \\
 & & & & & & 4 & 10 & 18 & 28 & \cdots & 0 \\
\hline
\end{array}
$$

We get $\gamma - \frac{1}{3}\gamma^{(0,-2)} - \frac{1}{12}\gamma^{(0,-3)}$. Continue ... !

$$
\begin{array}{ccccccccccc|c}
\cdots & \frac{1}{2} & \frac{1}{6} & & & & & & & & & 2 \\
\cdots & 1 & 1 & 1 & \frac{5}{6} & \frac{1}{2} & & & & & & 1 \\
 & & & & & & \frac{2}{3} & \frac{3}{2} & \frac{5}{2} & \frac{11}{3} & \cdots & 0 \\
\hline
\end{array}
$$

Idea of proof

$$\gamma - \frac{1}{3}\gamma^{(0,-2)} - \frac{1}{12}\gamma^{(0,-3)} - \frac{1}{30}\gamma^{(0,-4)}$$

$$\begin{array}{ccccccccccc|c}
\cdots & \frac{1}{10} & & & & & & & & & & 2 \\
\cdots & 1 & 1 & \frac{9}{10} & \frac{7}{10} & \frac{2}{5} & & & & & & 1 \\
 & & & & & & \frac{1}{2} & \frac{11}{10} & \frac{9}{5} & \frac{13}{5} & \cdots & 0 \\
\hline
\end{array}$$

Idea of proof

$$\gamma - \frac{1}{3}\gamma^{(0,-2)} - \frac{1}{12}\gamma^{(0,-3)} - \frac{1}{30}\gamma^{(0,-4)}$$

$\cdots$	$\frac{1}{10}$										2
$\cdots$	1	1	$\frac{9}{10}$	$\frac{7}{10}$	$\frac{2}{5}$						1
						$\frac{1}{2}$	$\frac{11}{10}$	$\frac{9}{5}$	$\frac{13}{5}$	$\cdots$	0

Proposition (Key claim)

All entries of the table stay non-negative through out this process.

Idea of proof

$$\gamma - \frac{1}{3}\gamma^{(0,-2)} - \frac{1}{12}\gamma^{(0,-3)} - \frac{1}{30}\gamma^{(0,-4)} - \dots - \frac{1}{168}\gamma^{(0,-7)}$$

$$\begin{array}{cccccccccccccc|c}
\cdots & \frac{1}{28} & & & & & & & & & & & & & 2 \\
\cdots & 1 & 1 & \frac{27}{28} & \frac{25}{28} & \frac{11}{14} & \frac{9}{14} & \frac{13}{28} & \frac{1}{4} & & & & & & 1 \\
 & & & & & & & & & \frac{2}{7} & \frac{17}{28} & \frac{27}{28} & \frac{19}{14} & \cdots & 0 \\
\hline
\end{array}$$

Proposition (Key claim)

All entries of the table stay non-negative through out this process.

Of course, our inequalities help to prove this.

Outlook

We can look at the Boij-Söderberg cone of cohomology tables of coherent sheaves (vector bundles) on an arbitrary ample polarized (smooth) variety.

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If $(X, \mathcal{O}_X(1))$ is a very ample polarized variety of dimension d then its Boij-Söderberg cone of cohomology tables coincides with the one for $(\mathbb{P}^d, \mathcal{O}(1))$.

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Conjecture

If $(X, \mathcal{O}_X(1))$ *is a very ample polarized variety of dimension d then its Boij-Söderberg cone of cohomology tables coincides with the one for* $(\mathbb{P}^d, \mathcal{O}(1))$.

Necessary and sufficient for this is that X has a sheaf whose cohomology table lies on the same ray as $\mathcal{O}_{\mathbb{P}^d}$.

The conjecture holds for curves and hypersurfaces. It remains true under the formation of Segre-products and transversal intersections.

Outlook

In another direction one could ask for

- graded modules over polynomial rings with different grading, e.g. $\mathbb{Z}^n$- graded,
- arbitrary graded rings,
  or
- cohomology tables of sheaves on varieties with respect to several polarization, for example vector bundles on $\mathbb{P}^1 \times \mathbb{P}^1$.

Little is kown in this area and beautiful things wait to be discovered.